Алгебра и анализ
Том. 18 (2006), № 4

St. Petersburg Math. J.
Vol. 18 (2007), No. 4, Pages 517–538
S 1061-0022(07)00959-4
Article electronically published on May 25, 2007

ON EDGE-REGULAR GRAPHS WITH $k \geq 3b_1 - 3$

I. N. BELOUSOV AND A. A. MAKHNEV

Abstract. An undirected graph on v vertices in which the degrees of all vertices are equal to k and each edge belongs to exactly λ triangles is said to be edge-regular with parameters (v, k, λ). It is proved that an edge-regular graph with parameters (v, k, λ) such that $k \geq 3b_1 - 3$ either has diameter 2 and coincides with the graph $P(2)$ on 20 vertices or with the graph $M(19)$ on 19 vertices; or has at most $2k + 4$ vertices; or has diameter at least 3 and is a trivalent graph without triangles, or the line graph of a quadrivalent graph without triangles, or a locally hexagonal graph; or has diameter 3 and satisfies $|\Gamma_3(u)| \leq 1$ for each vertex u.

Introduction

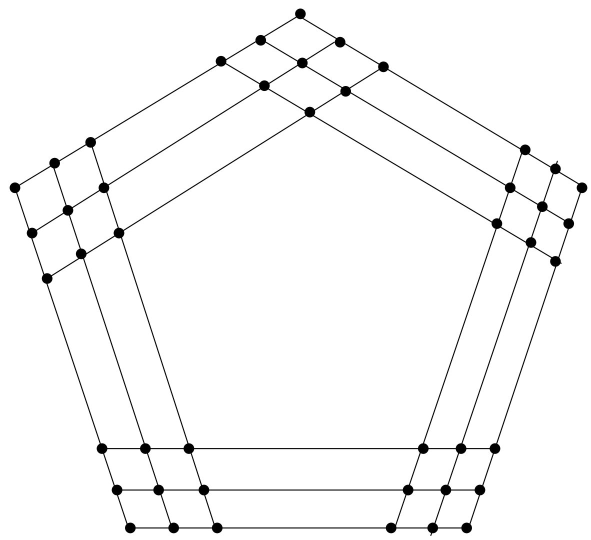
We consider undirected graphs without loops and multiple edges. If a and b are vertices of a graph Γ, then we denote by $d(a, b)$ the distance between a and b and by $\Gamma_i(a)$ the subgraph of Γ induced by the set of vertices that are at a distance of i from a in Γ. The subgraph $\Gamma_1(a)$ is called a *neighborhood* of a and is denoted by $[a]$. We denote by $a^\perp$ the subgraph that is the unit ball centered at a.

A graph Γ is said to be *regular of degree k* if $[a]$ contains exactly k vertices for each vertex a in Γ. A graph Γ is *edge-regular with parameters* (v, k, λ) if Γ has v vertices and is regular of degree k, and each edge of Γ lies in λ triangles. We say that a graph Γ is *amply regular with parameters* (v, k, λ, μ) if Γ is edge-regular with the corresponding parameters, and the subgraph $[a] \cap [b]$ contains μ vertices whenever $d(a, b) = 2$. An amply regular graph of diameter 2 is said to be *strongly regular*.

We denote by $K_{m_1,\dots,m_n}$ the complete n-partite graph with partite sets of orders $m_1, \dots, m_n$. If $m_1 = \dots = m_n = m$, then the corresponding graph is denoted by $K_{n\times m}$. The graph $K_{1,3}$ is called the 3-*claw*. A *triangle graph* $T(m)$ is a graph whose vertices are the unordered pairs of elements of X, $|X| = m$, and two pairs $\{a, b\}$ and $\{c, d\}$ are adjacent if and only if they have an element in common. A graph on a set $X \times Y$ of vertices is called an $(m \times n)$-*lattice* if $|X| = m$, $|Y| = n$, and two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $x_1 = x_2$ or $y_1 = y_2$. The vertices of the graph $M(19)$ are the elements of the field F_{19}. Two vertices are adjacent if their difference is a nonzero cube in F_{19}. This is a locally hexagonal graph of diameter 2. The graph $P(m)$ with $v = 5m^2$, $k = 4m - 2$, and $b_1 = 2m - 1$ is obtained by replacing the vertices of the pentagon with pairwise disjoint $(m \times m)$-lattices (the graph $P(3)$ is depicted below). A Taylor graph is an amply regular graph Γ of diameter 3 in which $\Gamma = u^\perp \cup w^\perp$ for any two vertices u and w with $d(u, w) = 3$. The Schläfli graph is a unique strongly regular graph with parameters (27, 16, 10, 8). We denote by $\mathcal{T}(k)$ the class of regular graphs of degree k without triangles, and by $\mathcal{E}(k)$ the class of line graphs for the graphs in $\mathcal{T}(k)$. The number of vertices in a subgraph Δ will be denoted by $|\Delta|$.

2000 *Mathematics Subject Classification.* Primary 05C60.

Key words and phrases. Undirected graph, edge-regular graph, locally hexagonal graph.
Supported by RFBR (grant no. 05-01-00046) and RFBR-NSFC (grant no. 05-01-39000).



The graph $P(3)$.

Suppose the distance between two vertices u and w in an edge-regular graph Γ is equal to 2. We say that the pair (u, w) is *good* if $\mu(u, w) = k - 2b_1 + 1$, and *almost good* if $\mu(u, w) = k - 2b_1 + 2$. By Lemma 1.1, the μ-subgraph corresponding to a good pair is a clique.

If the distance between u and w in Γ is i, then we denote by $b_i(u, w)$ (respectively, $c_i(u, w)$) the number of vertices in the intersection of $\Gamma_{i+1}(u)$ (respectively, $\Gamma_{i-1}(u)$) with $[w]$. We note that, in an edge-regular graph with parameters (v, k, λ), the value of $b_1(u, w)$ does not depend on the choice of an edge $\{u, w\}$ and is equal to $k - \lambda - 1$. For edge-regular graphs with $k \geq f(b_1)$ and some specific functions f, it is possible to obtain an estimate $v \leq g(k)$ (or to describe the graphs for which this estimate fails). Thus, in [1, Lemma 1.4.2], it was proved that if Γ is a connected incomplete edge-regular graph with parameters (v, k, λ) such that $k \geq 3b_1$, then the diameter of Γ is 2 and $v \leq 2k - 2$. In fact, it was proved that $v < k - 2 + 3b_1 + 3/(b_1 + 1)$. To sharpen the upper bound for the number of vertices, we need to describe the graphs with small values of b_1 (see Lemmas 1.2 and 1.3 below) and the graphs saturated by good pairs of vertices. In a corollary in [2], it was proved that if Γ is a connected edge-regular graph with parameters (v, k, λ) where $k \geq 3b_1 - 2$, then either Γ is a polygon, or the icosahedron graph, or $\Gamma \in \mathcal{E}_3$, or Γ is a graph of diameter 2 with at most $2k$ vertices, or the pentagon, or a (3×3)-lattice, or the triangle graph $T(7)$. The next step is the study of edge-regular graphs with $k \geq 3b_1 - 3$. The graphs of diameter 2 with $k \geq 3b_1 - 3$ were studied in [3].

Theorem. *Let Γ be a connected edge-regular graph with parameters (v, k, λ), let $b_1 = k - \lambda - 1$, and let $k \geq 3b_1 - 3$. Then one of the following statements is true*:

(1) *the diameter Γ is at most* 2, *and either the number v of vertices does not exceed* $2k + 4$, *or Γ coincides with the graph $P(2)$, or Γ coincides with the locally hexagonal graph on* 17 *or* 19 *vertices*;

(2) *the diameter of Γ is at least* 3, *and either $\Gamma \in \mathcal{T}(3) \cup \mathcal{E}(3) \cup \mathcal{E}(4)$, or Γ is a locally hexagonal graph*;

(3) *the graph Γ has diameter* 3 *and $|\Gamma_3(u)| \leq 1$ for each vertex u.*

Example. Let Φ_n be the graph in which the vertices are the 4-cycles of the symmetry group S_n and two vertices a and b are adjacent if ab is a 5-cycle. Then Φ_n is a 6-extension of the Johnson graph $J(n,4)$ and Φ_5 is an edge-regular graph of diameter 3 with parameters (30, 12, 6) such that $k = 3b_1 - 3$ and every vertex in [(1432)] forms a good pair with the vertex (1234) (if an edge-regular graph with $k \geq 3b_1 - 1$ is neither a polygon nor an icosahedron graph, then at most 2 vertices in $\Gamma_2(u)$ form good pairs with u; see [5]).

We do not know examples of graphs with $k \geq 3b_1 - 3$ that have vertices u and v such that $|\Gamma_3(u)| = 1$ and $|\Gamma_3(w)| = 0$.

Corollary. *Let Γ be a connected amply regular graph of diameter greater than 2 with parameters (v, k, λ, μ), where $k \geq 3b_1 - 3$. Then one of the following statements is true*:

(1) $\Gamma \in \mathcal{E}(4)$ *and* $\mu = b_1 - 2 = 1$;

(2) $\Gamma \in \mathcal{T}(3) \cup \mathcal{E}(3)$ *and* $\mu = b_1 - 1 = 1$;

(3) $\mu = b_1$ *and* Γ *is either an n-gon with $n \geq 6$, or a complete bipartite graph $K_{4,4}$ with the maximal matching removed, or the icosahedron graph, or the Johnson graph $J(6,3)$, or the locally Taylor graph $T(6)$ on 32 vertices, or a locally Schläfli graph on 56 vertices.*

The authors thank the referee for remarks that helped to improve the initial version of the present paper.

§1. Auxiliary results

Lemma 1.1. *Let Γ be an edge-regular graph with parameters (v, k, λ), and let $b_1 = k - \lambda - 1$. If the distance between vertices u and w in Γ is 2, then*:

(1) *the degree of each vertex in a μ-subgraph of Γ is at least $k - 2b_1$*;

(2) *a vertex d has degree α in the graph $[u] \cap [w]$ if and only if $[d]$ has $\alpha - (k - 2b_1)$ vertices outside $u^\perp \cup w^\perp$*;

(3) *if $\mu(u,w) = k - 2b_1 + 1$, then the subgraph $[u] \cap [w]$ is a clique and $[d] \subset u^\perp \cup w^\perp$ for each vertex $d \in [u] \cap [w]$*;

(4) *if $\Gamma - (u^\perp \cup w^\perp)$ has a unique vertex z, then $\mu(u,z) = \mu(w,z)$.*

Proof. Let $d \in [u] \cap [w]$. Then $|[d] - [u]| = |[d] - [w]| = b_1$. Therefore, at least $k - 2b_1$ vertices of $[d]$ belong to $[u] \cap [w]$. Statement (1) is proved.

Let $d \in [u] \cap [w]$, and let the degree of d in this μ-subgraph be equal to α. Then $k = \alpha + 2b_1 - |[d] - (u^\perp \cup w^\perp)|$. Therefore, $[d]$ contains $\alpha - (k - 2b_1)$ vertices outside $u^\perp \cup w^\perp$. Statement (2) is proved.

Statement (3) follows from (1) and (2).

Let $\{z\} = \Gamma - (u^\perp \cup w^\perp)$. Since the number of edges between $[u] - [w]$ and $[w] - [u]$ is equal to $b_1|[u] - [w]| - \mu(u,z)$, we obtain $\mu(u,z) = \mu(w,z)$. The lemma is proved. □

Lemma 1.2. *Let Γ be a connected edge-regular graph with parameters (v, k, λ), and let $b_1 = 2$. Then either $\Gamma \in \mathcal{T}(3) \cup \mathcal{E}(3)$, or Γ is one of the following graphs*:

(1) *the complete multipartite graph $K_{r \times 3}$*;

(2) *the (3×3)-lattice graph, or the triangle graph $T(5)$, or the Petersen graph*;

(3) *the icosahedron graph.*

Proof. This is Proposition 1 in [4]. □

Lemma 1.3. *Let Γ be a connected edge-regular graph with parameters (v, k, λ) and with $b_1 = 3$. Then either $\Gamma \in \mathcal{T}(4) \cup \mathcal{E}(4)$, or Γ is one of the following graphs*:

(1) *a locally hexagonal graph (including the Paley graph with parameters $(13, 6, 2, 3)$ and the Shrikhande graph)*;

(2) *the complete multipartite graph $K_{r \times 4}$*;

(3) *the triangle graph $T(6)$, or the Clebsch graph.*

Proof. This is Proposition 2 in [4]. □

Lemma 1.4. *Let* Γ *be an edge-regular graph, and let* $\mu(u,w) = k-2b_1+1$ *and* $\mu(u,z) = k-2b_1+2$ *for some vertices* w *and* z *in* $\Gamma_2(u)$*. Then* $|[u]\cap[w]\cap[z]| < 2$.

Proof. This statement follows from Lemmas 4 and 5 in [5]. □

Let $w, z \in \Gamma_2(u)$. We say that a triple (u,w,z) of vertices is *good* if $\mu(u,w)+\mu(u,z) \le 2k-4b_1+3$, and *almost good* if $\mu(u,w)+\mu(u,z) = 2k-4b_1+4$. We have considered the case of a good triple in Lemma 1.4. The case of an almost good triple will be treated in Lemma 1.5.

Lemma 1.5. *Let* Γ *be an edge-regular graph with* $k \ge 3b_1-3$*, let* $\mu(u,w)+\mu(u,z) = 2k-4b_1+4$ *for two vertices* w *and* z *of* $\Gamma_2(u)$*, let* $\Delta = [u]\cap[w]\cap[z]$*, and let* $\delta = |\Delta|$*. Then one of the following statements is true*:

(1) *the vertices* w *and* z *are not adjacent and* $\delta \le 1$;

(2) Δ *contains two nonadjacent vertices and* $\delta \le 2$;

(3) *the vertices* w *and* z *are adjacent,* Δ *is a clique, and if* $\delta > 1$*, then either*

(i) *the subgraph* Δ *contains a unique vertex* d *adjacent to a vertex outside* $u^\perp \cup [w]\cup[z]$*,* $\delta = 2$*, and, for* $e \in \Delta(d)$*, the subgraph* $[d]\cup[e]$ *contains* Δ *and* $[d]\cap[e]$ *is contained in* $\{u,w,z\}\cup([u]\cap([w]\cup[z]))\cup([w]\cap[z]-[u])$*, or*

(ii) *the subgraph* Δ *contains no vertices adjacent to a vertex outside* $u^\perp\cup[w]\cup[z]$*, and, for any two adjacent vertices* $d, e \in \Delta(d)$*, the subgraph* $[d]\cap[e]$ *contains* $\lambda-1+\gamma$ *vertices of* $\{u,w,z\}\cup([u]\cap([w]\cup[z]))\cup([w]\cap[z]-[u])$*, where* $\gamma = |\Delta-([d]\cup[e])|$.

Proof. If the vertices w and z are not adjacent, or Δ contains two nonadjacent vertices, or a subgraph of Δ contains a vertex adjacent to a vertex outside $u^\perp\cup[w]\cup[z]$, then the lemma follows from Theorem 1 in [6]. Therefore, we may assume that the vertices w and z are adjacent and Δ is a clique that does not contain vertices adjacent to a vertex outside $u^\perp\cup[w]\cup[z]$.

We say that a vertex d of $[u]\cap[w]\cap[z]$ has type (j) if $[d]$ contains j vertices of $([w]-[u]\cup[z])\cup([z]-[u]\cup[w])$. Obviously, $0 \le j \le 2$. If $\mu(u,w) \ne \mu(u,z)$, then, without loss of generality, we may assume that $\mu(u,w) = k-2b_1+1$ and $\mu(u,z) = k-2b_1+3$.

We prove that, for any two vertices $d, e \in \Delta(d)$, the subgraph $[d]\cap[e]$ contains $\lambda-1+\gamma$ vertices of $\{u,w,z\}\cup([u]\cap([w]\cup[z]))\cup([w]\cap[z]-[u])$, where $\gamma = |\Delta-([d]\cup[e])|$. This follows from analysis of all possible cases. We consider two cases in detail.

Suppose $\mu(u,w) = k-2b_1+1$ and $\mu(u,z) = k-2b_1+3$, and let d and e be vertices of type (1). Then $[d]\cap[e]$ contains u, w, z, $k-2b_1-1$ vertices of $[u]\cap[w]$, $k-2b_1+1-\delta$ vertices of $[u]\cap[z]-[w]$, and at least $2b_1-6-(k-b_1-1-\delta-\gamma)$ vertices of $[w]\cap[z]-[u]$. Altogether, we have $k-b_1-2+\gamma$ vertices.

Suppose $\mu(u,w) = \mu(u,z) = k-2b_1+2$, the vertex d is of type (1) (for definiteness, let $[d]$ contain a vertex of $[w]-[u]\cup[z]$), and e is of type (2). Then $[d]\cap[e]$ contains u, w, z, $k-2b_1-1$ vertices of $[u]\cap[w]$, $k-2b_1+2-\delta$ vertices of $[u]\cap[z]-[w]$, and at least $2b_1-7-(k-b_1-1-\delta-\gamma)$ vertices of $[w]\cap[z]-[u]$. Altogether, we have $k-b_1-2+\gamma$ vertices. The lemma is proved. □

§2. Reduction to graphs of diameter 3

In this section, Γ is an edge-regular graph with $k = 3b_1-3$. If the diameter of Γ is 2, then the conclusion of the theorem is valid by the results of [3].

Proposition 1. *Let* Γ *be a connected edge-regular graph of diameter greater than* 3 *and with parameters* (v,k,λ)*. If* $k = 3b_1-3$*, then* $\Gamma \in \mathcal{T}(3)\cup\mathcal{E}(4)$*, or* Γ *is a locally hexagonal graph.*

In this section, we assume that the diameter of Γ is at most 4. For each geodesic 3-path $abcd$, the subgraph $[b] \cap [d]$ lies in $[b] - [a]$, and therefore, $\mu(b,d) \leq b_1$. We fix a geodesic 4-path $uwxyz$ and put $\Delta = x^\perp - (u^\perp \cup z^\perp)$. If $b_1 \leq 3$, then the conclusion of the proposition is valid by Lemmas 1.2 and 1.3. Thus, we may assume that $b_1 \geq 4$.

Lemma 2.1. *The sum $\mu(u,x) + \mu(x,z)$ does not exceed $2b_1 - 3$.*

Proof. Let $\mu(u,x) = b_1$. Then $x^\perp \cap d^\perp = x^\perp - [u]$ for every vertex $d \in [x] \cap [z]$. If d and y are distinct vertices of $[x] \cap [z]$, then $d^\perp \cap y^\perp = \Delta \cup ([x] \cap [z])$, which contradicts the fact that the vertex z is adjacent to d and y. Hence, $\mu(x,z) = 1$ and $b_1 \leq 3$, a contradiction.

Let $\mu(u,x) = \mu(x,z) = b_1 - 1$. Then $|\Delta| = b_1$, and for each vertex $w \in [u] \cap [x]$ there is a unique vertex of $([u] \cap [x]) \cup \Delta$ that does not lie in $w^\perp$. The number of edges between $\Delta - \{x\}$ and $([u] \cap [x]) \cup ([x] \cap [z])$ is at least $2(b_1 - 1)(b_1 - 2)$. Therefore, each vertex in $\Delta - \{x\}$ is adjacent to exactly $2b_1 - 4$ vertices in $([u] \cap [x]) \cup ([x] \cap [z])$, and the subgraph $\Delta(x)$ is a $(b_1 - 1)$-coclique. Furthermore, the above-mentioned number of edges is equal to $2(b_1 - 1)(b_1 - 2)$, and the subgraphs $[u] \cap [x]$ and $[x] \cap [z]$ are cliques. For any two vertices $w, w' \in [u] \cap [x]$, the subgraph $[w] \cap [w']$ contains u, $b_1 - 3$ vertices of $[u] \cap [x]$, and exactly $b_1 - 2$ vertices of Δ.

Let d be a vertex of $[w] \cap [y]$ distinct from x. Since $|[u] \cap [x] \cap [d]| = b_1 - 2$ and each vertex of $[u] \cap [x] \cap [d]$ is adjacent to exactly $b_1 - 2$ vertices of $\Delta - \{x\}$, we have $\mu(u,d) = \mu(d,z) = b_1 - 1$. Since $b_1 = 4$, we see that $[d] \cap [x]$ contains at least 2 vertices of $[u]$. By symmetry, $[d] \cap [x]$ contains at least 2 vertices of $[z]$, which contradicts Lemma 1.5. ☐

Lemma 2.2. $\mu(u,x) = \mu(x,z) = b_1 - 2$.

Proof. We assume that $\mu(u,x) = b_1 - 2$ and $\mu(x,z) = b_1 - 1$. Then $|\Delta| = b_1 + 1$, and for each vertex $w \in [u] \cap [x]$ there is a unique vertex of Δ that does not lie in $[w]$. Therefore, $\Delta - \{x\}$ contains two vertices d and e that are adjacent to all vertices of $[u] \cap [x]$. Lemma 1.4 shows that $\mu(u,d) \geq b_1$ and $\mu(u,e) \geq b_1$. By Lemma 2.1, $d, e \in \Gamma_3(z)$. Now, for each vertex y of $[x] \cap [z]$, the subgraph $[x] \cap [y]$ contains $b_1 - 2$ vertices of $[x] \cap [z]$ and the same number of vertices of Δ. Therefore, $[x] \cap [z]$ is a clique, and, for distinct $y, y' \in [x] \cap [z]$, the subgraph $[y] \cap [y']$ contains z, $b_1 - 3$ vertices of $[x] \cap [z]$, and $b_1 - 1$ vertices of Δ, a contradiction. The lemma is proved. ☐

Now, we complete the proof of Proposition 1. By Lemmas 1.1 and 2.2, we have $\mu(u,d) = \mu(d,z) = b_1 - 2$ for every vertex d of $[w] \cap [y]$. In particular, $[w] - u^\perp$ and $[y] - z^\perp$ lie in $d^\perp$. Next, $|\Delta| = b_1 + 2$, and the number of edges between $([u] \cap [x]) \cup ([x] \cap [z])$ and $\Delta - \{x\}$ is equal to $2(b_1 - 2)(b_1 - 1)$. On the other hand, each vertex of $\Delta - \{x\}$ is adjacent to at most two vertices of $([u] \cap [x]) \cup ([x] \cap [z])$ by Lemma 1.4. Therefore, $2(b_1 - 2)(b_1 - 1) \leq 2(b_1 + 1)$ and $b_1 \leq 3$, a contradiction. Proposition 1 is proved.

§3. Graphs of diameter 3

Suppose Γ is a graph of diameter 3 that provides a counterexample to the theorem. Then $k = 3b_1 - 3$ and $\lambda = 2b_1 - 4$. By Lemma 1.1, the degree of each vertex in a μ-subgraph of Γ is at least $b_1 - 3$. If $b_1 = 2$, then $\Gamma \in \mathcal{T}(3)$ by Lemma 1.2. If $b_1 = 3$, then, by Lemma 1.3, the neighborhoods of vertices in Γ are either hexagons or consist of two isolated triangles. In any case, we obtain a contradiction with the choice of Γ. Therefore, $b_1 \geq 4$.

Proposition 2. *Let Γ be a connected edge-regular graph of diameter 3 and with parameters (v,k,λ). If $k = 3b_1 - 3$ and $b_1 \geq 4$, then $|\Gamma_3(u)| \geq 1$ for every vertex u.*

Let the conditions of Proposition 2 be fulfilled. We fix a geodesic 3-path $uwxy$. In Lemmas 3.1–3.11, we prove that $b_2(u,x)=1$. We have $[y]\cap\Gamma_3(w)\subset\Gamma_2(u)$ (see Lemmas 3.2–3.4). For $a\in\Gamma_2(u)$ and $\Delta(a)=[a]\cap\Gamma_3(u)$, the subgraph $\Delta(a)$ is a clique in $\Gamma_2(w)$ (see Lemma 3.5). Suppose that $[x]$ contains two vertices y and z of $\Gamma_3(u)$. Then $\mu(u,x)<b_1$ (see Lemma 3.1), $[y]-z^\perp\subset\Gamma_2(u)$ (see Lemma 3.6), and each vertex d of $[u]\cap\Gamma_2(y)$ is adjacent to a vertex of $[y]\cap[z]$ (see Lemma 3.7). Finally, each vertex of $[u]\cap\Gamma_2(y)$ is adjacent to at most one vertex of $[y]-z^\perp$ (see Lemmas 3.8–3.10).

Lemma 3.1. *If $[x]$ contains a vertex z of $\Gamma_3(u)-\{y\}$, then $\mu(u,x)<b_1$.*

Proof. Assuming that $[x]$ contains a vertex z of $\Gamma_3(u)-\{y\}$ and $\mu(u,x)=b_1$, we show that

(1) $x^\perp\cap y^\perp=x^\perp\cap z^\perp=y^\perp\cap z^\perp$.

Observe that $[x]\cap[u]$ contains b_1 vertices outside $y^\perp\cup z^\perp$. Therefore, $x^\perp\cap y^\perp=x^\perp\cap z^\perp$, the vertices y and z are adjacent, and $x^\perp\cap y^\perp=y^\perp\cap z^\perp$. Statement (1) is proved. Now we prove the following:

(2) if $a\in[w]\cap[y]-\{x\}$ and $\mu(u,a)=b_1$, then the vertices x and z are not adjacent to a.

Let $a\in[w]\cap[y]-\{x\}$ and $\mu(u,a)=b_1$. If a is adjacent to z, then, by statement (1), we have $a^\perp\cap y^\perp=a^\perp\cap z^\perp=y^\perp\cap z^\perp$, whence a is adjacent to x, a contradiction. Thus, the vertices x and z are not adjacent to a. Statement (2) is proved. Next, we show that

(3) if d is a vertex of $[w]\cap[y]$ adjacent to x, then $\{w\}=[u]\cap[d]\cap[x]$.

Let d be a vertex of $[w]\cap[y]$ adjacent to x. If $\mu(u,d)=b_1-2$, then the triple u,x,d is almost good. Since $[d]\cap[x]$ contains the vertices y and z, which are not adjacent to any vertex of $[u]\cap[x]\cap[d]$, we see that $\{w\}=[u]\cap[d]\cap[x]$ by Lemma 1.5.

Let $\mu(u,d)=b_1-1$. Then $[d]-y^\perp$ contains b_1-1 vertices of $[u]$ and a unique vertex c outside $[u]$. By symmetry, $[d]-z^\perp$ contains b_1-1 vertices of $[u]$ and a unique vertex e outside $[u]$. If $e=c$, then $d^\perp\cap y^\perp=d^\perp\cap z^\perp=y^\perp\cap z^\perp$, which contradicts the fact that $w\in[d]\cap[x]$. Thus, $e\neq c$, and $y^\perp\cap z^\perp$ contains $\lambda+1$ vertices of $d^\perp$. Therefore, $x^\perp\cap d^\perp$ contains $\lambda+1$ vertices outside $[u]$ and a unique vertex w of $[u]$. Statement (3) is proved. The next statement to be verified is

(4) the subgraph $[u]\cap[x]$ is a clique.

Suppose the degree of w in the graph $[u]\cap[x]$ is equal to b_1-3. Then $[w]-u^\perp\subset x^\perp$, and, by statement (1), we have $[w]\cap[y]=[w]\cap[z]=[w]-u^\perp$.

We assume that the subgraph $[w]\cap[y]$ contains two nonadjacent vertices d and d' and put $\delta=|[u]\cap[d]\cap[d']|$. Then $\mu(u,d)=\mu(u,d')=b_1-1$, and, by Lemma 1.5, we obtain $\delta\le 2$. If $\delta=1$, then $[u]\cap[w]$ contains b_1-2 vertices in each of the subgraphs $[d]$ and $[d']$, and, by statement (3), does not intersect $[x]$. This contradicts the fact that $b_1\le 3$ in this case. If $\delta=2$, then $[u]\cap[w]$ contains a vertex of $[d]\cap[d']$ and b_1-3 vertices in each of the subgraphs $[d]-[d']$ and $[d']-[d]$ of $[x]$. Hence, $b_1=4$. This contradicts the fact that $[w]\cap[w']$ contains u,d,d', and one vertex of each of $[u]\cap[d]$ and $[u]\cap[d']$. Thus, the subgraph $[w]\cap[y]$ is a b_1-clique.

Let $d,d'\in[w]\cap[y]-\{x\}$, and let $\mu(u,d)=\mu(u,d')=b_1-1$. Since $[d]\cap[d']$ contains the vertices y and z outside $[w]\cup[w']$, where w and w' are distinct vertices of $[u]\cap[d]\cap[d']$, Lemma 1.5 implies that $[u]\cap[d]\cap[d']=\{w\}$ for every two vertices d and d' of $[u]\cap[y]$.

If $b_1\ge 5$, then the degree of w in $[u]$ is at least $4(b_1-3)$, a contradiction. Thus, $b_1=4=\lambda$, which is impossible because the graph $[d]\cap[x]$ contains w,y,z, and two vertices of $[w]\cap[y]$.

Assume that the degree of w in the graph $[u]\cap[x]$ is b_1-2. Then $[u]\cap[x]-w^\perp$ contains a single vertex t, and the degree of x in the graphs $[w]\cap[y]$ and $[t]\cap[y]$ is b_1-2. Since $[x]$ contains the vertices y and z outside $w^\perp\cup t^\perp$, we see that the degree of x in the graph $[w]\cap[t]$ is at least b_1-1, and $[w]\cap[t]$ contains a vertex d of $\Gamma_2(u)\cap[x]$. Since

$[x] - y^{\perp} = [u] \cap [x]$, statement (1) shows that the vertex d is adjacent to y and z, and $\mu(u, d) < b_1$ by statement (2). Since $[u] \cap [d]$ contains the nonadjacent vertices w and t, we obtain $\mu(u, d) = b_1 - 1$, and $[u] \cap [d] \cap [x]$ contains the vertices w and t. This contradicts (3). Thus, the degree of each vertex w of the graph $[u] \cap [x]$ is $b_1 - 1$, so that $[u] \cap [x]$ is a clique. Statement (4) is proved. Finally, we prove the following statement:

(5) $\mu(u, d) = b_1 - 2$ whenever $w \in [u] \cap [x]$ and $d \in [w] \cap [x] \cap [y]$.

If $\mu(u, d) = b_1 - 1$ for a vertex $d \in [w] \cap [x] \cap [y]$, then, by statement (3), the subgraph $[x] - d^{\perp}$ contains $b_1 - 1$ vertices of $[u]$ and a vertex of $[y] \cap [z]$. Since $[w] \cap [u]$ contains $b_1 - 1$ vertices of $[x]$ and $b_1 - 3$ vertices of $[d]$, we see that $[u] \cap [d]$ contains a unique vertex w' that does not belong to $w^{\perp}$. Since $y^{\perp} \cap z^{\perp}$ lies in $x^{\perp}$, the triple w, y, z is almost good, d is adjacent to a vertex w' outside $[w] \cup y^{\perp} \cup z^{\perp}$, and $b_1 = 4$ by Lemma 1.5. Since $[d]$ contains the vertices y and z outside $w^{\perp} \cup (w')^{\perp}$, the degree of d in the graph $[w] \cap [w']$ is equal to 3, and $[w']$ contains a vertex r of $[w] \cap ([y] - [z])$ and a vertex s of $[w] \cap ([z] - [y])$. The graph $[d]$ contains the vertex w outside $y^{\perp} \cup z^{\perp} \cup (w')^{\perp}$. Therefore, the degree of d in the graphs $[y] \cap [w']$ and $[z] \cap [w']$ is equal to 3, and $[w']$ contains a vertex t adjacent to d, y, and z. We have $\mu(w', y) = \mu(w', z) = b_1 - 1$, and d is adjacent to the vertex w outside $(w')^{\perp} \cup [y] \cup [z]$ and $[d] \cap [t] = \{w', x, y, z\}$. Therefore, the vertices r and s are not adjacent to t, and t is adjacent to a vertex t' of $[x] \cap [z] - d^{\perp}$. Using Lemma 1.4, we obtain $\mu(t, w) \geq 3$, and $[t] \cap [w]$ contains a vertex of $[u] \cap [x]$. By Lemma 1.1, the subgraph $[x]$ contains a vertex outside $t^{\perp} \cup w^{\perp}$, which contradicts the fact that this vertex is in $[y] \cap [z] = \{d, x, t, t'\}$. Statement (5) is proved.

Now, we complete the proof of the lemma. By statement (5) and Lemma 1.4, we have $\{w\} = [u] \cap [d] \cap [e]$ for all vertices $d, e \in [w] \cap [x] \cap [y]$. If $b_1 \geq 5$, then $[w] \cap [x] \cap [y]$ contains at least two vertices and $2(b_1 - 3) + b_1 - 1 \leq 2b_1 - 4$, a contradiction. Thus, $b_1 = 4$, and, for each vertex $w \in [u] \cap [x]$, we have a unique vertex of $[w] \cap [z]$ adjacent to x. Since x is adjacent to three vertices of $[y] \cap [z]$, we see that there is a vertex d of $[x] \cap [y] \cap [z]$ adjacent to two vertices w and w' of $[u] \cap [x]$. But this is impossible because $[w] \cap [w']$ contains u, x, d, and 2 vertices of $[u] \cap [x]$. The lemma is proved. □

In Lemmas 3.2–3.3, we assume that $[y]$ contains a vertex z of $\Gamma_3(u) \cap \Gamma_3(w)$. For $a \in [w] \cap [y]$, let $\Sigma_a = a^{\perp} - ([u] \cup [z])$. We put $\Sigma = \Sigma_x$. Let $b \in ([u] \cap \Gamma_2(z)) \cup ([z] \cap \Gamma_2(u))$, and let $[b]$ contain i vertices of $\Gamma_2(z) \cap \Gamma_2(u)$. We say that the vertex b is *strong* if $i = 2$, and *weak* if $i = 1$.

Lemma 3.2. *Suppose* $[u] \cap [x]$ *contains a vertex adjacent to the vertices of* $[x] \cap [z]$. *Then the following statements are valid*:

(1) *if* $\mu(u, x) = b_1$, *then* $|\Sigma| = 2b_1 - 2 - \mu(x, z)$, $x^{\perp} - [u] = x^{\perp} \cap y^{\perp}$, *and each vertex of* $[x] \cap [z] - \{y\}$ *is adjacent to a vertex of* $[u] \cap [x]$;

(2) *if* $a \in [x] \cap [z] \cap \Gamma_2(u)$ *and the degree of* a *in the graph* $[x] \cap [z]$ *is equal to* $b_1 - 3$, *then* $\mu(u, x) = b_1$, *and the subgraph* $[u] \cap [a]$ *is a clique*;

(3) *if* e *is a strong vertex of* $[x] \cap [y] \cap [z]$ *and the degree of* e *in* $[x] \cap [z]$ *is* $b_1 - 3$, *then* $\mu(x, z) = b_1 - 2$;

(4) *each vertex of* $[x] \cap [z] \cap \Gamma_2(u)$ *is adjacent to* y.

Proof. We assume that some vertex of $[u] \cap [x]$ is adjacent to a vertex of $[x] \cap [z]$. Let $\mu(u, x) = b_1$. Then $[x] \cap [u]$ contains b_1 vertices outside $x^{\perp} \cap y^{\perp}$. Therefore, $x^{\perp} - [u] = x^{\perp} \cap y^{\perp}$. Next, $|\Sigma| = 2b_1 - 2 - \mu(x, z)$ and $x^{\perp} \cap y^{\perp} = \Sigma \cup ([x] \cap [z])$ for $y \in [x] \cap \Gamma_3(u)$. If a vertex a of $([x] \cap [z]) - \{y\}$ is not adjacent to a vertex of $[u] \cap [x]$, then $x^{\perp} \cap a^{\perp} = x^{\perp} - [u]$, $\Sigma \subset [a]$, and $a^{\perp} \cap y^{\perp} = \Sigma \cup ([x] \cap [z])$. This contradicts the fact that z is adjacent to a and y. Statement (1) is proved.

Suppose $a \in [x] \cap [z] \cap \Gamma_2(u)$ and the degree of a in $[x] \cap [z]$ is $b_1 - 3$. By Lemma 1.1, the subgraph $[a] - z^{\perp}$ lies in $x^{\perp}$. In particular, $[u] \cap [a] \subset [x]$. If $\mu(u, a) = b_1 - 2$, then

$\mu(u,x) = b_1$ by Lemma 1.4. Now, assume that $\mu(u,x) < b_1$. Then $\mu(u,a) = \mu(u,x) = b_1 - 1$ and $[u]\cap[a] = [u]\cap[x]$. This contradicts the fact that any vertex w of $[u]\cap[x]$ is not adjacent to a. Thus, $\mu(u,x) = b_1$.

If the subgraph $[u]\cap[a]$ is not a clique, then a is a weak vertex, and $[u]\cap[a] = [u]\cap[x] - \{w\}$ contains two nonadjacent vertices d and d'. Since $[a]$ contains the vertex d' outside $d^\perp \cup z^\perp$, the degree of a in the graph $[d]\cap[z]$ is $b_1 - 2$, $\mu(d,z) = b_1 - 1$, and $[d]\cap[x]$ contains $b_1 - 2$ vertices of each of $[u]$ and $[z]$. This contradicts the fact that $a^\perp$ contains only $b_1 - 3$ vertices of $[x]\cap[z] - \{y\}$. Statement (2) is proved.

Let $e \in [x]\cap[z]$, and let e be adjacent to a vertex x' of $\Gamma_2(u)\cap\Gamma_2(z) - \{x\}$. If the degree of e in the graph $[x]\cap[z]$ is $b_1 - 3$, then $[e] - z^\perp \subset x^\perp$, and $\mu(u,x) = b_1$ by Lemma 1.4. If $\mu(x,z) > b_1 - 2$, then the subgraph $[x]\cap[z] - e^\perp$ contains a vertex e'. First, we assume that $e' \in \Gamma_2(u)$. By Lemma 1.1, the subgraph $[u]\cap[e']$ does not intersect $[e]$. If e' is a weak vertex, then $[e']\cap[x]$ contains at most $b_1 - 2$ vertices of $[z]$ and at most $b_1 - 2$ vertices of $[u]$; but $2(b_1 - 2) \le b_1 - 1$, a contradiction. Thus, the vertex e' is strong and $[e']\cap[x]$ contains a vertex x'' of Σ, $b_1 - 2$ vertices of $[z]$, and $b_1 - 3$ vertices of $[u]$. Therefore, $2b_1 - 5 \le b_1 - 1$, $b_1 = 4$, and $\mu(u,x) = \mu(x,z) = 4$. Hence, $|\Sigma| = 2$ and $x' = x''$. Since the degree of x in $[e']\cap[w]$ is at least 2, we have $x' \in [w]$. Similarly, the degree of x' in $[e]\cap[w]$ is at least 2. Consequently, $[e]\cap[u]\cap[x']$ contains d, and the subgraph $[x]\cap[x']$ contains d, w, e, and e'. Therefore, the vertex x' is not adjacent to y. However, $[x] - ((x')^\perp \cup y^\perp)$ contains a vertex of each of $[e]\cap[z]$ and $[e']\cap[z]$, and the degree of x in the graph $[x']\cap[y]$ is at least 3, which is impossible because $|[x]\cap[x']| \ge 5$ in this case.

Thus, $[x]\cap[z] - e^\perp = \{y\}$, $\mu(x,z) = b_1 - 1$, and $|\Sigma| = b_1 - 1$. This contradicts the fact that $[x]\cap[y]$ contains $b_1 - 2$ vertices of Σ and at most $b_1 - 3$ vertices of $[z]$. Statement (3) is proved.

Suppose a belongs to $[x]\cap[z]\cap\Gamma_2(u)$ and is not adjacent to y. Then $\mu(x,z) \ge b_1 - 1$. If the degree of a in the graph $[x]\cap[z]$ is equal to $b_1 - 3$, then the vertex a is weak by (3), and the subgraph $[u]\cap[a]$ is a $(b_1 - 1)$-clique by (2). In particular, $\mu(u,x) = b_1$. We note that $[x] - (w^\perp \cup y^\perp)$ contains a, so that the degree of x in the graph $[w]\cap[y]$ is at least $b_1 - 2$, $|\Sigma| = b_1 - 1 = \mu(x,z)$, and $[w]\cap[y]$ contains Σ. This contradicts the fact that $[x]\cap[y]$ contains $b_1 - 2$ vertices of Σ and at most $b_1 - 3$ vertices of $[x]\cap[z]$. Thus, the degree of a in the graph $[x]\cap[z]$ is equal to $b_1 - 2$, and $\mu(x,z) = b_1$.

If the vertex a is weak, then, by (2), the subgraph $[u]\cap[a]$ is a $(b_1 - 1)$-clique and $|[u]\cap[a]\cap[x]| = b_1 - 2$. As above, $|\Sigma| = b_1 - 1 = \mu(x,z)$ and $[w]\cap[y]$ contains Σ. Hence, $[x]\cap[w]$ contains $b_1 - 2$ vertices in Σ and in $[u]\cap[x]$; it follows that $[u]\cap[x]$ is a clique. If d and d' are vertices in $[u]\cap[a]\cap[x]$, then $[d]\cap[d']$ contains a, u, w, x, $b_1 - 4$ vertices of $[u]\cap[a]\cap[x]$, and a vertex of $[u]\cap[a] - [x]$; in particular, we have $b_1 > 4$. If some vertex of $[u]\cap[a]\cap[x]$ is adjacent to a vertex outside $u^\perp \cup [a] \cup [x]$, then, by Lemma 1.5, we obtain $b_1 - 2 = 2$, a contradiction. Therefore, each vertex of $[u]\cap[a]\cap[x]$ is adjacent to a unique vertex of each of subgraphs $[x] - (a^\perp \cup [u])$ and $[a] - (x^\perp \cup [u])$. However, $[x] - (a^\perp \cup [u]) = \Sigma(x) \cup \{y\}$, so that each vertex of $[u]\cap[a]\cap[x]$ is strong. Next, the degree of a in the graph $[z]\cap(\bigcup_{d\in[u]\cap[a]\cap[x]}[d])$ is at least $3(b_1 - 3)$ and $b_1 = 5$. We put $\{d_1, d_2, d_3\} = [u]\cap[a]\cap[x]$ and $\{x_1, x_2, x_3\} = \Sigma(x)$; the vertices are numbered so that d_i is adjacent to x_i. The vertex d_i is adjacent to a unique vertex e_i of $([a]\cap[x]) - [u]$, and $\{e_1, e_2, e_3\} \cup \Sigma(x) = [x]\cap[y]$. Since $[e_1]\cap[x]$ contains a, d_1, y, and, possibly, also e_2, e_3 and at most one vertex of $\Sigma(x)$, we see that $e_2, e_3, x_1 \in [e_1]$ and $[d_1]\cap[y] = \{e_1, x, x_1\}$.

We put $\{d_1, d_1', d_1''\} = [u]\cap[e_1]$. Then $[d_1]\cap[w]$ contains u, x, x_1, d_2, d_3, and a vertex of $[u]\cap[e_1]$, say, d_1''. The subgraph $[x]\cap[x_1]$ contains w, y, d_1, e_1, and the vertices x_1 and x_2; in particular, Σ is a 4-clique. We note that $[x_1] - y^\perp$ lies in $d_1^\perp$ and $\mu(u,x_1) \ge 4$, whence $[x_1] \cap [u] = \{w, d, d_1', d_1''\}$. Replacing the triple x, a, y by x_1, d_1', w, we obtain a contradiction with the fact that the degree of d_1' in the graph $[u] \cap [x]$ is $2 = b_1 - 3$.

If a is a strong vertex, then $[a]$ contains two vertices x and x' of $\Gamma_2(u) \cap \Gamma_2(z)$, $\mu(u, a) = b_1 - 2$, and the degree of x in the graph $[w] \cap [a]$ is at least $b_1 - 2$. Therefore, a single vertex b of $[a] - (x^\perp \cup z^\perp)$ lies either in $[u]$ or in $\Gamma_2(u) \cap \Gamma_2(z)$. In the latter case, we have $|[u] \cap [a] \cap [x]| = b_1 - 2$ and $\mu(u, x) = b_1 - 1$, which contradicts Lemma 1.4. Thus, $b \in [u] \cap [a]$ and $x' \in [w] \cap [x]$, and $b_1 = 4$ by Lemma 1.4. Since $[x]$ contains a vertex a outside $w^\perp \cup y^\perp$, the degree of x in the graph $[w] \cap [y]$ is at least 2, and Σ contains at least three vertices of $[w] \cap [y]$.

We put $\{d, b\} = [u] \cap [a]$. If the degree of d in the graph $[u] \cap [x]$ is equal to 1, then $[d] - u^\perp \subset x^\perp \cap a^\perp$. In the case where the vertex d is weak, we obtain a clique $[d] \cap [z] = \{a, e, e'\}$, where $[e] \cap [e']$ contains a, d, x, z, and either x' or a vertex of $[u] \cap [d] - \{w, d'\}$, a contradiction. Thus, the vertex d is strong and, using (3), we obtain $\mu(u, x) = 2$ and $\Sigma = [w] - u^\perp$. Let $[x] \cap [z] = \{a, e, e', y\}$, where e is adjacent to d. If e is not adjacent to y, then $\Sigma \subset [y]$, the vertex e is strong, and $[x] \cap [x']$ contains d, w, a, e, and y, a contradiction. Thus, e is adjacent to y, $|\Sigma \cap [y]| = 3$, and $[x] \cap [e]$ contains a, d, y, and a vertex of $\{e', x'\}$. By Lemma 1.4, the vertex e is not adjacent to b. If e is adjacent to e', then e is weak and $[u] \cap [e] = \{d, c, c'\}$. In this case, $[d] \cap [x']$ contains a, w, x, and a vertex of $\{c, c'\}$, say, c. Since c is not adjacent to x and has degree 2 in the graph $[u] \cap [e]$, we conclude that c is weak and $[c] \cap [z]$ contains three vertices of $(e^\perp \cap [z]) - \{a, y\}$. This contradicts the fact that, in this case, $[e] \cap [e']$ contains a, c, x, y, and z. Thus, $e \in [x'] - [e']$, $[x] \cap [x'] = \{a, d, e, w\}$, x' is not adjacent to y, and $[d] \cap [y] = \{e, x\}$. By Lemma 1.4, we have $\mu(x', z) = 4$. Observe that $[d] - u^\perp \subset x^\perp \cap (x')^\perp$, whence the degree of d in the graph $[u] \cap [x']$ is 1 and $b, c \notin [x']$. Hence, $\mu(b, z) = \mu(c, z) = 4$, and Lemma 3.1 shows that the vertices b and c are not adjacent to w.

Now, $\mu(e, w) = \mu(a, w) = 3$ and $[e] \cap [w] = [a] \cap [w] = \{d, x, x'\}$. Since $\{d, x, x'\}$ is a clique, we see that, by Lemma 1.5, the subgraph $[a] - e^\perp$ contains the vertices b, e', and f, and the graph $[e] - a^\perp$ contains the vertices c, y, and g adjacent to d, x, and x', respectively. We put $\{r, s\} = ([w] \cap [x']) - \{d, x\}$. Then $[x'] \cap [r]$ contains the vertices w and s of $[u]$ and the vertices f and g outside $u^\perp$. By symmetry, $[x'] \cap [s]$ contains f and g. This contradicts the fact that, in this case, $[r] \cap [s]$ contains u, w, x', f, and g.

Thus, the degree of d in $[u] \cap [x]$ is equal to 2, and $[d] - u^\perp$ contains the vertices a and e of $[x] \cap [z]$ and the vertex e' of $[z] - [x]$. Therefore, the vertex d is weak and $[d] \cap [y] = \{x, e\}$. By Lemma 1.4, the vertex b is not adjacent to e, the triple u, x, e is almost good, and $[u] \cap [x] \cap [e]$ contains exactly two vertices d and d'. We note that x' is adjacent to b, otherwise b is adjacent to e' and $[a] \cap [x']$ contains x and three vertices of $[a] \cap [z] - \{e\}$; in particular, x' is adjacent to e'. This contradicts the fact that $\mu(u, e') = 3$ and $[u] \cap [a] \subset [e']$.

We put $[a] \cap [x] \cap [z] = \{e, f\}$ and assume that f is adjacent to x'. Then $[x] \cap [f]$ contains a, x', y, and d'. However, if f is adjacent to d', then $[e] \cap [f]$ contains a, d', x, y, and z. Therefore, the vertices e and f are not adjacent. Since the degrees of the vertices e and f in the graph $[d'] \cap [z]$ are equal to 1, we see that the subgraphs $[u] \cap [e]$ and $[u] \cap [f]$ are contained in $(d')^\perp$. Now, $[d']$ contains a vertex w outside $[e] \cup [f]$, so that $[u] \cap [e] \cap [f]$ contains the adjacent vertices d' and d''. Hence, the triple u, e, f is almost good, $[u] \cap [e] \cap [f]$ contains exactly two vertices d' and d'', and $[e] \cap [f]$ contains two vertices y and z none of which is adjacent to d or d'. This contradicts Lemma 1.5.

Thus, the vertices x' and d' are not adjacent to f and $[x] \cap [f]$ contains a, e, y, and a vertex x'' of Σ. Furthermore, $[x] \cap [x']$ contains a, w, y, and the vertex x''. Finally, $[x] \cap [d']$ contains d, e, w, and x'', which is impossible because $[x] \cap [x''] = \{d', x', w, f, y\}$. The lemma is proved. □

Lemma 3.3. *The subgraph* $[u] \cap [x]$ *contains no vertices adjacent to vertices of* $[x] \cap [z]$*, and* $\mu(u, x) < b_1$.

Proof. We assume that some vertex of $[u] \cap [x]$ is adjacent to a vertex of $[x] \cap [z]$. We prove that

(a) $\mu(x, z) > b_1 - 2$.

Suppose $\mu(x, z) = b_1 - 2$. Statement (3) of Lemma 3.2 implies that $\mu(u, x) = b_1$. By Lemma 3.1, we obtain $\Sigma \subset \Gamma_2(u)$. If e and e' are vertices of $[x] \cap [z] - \{y\}$, then the subgraphs $[u] \cap [e]$ and $[u] \cap [e']$ are contained in $[x]$, and they do not intersect because otherwise, for d in $[u] \cap [e] \cap [e']$, the subgraph $[x] \cap [d]$ contains two vertices e and e' of $[z]$, and $\mu(d, z) < b_1$. This contradicts Lemma 1.4. Then $1 + 2(b_1 - 2) \leq b_1$ and $b_1 \leq 3$, a contradiction. Thus, $[x] \cap [z] = \{e, y\}$, $b_1 = 4$, and $\Sigma \subset [y]$.

If the vertex e is weak, then e is adjacent to no vertex of $\Sigma(x)$ and $\mu(u, e) = 3$. By statement (2) of Lemma 3.2, the subgraph $[u] \cap [e]$ is a clique, and statement (4) of the same lemma shows that the subgraph $[u] \cap [e]$ lies in $[w]$. Therefore, for any two vertices $d, d' \in [u] \cap [e]$, the subgraph $[d] \cap [d']$ contains e, u, w, x, and a vertex of $[u] \cap [e]$, a contradiction.

Thus, the vertex e is strong, e is adjacent to a vertex x' of $\Sigma(x)$, and $\mu(u, e) = 2$. We have $\mu(x', z) = 4$ by Lemma 1.4. The subgraph $[e] \cap [x]$ contains y, x', and two vertices d and d' of $[u] \cap [e]$. Applying statement (4) of Lemma 3.2 to the path $zyxwu$, we see that the vertices d and d' are adjacent to w and $[d] \cap [d'] = \{u, w, x, e\}$. By Lemma 1.4, we have $\mu(w, e) = \mu(u, x) = 4$; in particular, w is adjacent to x'. If the vertices d and d' are not adjacent to x', then the degree of e in each of the subgraphs $[d] \cap [z]$ and $[d'] \cap [z]$ is equal to 2. Since $|[e] \cap [z] - \{y\}| = 3$, the triple z, d, d' is almost good, $[d] \cap [d'] \cap [z]$ contains at least 2 vertices, and $[d] \cap [d']$ contains two vertices u and w not adjacent to vertices of $[d] \cap [d'] \cap [z]$. This contradicts Lemma 1.5. Thus, we may assume that $x' \in [d] - [d']$, so that $[x] \cap [x'] = \{d, e, w, y\}$.

Let $c \in [u] \cap [x] - \{w, d, d'\}$. If c is not adjacent to w, then the degree of c in the graph $[u] \cap [x]$ is 1 and $[c] - u^{\perp} = \Sigma$, which contradicts the fact that the vertex x' is not adjacent to c. Thus, c is adjacent to w and $[w] \cap [x] = \{c, d, d', x'\}$. In particular, $[w] \cap [y] = \{x, x'\}$. Now, for $\{r, s\} = \Sigma - \{x, x'\}$, the subgraph $[r] \cap [x]$ contains c, d', y, and the vertex s. By symmetry, $[s] \cap [x]$ contains c, d', y, and the vertex r. This is impossible because d' is adjacent to three vertices of Σ. Statement (a) is proved. Now, we show that

(b) $[u] \cap [x]$ contains no weak vertices.

Let d be a weak vertex of $[u] \cap [x]$. We prove that the subgraph $[d] \cap [x] \cap [z]$ consists of strong vertices, the degree of d in the graph $[u] \cap [x]$ is equal to $b_1 - 1$, $\mu(u, x) = b_1$, and $b_1 = 4$.

If the subgraph $[d] \cap [x] \cap [z]$ contains a weak vertex e, then, by statement (3) of Lemma 3.2 and statement (a) above, the degree of e in the graph $[d] \cap [z]$ is equal to $b_1 - 2$. By symmetry, the degree of d in the graph $[u] \cap [e]$ is equal to $b_1 - 2$. Therefore, $[d] \cap [e]$ contains x, $b_1 - 2$ vertices of $[u]$, and $b_1 - 2$ vertices of $[z]$, a contradiction. Thus, $[d] \cap [x] \cap [z]$ consists of strong vertices.

The subgraph $[x] \cap [d]$ contains at most $b_1 - 1$ vertices of $[u]$ and at least $b_1 - 3$ vertices of $[z]$. If $[x] \cap [d]$ contains $b_1 - 2$ vertices of $[z]$, then the degree of d in the graph $[u]$ is at least $3(b_1 - 3) + 1$ provided $b_1 \geq 5$, a contradiction. If $b_1 = 4$, then the degree of each of the vertices $e, e' \in [d] \cap [z] \cap [x]$ in the graph $[d] \cap [z]$ is equal to 2, and $[e] \cap [e']$ contains d, x, y, z, and the third vertex of $[d] \cap [z]$. Thus, the degree of d in $[u] \cap [x]$ is $b_1 - 1$, and $\mu(u, x) = b_1$.

Suppose $[x] \cap [d]$ contains $b_1 - 3$ vertices of $[z]$ and $b_1 - 1$ vertices of $[u]$. If $b_1 \geq 6$, then the degree of d in $[u]$ is at least $3(b_1 - 3) + 1$, a contradiction. If $b_1 = 5$, then we put $[d] \cap [z] = \{e, e', a, a'\}$, where e and e' are adjacent to x. Then $[e] \cap [x]$ contains d, y, a vertex x' of Σ, one vertex of $[d] \cap [z]$, and two vertices of $([d] \cap [u]) \cup ([x] \cap [z] - \{y\})$. We have $[e] \cap [e'] = \{d, x, a, a', y, z\}$. If $\mu(u, a) < 5$, then $[u] \cap [d]$ contains w, two vertices of $[e]$, and two of $[e']$. By Lemma 1.4, the degree of d in $[u] \cap [a]$ is at most 1, a contradiction. Thus, $\mu(u, a) = \mu(u, a') = 5$, and, by Lemma 3.1, the vertices a and a' are not adjacent to y. Now, $[e] - y^{\perp}$ contains a, a', and three vertices of $[u]$. Therefore, $[y] \cap [z]$ contains e, e', two vertices adjacent to e, and two vertices adjacent to e', which implies that a vertex of $[x] \cap [z] - \{e, e', y\}$ is adjacent to y and to one of the vertices e and e', say, to e. In this case, $[u] \cap [e] - [x]$ contains f, and if f is not adjacent to w, then $\mu(e, w) < 5$, and $[u] \cap [w] \cap [e]$ contains d and x, which contradicts Lemma 1.4. Thus, $\mu(e, w) = 5$, $[w] \cap [d]$ contains u, f, x, and three vertices of $[u] \cap [x]$. The vertices f and w are adjacent by Lemma 3.1. We obtain $\mu(f, z) < 5$. If $\mu(x, z) = 5$, then $[u] \cap [e'] - [x]$ contains a vertex f' adjacent to w, which contradicts the fact that, in this case, $[d] \cap [w]$ contains u, f, f', x, and three vertices of $[u] \cap [x]$. Thus, $\mu(x, z) = 4$, and $[x] \cap [e']$ contains two vertices of $[z]$, a vertex of $\Sigma(x)$, and three vertices of $[u]$. We put $[u] \cap [e'] = \{d, g, g'\}$. Then both triples z, d, g and z, d, g' are almost good, and $[d] \cap [g] \cap [g']$ contains two vertices u and w that are adjacent to none of the vertices $\{e', a, a'\}$. Applying Lemma 1.5 to the above triples, we see that $[g] \cap [g']$ contains both vertices of $[e'] \cap [z] - \{a, a', e, y\}$. This contradicts Lemma 1.5 applied to the almost good triple z, g, g'.

Finally, if $b_1 = 4$, then the subgraph $[d] \cap [z]$ contains a vertex e of $[x]$ and two vertices, a and a', outside $[x]$. Then either $\mu(u, a) = 4$, or $[d] \cap [a]$ contains e, a', and two vertices of $[u] - (\{w\} \cup [e])$. If $\mu(u, a)$ and $\mu(u, a')$ do not exceed 3, then $[a] \cap [a']$ contains d, e, z, and two vertices of $([u] \cap [d]) - (\{w\} \cup [e])$, a contradiction. Thus, we may assume that $[a]$ contains $[u] \cap [e]$, and, in particular, $\mu(u, a) = 4$. We put $[u] \cap [e] = \{d, d'\}$. If d' and x are adjacent, then $[d] \cap [d']$ contains u, w, x, a, and e, a contradiction. Thus, d' is not adjacent to x, and $[a] \cap [x]$ contains d, e, and a vertex f of $[u]$ not adjacent to e. Since $[f]$ contains the vertex u, which lies outside $a^{\perp} \cup x^{\perp}$, we see that $[a] \cap [x]$ contains a vertex g of $[x] \cap [z] - [e]$ adjacent to f, $\mu(x, z) = 4$, and $|\Sigma| = 2$. Now $[x] \cap [x']$ contains e, w, y, and yet another strong vertex s. If $s \in [z]$, then the vertex r of $[x] \cap [z] - \{y, e, s\}$ is weak, which is impossible because $[u] \cap [r] \cap [x]$ contains a strong vertex. If $s \in [u]$, then $[x] \cap [u]$ contains two weak vertices lying in $[e]$, a contradiction. Statement (b) is proved.

Now, we complete the proof of the lemma. Let a vertex d of $[u] \cap [x]$ be adjacent to a vertex e of $[x] \cap [z]$ and to a vertex x' of $\Gamma_2(u) \cap \Gamma_2(z)$. Since $|([d] \cap [z]) - [x]| \leq 2$, we see that $[x] \cap [d]$ contains at least $b_1 - 4$ vertices of $[z]$. If $[d] \cap [x] \cap [z]$ contains three vertices, then the degree of d in $[u]$ is at least $3(b_1 - 3) + 1$, and $b_1 = 4$. This contradicts the fact that, in this case, $[d] - u^{\perp}$ contains five vertices. Suppose $[d] \cap [x] \cap [z]$ contains two vertices e and e'. Then $[x] \cap [x']$ contains d, e, e', w, and y, and $b_1 \geq 5$. If $b_1 = 6$, then, by Lemma 1.1, the subgraph $[d]$ contains five vertices of $[u] \cap [x]$ and two vertices in each of $[u] \cap [e] - [x]$ and $[u] \cap [e'] - [x]$. This contradicts the fact that $\lambda = 8$.

Thus, $b_1 = 5$. If each of the subgraphs $[e] \cap [x] \cap [u]$ and $[e'] \cap [x] \cap [u]$ contains one vertex, then $[d] \cap [u]$ contains three vertices of $[x]$ and two vertices of each of $[u] \cap [e] - [x]$ and $[u] \cap [e'] - [x]$. This contradicts the fact that $\lambda = 6$. Thus, we may assume that $[u] \cap [e] \cap [x]$ contains two vertices d and d'. By Lemma 1.4, the vertices d' and e' are not adjacent. Therefore, $[e']$ contains two vertices outside $x^{\perp} \cup z^{\perp}$, whence $\mu(x, z) = 5$ and $[x] \cap [z] \subset (e')^{\perp}$. By symmetry, $[d']$ contains two vertices outside $(x')^{\perp} \cup u^{\perp}$, whence $\mu(u, x') = 5$ and $[u] \cap [x'] \subset (d')^{\perp}$. In this case $[x] \cap [x']$ contains d, d', e, e', y, w, and a vertex of $[u] \cap [x]$, a contradiction.

Thus, $[d] \cap [x] \cap [z]$ contains a unique vertex e; in particular, $b_1 \leq 5$. If $b_1 = 5$, then, as above, $[e]$ contains two vertices outside $x^{\perp} \cup z^{\perp}$, so that $\mu(x, z) = 5$ and $[x] \cap [z] \subset e^{\perp}$. By symmetry, $[d]$ contains two vertices outside $u^{\perp} \cup x^{\perp}$, whence $\mu(u, x) = 5$ and $[u] \cap [x] \subset d^{\perp}$. Similarly, $\mu(u, x') = 5$, $[u] \cap [x'] \subset d^{\perp}$, and $\mu(x', z) = 5$, $[x'] \cap [z] \subset e^{\perp}$. In this case,

$[e]\cap[z]$ contains two vertices in each of $[x]\cap[x']$, $[x]-[x']$, and $[x']-[x]$. This is impossible because $[d]\cap[z]$ contains three vertices of $[x]$.

Thus, $b_1=4$, $[x]\cap[x']=\{d,e,w,y\}$, and the degree of d in each of the graphs $[u]\cap[x]$ and $[u]\cap[x']$ is equal to 3. We put $[d]\cap[z]=\{e,e'\}$ and $[u]\cap[e]=\{d,d'\}$. Since $[d]\cap[e']$ contains e and three vertices d', f, and g of $[d]\cap[u]$, the vertices f and g are adjacent to w. This contradicts the fact that $[d]\cap[w]$ contains u, f, g, x, and x'. $\square$

Lemma 3.4. *For every geodesic* 3*-path* $uwxy$, *the subgraph* $[y]\cap\Gamma_3(w)$ *lies in* $\Gamma_2(u)$.

Proof. We assume that $[y]$ contains a vertex z of $\Gamma_3(u)\cap\Gamma_3(w)$. By Lemma 3.3, both $\mu(u,x)$ and $\mu(x,z)$ are less than b_1.

Now, suppose that $\mu(u,x)=b_1-2$. Then $|\Sigma|$ is either b_1+1 or b_1+2. Let w and e be distinct vertices of $[u]\cap[x]$. Then $[e]-u^{\perp}$ contains b_1 vertices of Σ. Therefore, $[w]\cap[e]$ contains at least b_1-2 vertices of Σ. However, for $f\in[w]\cap[e]\cap\Sigma(x)$, we obtain a path $uwfy'z$, the subgraph $[f]$ contains two vertices of $[u]\cap[x]$, and $\mu(u,f)<b_1$, a contradiction.

Thus, we have $\mu(u,x)=\mu(x,z)=b_1-1$ for each vertex $x\in[w]\cap[y]$, and $|\Sigma|=b_1$. Therefore, for any $a\in[u]\cap[x]$, there is a unique vertex of $([u]\cap[x])\cup\Sigma$ that does not lie in $a^{\perp}$. By symmetry, $|\Sigma-y^{\perp}|\le 1$. If $[u]\cap[x]$ contains two nonadjacent vertices a and b, then $[u]\cap[x]\subset[d]$ for some vertex $d\in\Sigma(x)\cap[y]$. By Lemma 1.5, we obtain $|[u]\cap[x]\cap[d]|=2$, which contradicts the fact that $b_1\ge 4$. Thus, each vertex of $[u]\cap[x]$ is not adjacent to the only vertex of Σ; in particular, the subgraph $[u]\cap[x]$ is a clique. If two distinct vertices a and b of $[u]\cap[x]$ are not adjacent to one and the same vertex e of Σ, then $[a]\cap[b]$ contains u, x, b_1-3 vertices of $[u]\cap[x]$, and b_1-2 vertices of $\Sigma(x)$, a contradiction. Thus, every vertex of $\Sigma(x)$ is adjacent to b_1-2 vertices of each of the graphs $[u]\cap[x]$ and $[x]\cap[z]$. In particular, the subgraph $\Sigma(x)$ is a (b_1-1)-coclique.

If $b_1>4$, then $\Sigma(x)$ contains two vertices d and e of $[w]\cap[y]$, $\mu(u,d)=\mu(u,e)=b_1-1$, and $|[u]\cap[d]\cap[e]|=b_1-3$, which contradicts Lemma 1.5. Thus, $b_1=4$. For $d\in\Sigma(x)\cap[w]\cap[y]$, the subgraph $[u]\cap[d]\cap[x]$ contains two vertices a and w, and $[d]\cap[x]-[u]$ contains two vertices of $[x]\cap[z]$ that do not lie in $[a]\cup[w]$. This contradicts Lemma 1.5. The lemma is proved. $\square$

For $a\in\Gamma_2(u)$, we put $\Delta(a)=[a]\cap\Gamma_3(u)$.

Lemma 3.5. *The subgraph* $\Delta(x)$ *is a clique in* $\Gamma_2(w)$.

Proof. By Lemma 3.4, we have $\Delta(x)\subset\Gamma_2(w)$. Let y and z be two nonadjacent vertices of $\Delta(x)$. By Lemma 1.1, $\mu(w,y)$ and $\mu(w,z)$ do not exceed b_1-1. Next, $[x]-y^{\perp}$ contains z and at most b_1-1 vertices of $[u]$.

If $\mu(w,y)=b_1$, then $[w]-u^{\perp}=[w]\cap[y]$, and $[w]\cap[z]$ lies in $[y]$. Assume that $\mu(w,z)=b_1-1$. Let $d\in([w]\cap[y])-[z]$. By Lemma 1.1, the subgraph $[w]\cap[z]$ is a clique, and the subgraph $[x]$ lies in $\{y\}\cup w^{\perp}\cup z^{\perp}$ for each vertex x of $[w]\cap[z]$. We observe that if d is adjacent to a vertex e of $[w]\cap[z]$, then $[e]-z^{\perp}$ contains d, y, and b_1-2 vertices of $[u]$. Suppose d is adjacent to i vertices of $[w]\cap[z]$. If $b_1\ge 5$, then $i\ge 2$, and the degree of w in $[u]$ is at least $2(b_1-3)+(b_1-2)$, a contradiction. Thus, $b_1=4$ and $i=1$, because otherwise, for vertices $e,e'\in[w]\cap[z]$ adjacent to d, the subgraph $[e]\cap[e']$ contains d,w,y,z, and a vertex of $[w]\cap[z]-[d]$. We put $\{e,a,a'\}=[w]\cap[z]$, where only e is adjacent to d. Again by Lemma 1.1, the degree of w is 2 both in $[u]\cap[a]$ and in $[u]\cap[a']$, and $[a]\cap[a']$ contains e,y,z, and two vertices of $[u]$, a contradiction.

Thus, if $\mu(w,y)=b_1$, then $[w]\cap[y]=[w]\cap[z]$. Now, we assume that $[w]\cap[y]$ contains two nonadjacent vertices e and e'. Then the degree of w is equal to b_1-2 in each of the graphs $[u]\cap[e]$ and $[u]\cap[e']$, and $[u]\cap[e]\cap[e']=\{w\}$ by Lemma 1.5. Let $a\in[w]\cap[y]-\{e,e'\}$. Then either $[u]\cap[e]$, or $[u]\cap[e']$ contains a vertex d of $[w]\cap[a]$. For definiteness, let $d \in [e]$. It follows that $[a] \cap [e]$ contains two vertices y and z that do not lie in $[w] \cup [d]$. This contradicts Lemma 1.5. Thus, $[w] \cap [y]$ is a clique. If $|[u] \cap [a] \cap [b]| \geq 2$ for distinct vertices a and b of the graph $[w] \cap [y]$, then $[a] \cap [e]$ contains two vertices y and z that do not belong to the union of neighborhoods of two vertices in $[u] \cap [a] \cap [b]$, which contradicts Lemma 1.5. Thus, $4(b_1 - 2) \leq 2b_1 - 4$, a contradiction.

Therefore, we have $\mu(w, y) = \mu(w, z) = b_1 - 1$ for every vertex $w \in [u] \cap [x]$. In particular, this implies that the subgraph $[u] \cap [x]$ is a clique. Indeed, otherwise $[u] \cap [x]$ contains nonadjacent vertices w and w', the vertex x is adjacent to two vertices w' and z outside $w^\perp \cup y^\perp$, and $\mu(w, y) = b_1$ by Lemma 1.1. By Lemma 1.5, we obtain $[w] \cap [y] \cap [z] = \{x\}$, and $[w] \cap [x]$ contains $2b_1 - 4$ vertices of $[y] \cup [z]$, which contradicts the fact that $\mu(u, x) = 1$. The lemma is proved. □

In Lemmas 3.6–3.8 below, we assume that $\Delta(x)$ contains distinct vertices y and z. By Lemma 3.5, the subgraph $\Delta(x)$ is a clique. Therefore, the vertices y and z are adjacent. By Lemma 3.1, we have $\mu(u, x) < b_1$.

Lemma 3.6. *The following is true*:

(1) *if* $t \in [y] \cap [z] \cap \Gamma_2(u)$ *and* t *is adjacent to exactly* j *vertices of* $[y] - z^\perp$, *then* t *is not adjacent to exactly* j *vertices of* $([y] \cap [z]) - \{t\}$, $j \leq 2$, *and* $[t] \cap ([y] - z^\perp) \subset \Gamma_2(u)$;

(2) *for any adjacent vertices* p *and* q *of* $\Gamma_3(u)$, *we have* $[u] \cap \Gamma_2(p) = [u] \cap \Gamma_2(q)$;

(3) $[y] - z^\perp \subset \Gamma_2(u)$.

Proof. Observe that $[t] \cap [y]$ contains z, j vertices of $[y] - z^\perp$, and $2b_1 - 5 - j$ vertices of $[y] \cap [z]$. Therefore, t is not adjacent to exactly j vertices of $[y] \cap [z]$. Since $j + \mu(t, u) \leq b_1$, we have $j \leq 2$. By Lemma 3.5 (applied to t in the role of x), we obtain $[t] \cap ([y] - z^\perp) \subset \Gamma_2(u)$. Statement (1) is proved.

Suppose $[u] \cap \Gamma_2(p) \neq [u] \cap \Gamma_2(q)$. We may assume that $\Gamma_2(p) \cap \Gamma_3(q)$ contains a vertex d nonadjacent to u. Then $u \in [d] \cap \Gamma_3(p)$, and Lemma 3.4 shows that $u \in \Gamma_2(q)$, a contradiction. Statement (2) is proved.

Now, assume that $r \in ([y] - z^\perp) - \Gamma_2(u)$. Then $[r] \cap [z]$ lies in $\Gamma_3(u)$, because otherwise, for $a \in [r] \cap [z] \cap \Gamma_2(u)$, the subgraph $\Delta(a)$ is not a clique, which contradicts Lemma 3.5. By statement (2), we have $[u] \cap \Gamma_2(r) = [u] \cap \Gamma_2(o) = [u] \cap \Gamma_2(z)$ for every vertex $o \in [r] \cap [z]$. If $d \in [u] \cap \Gamma_2(r)$, then each of the graphs $[d] \cap [z]$ and $[d] \cap [r]$ contains at least $b_1 - 2$ vertices. Hence, $b_1 = 4$ and $\mu(d, z) = \mu(d, r) = 2$. We put $\{e, f\} = [d] \cap [z]$ and $\{g, h\} = [d] \cap [r]$. By Lemma 1.4, either $[d] \cap [y] = \{e, f, g, h\}$, or we may assume that $[d] \cap [y] = \{e, g\}$. Furthermore, we have $([y] - z^\perp) \cap \Gamma_3(u) = \{r\}$, because otherwise, if a vertex r' of this subgraph is different from r, then $[d] \cap [r] = [d] \cap [r']$, which contradicts Lemma 1.4. By symmetry, $([y] - r^\perp) \cap \Gamma_3(u) = \{z\}$.

We prove that $[y] \cap \Gamma_2(u) \subset [r] \cup [z]$. Assume the contrary. Then, for $a \in ([y] \cap \Gamma_2(u)) - ([r] \cup [z])$ and $b \in [u] \cap [a]$, we have $b \in \Gamma_2(r) \cap \Gamma_2(z)$ by statement (2). This contradicts the inclusion $[b] - u^\perp \subset [r] \cup [z]$. Now, by Lemma 1.1, the degree of y in the graph $[r] \cap [z]$ is equal to 1. If y' is a vertex of $[r] \cap [z]$ adjacent to y, then we can take this vertex in place of y, and $\{y, y'\}$ is a connected component of the subgraph $[r] \cap [z]$. The subgraph $[y] \cap [y']$ contains r, z, and two vertices of $\Gamma_2(u)$. Similarly, $[r] \cap [y]$ contains y' and three vertices of $\Gamma_2(u)$. Thus, each of the vertices y and y' is adjacent to six vertices of $\Gamma_2(u)$. Therefore, $\Gamma_2(u)$ contains ten vertices, each being adjacent to at least two vertices of $\{r, y, y', z\}$. We note that $[r] - y^\perp$ does not intersect $\Gamma_3(u)$; indeed, otherwise $s \in ([r] - y^\perp) \cap \Gamma_3(u)$, the degree of the vertex r in $[y] \cap [s]$ is equal to 1, and s is adjacent to y'. Since $([y'] - z^\perp) \cap \Gamma_3(u) = \{r\}$, we see that s is adjacent to r, a contradiction.

Let $e \in [r] - ([y] \cup [y'])$. Then $e \in \Gamma_2(u)$, and, for $d \in [u] \cap [e]$, the subgraph $[d] \cap [r]$ contains a vertex s adjacent to y and y', and we have $\mu(d, y) = \mu(d, y') = 2$. Also, $[s] - d^\perp$ contains a vertex of $[y] \cap [y'] \cap [r]$. Therefore, both vertices of $[y] \cap [y'] \cap \Gamma_2(u)$ lie in $[r]$.

On the other hand, if $e' \in [z] - ([y] \cup [y'])$ and $d' \in [u] \cap [e']$, then the subgraph $[d'] \cap [z]$ contains a vertex s' adjacent to y and y', a contradiction. □

Lemma 3.7. *Every vertex d of $[u] \cap \Gamma_2(y)$ is adjacent to a vertex of $[y] \cap [z]$.*

Proof. We assume that a vertex d of $[u] \cap \Gamma_2(y)$ is adjacent to no vertex of $[y] \cap [z]$. Then each of the subgraphs $[d] \cap [y]$ and $[d] \cap [z]$ contains at least $b_1 - 2$ vertices. It follows that $b_1 = 4$ and $\mu(d, y) = \mu(d, z) = 2$. We put $\{e, f\} = [d] \cap [z]$, $\{g, h\} = [d] \cap [y]$, and $\{s_1, \dots, s_4\} = [d] \cap [u]$. By Lemma 1.1, a neighborhood of each vertex of $\{e, f\}$ lies in $d^\perp \cup z^\perp$.

Suppose that g and h are not adjacent to e. Then $[d] \cap [e]$ contains f and three vertices of $\{s_1, \dots, s_4\}$. In particular, $\mu(e, u) = 4$, and we may assume that e is adjacent to s_1, s_2, and s_3. Since d is adjacent to the vertex e outside $u^\perp \cup g^\perp$, the degree of d in the graph $[u] \cap [g]$ is at least 2 by Lemma 1.1. We note that each vertex s_i, $i = 1, 2, 3$, is adjacent to at most one vertex of $\{g, h\}$ (otherwise, $[s_i] \cap [d]$ contains u, e, g, h, and two vertices of $[u] \cap [e]$). If each vertex s_i, $i = 1, 2, 3$, is adjacent to g or to h (for definiteness, let $s_1, s_2 \in [g]$ and $s_3 \in [h]$), then the degree of s_i in the graph $[e] \cap [u] - \{d\}$ is at least 1. Therefore, some vertex s_i has degree 2 in the graph $\{s_1, \dots, s_3\}$. This contradicts the fact that the subgraph $[d] \cap [s_i]$ contains u, e, two vertices of $[e] \cap [u]$, and a vertex of $\{g, h\}$. Thus, one of the vertices $\{s_1, \dots, s_3\}$ does not belong to $[g] \cup [h]$.

If f is not adjacent to g, then, repeating the argument for g as e, we see that $[g]$ contains $\{s_1, s_2, s_4\}$, s_3 is not adjacent to h, and s_4 is not adjacent to f. However, in this case $[d] \cap [h]$ contains f, g, s_4, and a vertex of $\{s_1, s_2\}$, a contradiction. Thus, if e is not adjacent to g and to h, then $[f]$ contains g and h. Since $[d] \cap [g]$ contains f, h, and two vertices of $\{s_1, \dots, s_4\}$, and $[d] \cap [h]$ contains two vertices of $\{s_1, \dots, s_4\}$, we may assume that g is adjacent to s_1 and to s_4 and h is adjacent to s_2 and to s_4. Observe that $[u] \cap [g] \cap [h]$ contains two vertices d and s_4 such that d is adjacent to e outside $u^\perp \cup [g] \cup [h]$. By Lemma 1.5, the subgraph $[d] \cup [s_4]$ contains $[g] \cap [h]$, which contradicts the fact that the vertex y of $[g] \cap [h]$ does not belong to $[d] \cup [s_4]$.

Thus, every vertex of $\{e, f\}$ is adjacent to a vertex of $\{g, h\}$. We assume that $g, h \in [e]$. There is no loss of generality in assuming that f is adjacent to g. Then $[d] - u^\perp$ lies in $e^\perp \cap g^\perp$. Therefore, $\mu(u, e) = \mu(u, g) = 2$, and the subgraph $[e] \cap [g]$ contains d, f, h, and either a vertex t of $[y] \cap [z]$ or a vertex s_i. In the latter case, we obtain a contradiction with Lemma 1.4.

Thus, the subgraph $[e] \cap [g]$ contains a vertex t of $[y] \cap [z]$. For definiteness, assume that e is adjacent to s_1 and g is adjacent to s_2. If s_1 is adjacent to f, then the degrees of e and f in the graph $[g] \cap [z]$ are at least 2 (because $s_1 \notin g^\perp \cup z^\perp$). However, the vertices e and f are not adjacent to the vertex y of $[g] \cap [z]$. Therefore, $[e] \cap [f]$ contains d, g, s_1, z, and a vertex of $[g] \cap [z]$, a contradiction. Thus, s_1 is not adjacent to f and, by symmetry, s_2 is not adjacent to h.

If s_1 is adjacent to h, then, by Lemma 1.4, we have $\mu(u, h) = 4$. Since $[h] - y^\perp \subset d^\perp$, we see that $[d] \cap [h]$ contains e, g, s_1, s_3, and s_4, a contradiction. Thus, s_1 is not adjacent to h and s_2 is not adjacent to f. The subgraph $[s_1] - u^\perp$ contains three vertices p, p', and t of $[e] - \{f, g, h, z\}$. By symmetry, $[s_2] - u^\perp = \{q, q', t\}$. Furthermore, $[t]$ contains a vertex of $[s_1] \cap [y]$ and a vertex of $[s_2] \cap [z]$. This allows us to assume that $p, q \in [t]$. Since $[t] \cap [e] = \{g, p, s_1, z\}$, the degree of t in the graph $[s_1] \cap [y]$ is equal to 1. However, $[s_1]$ contains a vertex off $u^\perp \cup t^\perp$. Therefore, the subgraph $[u] \cap [t]$ is a 3-clique $\{s_1, s_2, r\}$. Since $[p] \cap [t] = \{e, s_1, y, z\}$ and $[g] \cap [t] = \{e, s_2, y, z\}$, the graph $[r] \cap [t]$ contains only s_1 and s_2, a contradiction.

Thus, we may assume that $e \in [g] - [h]$, $f \in [h] - [g]$, and $|[o] \cap [u]| = 3$ for every vertex o of $\{e, f, g, h\}$. Suppose that the vertex s_1 is adjacent to three vertices of $\{e, f, g, h\}$; say, s_1 is adjacent to e, f, and g. Then the vertices s_2, s_3, s_4, and h are not adjacent to

s_1. This contradicts the fact that the degree of s_1 in the graph $[u] \cap [g]$ is at least 2. Thus, every vertex s_i is adjacent to exactly two vertices of $\{e, f, g, h\}$, and $\{s_1, \dots, s_4\}$ is a quadrangle. This is impossible because $[d] \cap [s_1]$ contains u, two vertices among $\{s_1, \dots, s_4\}$, and two among $\{e, f, g, h\}$. □

Lemma 3.8. *Suppose a vertex d of $[u] \cap \Gamma_2(y)$ is adjacent to α vertices $g_1, \dots, g_\alpha$ of $[y] - z^\perp$. Then $\alpha \leq 2$, and the following is true if $\alpha = 2$:*

(1) *each vertex t of $[d] \cap [y] \cap [z]$ is adjacent to a vertex of $\{g_1, g_2\}$;*
(2) *each of the vertices g_1 and g_2 is not adjacent to at most one vertex of $[d] \cap [y] \cap [z]$;*
(3) *at most one vertex of $[d] \cap [y] \cap [z]$ does not belong to $[g_1] \cap [g_2]$.*

Proof. Since $\mu(d, z) \leq b_1 - \alpha$, we have $\alpha \leq 2$. Let $\alpha = 2$. Then d, z is a good pair. By Lemma 3.7, the vertex d is adjacent to a vertex t of $[y] \cap [z]$.

If $\mu(d, y) < b_1$, then, by Lemma 1.4, we obtain $\{t\} = [d] \cap [y] \cap [z]$, whence $\mu(d, y) + b_1 - 3 \leq b_1$. In particular, $b_1 \leq 5$, and $b_1 \leq 4$ if d, y is an almost good pair. In this case, all statement of the lemma are true. Thus, we may assume that $\mu(d, y) = b_1$ and $[d] \cap [z] \subset [y]$.

If t is not adjacent to the vertices of $\{g_1, g_2\}$, then $[t]$ contains $b_1 - 3$ vertices of $[d] \cap [y]$ and $b_1 - 1$ vertices of $[d] - y^\perp$ (and the latter vertices lie in $[u]$). Now, $[u] \cap [t]$ contains d and $b_1 - 1$ vertices of $[d] - [y]$. This contradicts Lemma 3.1. Statement (1) is proved.

Suppose t is not adjacent to g_1. Let $[d] \cap [z]$ contain a vertex e distinct from t and not adjacent to g_1. Then $[g_2]$ contains the vertices t and e of $[d] \cap [z]$, so that $\mu(g_2, z) \geq b_1$. By Lemma 3.1, the triple (u, t, e) is almost good, and $[t] \cap [e]$ contains two vertices y and z not adjacent to the vertices of $[u] \cap [t] \cap [e]$. By Lemma 1.5, we have $[u] \cap [t] \cap [e] = \{d\}$. Observe that $[t] \cup [e]$ contains $[d] - (\{u\} \cup [y])$. If $b_1 > 4$, then $[d] \cap [z]$ contains a third vertex f. We may assume that $[f] \cap [e]$ contains at least $(b_1 - 3)/2$ vertices of $[d] \cap [u]$, and $b_1 \leq 3$ by Lemma 1.4, a contradiction. Thus, $b_1 = 4$, and $[d] \cap [g_1]$ contains g_2 and three vertices of $[u]$. In particular, $\mu(u, g_1) = 4$. Now we restore $[y]$. This subgraph contains the $K_{1,1,2}$-subgraph $\{t, e, g_2, z\}$ and two vertices of $[y] - ([d] \cup [z])$ belonging to $X_0(\{t, e, g_2, z\})$. It follows that $[y] - \{t, e, g_2, z\}$ contains the $K_{1,1,3}$-subgraph (which, possibly, is not induced). However, $[g_1] \cap [g_2]$ contains d, y, at most one vertex of $[d] \cap [u]$, and a vertex of $[y]$. Therefore, $X_0(\{t, e, g_2, z\})$ contains a 4-clique K, and the vertex g_2 of $[y] - K$ is adjacent to two vertices of K. But $[y] - (\{g_2\} \cup K)$ is a 4-clique L, and $[t] \cap [e]$ contains d, y, z, g_2, and one more vertex of L, a contradiction. Statement (2) is proved.

Now, we assume that (3) fails. Then $[d] \cap [y] \cap [z]$ contains vertices t and e not adjacent to g_1 and g_2, respectively. As above, $[t] \cap [e]$ contains a unique vertex d of $[u]$, and every vertex of $[d] \cap [u]$ is adjacent to t or to e.

Suppose $[d] \cap [y] \cap [z]$ contains a vertex f distinct from t and e. Then f is adjacent to g_1 and g_2, and $|[u] \cap [f]| = b_1 - 2 = 1$, a contradiction.

Thus, $[d] \cap [y] \cap [z] = \{t, e\}$ and $b_1 = 4$. Suppose that the vertices g_1 and g_2 are not adjacent. Then the degree of g_i in the graph $[d] \cap [y]$ is equal to $b_1 - 3$, and $[g_1] \cap [g_2]$ contains a vertex r of $[d] \cap [u]$. There is no loss of generality in assuming that r is adjacent to t; then $[d] \cap [r]$ contains u, t, g_1, g_2, and a vertex of $[u] \cap [g_1]$, a contradiction. Thus, the vertices g_1 and g_2 are adjacent. If $[g_1] \cap [g_2]$ contains a vertex r of $[d] \cap [u]$ (for definiteness, let r be adjacent to t), then $[d] \cap [r]$ contains u, t, g_1, g_2, and a vertex of $[u] \cap [t]$, a contradiction. Thus, $[g_1] \cap [g_2]$ does not intersect $[d] \cap [u]$.

Let $\{r, s\} = [d] \cap [u] \cap [t]$, and let f and h be vertices of $([y] \cap [z]) - [d]$ and $[z] - y^\perp$, respectively, adjacent to t. If r is adjacent to g_1, then $[d]$ contains the $K_{3,3}$-subgraph $\{e, g_2, r; t, g_1, r'\}$. Then the degree of r in the graph $[t] \cap [u]$ is equal to 2, and $[d] \cap [r]$ contains u, t, s, g_1, and r', a contradiction. Thus, $\{r, s\} = [d] \cap [u] \cap [g_2]$ and $[t] \cap [g_2] = \{d, r, s, y\}$. If the vertices r and s are not adjacent, then the degree of r in the graph

$[t] \cap [u]$ is equal to 1 and $[r] \cap [t]$ contains two vertices of $[z] - [g_2]$. By Lemma 1.1, the degree of r in the graph $[u] \cap [g_2]$ is equal to 3, which contradicts the fact that r is not adjacent to s. Thus, the vertices r and s are adjacent and $[r] \cap [s] = \{d, u, t, g_2\}$. Now, at least one of the vertices r or s (for definiteness, let it be r) is not adjacent to f, the degree of t in the graph $[r] \cap [y]$ is equal to 1, and r is adjacent to h. Hence, $[t] \cap [s] = \{d, r, f, g_2\}$. Since $[t]$ is a regular graph of degree 4, the vertex h is adjacent to f and to e.

Let o and h be vertices of $([y] \cap [z]) - [d]$ and $[z] - y^{\perp}$, respectively, adjacent to e. As above, h is adjacent to o and $\mu(o, u) = 3$. It follows that there is a vertex p of $[z] - y^{\perp}$ adjacent to no vertex of $[y] \cap [z]$. This contradicts the fact that, in this case, the vertex z is isolated in $[p] \cap [y]$. □

Lemma 3.9. *Suppose a vertex d of $[u] \cap \Gamma_2(y)$ is adjacent to two vertices g_1 and g_2 of $[y] - z^{\perp}$. Then every vertex of $[d] \cap [y] \cap [z]$ is adjacent to both g_1 and g_2.*

Proof. By assumption, the pair d, z is good. Suppose that $[d] \cap [y] \cap [z]$ contains a unique vertex t outside $[g_1] \cap [g_2]$. Without loss of generality, we may assume that t is not adjacent to g_1.

Let $\mu(d, y) = b_1$. By Lemma 3.8, every vertex of $[d] \cap [y] \cap [z] - \{t\}$ is adjacent to g_1 and g_2 and forms a good pair with u. Furthermore, the pair u, t is almost good, and the degree of d in the graph $[u]$ is at least $(b_1 - 2) + (b_1 - 3)^2$. Hence, $b_1^2 - 5b_1 + 7 \leq 2b_1 - 4$ and $b_1 = 4$. We put $[d] \cap [z] = \{e, t\}$. Then $[e] \cap [y] = \{g_1, g_2, t, z\}$. If the vertices g_1 and g_2 are not adjacent, then the degree of g_1 in the graph $[d] \cap [y]$ is equal to $b_1 - 3 = 1$. Therefore, $[g_1] \cap [e] \cap [z] = \{y\}$, which is impossible because e is adjacent to the vertex g_2 lying outside $g_1^{\perp} \cup z^{\perp}$. Thus, the vertices g_1 and g_2 are adjacent. Then $[e] \cap [z]$ contains two vertices h and h' outside $y^{\perp}$. On the other hand, $[e] \cap [g_1]$ contains the vertices d, g_2, y and at most one vertex of $\{h, h'\}$. For definiteness, let g_1 be not adjacent to h. Since $[e] \cap [g_2] = \{d, g_1, t, y\}$ and $[e] \cap [t] = \{d, g_2, y, z\}$, we see that $[e] - h^{\perp}$ contains d, g_1, g_2, t, and y, a contradiction.

Let $\mu(d, y) = b_1 - 1$. By Lemma 1.4, we have $[d] \cap [y] \cap [z] = \{t\}$ and $b_1 = 4$. Let $[d] \cap [z] = \{e, t\}$. If e is adjacent to g_1 and g_2, then u, e is a good pair. The graph $[d] \cap [t] - \{e, g_1, g_2, t\}$ contains two vertices belonging to $[u]$. Therefore, u, t is an almost good pair. Finally, $[d] \cap [g_1]$ contains two vertices of $[u]$, and u, g_1 is an almost good pair. This contradicts the fact that, by Lemma 1.4, the degree of d in the graph $[u]$ is equal to 5. If g_1 and g_2 are not adjacent to e, then $[d] \cap [e] \cap [g_1]$ contains at least two vertices lying in $[u]$. First, suppose that $[d] \cap [e] \cap [g_1] = \{r, s\}$. Then $[u] \cap [e] = \{d, r, s, e'\}$ and $[u] \cap [g_1] = \{d, r, s, g_1'\}$. By Lemma 1.1, the degrees of r and s in the graphs $[u] \cap [e]$ and $[u] \cap [g_1]$ are at least 2. If the vertices r and s are not adjacent, then $[d] \cap [r]$ contains u, e, e', g_1, and g_1', a contradiction. Thus, the vertices r and s are adjacent and $[d] \cap [g_2] = \{t, e', g_1, g_1'\}$, which contradicts the fact that, in this case, the degree of e' in the graph $[u] \cap [e]$ is at least 2 by Lemma 1.1. Now, let $[d] \cap [e] \cap [g_1] = \{q, r, s\}$. Then $[u] \cap [e] = \{d, q, r, s\} = [u] \cap [g_1]$. By Lemma 1.1, the degrees of the vertices q, r, and s in the graph $[u] \cap [e] - \{d\}$ are at least 1. This contradicts the fact that none of the vertices q, r, or s has degree 5 in the graph $[d]$.

Let $e \in [g_2] - [g_1]$, and let $[e] \cap [u] = \{d, r, s\}$. Without loss of generality, we may assume that the vertex g_1 is adjacent to r. By Lemma 1.1 applied to the graph $[u] \cap [e]$, the vertex r is adjacent to s. Similarly, $[u] \cap [t]$ is a triangle containing d and yet another vertex t' adjacent to g_1 and distinct from r. If the vertices t and s are adjacent, then $[d] \cap [s] = \{e, r, t, u\}$, which is impossible because $s^{\perp}$ contains $[u] \cap [t]$. Thus, the vertices t and s are not adjacent. If g_1 is not adjacent to s, then $[d] \cap [r]$ contains e, g_1, s, u, and a vertex of $[u] \cap [g_1]$, a contradiction. We have $[d] \cap [g_1] = \{g_2, r, s, t'\}$, so that g_1 is not adjacent to a vertex of $[u] \cap [t] - \{t', d\}$. Thus, t' is adjacent to the vertex t outside $g_1^{\perp} \cup u^{\perp}$. By Lemma 1.1, the vertex t' must be adjacent to r or to s, a contradiction.

Let $e \in [g_1] - [g_2]$. Since u, g_1, t is an almost good triple and the vertices t and g_1 are not adjacent, we have $[u] \cap [t] \cap [g_1] = \{d\}$ by Lemma 1.5. If both vertices of $[u] \cap [g_1] - \{d\}$ are not adjacent to e, then, by Lemma 1.1, the degree of g_1 in the graph $[e] \cap [y]$ is equal to 3, which contradicts the fact that $[e] \cap [t]$ contains three vertices of $[u]$ and at least two vertices of $z^{\perp}$.

We put $\{r_1, r_2\} = [d] \cap [u] \cap [t]$, $\{s_1, s_2\} = [d] \cap [u] \cap [g_1]$, $\{o_1, o_2\} = [y] \cap [z] \cap [t]$, and $\{f\} = [y] \cap [z] - t^{\perp}$. Assume that the vertex e is adjacent to s_1 and s_2. Then $[t] \cap [e] = \{d, o_1, o_2, z\}$, and the degree of t in the graph $[r_i] \cap [y]$ is equal to 2. If r_1 is not adjacent to g_2, then $o_1, o_2 \in [r_1]$ by Lemma 1.1, and r_2 is adjacent to some vertex o_i. Since $[o_i] \cap [t]$ contains e, r_1, r_2, y, and z, we arrive at a contradiction. Thus, if the vertex e is adjacent to both s_1 and s_2, then g_2 is adjacent to both r_1 and r_2. In this case, the vertices r_1 and r_2 are adjacent, because otherwise the degree of t in the graph $[r_i] \cap [y]$ is at least 3 and again $|[o_i] \cap [t]| = 5$. Thus, we may assume that r_i is adjacent to o_i. Since $[t] \subset r_1^{\perp} \cup o_2^{\perp}$, we see that o_1 is not adjacent to o_2. Therefore, $[o_i] \cap [u]$ contains r_i and two vertices of $([u] - d^{\perp}) \cap [r_i]$. Furthermore, $[o_i] - y^{\perp}$ contains e and three vertices of $[u]$. Similarly, $[o_i] - z^{\perp}$ contains three vertices of $[u]$. Thus, $[o_i] \cap [y]$ contains f, t, z, and a vertex h_i of $[y] - z^b ot$. Now, $[r_i] \cap [o_i]$ contains t, two vertices of $[u]$, and a vertex of $\{h_i, f\}$. However, $f \notin [r_1] \cap [r_2]$, so that some vertex r_i is adjacent to h_i. Replacing the triple (d, g_1, g_2) by (r_i, h_i, g_2), we obtain a contradiction with statement (3) of Lemma 3.8.

Without loss of generality, we may assume that the vertex e is adjacent to r_1, s_1, and o_1. Then $\mu(r_1, y) = 3$, and $[r_1] \cap [d]$ contains e, t, u, and at most one vertex of $\{r_2, s_1\}$. First, suppose that the vertices r_1 and r_2 are not adjacent. Then $[r_i] \subset u^{\perp} \cup t^{\perp}$. Since the degree of t in the graph $[r_1] \cap [y]$ is equal to 2, we see that $[r_1]$ contains two vertices of $\{g_2, o_1, o_2\}$. Since the degree of t in the graph $[r_2] \cap [y]$ is equal to 3, the graph $[r_2]$ contains the vertices g_2, o_1, and o_2. On the other hand, $[r_1]$ contains $[d] \cap [z]$. By Lemma 1.4, we have $\mu(r_1, z) = 4$, and r_1 is adjacent to o_1 and o_2. This contradicts the fact that $[o_1] \cap [t]$ contains e, r_1, r_2, y, and z.

Thus, $r_1 \in [r_2] - [s_1]$, and the degree of r_1 in the graph $[e] \cap [u]$ is equal to 1. Since $[t] \subset r_1^{\perp} \cup y^{\perp}$, the degree of t in the graph $[r_1] \cap [y]$ is equal to 1. On the other hand, $[r_1]$ contains $[d] \cap [z]$. By Lemma 1.4, we obtain $\mu(r_1, z) = 4$, and r_1 is not adjacent to g_2. If r_1 is adjacent to a vertex h of $[z] - (\{e\} \cup y^{\perp})$, then, replacing the triple (d, g_1, g_2) with (r_1, h, e) and applying Lemma 3.8, we conclude that $o_1 \in [r_1] \cap [h]$. This case was analyzed in the second paragraph of the proof of the lemma.

Thus, $[r_1] \cap [z] = \{e, f, o_1, t\}$. Since the degree of s_1 in the graph $[u] \cap [e]$ is equal to 1, we have $[s_1] - u^{\perp} \subset e^{\perp}$. If r_2 is adjacent to o_1, then we obtain a contradiction with the fact that $[o_1] \cap [t]$ contains e, r_1, r_2, y, and z. Therefore, $[t] \cap [r_2] = \{d, g_2, o_2, r_1\}$, and $[f] \cap [u]$ contains three vertices of $r_1^{\perp}$. The subgraph $[u] \cap [o_1]$ contains a unique vertex r_1 of $[d]$. Furthermore, the degree of e in the graph $[g_1] \cap [z]$ is at least 2 ($[e]$ contains the vertex r_1 outside $g_1^{\perp} \cup z^{\perp}$). Since the degrees of the vertices g_1 and e in the graphs $[d] \cap [y]$ and $[d] \cap [z]$ (respectively) are equal to 1, we see that $[g_1] \cap [e] \cap [z]$ lies in $[y]$ and therefore, contains f and o_1. On the other hand, $[o_1]$ contains the vertex r_1 and a vertex of $[u] - d^{\perp}$. Thus, the degree of o_1 in the graph $[g_1] \cap [z]$ is at least 3, and, in particular, o_1 is adjacent to f. This contradicts the fact that $[f] \cap [o_1]$ contains e, g_1, r_1, y, and z. The lemma is proved. □

Lemma 3.10. *Every vertex of $[u] \cap \Gamma_2(y)$ is adjacent to at most one vertex of $[y] - z^{\perp}$.*

Proof. Suppose that a vertex d of $[u] \cap \Gamma_2(y)$ is adjacent to two vertices g_1 and g_2 of $[y] - z^{\perp}$. Then $\mu(d, z) = b_1 - 2$, and $[t] \subset d^{\perp} \cup z^{\perp}$ for $t \in [d] \cap [z]$. First, we consider the case where $[d]$ contains the vertices $e_1, \ldots, e_{b_1-2}$ of $[y] \cap [z]$. By Lemma 3.9, each vertex e_i is adjacent to g_1 and g_2, and $\mu(u, e_i) = b_1 - 2$. For $i \neq j$, the subgraphs $[u] \cap [e_i]$ and $[u] \cap [e_j]$ contain a unique vertex d in common. Therefore, $(b_1 - 3)(b_1 - 2) \leq 2b_1 - 4$ and $b_1 \leq 5$. If $b_1 = 5$, then $[d] \cap [z] = \{e_1, e_2, e_3\}$ is a triangle, and $[e_1] \cap [e_2] = \{e_3, g_1, g_2, d, y, z\}$. Since $e_i \in [g_1] \cap [g_2]$, Lemma 1.1 shows that the subgraph $[e_i]$ contains two vertices of $[z] - ([d] \cup y^{\perp})$. This is impossible, because $[z]$ contains seven vertices of $y^{\perp}$ and six vertices outside $y^{\perp}$. If $b_1 = 4$, then $[e_1] \cap [e_2] = \{g_1, g_2, d, y, z\}$, a contradiction.

Thus, $[d]$ contains a vertex h of $[z] - y^{\perp}$. Then the pair d, z is good and the pair d, y is almost good. By Lemma 1.4, we obtain $|[d] \cap [y] \cap [z]| \leq 1$. Hence, $b_1 = 4$ and $[d] \cap [y] \cap [z] = \{e\}$. By Lemma 3.9, the vertex e is adjacent to g_1 and g_2, whence $\mu(u, e) = 2$. We put $[e] \cap [y] \cap [z] = \{o\}$. Let r and s be vertices of $[u] \cap [e] - \{d\}$ and $[e] \cap [z] - \{h, o, y\}$, respectively.

Suppose that the vertices g_1 and g_2 are not adjacent. Then $g_i^{\perp} \subset d^{\perp} \cup y^{\perp}$. If g_1 and g_2 are adjacent to h, then $[g_i] \cap [d]$ contains two vertices of $[u]$. Therefore, the pairs u, g_1 and u, g_2 are almost good, and $[u] \cap [g_1] \cap [g_2] = \{d\}$ by Lemma 1.5. On the other hand, $[h] \cap [u]$ contains a vertex p of $[d]$ adjacent to g_i, so that $\{p, d\} \subset [u] \cap [h] \cap [g_i]$. This contradicts Lemma 1.4. If $h \in [g_1] - [g_2]$, then the degree of e in the graph $[g_2] \cap [z]$ is at least 2 ($[e]$ contains the vertex g_1 outside $g_2^{\perp} \cup z^{\perp}$). In this case, we have $o \in [g_1]$. Applying Lemma 1.4 to the triples u, g_1, g_2 and u, g_1, h, we see that r is adjacent neither to g_1 nor to h. Therefore, $[e] \cap [g_1]$ contains d, h, y, and o. Since the degree of e in the graph $[r] \cap [z]$ is at least 2 ($[e]$ contains g_1), we see that r is adjacent to o and s. This contradicts the fact that $[e] \cap [o]$ contains g_1, g_2, r, y, and z. If g_1 and g_2 are not adjacent to h, then $[d] \cap [x]$ contains three vertices of $[u]$ for $x \in \{g_1, g_2, h\}$. Therefore, $[u] \cap [d]$ contains the vertex r adjacent to g_1, g_2, and h. By Lemma 1.1, the degree of r in the graph $[u] \cap [h]$ is at least 3 and $[h] \cap [r] \subset d^{\perp}$. This contradicts the fact that $[d] \cap [r]$ contains u, g_1, g_2, h, and two more vertices.

Thus, the vertices g_1 and g_2 are adjacent. We prove that $r \in [g_1] \cup [g_2]$. Assuming the contrary, we have $[e] \cap [r] = \{d, h, o, s\}$. Replacing d with r, we obtain $[r] \cap [y] \cap [z] = \{e, o\}$, which contradicts the statement obtained in the first paragraph of the proof. We may assume that $r \in [g_1]$.

If h is adjacent to r, then $[d] \cap [r] \cap [z] = \{e, h\}$. Since the pair d, z is good, we have $\mu(r, z) = 4$ by Lemma 1.4. In this case, we have $|[r] \cap ([y] \cup [z])| = 5$, which contradicts the fact that $|[r] - u^{\perp}| = 4$. Thus, h is not adjacent to r. Now, $[e]$ contains the vertex h outside $r^{\perp} \cup y^{\perp}$, and the degree of e in the graph $[r] \cap [y]$ is at least 2. If the degree of e in $[r] \cap [y]$ is at least 3, then $[r]$ contains two vertices of each of the graphs $[y] - z^{\perp}$ and $[y] \cap [z]$. Again, this contradicts the statement proved in the first paragraph of the proof. Thus, $[e] \cap [r]$ contains d, g_1, exactly one vertex of $\{g_2, o\}$, and s.

Assume that $o \in [r] \cap [h]$. Then $[o]$ contains the vertex h outside $r^{\perp} \cup y^{\perp}$, the degree of o in the graph $[r] \cap [y]$ is at least 2, and o is adjacent to g_1. In this case, $[e] \cap [g_1]$ contains d, g_2, o, r, and y, a contradiction. Thus, $o \notin [r] \cap [h]$. Again, $[e]$ contains the vertex y outside $r^{\perp} \cup h^{\perp}$, and the degree of e in the graph $[r] \cap [h]$ is at least 2. Therefore, $[e] \cap [r] \cap [h] = \{d, s\}$ (we recall that $[e] \cap [g_1] = \{d, g_2, r, y\}$).

If s is adjacent to g_2, then r and h are not adjacent to g_2 because $[e] \cap [g_2] = \{d, g_1, s, y\}$. It follows that $[e] \cap [r] = \{d, g_1, s, o\}$ and $[e] \cap [h] = \{d, s, o, z\}$, which is incompatible with $o \notin [r] \cap [h]$. Thus, s is not adjacent to g_2, whence $[e] \cap [s] = \{h, o, r, z\}$.

First, assume that r is adjacent to g_2. Then $[u] \cap [g_1] \cap [d] = \{r, p_1\}$ and $[u] \cap [g_2] \cap [d] = \{r, p_2\}$. Here, $p_1 \neq p_2$ since $[g_1] \cap [g_2] = \{d, e, r, y\}$. In this case, $[d] - h^{\perp} = \{g_1, g_2, r, u\}$, so that $p_1, p_2, q \in [h]$, where $q \in [d] - (e^{\perp} \cup \{u, p_1, p_2\})$. Therefore, $[d] \cap [q] = \{h, p_1, p_2, u\}$, and the graph $[d]$ is constructed. But now the vertices p_1 and p_2 are not adjacent, $[h]$ contains the vertices e and z outside $p_1^{\perp} \cup p_2^{\perp}$, and the degree of h in the graph

$[p_1] \cap [p_2]$ is at least 3. Therefore, $[p_1] \cap [p_2] \cap [h]$ contains d, q, and a vertex of $[z] - d^{\perp}$. Thus, p_1 and p_2 are nonadjacent vertices forming good or almost good pairs with z, and $|[p_1] \cap [p_2] \cap [z]| \geq 2$. This contradicts Lemma 1.5.

Consequently, r is not adjacent to g_2. Therefore, r is adjacent to o, and o is not adjacent to h. The subgraph $[r] \cap [z] = \{e, o, s\}$ is a triangle, and $[r] \cap [s]$ contains e, o, and two vertices of $[r] \cap [u]$. Similarly, $[r] \cap [o]$ contains e, s, and two vertices of $[r] \cap [u]$. Since the vertex d of $[r] \cap [u]$ does not belong to $[o] \cup [s]$, we see that $[o] \cap [s]$ contains a vertex t of $[r] \cap [u]$. If t is adjacent to g_1, then the degree of t in the graph $[o] \cap [u]$ is at least 2. On the other hand, $[r] \cap [t] = \{g_1, o, s, u\}$, and the degree of o in the graph $[r] \cap [y]$ is equal to 1. Therefore, $[o] \cap [u] \subset [o] - y^{\perp} \subset r^{\perp}$, which contradicts the fact that $[o] \cap [u] \cap [t]$ lies in $\{r\}$. Thus, t is not adjacent to g_1. By the construction of r, the subgraph $[o] \cap [g_1]$ contains a unique vertex p of $[r] \cap [u]$. Now, $[o]$ contains the vertex s outside $p^{\perp} \cup y^{\perp}$ (because $[o] \cap [s] = \{e, p, r, z\}$), and the degree of o in the graph $[p] \cap [y]$ is at least 2. However, $g_1 \in [p] - [o]$, whence $\mu(p, y) = 4$. If $|[p] \cap [y] \cap [z]| = 2$, then, replacing d with p, we obtain a contradiction with the statement established in the first paragraph of the proof. Thus, $|[p] \cap [y] \cap [z]| = 3$. Since $[o] - r^{\perp} \subset y^{\perp}$ and the degree of o in $[r] \cap [z]$ is 2, we conclude that $[o] - (r^{\perp} \cup z^{\perp})$ contains a vertex q (of $[y]$). Also, $[e] \cap [g_1] = \{d, g_2, r, y\}$. Therefore, p is not adjacent to e. This contradicts the fact that $[o] \cap [y]$ contains e, q, z, and two vertices of $[p] \cap [y] \cap [z]$. The lemma is proved. □

Lemma 3.11. *For any two vertices u and x at a distance of 2 in the graph Γ, we have $b_2(u, x) \leq 1$.*

Proof. By Lemma 3.10, each vertex of $[u] \cap \Gamma_2(y)$ is adjacent to at most one vertex of $[y] - z^{\perp}$. It follows that the number of edges between $[u] \cap \Gamma_2(y)$ and $[y] - z^{\perp}$ is at least $b_1(b_1 - 2)$ and at most $3b_1 - 3$. Then $b_1^2 - 5b_1 + 3 \leq 0$ and $b_1 = 4$. We put $\{g_1, \ldots, g_4\} = [y] - z^{\perp}$ and $\{h_1, \ldots, h_4\} = [z] - y^{\perp}$. Either each vertex g_i is adjacent to exactly two vertices of $[u]$ and some vertex t of $[u]$ is not adjacent to any of the vertices $\{g_1, \ldots, g_4\}$, or each vertex $g_1, \ldots, g_3$ is adjacent to exactly two vertices of $[u]$ and g_4 is adjacent to exactly three vertices of $[u]$.

We assume that, in the first case, the vertex t belongs to $\Gamma_2(y)$. If the vertex e of $[y] \cap [z]$ is adjacent to the vertex g_i of $[y] - z^{\perp}$, then the degree of e in the graph $[t] \cap [y]$ is equal to 2, and $[e] \cap [y]$ contains g_i, z, and two vertices of $[t] \cap [y]$.

If the degrees of two distinct vertices e_1 and e_2 of $[t] \cap [y]$ in the graph $[t] \cap [y]$ are equal to 1, then $[e_i] \subset t^{\perp} \cup y^{\perp}$. Consequently, e_i is not adjacent to a vertex of $[y] - z^{\perp}$, and $[e_1] \cap [e_2]$ contains t, y, z, and two vertices of $[y] \cap [z]$, a contradiction. If $\mu(t, y) = 2$, then $|[t] \cap ([z] - [y])| \leq 1$ by Lemma 3.10, but this contradicts Lemma 1.4. If $\mu(t, y) = 3$, then the subgraph $[t] \cap [y] = \{e_1, e_2, e_3\}$ is a clique. In this case, $[e_i] \cap [e_j] = \{t, y, z, e_{6-i-j}\}$ and precisely one vertex w of $[t] - (u^{\perp} \cup y^{\perp})$ is adjacent to at most one vertex of $\{e_1, e_2, e_3\}$. There is no loss of generality in assuming that e_1 and e_2 are not adjacent to w. Then $\mu(u, e_1) = \mu(u, e_2) = 3$ and the degree of t in the graph $[u] \cap [e_i]$, $i = 1, 2$, is equal to 2. By Lemma 1.5, we have $[u] \cap [t] \subset [e_1] \cup [e_2]$. However, $[u] \cap [e_3]$ contains a vertex of $[t]$ lying in $[e_i]$, and $|[u] \cap [e_i] \cap [e_3]| \geq 2$, which contradicts Lemma 1.5.

Thus, $\mu(t, y) = 4$. If the subgraph $[t] \cap [y]$ contains a vertex e of degree 1, then $[e] \cap [y]$ contains z, a vertex of $[t] \cap [y]$, and two vertices of $[y] - z^{\perp}$, which contradicts Lemma 1.1. If the subgraph $[t] \cap [y]$ contains two vertices d and e of degree 3, then $[d] \cap [e]$ contains t, y, z, and two vertices of $[y] \cap [z]$. Thus, $[t] \cap [y]$ is a quadrangle. Therefore, the subgraph $[y] - z^{\perp} = \{g_1, \ldots, g_4\}$ is a clique. Furthermore, each vertex e of $[y] \cap [z]$ is adjacent to exactly two vertices of $[t] \cap [u]$, and each vertex w of $[t] \cap [u]$ is adjacent to g_i, h_j, and two vertices of $[y] \cap [z]$. On the other hand, each vertex g_i is adjacent to exactly one vertex of $[y] \cap [z]$. This contradicts the fact that $[w] - u^{\perp} \subset [g_i]$.

Thus, $t \in \Gamma_3(y) \cap \Gamma_3(z)$. Hence, $[u] \cap [g_i] = \{d_i, w_i\}$, where $d_i \in [u] - t^\perp$, $w_i \in [t] \cap [u]$, $i = 1, \ldots, 4$. By symmetry, we have $[z] \cap [d_i] = \{h_i, e_i\}$, where $e_i \in [y] \cap [z]$, $i = 1, \ldots, 4$, and, by Lemma 1.4, the vertex g_i is adjacent to e_i and h_i. Since $[w_i] \subset u^\perp \cup g_i^\perp$, we see that w_i is adjacent to a unique vertex r_i of $[t] - u^\perp$. Now, the subgraph $\{t, u, y, z, g_i, e_i, h_i, d_i, w_i, r_i \mid i = 1, \ldots, 4\}$ is a connected 28-vertex component of the graph Γ. But this is impossible because, in this case, for any two adjacent vertices e_i, e_j, the subgraph $[e_i] \cap [e_j]$ coincides with $\{y, z\}$.

Thus, each of the vertices g_1, g_2, g_3 is adjacent to exactly two vertices of $[u]$, and g_4 is adjacent to three vertices p_1, p_2, and p_3 of $[u]$. By symmetry, each of the vertices h_1, h_2, h_3 is adjacent to exactly two vertices of $[u]$, and h_4 is adjacent to three vertices of $[u]$. If p_1 is adjacent to a vertex h_i for $i \leq 3$, then $[p_1] \subset u^\perp \cup h_i^\perp$. In particular, h_i is a unique vertex of $\{h_1, \ldots, h_4\}$ adjacent to g_4. Then $p_2, p_3 \in [h_4]$, and the vertices g_4 and h_4 are adjacent by Lemma 1.5, a contradiction.

So, $p_1, p_2, p_3 \in [h_4]$; Lemma 1.5 shows that the vertices g_4 and h_4 are adjacent. Since $[u] \cap [h_1]$ contains two vertices adjacent to only one vertex of $\{g_1, g_2, g_3\}$, we see that $|[h_1] \cap \{g_1, g_2, g_3\}| = 2$; in particular, $[h_i] \subset [u] \cup [y] \cup [z]$ for $i \in \{1, \ldots, 4\}$. The number of edges between $[u]$ and $\Gamma_2(u)$ is equal to 36, and at most 30 of these edges are incident to vertices of $[y] \cup [z]$. However, if r is a vertex of $\Gamma_2(u) - ([y] \cup [z])$, then, by Lemma 1.4, the subgraph $[u] \cap [r]$ lies in $[g_4]$. We obtain $|[u] \cap [g_4] \cap [r]| \geq 2$, which contradicts Lemma 1.8. The lemma is proved. □

Lemma 3.12. *For every vertex u, we have $|\Gamma_3(u)| \leq 1$.*

Proof. Let y and z be distinct vertices of $\Gamma_3(u)$, and let $uwxy$ be a geodesic 3-path. By Lemma 3.11, the subgraph $[y] \cap \Gamma_2(u)$ does not intersect $[z]$. We assume that the vertices y and z are adjacent. We have $z \in \Gamma_2(w)$ by Lemma 3.4. Furthermore, $[y] \cap [z]$ contains $2b_1 - 4$ vertices of $\Gamma_3(u)$. By Lemma 3.11, $|[w]| = 3b_1 - 3 \geq (b_1 - 2)(2b_1 - 2)$. Hence, $b_1 \leq 3$, a contradiction.

If the distance between the vertices y and z is 2, then $[y] \cap [z]$ contains at least $b_1 - 2$ vertices of $\Gamma_3(u)$. This contradicts the fact proved in the preceding paragraph. Thus, the distance between the vertices y and z is equal to 3, and $kb_1 \geq 2k(b_1 - 2)$. Consequently, $b_1 = 4$, $\Gamma_2(u) = [y] \cup [z]$, and $\mu(u, r) = 2$ for every vertex r of $\Gamma_2(u)$. Since the subgraph $[w] - u^\perp$ lies in $x^\perp$, the graph $[x] \cap ([u] \cup [z])$ is a 4-clique. If x and x' are distinct vertices of $[w] \cap [y]$, then $[w] - u^\perp$ is a 4-clique, and $[x] \cap [x']$ contains y and four vertices of $[w] \cap [y]$, a contradiction. The lemma and Proposition 2 are proved. □

The theorem follows from [2, 3] and Propositions 1 and 2. We prove the corollary. Let Γ be a connected amply regular graph of diameter exceeding 2 and with parameters (v, k, λ, μ), and let $k \geq 3b_1 - 3$. Then $b_1 - 2 \leq \mu \leq b_1$.

Lemma 3.13. *If $\mu = b_1 - 2$, then $\Gamma \in \mathcal{E}(4)$.*

Proof. Let $\mu = b_1 - 2$. Then any two vertices the distance between which is equal to 2 form a good pair. Therefore, Γ is a Terwilliger graph without 3-claws. By [7], either $\mu = 1$, or Γ is the icosahedron graph (and $\mu = b_1$). In the case where $\mu = b_1 - 2 = 1$, we obtain $b_1 = 3$ and $\Gamma \in \mathcal{E}(4)$. □

Lemma 3.14. *If $\mu = b_1$, then Γ is either an n-gon with $n \geq 6$, or the complete bipartite graph $K_{4,4}$ with a maximal matching removed, or the icosahedron graph, or the Johnson graph $J(6,3)$, or the locally Taylor graph $T(6)$ on 32 vertices, or the locally Schläfli graph on 56 vertices.*

Proof. Let $\mu = b_1$. By [1, Theorem 1.5.5], the graph Γ is either a polygon or a Taylor graph. In the latter case, either $\lambda = 0$ and Γ is a complete bipartite graph $K_{k+1,k+1}$

with a maximal matching removed, or a neighborhood of each vertex of Γ is a strongly regular graph with parameters (v', k', λ', μ') and $k' = 2\mu'$. If $\lambda = 0$, then $k = b_1 + 1$, and the condition $k \geq 3b_1 - 3$ implies that $b_1 \leq 2$ (if $b_1 = 1$, the graph Γ is a hexagon).

Now, let $v' = k, k' = \lambda = 2\mu'$. Then $b_1 = 2(k' - \lambda' - 1)$. Since $k \geq 3b_1 - 3$, we have $\mu' \geq b_1 - 2$. If $\mu' = b_1 - 1$, then the subgraph $[a]$ is a strongly regular graph with parameters $(v', 2b_1 - 2, 3b_1/2 - 3, b_1 - 1)$. In the half case, we obtain $b_1 = 2$, and Γ is the icosahedron graph. If $(\lambda' - \mu')^2 + 4(k' - \mu')$ is a square, then $b_1 = 2s$ and $s(s+4)$ is a square, a contradiction.

If $\mu' = b_1 - 2$, then the subgraph $[a]$ is a strongly regular graph with parameters $(v', 2b_1 - 4, 3b_1/2 - 5, b_1 - 2)$. In the half case, we obtain $b_1 = 2$, the graph $[a]$ is the (3×3)-lattice, and Γ is the Johnson graph $J(6,3)$. If $(\lambda' - \mu')^2 + 4(k' - \mu')$ is a square, then $n' = b_1/2 - 1$, and the nonprincipal eigenvalues of the graph $[a]$ are equal to $b_1/2 - 2$ and -2. If $b_1 = 6$, the graph $[a]$ is the triangular graph $T(6)$, and Γ is the Taylor graph on 32 vertices.

Since a Seidel graph with $\mu \geq 6$ is either the Clebsch graph or a Schläfli graph and, for $b_1 = 8$, the graph $[a]$ must have the parameters (21, 12, 7, 6), we see that Γ is the locally Taylor Schläfli graph on 56 vertices. □

Lemma 3.15. *If* $\mu = b_1 - 1$, *then* $\mu = 1$ *and* $\Gamma \in \mathcal{T}(3) \cup \mathcal{E}(3)$.

Proof. Let $\mu = b_1 - 1$. First, we assume that $k = 3b_1 - 3$. Then $\mu = k - 2b_1 + 2$. The following statement was proved in [8].

Corollary. *Let* Γ *be an amply regular graph with parameters* (v, k, λ, μ), *and let* $\mu = k - 2b_1 + 2$. *Then* Γ *is either a Seidel graph or a trivalent graph without triangles of diameter exceeding* 2, *and with* $\mu = 1$.

Let $k \geq 3b_1 - 2$. In [2] it was proved that if Γ is a connected edge-regular graph with parameters (v, k, λ), and if $k \geq 3b_1 - 2$, then either Γ is a polygon, or Γ is the icosahedron graph, or $\Gamma \in \mathcal{E}(3)$, or Γ is a graph of diameter 2. The lemma is proved. □

The corollary follows from Lemmas 3.13–3.15.

References

[1] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-regular graphs*, Ergeb. Math. Grenzgeb. (3), vol. 18, Springer-Verlag, Berlin, 1989. MR1002568 (90e:05001)

[2] A. A. Makhnev and I. M. Minakova, *On a class of edge-regular graphs*, Izv. Gomel. Gos. Univ., Voprosy Algebry **3** (2000), 145–154. (Russian)

[3] S. P. Zaripov, A. A. Makhnev, and I. P. Yablonko, *Edge-regular graphs of diameter* $2c\lambda \geq 2k/3 - 2$, Proc. Ukrain. Math. Congr. (Kiev, 2001), Sect. 1: Algebra and Number Theory, Inst. Mat. Nats. Akad. Nauk Ukrainy, Kiev, 2003, pp. 46–61. (Russian)

[4] A. A. Makhnev, *On the strong regularity of some edge-regular graphs*, Izv. Ross. Akad. Nauk Ser. Mat. **68** (2004), no. 1, 159–182; English transl., Izv. Math. **68** (2004), no. 1, 159–180. MR2096940 (2005f:05171)

[5] A. A. Makhnev, A. A. Vedenev, A. N. Kuznetsov, and V. V. Nosov, *On good pairs in edge-regular graphs*, Diskret. Mat. **15** (2003), no. 1, 77–97; English transl., Discrete Math. Appl. **13** (2003), no. 1, 85–104. MR1996746 (2004d:05205)

[6] I. N. Belousov, E. I. Gurskiĭ, A. S. Dergach, and A. A. Makhnev, *On almost good pairs in edge-regular graphs*, Problems Theor. and Appl. Math. (Proc. Youthful Conf., Ekaterinburg 2004), pp. 9–11 (Russian)

[7] V. V. Kabanov and A. A. Makhnev, *On separable graphs with some regulatory conditions*, Mat. Sb. **187** (1996), no. 10, 73–86; English transl., Sb. Math. **187** (1996), no. 10, 1487–1501. MR1438977 (98d:05154)

[8] I. N. Belousov and A. A. Makhnev, *On almost good vertex pairs in edge-regular graphs*, Izv. Ural. Gos. Univ. Mat. Mekh. No. 7 (36) (2005), 35–48. (Russian) MR2190940 (2006h:05241)

Institute of Mathematics and Mechanics, Ural Branch of RAS, 16 Kovalevskaya Street, Ekaterinburg, Russia 620219

Institute of Mathematics and Mechanics, Ural Branch of RAS, 16 Kovalevskaya Street, Ekaterinburg, Russia 620219

E-mail address: makhnev@imm.uran.ru

Received 27/JUN/2005

Translated by B. M. BEKKER