ON LOCALLY $GQ(s, t)$ GRAPHS WITH STRONGLY REGULAR μ -SUBGRAPHS

V. I. KAZARINA AND A. A. MAKHNEV

ABSTRACT. The connected locally $GQ(s, t)$ graphs are studied in which every μ -subgraph is a known strongly regular graph (i.e., $K_{m,m}$ for a positive integer m , the Moore graph with parameters $(k^2 + 1, k, 0, 1)$, $k = 2, 3$, or 7 , the Clebsch graph, the Gewirtz graph, the Higman–Sims graph, or the second neighborhood (with parameters $(77, 16, 0, 4)$) of a vertex in the Higman–Sims graph). It is proved that if Γ is a strongly regular locally $GQ(s, t)$ graph in which every μ -subgraph is isomorphic to a known strongly regular graph Δ , then one of the following statements is true: (1) $\Delta = K_{t+1, t+1}$ and either $s = 1$ and $\Gamma = K_{3 \times (t+1)}$, or $s = 4, t = 1$, and Γ is a quotient of the Johnson graph $\overline{J}(10, 5)$, or $s = t = 1, 2, 3, 8, 13$; (2) Δ is a Petersen graph and Γ is a unique locally $GQ(2, 2)$ graph with parameters $(28, 15, 6, 10)$; (3) Δ is the Gewirtz graph and Γ is the McLaughlin graph.

INTRODUCTION

We consider nonoriented graphs with no loops and no multiple edges. For a vertex a of a graph Γ , we denote by $\Gamma_i(a)$ the i -neighborhood of a , i.e., the subgraph induced by Γ on the set of all vertices lying at a distance of i from a . We put $\Gamma(a) = \Gamma_1(a)$ and $a^\perp = \{a\} \cup \Gamma(a)$. If the graph Γ is fixed, then we write $[a]$ instead of $\Gamma(a)$. Unless otherwise specified, by a subgraph we mean an induced subgraph. Largely, we use the same notation as in [1].

Let $\mathcal{F}$ be a class of graphs. We say that a graph Γ is a *locally $\mathcal{F}$ graph* if $[a]$ lies in $\mathcal{F}$ for each vertex a of Γ . If the class $\mathcal{F}$ consists of graphs isomorphic to a graph Δ , then we say that Γ is a *locally Δ graph*.

Let Γ be a graph, and let $a, b \in \Gamma$. The number of vertices in $[a] \cap [b]$ is denoted by $\mu(a, b)$ if the distance between a and b in Γ is 2 and by $\lambda(a, b)$ if a and b are adjacent in Γ . In the sequel, the induced subgraph $[a] \cap [b]$ is called a μ -subgraph (respectively, a λ -subgraph).

By the *degree of a vertex* we mean the number of vertices in the neighborhood of that vertex. A graph Γ is said to be *regular* of degree k if the degree of every vertex a in Γ is equal to k . A graph Γ is *edge regular* with parameters (v, k, λ) if it has v vertices, is regular of degree k , and each edge of it lies in λ triangles. A graph Γ is *completely regular* with parameters (v, k, λ, μ) if it is edge regular with the corresponding parameters and $[a] \cap [b]$ contains μ vertices for any two vertices a and b that are 2 apart in Γ . A completely regular graph is said to be *strongly regular* if it has diameter 2. By $K_{m_1, \dots, m_n}$ we denote a complete multipartite graph $\{M_1, \dots, M_n\}$ with parts M_i of order m_i . If $m_1 = \dots = m_n = m$, then this graph is denoted by $K_{n \times m}$. For a subgraph Δ of Γ , we

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denote by $X_i(\Delta)$ the set of all vertices in $\Gamma - \Delta$ that are adjacent to exactly i vertices of Δ .

A *geometry* G of rank 2 is an incidence system with a set P of points and set $\mathcal{B}$ of blocks, and without multiple blocks. Here, each block can be identified with the set of points incident to it, and incidence becomes the usual inclusion. Two points in P are *collinear* if they lie in the same block. The *point graph* of a geometry G is the graph on the set P in which two points are adjacent if they are distinct and collinear. The *block graph* is defined similarly. We say that a geometry is *connected* if its point graph is connected.

For $a \in P$, we define the *residue* G_a as the geometry with the set of points P_a collinear to a and the set of blocks $\mathcal{B}_a = \{B - \{a\} \mid a \in B \in \mathcal{B}\}$. Suppose $a \in P$ and $B \in \mathcal{B}$. If $a \notin B$ (respectively, $a \in B$), then the pair (a, B) is called an *antiflag* (respectively, a *flag*).

If any two blocks in $\mathcal{B}$ intersect at one point at most, then the set of blocks is called the *set of lines* and is denoted by $\mathcal{L}$, and the geometry $(P, \mathcal{L})$ is called a *partial space of lines*. A partial space of lines has order (s, t) if each line contains $s + 1$ points and each point lies on $t + 1$ lines.

A partial space of lines of order (s, t) is called an α -*partial geometry* if for each antiflag (a, L) there exist exactly α lines passing through a and intersecting L (notation: $pG_\alpha(s, t)$). For $\alpha = 1$, such a geometry is called a *generalized quadrangle* and is denoted by $GQ(s, t)$. In the point graph for $GQ(s, t)$, a coclique consisting of $st + 1$ points is called an *ovoid*. By a *spread* in $GQ(s, t)$ we mean a set of $st + 1$ lines forming a partition of the set of points.

The point graph of the geometry $pG_\alpha(s, t)$ is strongly regular, with $v = (s + 1)(1 + st/\alpha)$, $k = s(t + 1)$, $\lambda = s - 1 + t(\alpha - 1)$, and $\mu = \alpha(t + 1)$. Any strongly regular graph with these parameters is called a *pseudo-geometric graph* for $pG_\alpha(s, t)$.

A geometry G is called an *extension* of α -partial geometries if it is connected and the residue at each point coincides with $pG_\alpha(s, t)$ for appropriate (s, t) (notation: EpG_α). By connectedness, the order (s, t) of a geometry does not depend on the choice of a residue, and such an extension is denoted by $EpG_\alpha(s, t)$. The geometry EpG_α is said to be *triangular* if arbitrary three pairwise collinear points lie in the same block (which is necessarily unique). The study of triangular extensions of geometries G of class pG_α is based on the study of locally connected graphs $\Gamma(G_a)$ (and is equivalent to such a study if $\alpha = 1$).

A subset Λ of a generalized triangle is called a *hyperoval* if each line intersects Λ at 0 or 2 points. In other words, a hyperoval in $GQ(s, t)$ is a regular subgraph of degree $t + 1$, without triangles, and having an even number of vertices. It is well known [2] that the μ -subgraphs in locally $GQ(s, t)$ graphs are hyperovals.

If a regular graph of degree k and diameter d has v vertices, then we have the following inequality proved by Moore:

$$v \leq 1 + k + k(k - 1) + \dots + k(k - 1)^{d-1}.$$

A graph for which this inequality becomes equality is called a *Moore graph*. A simple example of a Moore graph is a $(2d + 1)$ -gon. Damerell [3] proved that a Moore graph of degree $k \geq 3$ has diameter 2. In this case, we have $v = k^2 + 1$, the graph is strongly regular with $\lambda = 0$ and $\mu = 1$, and the degree k can be equal to 3 (the Petersen graph), to 7 (the Hoffman–Singleton graph), or to 57. Actually, it is not known whether a Moore graph of degree $k = 57$ really exists.

The Clebsch graph is defined on the set of vectors of the four-dimensional linear space V over the field of two elements; in this case, two vectors are adjacent if the Hamming distance between them is 1 or 4. This is the only strongly regular graph with parameters

(16, 5, 0, 2). The Gewirtz, Higman–Sims, and McLaughlin graphs are the graphs of rank 3 of the group $L_3(4)$, the Higman–Sims group, and the McLaughlin group on 56, 100, and 275 vertices, respectively.

The Johnson graph $J(n, m)$ is defined on the set of m -element subsets of a given n -element set X , and two m -element subsets a and b are adjacent if $|a \cap b| = m - 1$. A quotient Johnson graph $J^\sigma(2m, m)$ is the factor-graph the vertices of which are pairs $(a^\sigma, X - a)$, where σ is a permutation on X of order at most 2 that has at least 8 fixed points. If $\sigma = 1$, then the quotient is said to be *standard* and is denoted by $\overline{J}(2m, m)$.

Let Γ be a connected locally $GQ(s, t)$ graph, and let u and w be its vertices with $d(u, w) = 2$. It is well known that $\mu(u, w)$ is even and $\max\{2(t+1), (s+1)(t+2-s)\} \leq \mu(u, w) \leq 2(st+1)$.

In the present paper, we study the connected locally $GQ(s, t)$ graphs in which the μ -subgraphs are known strongly regular graphs. Let Δ be a known strongly regular graph without triangles. Then one of the following possibilities is realized:

- (1) $\Delta = K_{m,m}$ for some positive integer m ;
- (2) Δ is a Moore graph with parameters $(k^2 + 1, k, 0, 1)$, $k = 2, 3$, or 7 ;
- (3) Δ is the Clebsch graph with parameters $(16, 5, 0, 2)$ or the Gewirtz graph with parameters $(56, 10, 0, 2)$;
- (4) Δ is the Higman graph with parameters $(100, 22, 0, 6)$;
- (5) Δ is the second neighborhood (with parameters $(77, 16, 0, 4)$) of a vertex in a Higman–Sims graph.

For each collection of parameters among those listed above, there exists a unique strongly regular graph with these parameters.

Theorem. *Let Γ be a completely regular locally $GQ(s, t)$ graph in which every μ -subgraph is isomorphic to a known strongly regular graph Δ . Then one of the following statements is valid:*

- (1) $\Delta = K_{t+1, t+1}$ and $t+1$ divides $s^2(s^2 - 1)$;
- (2) Δ is a Petersen graph, and Γ is a unique locally $GQ(2, 2)$ graph with parameters $(28, 15, 6, 10)$;
- (3) Δ is a Hoffman–Singleton graph, $t = 6$, and $s = 9, 14, 15, 24, 29$, or 30 ;
- (4) Δ is a Clebsch graph, $t = 4$, and $s = 2, 4, 6, 8, 11, 12$, or 16 ;
- (5) Δ is a Gewirtz graph, $t = 9$, and $s = 3, 7, 27, 31$, or 63 ;
- (6) Δ is a graph with parameters $(77, 16, 0, 4)$, $t = 15$, and $s = 21, 41, 55, 153$, or 195 ;
- (7) Δ is a Higman–Sims graph, $t = 21$, and $s = 19, 24, 34, 35, 39, 45, 49, 69, 84, 89, 99, 105, 115, 119, 144, 159, 175, 189, 199, 210, 214, 259, 294, 309, 339, 364, 375, 399, 419$, or 420 .

For $t = 1$, the Johnson graphs $J(2m, m)$ and their quotients (for $m > 4$) are locally $GQ(m-1, 1)$ graphs in which every μ -subgraph is isomorphic to a quadrangle. For $t > 1$, it is not known whether there exists a graph as in the conclusion of the theorem and having diameter greater than 2.

Corollary. *Let Γ be a strongly regular locally $GQ(s, t)$ graph in which every μ -subgraph is isomorphic to a known strongly regular graph Δ . Then one of the following statements is valid:*

- (1) $\Delta = K_{t+1, t+1}$ and either $s = 1$ and $\Gamma = K_{3 \times (t+1)}$, or $s = 4, t = 1$, and Γ is a quotient of the Johnson graph $\overline{J}(10, 5)$, or else $s = t = 1, 2, 3, 8$, or 13 ;
- (2) Δ is a Petersen graph and Γ is a unique locally $GQ(2, 2)$ graph with parameters $(28, 15, 6, 10)$;
- (3) Δ is a Gewirtz graph and Γ is a McLaughlin graph.

The existence and uniqueness are known for strongly regular locally $GQ(t, t)$ graphs with μ -subgraphs isomorphic to $K_{t+1, t+1}$ for $t = 1$ (Γ is a quotient of the Johnson graph

$\overline{J}(10, 5)$), $t = 2$ (Γ is a unique locally $GQ(2, 2)$ graph with parameters $(28, 15, 6, 10)$), and $t = 3$ (Γ is a graph of rank 3 with parameters $(176, 40, 12, 8)$ for the group $U_4(2)$).

We note that the completely regular locally $GQ(4, 2)$ graphs were classified in [6]. In the paper [7], the connected locally $GQ(3, t)$ graphs are described.

In §1 we present some auxiliary results. In §2, we consider the cases where Δ is one of the Moore graphs. In §3 we assume that Δ is a Clebsch graph or a Gewirtz graph. In §4, we consider the case where Δ is a graph with parameters $(77, 16, 0, 4)$ or a Higman–Sims graph.

§1. AUXILIARY RESULTS

Lemma 1.1. *Let Γ be a locally $GQ(s, t)$ graph. Then the maximal cliques in Γ consist of $s + 2$ points (such cliques will be called blocks), each point lies in $(t + 1)(st + 1)$ blocks, any two points lie in $t + 1$ common blocks, and any two blocks intersect at two points at most.*

Proof. All the statements follow from the definition of the extension and the properties of GQ [2]. $\square$

Lemma 1.2. *Let Λ be a hyperoval of a generalized quadrangle $GQ(s, t)$, and let $\mu = |\Lambda|$. Then μ is even and $\mu_* \leq \mu \leq \mu^*$, where $\mu_* = \max\{2(t + 1), (s + 1)(t + 2 - s)\}$ and $\mu^* = 2(st + 1)$. If $\mu = (s + 1)(t + 2 - s)$ (respectively, $\mu = \mu^*$), then, for each point $a \notin \Lambda$, exactly $(t + 2 - s)/2$ lines (respectively, exactly $t + 1$ lines) containing the point a intersect Λ at two points.*

Proof. The estimates for μ and the fact that μ is even follow from Lemmas 3.9 and 3.11 in [2]. If $\mu = (s + 1)(t + 2 - s)$, then from the proof of Lemma 3.11 in [2] it follows that, for $a \notin \Lambda$, the number of lines in $a^\perp$ that do not intersect Λ is equal to $(s + t)/2$.

If $\mu = \mu^*$, then, by Lemma 3.9 (b) in [2], each line containing a intersects Λ (obviously, at two points). $\square$

Lemma 1.3. *Let Γ be a strongly regular graph with parameters (v, k, λ, μ) . Then:*

- (1) $k(k - \lambda - 1) = \mu(v - k - 1)$ (rectangular relation);
- (2) either Γ has parameters $(4\mu + 1, 2\mu, \mu - 1, \mu)$ (the half case), or the secondary eigenvalues $n - m$ and $-m$ of the graph Γ are integers, where $n^2 = (\lambda - \mu)^2 + 4(k - \mu)$, $n - \lambda + \mu = 2m$, and the multiplicity of the eigenvalue $n - m$ is equal to $\frac{k(m-1)(k+m)}{\mu n}$;
- (3) if m is an integer greater than 1, then $m - 1$ divides $k - \lambda - 1$ and

$$\mu = \lambda + 2 + (m - 1) - \frac{k - \lambda - 1}{m - 1}, \quad n = m - 1 + \frac{k - \lambda - 1}{m - 1}.$$

Proof. Statements (1) and (2) are well known (see, e.g., Chapter 1 in [1]). Statement (3) is Lemma 3.1 in [4]. $\square$

The requirement that $\frac{k(m-1)(k+m)}{\mu n}$ be an integer is called the *integer condition*.

Lemma 1.4. *Let Γ be a point graph of a generalized quadrangle $GQ(s, t)$. Then Γ is a strongly regular graph with parameters $v = (s + 1)(st + 1)$, $k = s(t + 1)$, $\lambda = s - 1$, $\mu = t + 1$, and $s + t$ divides $st(s + 1)(t + 1)$.*

Proof. The relations for the parameters follow from the definition of GQ . The integer condition for Γ takes the following form: $s + t$ divides $st(s + 1)(t + 1)$. $\square$

Lemma 1.5. *Let Γ be a strongly regular locally $GQ(s, t)$ graph with parameters (v, k, λ, μ) . Then:*

- (1) $k = (s + 1)(st + 1)$, $\lambda = s(t + 1)$, and $s + t$ divides $st(s + 1)(t + 1)$;

- (2) $t \leq m - 1 \leq st$, $m - 1$ divides s^2t , $\mu = s(t + 1) + m + 1 - s^2t/(m - 1)$, and $n = m - 1 + s^2t/(m - 1)$;
 (3) $\mu \cdot n$ divides $(m - 1)(s + 1)(st + 1)((s + 1)(st + 1) + m)$;
 (4) $s + 2$ divides $2t(t + 1)(2t - 1)(4t + \mu - 2)$.

Proof. Statement (1) follows from Lemma 1.4. Statements (2) and (3), except for the inequality $t \leq m - 1 \leq st$, follow from the definition of GQ and Lemma 1.3. The inequality itself follows from the fact that $(s + 1)(t + 2 - s) \leq \mu \leq 2(st + 1)$.

The number of blocks in Γ is equal to $v(t + 1)(st + 1)/(s + 2)$. This implies statement (4). The lemma is proved. $\square$

§2. THE CASE OF MOORE GRAPHS

In what follows, Γ is a completely regular locally $GQ(s, t)$ graph in which every μ -subgraph is isomorphic to a known strongly regular graph Δ . Then $k = (s + 1)(st + 1)$ and $\lambda = s(t + 1)$. Let $\Delta = [a] \cap [b]$. We note that each line L of the generalized quadrangle $[a]$ intersects Δ at 0 or 2 points. Indeed, if $c \in L \cap \Delta$, then $M = (L \cup \{a\}) - \{c\}$ is a line in $[c]$, and, by the definition of a generalized quadrangle, $[b]$ contains a unique point d in M . Thus, $\{c, d\} = L \cap \Delta$.

In this section, we consider the case where Δ is a complete bipartite graph or a Moore graph.

Lemma 2.1. *If $\Delta = K_{t+1, t+1}$, then $t + 1$ divides $s^2(s^2 - 1)$.*

Proof. From the rectangular relation $k(k - \lambda - 1) = k_2\mu$, where $k_2 = |\Gamma_2(a)|$ for every vertex a , we conclude that $2(t + 1)$ divides $(s + 1)(st + 1)s^2t$.

Since $t + 1$ is prime to t , we have $(t + 1, st + 1) = (t + 1, s - 1)$, and $t + 1$ divides $s^2(s^2 - 1)$. The lemma is proved. $\square$

Lemma 2.2. *If $\Delta = K_{t+1, t+1}$ and Γ is strongly regular, then either $s = 1$, or $s = 4$ and $t = 1$, or else $s = t = 2, 3, 8$, or 13 .*

Proof. By [8, Theorem 3.1], either the conclusion of the lemma is valid, or $s = t = 4$. However, by [9], in the latter case there are no such graphs. $\square$

We note that a locally $GQ(1, t)$ graph coincides with $K_{3 \times (t+1)}$. Let $s > 1$. For small values of t , some additional information is known. If $t = 1$, then the neighborhoods of the vertices in Γ are $(s + 1) \times (s + 1)$ lattices, and by [5], Γ is a Johnson graph or its quotient. If $t = 2$, then $s = 2$ or 4 . In the case where $s = 2$, the graph Γ is a unique locally $GQ(2, 2)$ graph with parameters $(36, 15, 6, 6)$. If $s = 4$, then, by [6], Γ is a unique distance-regular locally $GQ(4, 2)$ graph on 378 vertices with the intersection array $(45, 32, 12, 1; 1, 6, 32, 45)$.

If $t = 3$, then $s = 3, 5$, or 9 . If $s = 3$, then, by [7], Γ is a unique locally $GQ(3, 3)$ graph with parameters $(176, 40, 12, 8)$. If $t = 4$, then 5 divides $s^2(s^2 - 1)$, so that $s = 4, 6, 11$, or 16 . For $s = 4$, the graph Γ is strongly regular and, by a result in [9], does not exist.

Let Δ be a Moore graph. If Δ is a pentagon, then $t = 1$ and Γ is locally an $(s + 1) \times (s + 1)$ lattice. However, in this case the connected components of the μ -subgraphs are cycles of even length, a contradiction.

Lemma 2.3. *If Δ is a Petersen graph, then Γ is a unique strongly regular locally $GQ(2, 2)$ graph with parameters $(28, 15, 6, 8)$.*

Proof. If Δ is a Petersen graph, then $t = 2$ and $s = 1, 2$, or 4 . For $s = 1$, we obtain $\mu = 6$, a contradiction. If $s = 2$, then the conclusion of the lemma is valid (see, e.g., [2]).

Let $s = 4$. Then, by [6], for a completely regular locally $GQ(4, 2)$ graph we have $\mu = 6$ or $\mu = 18$. The lemma is proved. $\square$

For the remainder of this section, we assume that Δ is a Hoffman–Singleton graph. Then $t = 6$. By the integer condition, $s + 6$ divides $42s(s + 1)$, and $s = 3, 4, 6, 8, 9, 12, 14, 15, 22, 24, 29, 30$, or 36 . Since $\mu = 50$ divides $6s^2(s + 1)(6s + 1)$, we have $s = 4, 9, 14, 15, 24, 29$, or 30 .

Lemma 2.4. *Let a and b be vertices in Γ with $d(a, b) = 2$, and let $\Delta = [a] \cap [b]$ be a Hoffman–Singleton graph. Then for $x_i = |X_i(\Delta) \cap [b]|$ we have:*

- (1) $x_i = 0$ for odd i and for $i > 8$;
- (2) $\sum x_i = (s + 1)(6s + 1) - 50$ and $\sum ix_i = 50(7s - 7)$.

Proof. If a vertex c in $[b] - [a]$ is adjacent to a vertex in Δ , then $[c] \cap \Delta$ consists of isolated edges. Observe that, for an edge $\{u, w\}$ of Δ , the subgraph $\Delta - ([u] \cup [w])$ consists of three isolated edges. Therefore, $x_i = 0$ for odd i and for $i > 8$. Statement (1) is proved.

Next, $[b] - \Delta$ contains $(s + 1)(6s + 1) - 50$ vertices, and each vertex of Δ is adjacent to $7s - 7$ vertices of $\Gamma - \Delta$. This implies (2). $\square$

Lemma 2.5. *The parameter s is not equal to 4.*

Proof. If $s = 4$, then $\mu = 2(st + 1) = 50$, and by Lemma 1.2, the graph Γ is strongly regular with parameters $(366, 125, 28, 50)$. Now, $n^2 = (\lambda - \mu)^2 + 4(k - \mu) = 28^2$, and the secondary eigenvalues of the graph are $n - m = 3$ and $-m = -25$. By the integer condition, $n\mu = 28 \cdot 50$ divides $(m - 1)k(k + m) = 24 \cdot 125 \cdot 150$, a contradiction. $\square$

Lemma 2.6. *The graph Γ is not strongly regular.*

Proof. If Γ is a strongly regular graph with the smallest eigenvalue $-m$, then $m - 1$ divides $6s^2$ and $\mu = 50 = \lambda + 2 + (m - 1) - 6s^2/(m - 1)$.

If $s = 9$, then $\lambda = 63$ and $15 + (m - 1) = 6 \cdot 9^2/(m - 1)$, a contradiction. If $s = 14$, then $\lambda = 98$ and $50 + (m - 1) = 6 \cdot 14^2/(m - 1)$. If $m - 1$ is not divisible by 7, then $50 + (m - 1)$ is divisible by 49, a contradiction. If $m - 1$ is divisible by 7, then $(m - 1)$ is divisible by 49, a contradiction.

If $s = 15$, then $\lambda = 105$ and $57 + (m - 1) = 6 \cdot 15^2/(m - 1)$. Therefore, $m - 1 = 3r$ and $19 + r = 450/r$. If r is divisible by 5, then r is divisible by 25 and $19 + r > 450/r$, a contradiction. Thus, r is not divisible by 5 and $19 + r$ is divisible by 25. Therefore, $r = 6$ and $25 \neq 450/6$, a contradiction.

If $s = 24$, then $\lambda = 168$ and $120 + (m - 1) = 6 \cdot 24^2/(m - 1)$. Therefore, $m - 1 = 3r$ and $40 + r = 2 \cdot 24^2/r$. If r is divisible by 3, then r is divisible by 9 and $40 + r$ is a power of 2, a contradiction. Thus, r is a power of 2 and $40 + r$ is divisible by 9, so that $r = 32$ and $72 \neq 36$, a contradiction.

If $s = 29$, then $\lambda = 203$ and $155 + (m - 1) = 6 \cdot 29^2/(m - 1)$. If $m - 1$ is divisible by 29, then $m - 1$ is divisible by 29^2 , a contradiction. If $m - 1$ is not divisible by 29, then $155 + (m - 1) < 6 \cdot 29^2/(m - 1)$.

Finally, if $s = 30$, then $\lambda = 210$ and $162 + (m - 1) = 6 \cdot 30^2/(m - 1)$. Therefore, $m - 1 = 3^r l$. If $r \leq 1$, then $162 + (m - 1)$ is not divisible by 9 and $6 \cdot 30^2/(m - 1)$ is divisible by 9. If $r \geq 2$, then $162 + (m - 1)$ is divisible by 9 and $6 \cdot 30^2/(m - 1)$ is not divisible by 9. Thus, we obtain a contradiction in all cases. The lemma is proved. $\square$

§3. THE CASE OF CLEBSCH GRAPHS AND GEWIRTZ GRAPHS

In this section, we consider the case where Δ is either a Clebsch graph or a Gewirtz graph.

Lemma 3.1. *If Δ is a Clebsch graph, then $t = 4$ and $s = 2, 4, 6, 8, 11, 12$, or 16 . If $s = 2$, then Γ is a locally $GQ(2, 4)$ Taylor graph on 56 vertices.*

Proof. If Δ is a Clebsch graph, then $t = 4$ and $s = 2, 4, 6, 8, 11, 12$, or 16 . In all cases, 16 divides $4(s+1)(4s+1)s^2$, and the rectangular relation imposes no new restrictions.

If $s = 2$, then Γ is a locally $GQ(2, 4)$ Taylor graph on 56 vertices (see [2]). $\square$

Lemma 3.2. *Let a and b be vertices in Γ with $d(a, b) = 2$, and let $\Delta = [a] \cap [b]$ be a Clebsch graph. We denote by X_i the set of vertices in $[b] - [a]$ adjacent to exactly i vertices in Δ , $x_i = |X_i|$. Then:*

- (1) $x_i = 0$ for odd i and for $i > 8$;
- (2) $\sum x_i = (s+1)(4s+1) - 16$ and $\sum ix_i = 16(5s-5)$;
- (3) if $s = 4$, then the diameter of Γ is equal to 3 .

Proof. If a vertex c of $[b] - [a]$ is adjacent to a vertex in Δ , then $[c] \cap \Delta$ consists of isolated edges. Observe that, for an edge $\{u, w\}$ of Δ , the subgraph $\Delta - ([u] \cup [w])$ consists of three isolated edges. Therefore, $x_i = 0$ for odd i and for $i > 8$. Statement (1) is proved.

Next, $\Gamma - \Delta$ has $(s+1)(4s+1) - 16$ vertices, and each vertex of Δ is adjacent to $5s - 5$ vertices of $\Gamma - \Delta$. This implies (2).

Let $s = 4$. Then $k = 85$, $\lambda = 20$, and for each vertex a we have $|\Gamma_2(a)| = 340$. Also, $x_0 + x_2 + x_4 + x_6 + x_8 = 69$ and $x_2 + x_4 + 3x_6 + 4x_8 = 120$.

If Γ is a strongly regular graph, then its parameters are $(426, 85, 20, 16)$. However, in this case we have $n^2 = (\lambda - \mu)^2 + 4(k - \mu) = 292$, a contradiction.

If the diameter of Γ is at least 4 and $acbde$ is a geodesic path in Γ , then $[b]$ contains some Clebsch subgraphs $\Delta_1 = [a] \cap [b]$ and $\Delta_2 = [b] \cap [e]$ such that each vertex of Δ_1 is at a distance of 2 from every vertex of Δ_2 . We note that the number of lines in $[b]$ is 85 , and 40 of them are secants for Δ_1 and do not intersect Δ_2 . By symmetry, 40 lines in $[b]$ are secants for Δ_2 and do not intersect Δ_1 . Thus, exactly 5 lines $L_1, \dots, L_5$ in $[b]$ do not intersect either Δ_1 or Δ_2 .

For a vertex f of $[b] - (\Delta_1 \cup \Delta_2)$, we put $\delta_f = |[f] \cap (\Delta_1 \cup \Delta_2)|$. Then $\delta_f \leq 10$, and for each secant M of Δ_2 we obtain $\sum_{f \in M} |[f] \cap \Delta_2| = 12$ and $\sum_{f \in M} |[f] \cap \Delta_1| = 16$. Therefore, $\sum_{f \in M} \delta_f = 28$. Thus, for two points $g \in M$ we have $\delta_g = 10$ (and these points do not lie on the lines L_i), and for one point $f \in M$ we have $\delta_f = 8$ (and f belongs to a unique line L_i).

Thus, either there is a vertex z in $[b]$ lying on each of the five lines $L_1, \dots, L_5$, or the lines L_i and L_j are pairwise disjoint. However, in the latter case, the line passing through points lying on L_1 and L_2 is a secant for one of the subgraphs Δ_1 or Δ_2 and contains two points f with $\delta_f = 8$, a contradiction.

We put $\Sigma = [b] - (z^\perp \cup \Delta_1 \cup \Delta_2)$, $Y_i = X_i(\Delta_1) \cap \Sigma$, and $y_i = |Y_i|$. Then Σ is a regular graph on 32 vertices, Σ is of degree 5 and without triangles, $\sum y_i = 32$, and $\sum iy_i = 160$.

Now, we describe the types relative to Δ_1 of the lines that do not pass through z . If a line M does not intersect Δ_1 , then this line is a secant for Δ_2 and contains one point from each of $X_i(\Delta_1)$, $X_j(\Delta_1)$, and $X_l(\Delta_1)$, so that $i + j + l = 16$. We assume that the point in $M \cap X_i(\Delta_1)$ belongs to $[z]$ and $j \leq l$. In this case, we say that the line M is of type (i, j, l) . The structure of a Clebsch graph shows that $\{i, j, l\} \neq \{4, 6, 6\}$. Therefore, the lines that do not intersect Δ_1 are of types $(0, 8, 8)$, $(2, 6, 8)$, $(4, 4, 8)$, or $(6, 2, 8)$. Similarly, a secant L for Δ_1 contains one point from each of $[z] \cap X_i(\Delta_1)$, $X_j(\Delta_1)$, and $X_l(\Delta_1)$, and $j \geq l$ and $i + j + l = 12$. We say that the line L is of type (i, j, l) .

If a line is of type (i, j, l) relative to Δ_1 , then it is of type $(8-i, 10-j, 10-l)$ relative to Δ_2 ; therefore, no secant for Δ_1 can be of type $(4, 4, 4)$ or $(2, 6, 4)$. Thus, the secants for Δ_1 are of types $(8, 2, 2)$, $(6, 4, 2)$, $(4, 6, 2)$, or $(2, 8, 2)$.

Thus, a neighborhood of a point in Y_2 contains 4 points of Y_8 and 1 point of $Y_2 \cup Y_4 \cup Y_6 \cup Y_8$, a neighborhood of a point in Y_4 contains 3 points of Y_8 and 2 points of Y_2 , a neighborhood of a point in Y_6 contains 2 points of Y_8 and 3 points of Y_2 , and a

neighborhood of a point in Y_8 contains 4 points of Y_2 and 1 point of $Y_0 \cup Y_2 \cup Y_4 \cup Y_6$. Now, the number of edges between Y_2 and Y_8 that lie on lines that do not intersect Δ_1 is equal to $4y_2$ and does not exceed y_8 . Similarly, the number of edges between Y_2 and Y_8 that lie on lines intersecting Δ_1 is equal to $4y_8$ and does not exceed y_2 . Hence, $y_2 = y_8 = 0$. This contradicts the fact that, in this case, we also have $y_4 = y_6 = 0$. $\square$

Lemma 3.3. *If Δ is a Clebsch graph, then Γ is not strongly regular.*

Proof. Let Δ be a Clebsch graph, and let Γ be a strongly regular graph with the smallest eigenvalue $-m$. By Lemma 1.3, the number $m - 1$ divides $4s^2$ and $\mu = 16 = \lambda + 2 + (m - 1) - 4s^2/(m - 1)$.

If $s = 6$, then $\lambda = 30$ and $16 + (m - 1) = 4 \cdot 6^2/(m - 1)$. Therefore, $m - 1$ is a power of 2 and $16 + (m - 1)$ is divisible by 9. But $18 \neq 72$, a contradiction.

If $s = 8$, then $\lambda = 40$ and $26 + (m - 1) = 4 \cdot 8^2/(m - 1)$, a contradiction. If $s = 11$, then $\lambda = 55$ and $41 + (m - 1) = 4 \cdot 11^2/(m - 1)$. Therefore, $m - 1$ is not divisible by 11 and $41 + (m - 1)$ is divisible by 11^2 , a contradiction.

If $s = 12$, then $\lambda = 60$ and $46 + (m - 1) = 4 \cdot 12^2/(m - 1)$. If $m - 1$ is divisible by 3, then either $m - 1 = 9$ and $55 \neq 64$, or $m - 1 = 18$ and $64 \neq 32$. Thus, $m - 1$ is a power of 2 and $46 + (m - 1)$ is divisible by 9. Therefore, $m - 1 = 8$ and $54 \neq 72$, a contradiction.

If $s = 16$, then $\lambda = 80$ and $66 + (m - 1) = 4 \cdot 16^2/(m - 1)$, a contradiction. $\square$

Lemma 3.4. *If Δ is a Gewirtz graph, then $t = 9$ and $s = 3, 7, 27, 31$, or 63 . If $s = 3$, then Γ is a McLaughlin graph.*

Proof. If Δ is a Gewirtz graph, then $t = 9$ and $s = 3, 6, 7, 9, 11, 15, 18, 21, 27, 31, 36, 39, 45, 51, 63, 71, 72$, or 81 . Since 56 divides $9s^2(s + 1)(9s + 1)$, we have $s = 3, 7, 27, 31$, or 63 .

If $s = 3$, then, by [7], Γ is a McLaughlin graph. $\square$

Lemma 3.5. *If Δ is a Gewirtz graph and Γ is strongly regular, then Γ is a McLaughlin graph.*

Proof. Let Δ be a Gewirtz graph, and let Γ be strongly regular with the smallest eigenvalue $-m$. Then, by Lemma 1.3, the number $m - 1$ divides $9s^2$ and $\mu = 56 = \lambda + 2 + (m - 1) - 9s^2/(m - 1)$. By Lemma 3.4, we have $s = 3, 7, 27, 31$, or 63 .

If $s = 7$, then $\lambda = 70$ and $16 + (m - 1) = 9 \cdot 7^2/(m - 1)$. If $m - 1$ is divisible by 7, then $m - 1$ is divisible by 49 and $65 \neq 9$. Thus, $m - 1$ is a power of 3 and $16 + (m - 1)$ is divisible by 49, a contradiction.

If $s = 27$, then $\lambda = 270$ and $216 + (m - 1) = 9 \cdot 27^2/(m - 1)$, a contradiction. If $s = 31$, then $\lambda = 310$ and $256 + (m - 1) = 9 \cdot 31^2/(m - 1)$. Therefore, $m - 1$ is not divisible by 31 and $256 + (m - 1)$ is divisible by 31^2 , a contradiction.

If $s = 63$, then $\lambda = 630$ and $576 + (m - 1) = 9 \cdot 63^2/(m - 1)$. Therefore, $m - 1 = 9r$ and $64 + r = 63^2/r$. If r is divisible by 7, then r is divisible by 49 and $64 + 49 \neq 81$. If r is not divisible by 7, then r is a power of 3 and $64 + r$ is divisible by 49, a contradiction. The lemma is proved. $\square$

§4. THE CASE WHERE Δ IS A GRAPH WITH $\mu > 2$

In this section, we consider the cases in which Δ is a graph with $\mu > 2$.

Lemma 4.1. *Let Δ be a strongly regular graph with parameters $(77, 16, 0, 4)$. Then $t = 15$ and $s = 21, 41, 55, 154$, or 195 . Moreover, Γ is not strongly regular.*

Proof. Let Δ be a strongly regular graph with parameters $(77, 16, 0, 4)$. Then $t = 15$, $s + 15$ divides $15 \cdot 16s(s + 1)$, and 77 divides $15s^2(s + 1)(15s + 1)$. Hence, $s = 21, 41, 55, 154$, or 195 .

If Γ is strongly regular with the smallest eigenvalue $-m$, then, by Lemma 1.3, $m-1$ divides $15s^2$ and $\mu = 77 = \lambda + 2 + (m-1) - 15s^2/(m-1)$.

If $s = 21$, then $\lambda = 336$ and $261 + (m-1) = 15 \cdot 21^2/(m-1)$. Therefore, $15 \cdot 21^2/(m-1) - (m-1)$ is divisible by 9, a contradiction.

If $s = 41$, then $\lambda = 656$ and $581 + (m-1) = 15 \cdot 41^2/(m-1)$. Therefore, $m-1$ is not divisible by 41 and $581 + (m-1)$ is divisible by 41^2 , a contradiction.

If $s = 55$, then $\lambda = 880$ and $803 + (m-1) = 15 \cdot 55^2/(m-1)$. Therefore, $m-1 = 11r$ and $71 + r = 3 \cdot 5^3/r$. We see that r is not divisible by 5 and $71 + r$ is divisible by 125, a contradiction.

If $s = 154$, then $\lambda = 2464$ and $2387 + (m-1) = 15 \cdot 154^2/(m-1)$. Therefore, $m-1 = 11r$ and $217 + r = 15 \cdot 14^2/r$. We have $r = 7l$ and $31 + l = 60/l$, a contradiction.

If $s = 195$, then $\lambda = 3120$ and $3043 + (m-1) = 15 \cdot 195^2/(m-1) = 3^3 \cdot 5^3 \cdot 13/(m-1)$. If $m-1$ is divisible by 5, then $m-1$ is divisible by 5^3 and $3043 + (m-1) > 15 \cdot 195^2/(m-1)$. If $m-1$ is divisible by 3, then $m-1$ is divisible by 3^3 and again we have $3043 + (m-1) > 15 \cdot 195^2/(m-1)$. Thus, $m-1$ is equal to 1 or 13, and $3043 + (m-1)$ is not divisible by 5, a contradiction. $\square$

Lemma 4.2. *Let Δ be a Higman–Sims graph. Then $t = 21$, $s = 19, 24, 34, 35, 39, 45, 49, 69, 84, 89, 99, 105, 115, 119, 144, 159, 175, 189, 199, 210, 214, 259, 294, 309, 339, 364, 375, 399, 419$, or 420 . Moreover, Γ cannot be strongly regular.*

Proof. Let Δ be a Higman–Sims graph with parameters $(100, 22, 0, 6)$. Then $t = 21$, $s + 21$ divides $21 \cdot 22s(s+1)$ and 100 divides $21s^2(s+1)(21s+1)$. Thus, s or $s+1$ is divisible by 5 and $s+21$ divides $2^3 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11$. Hence, $s = 9, 14, 15, 19, 24, 34, 35, 39, 45, 49, 69, 84, 89, 99, 105, 115, 119, 144, 159, 175, 189, 199, 210, 214, 259, 294, 309, 339, 364, 375, 399, 419$, or 420 .

We recall that if $t > s$, then $\mu \geq (s+1)(t+2-s)$, so that $s \geq 19$. If Γ is strongly regular and $-m$ is its smallest eigenvalue, then, by Lemma 1.3, $m-1$ divides $21s^2$ and $\mu = 100 = \lambda + 2 + (m-1) - 21s^2/(m-1)$.

Next, $n^2 = (\lambda - \mu)^2 + 4(k - \mu) = (22s - 100)^2 + 4((s+1)(21s+1) - 100)$. Therefore, $x = (11s - 50)^2 + (s+1)(21s+1) - 100$ is a square. If s is odd, then $x \equiv 1 \pmod{8}$ and $s+1 \equiv 2 \pmod{4}$. If s is even, then again $x \equiv 1 \pmod{8}$ and $s \equiv 0 \pmod{4}$. Consequently, $s = 24, 45, 49, 69, 84, 89, 105, 144, 189, 309, 364$, or 420 .

Substituting these values of s , we see that in all cases $(\lambda - \mu)^2 + 4(k - \mu)$ is not a square. $\square$

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INSTITUTE OF MATHEMATICS AND MECHANICS, URAL BRANCH OF THE RUSSIAN ACADEMY OF SCIENCES,
EKATERINBURG, RUSSIA

INSTITUTE OF MATHEMATICS AND MECHANICS, URAL BRANCH OF THE RUSSIAN ACADEMY OF SCIENCES,
EKATERINBURG, RUSSIA

E-mail address: makhnev@imm.uran.ru

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