ABSOLUTE CONTINUITY OF THE “EVEN” PERIODIC SCHRÖDINGER OPERATOR WITH NONSMOOTH COEFFICIENTS

M. TIKHOMIROV AND N. FILONOV

§1. INTRODUCTION

The question about the absolute continuity of the spectrum of the Schrödinger operator is actively studied (see [2, 3, 6, 8, 13, 10]). The operator in question can be described in detail as follows. For any set $\Xi \subset \mathbb{R}^d$, let $h_{\Xi}[u, v]$ denote the integral

$$(1) \quad h_{\Xi}[u, v] = \int_{\Xi} (\langle g(i\nabla + A)u, (i\nabla + A)v \rangle + Vu\bar{v}) \, dx, \quad u, v \in H^1(\Xi),$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{C}^d$. In $L_2(\mathbb{R}^d)$, consider the bilinear form $h_{\mathbb{R}^d}$ defined on the Sobolev space $\text{Dom } h_{\mathbb{R}^d} = H^1(\mathbb{R}^d)$. We assume that the coefficients satisfy the following conditions.

Condition 1. *The functions g , A , and V are real-valued and 2π -periodic in all variables.*

Condition 2. a) *The matrix-valued function g (metric) is positive and bounded,*

$$c_0 1 \leq g(x) \leq c_1 1, \quad 0 < c_0 \leq c_1 < \infty.$$

b) *The magnetic potential A and the electric potential V belong to the following classes:*

$$A \in L_{q,\text{loc}}, \quad V \in L_{q/2,\text{loc}}, \quad \text{where } q = d \text{ if } d \geq 3, \quad \text{and } q > 2 \text{ if } d = 2.$$

Under Conditions 1 and 2, $h_{\mathbb{R}^d}$ is a semibounded closed form. The selfadjoint operator H corresponding to $h_{\mathbb{R}^d}$ will be called the *Schrödinger operator*.

At present, the absolute continuity of the spectrum of H is known under the following assumptions. For $d = 2$, it suffices to assume that $\det g \in W_{q,\text{loc}}^1$, $q > 2$ (see [13]). For $d \geq 3$, absolute continuity was proved if $g(x) = a(x)1$, where a is a scalar function, $a \in C^1$, $A \in H_{\text{loc}}^s$, $s > (3d - 2)/2$, and $V \in L_{p,\text{loc}}$, $p = \max\{d/2, d - 2\}$ (see [9]).

In [12], Friedlander considered the situation under an additional condition of “evenness”.

Condition 3. *The operator H is invariant under the inversion $x_1 \mapsto -x_1$. In terms of its coefficients, this means that*

$$\begin{aligned} g_{11}(-x_1, x') &= g_{11}(x_1, x'), & g_{\alpha\beta}(-x_1, x') &= g_{\alpha\beta}(x_1, x'), \\ A_{\alpha}(-x_1, x') &= A_{\alpha}(x_1, x'), & V(-x_1, x') &= V(x_1, x'), \\ g_{1\alpha}(-x_1, x') &= -g_{1\alpha}(x_1, x'), & A_1(-x_1, x') &= -A_1(x_1, x'), \\ \alpha, \beta &= 2, \dots, d. \end{aligned}$$

2000 *Mathematics Subject Classification.* Primary 35Q40.

Key words and phrases. Absolute continuity of the spectrum, periodic Schrödinger operator.

The first author was supported by RFBR (grants nos. 01-01-00218 and 02-01-00798).

Friedlander showed that the absolute continuity of H follows from Conditions 1, 2a), and 3 if g , A , and V are infinitely smooth. Our goal in this paper is to generalize his result to the case of nonsmooth coefficients. For this, we only need to require that H satisfy the *unique continuation property*.

Condition 4. For any $\Xi \subset \mathbb{R}^d$, if $u \in H^1(\Xi)$, $h_\Xi[u, v] = 0$ for all $v \in \dot{H}^1(\Xi)$, and $u|_B = 0$, where $B \subset \Xi$ is a ball, then $u = 0$ in Ξ .

§2. FORMULATION OF THE RESULT

Let $\mathbb{T}^{d-1}$ be a $(d-1)$ -dimensional torus, $\Omega = (-\pi, \pi) \times \mathbb{T}^{d-1}$ a toroidal cylinder, $\Gamma_\pm = \{\pm\pi\} \times \mathbb{T}^{d-1}$ the left and right edges of Ω , $\partial\Omega = \Gamma_- \cup \Gamma_+$. For the set Ω , we omit the index in the form (1), $h[u, v] := h_\Omega[u, v]$.

Theorem 2.1. Suppose the coefficients g , A , V of the form (1) satisfy Conditions 1–4. Then the set of points $\zeta \in \mathbb{C}$ satisfying the following properties is finite:

- 1) $\operatorname{Im} \zeta \neq 0$, $|\zeta| \neq 1$;
- 2) there exists $u_\zeta \in H^1(\Omega) \setminus \{0\}$ such that $u_\zeta|_{\Gamma_+} = \zeta u_\zeta|_{\Gamma_-}$ and

$$h[u_\zeta, v] = 0 \quad \text{for all } v \in H^1(\Omega) \text{ with } \bar{\zeta}v|_{\Gamma_+} = v|_{\Gamma_-}.$$

This theorem implies by standard methods (see [3, 7]) that the number $\lambda = 0$ cannot be an eigenvalue of the operator H .

The unique continuation property for the solutions of elliptic equations is well investigated.

Theorem 2.2 (see [14]). Let $d = 2$, and let $g, V \in L_\infty$, $A = 0$. Then H has the unique continuation property.

Theorem 2.3 (see [5]). Suppose $d \geq 3$, $g \in \operatorname{Lip}$, $A \in L_{q,\operatorname{loc}}$, $q > d$, and $V \in L_{d/2,\operatorname{loc}}$. Then H has the unique continuation property.

Under these assumptions about the coefficients, we shall see that $\lambda = 0$ is not an eigenvalue of H . Since the conditions on the potential V are invariant with respect to the shift by a constant, the point spectrum of H is empty. So, the following statement is true.

Theorem 2.4. Suppose that the coefficients of the operator H satisfy Conditions 1, 2a), and 3, and that

- a) $V \in L_\infty$, $A = 0$ if $d = 2$;
- b) $g \in \operatorname{Lip}$, $A \in L_{q,\operatorname{loc}}$, $q > d$, $V \in L_{d/2,\operatorname{loc}}$ if $d \geq 3$.

Then the spectrum of H corresponding to the form (1) with $\Xi = \mathbb{R}^d$ is absolutely continuous.

Remark 2.1. From the unique continuation theorem (see [5]) it follows that in the two-dimensional case with $g \in \operatorname{Lip}$ and $A \in L_{q,\operatorname{loc}}$, $V \in L_{q/2,\operatorname{loc}}$, $q > 2$, the spectrum of the “even” operator H is absolutely continuous. But this result has already been known without the assumption of “evenness” (see [13]).

Remark 2.2. The assumptions in b) are close to optimal. If $d \geq 3$, then there exists a Schrödinger operator H with $g \in \bigcap_{\alpha < 1} C^\alpha$, $A = 0$, and $V = 0$ that does not possess the unique continuation property and has an eigenvalue of infinite multiplicity (see [11]).

Remark 2.3. In the same way, a similar statement for a waveguide can be proved. Let U be a bounded domain in $\mathbb{R}^k$, $k < d$, with Lipschitz boundary. Suppose that g , A , V satisfy the assumptions of Theorem 2.4 in $\Xi = \mathbb{R}^{d-k} \times U$ (g , A , V are periodic in the first $(d-k)$ variables). Then the spectrum of the operator H corresponding to the form

(1) and defined on $\text{Dom } h_{\Xi} = \dot{H}^1(\Xi)$ (the Dirichlet problem) or $\text{Dom } h_{\Xi} = H^1(\Xi)$ (the Neumann problem) is absolutely continuous.

The remaining part of the paper is devoted to the proof of Theorem 2.1. Mostly, we use the method of [12], where the Dirichlet-to-Neumann operators play a crucial role. We do not use the pseudodifferential technique, working with Dirichlet–Neumann forms (defined on the trace space $H^{1/2}(\Gamma_{\pm})$) instead of operators; this approach permits us to handle nonsmooth coefficients.

In §3, the solvability of various boundary value problems in the cell Ω is discussed. In §4, we prove an auxiliary estimate. In §5 we define the Dirichlet–Neumann forms and investigate their properties. The proof of Theorem 2.1 will be completed in §6.

§3. BOUNDARY-VALUE PROBLEMS IN THE CELL

Consider the kernel of the Dirichlet problem

$$\ker_D := \{w \in \dot{H}^1(\Omega) : h[w, v] = 0, v \in \dot{H}^1(\Omega)\}.$$

It is easily seen that $\dim \ker_D < \infty$.

Let $B : H^{1/2}(\mathbb{T}^{d-1}) \rightarrow H^1(\Omega)$ be a bounded linear extension operator such that $(B\varphi)|_{\Gamma_-} = \varphi$, $(B\varphi)|_{\Gamma_+} = 0$. By definition, we put

$$\text{Dom} := \{\varphi \in H^{1/2}(\mathbb{T}^{d-1}) : h[B\varphi, w] = 0, w \in \ker_D\}.$$

Remark 3.1. 1) The definition of Dom does not depend on the choice of the operator B .
2) The codimension of Dom in $H^{1/2}$ is finite.

Lemma 3.1. *The problem*

$$(2) \quad \begin{cases} h[u, v] = 0 & \text{for all } v \in \dot{H}^1(\Omega), \\ u|_{\Gamma_-} = \varphi, & u|_{\Gamma_+} = 0, \end{cases}$$

has a solution in $H^1(\Omega)$ if and only if $\varphi \in \text{Dom}$. If $\varphi \in \text{Dom}$, then there exists a unique solution u of (2) such that $u \perp \ker_D$.

Proof. Obviously, if (2) is solvable, then $\varphi \in \text{Dom}$. Conversely, let $\varphi \in \text{Dom}$. The Fredholm theory shows that there exists $u_0 \in \dot{H}^1(\Omega)$ such that

$$h[u_0, v] = h[B\varphi, v] \quad \text{for all } v \in \dot{H}^1(\Omega).$$

Then $u = B\varphi - u_0$ is a solution of (2). The final statement of the lemma is obvious. $\square$

Definition 1. By the Poisson operator $\mathcal{P}$ we shall mean the operator $\mathcal{P} : \text{Dom} \rightarrow H^1(\Omega)$ that takes $\varphi \in \text{Dom}$ to the solution u of (2) orthogonal to $\ker_D$.

Remark 3.2. The two-sided estimate

$$(3) \quad c_2 \|\varphi\|_{H^{1/2}} \leq \|\mathcal{P}\varphi\|_{H^1} \leq c_3 \|\varphi\|_{H^{1/2}}$$

is true.

Also, we introduce the inversion operator

$$(Ju)(x_1, x') := u(-x_1, x')$$

and the operators $\tilde{B} := JB$, $\tilde{\mathcal{P}} := J\mathcal{P}$. By Condition 3, we have

$$h[J u, J v] = h[u, v], \quad u, v \in H^1(\Omega),$$

and $J \ker_D = \ker_D$.

Lemma 3.2. *Let $\zeta \neq \pm 1$. The problem*

$$(4) \quad \begin{cases} h[u, v] = 0 & \text{for all } v \in \dot{H}^1(\Omega), \\ u|_{\Gamma_-} = \varphi, & u|_{\Gamma_+} = \zeta\varphi, \end{cases}$$

has a solution in H^1 if and only if $\varphi \in \text{Dom}$. In this case

$$(5) \quad u = \mathcal{P}\varphi + \zeta\tilde{\mathcal{P}}\varphi + w, \quad \text{where } w \in \ker_D.$$

Proof. Let u be a solution of (4). Then $(1 - \zeta^2)^{-1}(u - \zeta Ju)$ is a solution of (2), whence $\varphi \in \text{Dom}$. Conversely, let $\varphi \in \text{Dom}$. Then the function $u = \mathcal{P}\varphi + \zeta\tilde{\mathcal{P}}\varphi$ is a solution of (4). The final statement of the lemma is obvious. $\square$

§4. THE SPACE $R(\mathcal{P})$

The image of the Poisson operator can be described explicitly:

$$R(\mathcal{P}) = \{u \in H^1(\Omega) : u|_{\Gamma_+} = 0, h[u, v] = 0 \forall v \in \dot{H}^1(\Omega), u \perp \ker_D\}.$$

Thus, $R(\mathcal{P})$ is a closed subspace of $H^1(\Omega)$, and $R(\mathcal{P}) \cap \dot{H}^1(\Omega) = \{0\}$.

Lemma 4.1. *Let W be a closed subspace of $H^1(\Omega)$ with $W \cap \dot{H}^1(\Omega) = \{0\}$. Let $\psi \in L_r(\Omega)$, where $r > 1$ if $d = 2$ and $r = d/2$ if $d \geq 3$. Then for any $\varepsilon > 0$ there exists $c(\varepsilon)$ such that*

$$\int_{\Omega} |\psi u^2| dx \leq \varepsilon \int_{\Omega} |\nabla u|^2 dx + c(\varepsilon) \int_{\partial\Omega} |u|^2 dS, \quad u \in W.$$

Remark 4.1. Of course, this statement is valid for any bounded domain with Lipschitz boundary.

Proof. Suppose that for some $\varepsilon > 0$ there exists a sequence $\{u_n\} \subset W$ such that

$$\int_{\Omega} |\psi u_n^2| dx = 1 \geq \varepsilon \|\nabla u_n\|_{L_2(\Omega)}^2 + n \|u_n|_{\partial\Omega}\|_{L_2(\partial\Omega)}^2.$$

Without loss of generality we may assume that $u_n \xrightarrow{H^1} u$. It is well known that the embedding

$$H^1(\Omega) \subset L_2(\Omega, |\psi| dx)$$

is compact (for $d = 2$ this is a consequence of the fact that the embedding $H^1 \subset L_{2r/(r-1)}$ is compact; for $d \geq 3$ we refer the reader to [1]). Thus, $\int_{\Omega} |\psi u^2| dx = 1$. On the other hand, since $\|u_n|_{\partial\Omega}\|_{L_2} \rightarrow 0$ and the embedding $H^1(\Omega) \subset L_2(\partial\Omega)$ is also compact, we have $u|_{\partial\Omega} = 0$. Therefore,

$$u \in \dot{H}^1(\Omega) \cap W \implies u = 0,$$

a contradiction. $\square$

Corollary 1. *Suppose g , A , and V satisfy Condition 2. Then for any $\varepsilon > 0$ there exists $C(\varepsilon)$ such that*

$$\begin{aligned} & \int_{\Omega} (\langle gA, A \rangle + |V|) |u|^2 dx + 2 \int_{\Omega} |\langle gAu, \nabla u \rangle| dx \\ & \leq \varepsilon \int_{\Omega} |\nabla u|^2 dx + C(\varepsilon) \int_{\Gamma_-} |u|^2 dS, \quad u \in R(\mathcal{P}). \end{aligned}$$

§5. DIRICHLET–NEUMANN FORMS

In $L_2(\mathbb{T}^{d-1})$, consider the subspace $\mathcal{H} := \overline{\text{Dom}}^{L_2}$ (if $\ker_D = \{0\}$, then $\mathcal{H}$ coincides with L_2).

Definition 2. The forms n_0, n_1 in the Hilbert space $\mathcal{H}$ that are defined on the domain

$$\text{Dom } n_0 = \text{Dom } n_1 = \text{Dom}$$

by the formulas

$$n_0[\varphi, \psi] = h[\mathcal{P}\varphi, \mathcal{P}\psi], \quad n_1[\varphi, \psi] = h[\mathcal{P}\varphi, \tilde{\mathcal{P}}\psi],$$

will be called the Dirichlet–Neumann forms.

It is easily seen that n_0 is Hermitian. The form n_1 is also Hermitian, which follows from Condition 3:

$$n_1[\varphi, \psi] = h[\mathcal{P}\varphi, \tilde{\mathcal{P}}\psi] = h[\tilde{\mathcal{P}}\varphi, \mathcal{P}\psi] = \overline{n_1[\psi, \varphi]}.$$

Lemma 5.1. *The form n_0 is semibounded and closed.*

Proof. Let $\varphi \in \text{Dom}$, $\mathcal{P}\varphi = u$. By the corollary to Lemma 4.1, we get

$$(6) \quad \begin{aligned} n_0[\varphi, \varphi] &= \int_{\Omega} (\langle g(i\nabla + A)u, (i\nabla + A)u \rangle + V|u|^2) dx \\ &\geq \frac{c_0}{2} \int_{\Omega} |\nabla u|^2 dx - c_4 \int_{\Gamma_-} |\varphi|^2 dS. \end{aligned}$$

Thus, the form n_0 is semibounded. Combining (6) and (3), we obtain

$$c_5 \|\varphi\|_{H^{1/2}}^2 \leq n_0[\varphi, \varphi] + c_4 \|\varphi\|_{L_2}^2 \leq c_6 \|\varphi\|_{H^{1/2}}^2,$$

so that n_0 is closed. $\square$

To prove that $\ker n_1 = \{0\}$, we need the following facts.

Lemma 5.2 (see, e.g., [4]). *Let $l, l_1, \dots, l_n$ be bounded linear functionals on a Hilbert space. If $\bigcap_{k=1}^n \ker l_k \subset \ker l$, then l is a linear combination of $l_1, \dots, l_n$, i.e., $l = \sum_{k=1}^n \alpha_k l_k$.*

Lemma 5.3. *Suppose the coefficients of the form h satisfy Condition 4. If the function $w \in H^1(\Omega)$ is such that*

- 1) $w|_{\Gamma_+} = 0$,
- 2) $h[v, w] = 0$ for any $v \in H^1(\Omega)$ with $v|_{\Gamma_-} = 0$,

then $w = 0$.

Proof. Consider the cylinder $\Xi = (-\pi, 4) \times \mathbb{T}^{d-1}$ and the function $\hat{w}(x) \in H^1(\Xi)$ such that $\hat{w}(x) = w(x)$ if $x_1 \leq \pi$ and $\hat{w}(x) = 0$ if $x_1 > \pi$. Then $\hat{w} \in H^1(\Xi)$ and

$$h_{\Xi}[\hat{w}, v] = h[w, v|_{\Omega}] = 0, \quad v \in \dot{H}^1(\Xi).$$

Now, the identity $w = 0$ follows from Condition 4. $\square$

Theorem 5.1. *The kernel of the form n_1 is trivial, $\ker n_1 = \{0\}$.*

Proof. Let $\varphi \in \ker n_1$, $\mathcal{P}\varphi = u$. Then

$$h[v, u] = 0 \quad \text{for all } v \in R(\tilde{\mathcal{P}}).$$

Let $\{u_k\}$ be a basis in $\ker_D$. Consider the linear functionals

$$l(\psi) = h[\tilde{B}\psi, u], \quad l_k(\psi) = h[\tilde{B}\psi, u_k]$$

on $H^{1/2}(\mathbb{T}^{d-1})$. By definition, $\bigcap_k \ker l_k = \text{Dom}$. If $\psi \in \text{Dom}$, then $h[\tilde{B}\psi, u] = h[\tilde{\mathcal{P}}\psi, u] = 0$, whence $\text{Dom} \subset \ker l$. By Lemma 5.2, there exists $u_0 \in \ker_D$ such that

$$h[\tilde{B}\psi, u - u_0] = 0 \quad \text{for all } \psi \in H^{1/2}(\mathbb{T}^{d-1}).$$

If $w = u - u_0$, then $h[v, w] = 0$, $v \in \dot{H}^1(\Omega)$. Consequently,

$$h[v, w] = 0 \quad \text{for all } v \in H^1(\Omega) : v|_{\Gamma_-} = 0.$$

Lemma 5.2 yields $w = 0$, so that $u = u_0 \in \dot{H}^1(\Omega) \iff u = 0$. $\square$

§6. PROOF OF THEOREM 2.1

We shall use the following abstract statement.

Lemma 6.1. *Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{F}$ be a dense linear set in $\mathcal{H}$, and let n_0, n_1 be Hermitian bilinear forms such that $\text{Dom } n_0 = \text{Dom } n_1 = \mathcal{F}$. Assume that n_0 is semibounded from below and closed, $\ker n_1 = \{0\}$. If the embedding of $\mathcal{F}$ with norm $\|f\|_{\mathcal{F}}^2 = n_0[f, f] + \gamma\|f\|_{\mathcal{H}}^2$ in $\mathcal{H}$ is compact, then the set of nonreal z such that $\ker(n_0 + zn_1) \neq \{0\}$ is finite.*

Proof. Without loss of generality we can assume that $\text{Im } z > 0$. Consider the set

$$M = \{z \in \mathbb{C} : \text{Im } z > 0, \ker(n_0 + zn_1) \neq \{0\}\},$$

and let

$$\mathcal{M} = \sum_{z \in M} \ker(n_0 + zn_1)$$

be the linear span of such kernels. For $z_1, z_2 \in M$ (including the case where $z_1 = z_2$) and $\varphi_k \in \ker(n_0 + z_k n_1)$, $k = 1, 2$, we have

$$n_0[\varphi_1, \varphi_2] + z_1 n_1[\varphi_1, \varphi_2] = 0, \quad n_0[\varphi_1, \varphi_2] + \bar{z}_2 n_1[\varphi_1, \varphi_2] = 0.$$

Therefore, $n_0[\varphi_1, \varphi_2] = n_1[\varphi_1, \varphi_2] = 0$, whence $n_0[\varphi, \psi] = n_1[\varphi, \psi] = 0$, $\varphi, \psi \in \mathcal{M}$. Combining this and the compactness of the embedding $\mathcal{F} \subset \mathcal{H}$, we see that $\dim \mathcal{M} < \infty$.

Next, there exist operators $N_0, N_1 : \mathcal{M} \rightarrow \mathcal{F}$ such that

$$(N_k \varphi, \psi)_{\mathcal{F}} = n_k[\varphi, \psi], \quad \varphi \in \mathcal{M}, \quad \psi \in \mathcal{F}.$$

Obviously, $\ker N_1 = \{0\}$. If $\varphi \in \ker(n_0 + zn_1)$, then $N_0 \varphi = -z N_1 \varphi$, whence $R(N_0) = R(N_1)$, and we can define the operator $N_1^{-1} N_0$ acting in $\mathcal{M}$. Then the cardinality of M does not exceed the dimension of $\mathcal{M}$, because $M \subset \sigma(-N_1^{-1} N_0)$. $\square$

Proof of Theorem 2.1. Suppose that $\zeta \in \mathbb{C}$, $\text{Im } \zeta \neq 0$, $|\zeta| \neq 1$ and there exists $u_\zeta \in H^1(\Omega)$, $u_\zeta \neq 0$, such that $u_\zeta|_{\Gamma_+} = \zeta u_\zeta|_{\Gamma_-}$ and

$$h[u_\zeta, v] = 0 \quad \text{for all } v \in H^1(\Omega) \text{ with } \bar{\zeta} v|_{\Gamma_+} = v|_{\Gamma_-}.$$

By Lemma 3.2 (formula (5)), we have $u_\zeta = \mathcal{P}\varphi + \zeta \tilde{\mathcal{P}}\varphi + w$, where $w \in \ker_D$, $\varphi = u_\zeta|_{\Gamma_-}$.

First, suppose $\varphi = 0$. Then $u_\zeta \in \ker_D$. If $f \in H^1(\Omega)$ and $f|_{\Gamma_-} = 0$, then

$$h[\zeta J u_\zeta + u_\zeta, f] = h[u_\zeta, \bar{\zeta} J f + f] = 0.$$

By Lemma 5.3, $J u_\zeta = -\zeta^{-1} u_\zeta$. Since the spectrum of J consists of two numbers 1 and -1 , we have $u_\zeta = 0$, a contradiction.

Now, let $\varphi \neq 0$. Then, for any $\psi \in \text{Dom}$, choosing $v = \tilde{\mathcal{P}}\psi + \bar{\zeta} \mathcal{P}\psi$, we get

$$\begin{aligned} 0 &= h[u_\zeta, v] = h[\mathcal{P}\varphi + \zeta \tilde{\mathcal{P}}\varphi + w, \tilde{\mathcal{P}}\psi + \bar{\zeta} \mathcal{P}\psi] = (1 + \zeta^2) n_1[\varphi, \psi] + 2\zeta n_0[\varphi, \psi] \\ &\implies n_0[\varphi, \psi] + z n_1[\varphi, \psi] = 0, \end{aligned}$$

where $z = (1 + \zeta^2)/(2\zeta)$. Recall that $z \notin \mathbb{R}$ by assumption. The compactness of the embedding $\mathcal{F} = \text{Dom} \subset \mathcal{H}$ follows from the compactness of the embedding $H^{1/2}(\mathbb{T}^{d-1}) \subset L_2(\mathbb{T}^{d-1})$, which allows us to apply Lemma 6.1. So, the number of such ζ is finite. $\square$

REFERENCES

- [1] M. Sh. Birman and M. Z. Solomyak, *Schrödinger operator. Estimates for the number of bound states as function-theoretical problem*, Talks Given at XIV School on Operators in Function Spaces. Pt. 1, Novgorod. Ped. Inst., Novgorod, 1990, 99 pp.; English transl., Spectral Theory of Operators (Fourteenth School on Operators in Functional Spaces, Novgorod, 1989), Amer. Math. Soc. Transl. Ser. 2, vol. 150, Amer. Math. Soc., Providence, RI, 1992, pp. 1–54. MR1157648 (93j:35120)
- [2] M. Sh. Birman and T. A. Suslina, *Absolute continuity of the two-dimensional periodic magnetic Hamiltonian with discontinuous vector-valued potential*, Algebra i Analiz **10** (1998), no. 4, 1–36; English transl., St. Petersburg Math. J. **10** (1999), no. 4, 579–601. MR1654063 (99k:81060)
- [3] ———, *Periodic magnetic Hamiltonian with variable metric. The problem of absolute continuity*, Algebra i Analiz **11** (1999), no. 2, 1–40; English transl., St. Petersburg Math. J. **11** (2000), no. 2, 203–232. MR1702587 (2000i:35026)
- [4] A. N. Kolmogorov and S. V. Fomin, *Elements of the theory of functions and functional analysis*, 6th ed., “Nauka”, Moscow, 1989; English transl. of 1st ed., Vol. 1, Graylock Press, Rochester, NY, 1957; Vol. 2, Graylock Press, Albany, NY, 1961. MR1025126 (90k:46001); MR0085462 (19:44d); MR0118796 (22:9566a)
- [5] H. Koch and D. Tataru, *Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients*, Comm. Pure Appl. Math. **54** (2001), 339–360. MR1809741 (2001m:35075)
- [6] A. Morame, *Absence of singular spectrum for a perturbation of a two-dimensional Laplace–Beltrami operator with periodic electro-magnetic potential*, J. Phys. A **31** (1998), 7593–7601. MR1652918 (99i:81039)
- [7] M. Reed and B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Acad. Press, New York–London, 1978. MR0493421 (58:12429c)
- [8] A. Sobolev, *Absolute continuity of the periodic magnetic Schrödinger operator*, Invent. Math. **137** (1999), 85–112. MR1703339 (2000g:35028)
- [9] T. A. Suslina and R. G. Shterenberg, *Absolute continuity of the spectrum of the Schrödinger operator with the potential concentrated on a periodic system of hypersurfaces*, Algebra i Analiz **13** (2001), no. 5, 197–240; English transl., St. Petersburg Math. J. **13** (2002), no. 5, 859–891. MR1882869 (2002m:35172)
- [10] L. Thomas, *Time dependent approach to scattering from impurities in a crystal*, Comm. Math. Phys. **33** (1973), 335–343. MR0334766 (48:13084)
- [11] N. Filonov, *Second-order elliptic equation of divergence form having a compactly supported solution*, Probl. Mat. Anal., vyp. 22, S.-Peterburg. Gos. Univ., St. Petersburg, 2001, pp. 246–257; English transl., J. Math. Sci. **106** (2001), no. 3, 3078–3086. MR1906035 (2003h:35047)
- [12] L. Friedlander, *On the spectrum of a class of second order periodic elliptic differential operators*, Comm. Math. Phys. **229** (2002), no. 1, 49–55. MR1917673 (2003k:35179)
- [13] R. Shterenberg, *Absolute continuity of the spectrum of a two-dimensional periodic magnetic Schrödinger operator with positive electric potential*, Trudy S.-Peterburg. Mat. Obshch. **9** (2001), 199–233. (Russian)
- [14] F. Schulz, *On the unique continuation property of elliptic divergence form equations in the plane*, Math. Z. **228** (1998), 201–206. MR1630571 (99e:35035)

DEPARTMENT OF PHYSICS, ST. PETERSBURG STATE UNIVERSITY, ULYANOVSKAYA 1, PETRODVORETS,
ST. PETERSBURG 198504, RUSSIA
E-mail address: misha@mt5788.spb.edu

DEPARTMENT OF PHYSICS, ST. PETERSBURG STATE UNIVERSITY, ULYANOVSKAYA 1, PETRODVORETS,
ST. PETERSBURG 198504, RUSSIA
E-mail address: filonov@mph.phys.spbu.ru

Received 1/SEP/2003

Translated by M. TIKHOMIROV