HOCHSCHILD COHOMOLOGY OF THE LIU–SCHULZ ALGEBRAS

A. I. GENERALOV AND N. YU. KOSOVSKAYA

ABSTRACT. A description of the Hochschild cohomology algebra for the symmetric 8-dimensional algebras found by Liu and Schulz is given in terms of generators and relations. The dimensions of the cohomology groups are calculated.

§1. INTRODUCTION

Let R be a finite-dimensional algebra over a field K , and let $\Lambda = R^e = R \otimes_K R^{\text{op}}$ be its enveloping algebra. Then the n th Hochschild cohomology group of the algebra R with coefficients in an R -bimodule M is defined as follows: $\text{HH}^n(R, M) = \text{Ext}_{\Lambda}^n(R, M)$. If $M = R$, we use the notation $\text{HH}^n(R) = \text{HH}^n(R, R)$. On the vector space

$$\text{HH}^*(R) = \bigoplus_{n \geq 0} \text{HH}^n(R) = \bigoplus_{n \geq 0} \text{Ext}_{\Lambda}^n(R, R)$$

we can introduce a $\smile$ -product with respect to which it becomes an associative K -algebra (see [1, §5], [2, Chapter XI], and [3]); this algebra is called the *Hochschild cohomology algebra*. It is known that the $\smile$ -product coincides with the Yoneda product on the Ext-algebra $\bigoplus_{n \geq 0} \text{Ext}_{\Lambda}^n(R, R)$ of the Λ -module R (see, e.g., [4, p. 120]). Moreover, in [3] it was proved that $\text{HH}^*(R)$ is a graded commutative algebra.

Let K be an arbitrary field. For a nonzero $\rho \in K$, we consider the algebra (see [5])

$$(1.1) \quad R = R_{\rho} = K \langle X_0, X_1, X_2 \rangle / (\{X_i^2, X_{i+1}X_i + \rho X_i X_{i+1}\}_{i=0,1,2})$$

(we always identify X_3 with X_0). We denote the coset containing X_i by x_i . It is easily seen that R is an 8-dimensional local K -algebra:

$$R = {}_K \langle 1, x_0, x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2 \rangle.$$

Furthermore, the algebras R_{ρ} are symmetric [6], and they have wild representation type (cf., e.g., [7, I.10.10(a)]).

In the case where $\rho \in K^* = K \setminus \{0\}$ is not a root of unity, the algebras R_{ρ} were defined originally in [6] in a form different from (1.1) and were used to produce an example of an indecomposable R -module M for which the syzygy modules $\Omega^n(M)$ for $n > 0$ are 4-dimensional but pairwise nonisomorphic; see [8]. Such examples do not exist for some other classes of algebras, e.g., for the group algebras of finite groups. It is well known [9, 10, 11] that the cohomology ring $H^*(G, K)$ of a finite group G is a finitely generated K -algebra (this is the “Golod–Venkov–Evens theorem”). Therefore, it is of interest to calculate the Yoneda algebra of the algebras R_{ρ} as in (1.1). This was done by the authors in [12]; in particular, we proved that these Yoneda algebras are finitely generated.

In the present paper, we use a similar technique (cf. [13]) to give a description of the Hochschild cohomology algebra $\text{HH}^*(R_{\rho})$. For this, we use a minimal Λ -projective

2000 *Mathematics Subject Classification*. Primary 16E40, 16E05.

Key words and phrases. Symmetric algebra, Hochschild cohomology.

Supported by RFBR (grant no. 06-01-00200).

resolution of the module R_ρ . This resolution was constructed by the second author in [14]. The main result of this paper is a description of the algebra $\mathrm{HH}^*(R_\rho)$ in terms of generators and relations (Theorem 2.1). This theorem implies that if ρ has finite order in K^* , then $\mathrm{HH}^*(R_\rho)$ is finitely generated. But if ρ is not a root of unity, then the algebra $\mathrm{HH}^*(R_\rho)$ is not finitely generated, and in that case we present a countable set of generators of $\mathrm{HH}^*(R_\rho)$. Note also that even if $\mathrm{HH}^*(R_\rho)$ is not finitely generated, the dimensions $\dim_K \mathrm{HH}^n(R_\rho)$ are uniformly bounded for all $n > 0$.

§2. STATEMENT OF MAIN RESULTS

Let $K\langle \mathcal{T} \rangle$ denote a free associative K -algebra (with unit) generated by a set $\mathcal{T}$ of free generators. As usual, by $K[\mathcal{T}]$ we denote a polynomial ring generated by $\mathcal{T}$. Let s be the order of ρ in the group K^* . Note that the case of $s = \infty$ is allowed. To describe the structure of the algebra $\mathrm{HH}^*(R)$, we introduce several graded K -algebras.

(I) Suppose $\mathrm{char} K \neq 2$.

(I.1) Put

$$\mathcal{X}_1 = \{h, h_i, y_i, t_i, l_i, p_i, q_i\}_{i \in \{0,1,2\}}.$$

On the algebra $K\langle \mathcal{X}_1 \rangle$, we introduce a grading such that

$$\deg h = \deg h_i = 0, \quad \deg y_i = \deg t_i = \deg l_i = 1, \quad \deg p_i = \deg q_i = 2$$

for all $i \in \{0,1,2\}$. Then we consider the graded K -algebra $\mathcal{A}_1 = K\langle \mathcal{X}_1 \rangle / I_1$, where the ideal I_1 of the algebra $K\langle \mathcal{X}_1 \rangle$ is generated by the elements

$$(2.1) \quad ab - (-1)^{\deg a \cdot \deg b} ba$$

for all $a, b \in \mathcal{X}_1$ and by the following homogeneous elements (for all $i, j, k, k_1, k_2 \in \{0,1,2\}$):

$$(2.2) \quad \begin{cases} h^2, hh_i, hy_i, ht_i, hl_i, hp_i, hq_i, \\ h_i h_j, y_i t_i, y_i l_i, t_i l_i, \\ h_i y_j, h_i t_j, h_i l_j \quad \text{for } i \neq j; \end{cases}$$

$$(2.3) \quad q_i^2 - p_{i+1}p_{i+2}, \quad q_{i+1}q_{i+2} - p_i q_i;$$

$$(2.4) \quad \begin{cases} y_i p_{i+1} - l_i q_{i+2}, & t_i p_i - y_i q_{i+1}, & l_i p_{i+2} - t_i q_i, \\ y_i p_{i+2} - t_i q_{i+1}, & t_i p_{i+1} - l_i q_i, & l_i p_i - y_i q_{i+2}; \end{cases}$$

$$(2.5) \quad y_i q_i - l_i q_{i+1}, \quad y_i q_i - t_i q_{i+2};$$

$$(2.6) \quad \begin{cases} y_{i+1} t_{i+2} - h_i p_{i+1}, & y_{i+1} y_{i+2} - h_i q_i, & y_{i+1} y_{i+2} - l_{i+1} t_{i+2}, \\ t_{i+1} l_{i+2} - h_i p_i, & y_{i+1} l_{i+2} - h_i q_{i+2}, & t_{i+1} t_{i+2} - y_{i+1} l_{i+2}, \\ l_{i+1} y_{i+2} - h_i p_{i+2}, & t_{i+1} y_{i+2} - h_i q_{i+1}, & l_{i+1} l_{i+2} - t_{i+1} y_{i+2}; \end{cases}$$

$$(2.7) \quad y_i h_i - t_{i+1} h_{i+1}, \quad y_i h_i - l_{i+2} h_{i+2}.$$

Note that we always treat the indices as elements of the group $\mathbb{Z}_3$.

(I.2) Put

$$\mathcal{X}_2 = \{u_i, y_i, p_i\}_{i \in \{0,1,2\}}.$$

On the algebra $K\langle \mathcal{X}_2 \rangle$, we introduce a grading such that

$$\deg u_i = 0, \quad \deg y_i = 1, \quad \deg p_i = 2$$

for all $i \in \{0,1,2\}$. By definition, we put $\mathcal{A}_2 = K\langle \mathcal{X}_2 \rangle / I_2$, where I_2 is the ideal of the algebra $K\langle \mathcal{X}_2 \rangle$ generated by the elements (2.1) (for all $a, b \in \mathcal{X}_2$) and by the following elements (for all $i \in \{0,1,2\}$):

$$(2.8) \quad u_i^2, \quad u_i y_i, \quad u_i p_i.$$

(I.3) Let $s = 2s' + 1$ with $s' \in \mathbb{N}$. Consider the set

$$(2.9) \quad \mathcal{X}_3 = \{h, h_i, y_i, t_i, l_i, z_i^{(m)}, p, p_i, q_i, r_i\}_{i \in \{0,1,2\}, m \in \{1, \dots, s-1\}}.$$

On the algebra $K\langle \mathcal{X}_3 \rangle$, we introduce a grading such that

$$\begin{aligned} \deg h = \deg h_i = 0, \quad \deg y_i = 1, \quad \deg t_i = \deg l_i = 3s - 2, \\ \deg z_i^{(m)} = 2m, \quad \deg p = 6, \quad \deg p_i = \deg q_i = 2s, \quad \deg r_i = s + 3 \end{aligned}$$

for all $i \in \{0, 1, 2\}$, $m \in \{1, \dots, s-1\}$. Put $\mathcal{A}_3 = K\langle \mathcal{X}_3 \rangle / I_3$, where I_3 is the ideal of the algebra $K\langle \mathcal{X}_3 \rangle$ generated by the elements (2.1) (for all $a, b \in \mathcal{X}_3$) and by the following homogeneous elements (for all $i, j \in \{0, 1, 2\}$, $m, m_1, m_2 \in \{1, \dots, s-1\}$):

$$(2.10) \quad \left\{ \begin{aligned} &h^2, hh_i, hy_i, ht_i, hl_i, hz_i^{(m)}, hp, hp_i, hq_i, hr_i, \\ &h_ih_j, h_it_j, h_il_j, h_iz_j^{(m)}, h_ip, h_iq_j, h_ir_j, \\ &h_iy_j, h_ip_j, y_iz_j^{(m)}, t_iz_j^{(m)}, l_iz_j^{(m)} \quad \text{for } i \neq j, \\ &y_it_i, y_il_i, t_il_i, z_i^{(m_1)}z_j^{(m_2)}, \\ &t_iz_i^{(m)}, l_iz_i^{(m)}, z_i^{(m)}p, z_i^{(m)}q_j, z_i^{(m)}r_j \quad \text{for } m \neq s', \\ &z_i^{(m)}p_j \quad \text{for } i \neq j, m \neq s'; \end{aligned} \right.$$

$$(2.11) \quad \left\{ \begin{aligned} &q_i^2 - p_{i+1}p_{i+2}, \quad r_i^2 - \rho^3 p p_i, \\ &q_{i+1}q_{i+2} - p_iq_i, \quad r_{i+1}r_{i+2} - \rho^3 p q_i, \\ &q_i r_i - \rho^{s'+2} p^{s'+1}, \\ &q_{i+1}r_i - p_i r_{i+2}, \quad q_{i+2}r_i - p_i r_{i+1}; \end{aligned} \right.$$

$$(2.12) \quad \left\{ \begin{aligned} &p^s - p_0 p_1 p_2, \\ &p^{s'+1} q_i - \rho^{s'-1} p_{i+1} p_{i+2} r_i, \\ &p^{s'} r_i - \rho^{s'+2} p_i q_i; \end{aligned} \right.$$

$$(2.13) \quad \left\{ \begin{aligned} &t_i p_i - \rho^{s'-3} y_i p^{s'} q_{i+1}, \quad l_i p_i - \rho^{s'-3} y_i p^{s'} q_{i+2}, \\ &t_i p - \rho^{-5} y_i p_{i+2} r_{i+1}, \quad l_i p - \rho^{-5} y_i p_{i+1} r_{i+2}, \\ &t_i q_{i+1} - \rho^{s'-3} y_i p^{s'} p_{i+2}, \quad l_i q_{i+1} - \rho^{s'-3} y_i p^{s'} q_i, \\ &t_i q_{i+2} - \rho^{s'-3} y_i p^{s'} q_i, \quad l_i q_{i+2} - \rho^{s'-3} y_i p^{s'} p_{i+1}, \\ &t_i r_i - \rho^{-2} y_i p_{i+2} q_{i+2}, \quad l_i r_i - \rho^{-2} y_i p_{i+1} q_{i+1}, \\ &t_i r_{i+1} - \rho^{-2} y_i p_{i+1} p_{i+2}, \quad l_i r_{i+1} - \rho^{-2} y_i p_{i+1} q_i, \\ &t_i r_{i+2} - \rho^{-2} y_i p_{i+2} q_i, \quad l_i r_{i+2} - \rho^{-2} y_i p_{i+1} p_{i+2}; \end{aligned} \right.$$

$$(2.14) \quad t_i q_i - l_i p_{i+2}, \quad l_i q_i - t_i p_{i+1};$$

$$(2.15) \quad \left\{ \begin{aligned} &t_{i+1} t_{i+2} - \rho^{s'-8} y_{i+1} y_{i+2} p^{s'-1} p_i r_{i+1}, \\ &l_{i+1} l_{i+2} - \rho^{s'-8} y_{i+1} y_{i+2} p^{s'-1} p_i r_{i+2}, \\ &t_{i+1} l_{i+2} - \rho^{s'-8} y_{i+1} y_{i+2} p^{s'-1} p_i r_i, \\ &l_{i+1} t_{i+2} - \rho^{s'-8} y_{i+1} y_{i+2} p^{s'-1} p_i q_i; \end{aligned} \right.$$

$$(2.16) \quad \begin{cases} z_i^{(s')} p - \rho^{-5} y_{i+1} y_{i+2} r_i, \\ z_i^{(s')} q_i - \rho^{s'-3} y_{i+1} y_{i+2} p^{s'}, \\ z_i^{(s')} r_i - \rho^{-2} y_{i+1} y_{i+2} p_i, \\ z_i^{(s')} r_{i+1} - \rho^{-2} y_{i+1} y_{i+2} q_{i+2}, \\ z_i^{(s')} r_{i+2} - \rho^{-2} y_{i+1} y_{i+2} q_{i+1}; \end{cases}$$

$$(2.17) \quad \begin{cases} z_i^{(s')} p_{i+1} - y_{i+1} t_{i+2}, & z_i^{(s')} p_{i+2} - l_{i+1} y_{i+2}, \\ z_i^{(s')} q_{i+1} - t_{i+1} y_{i+2}, & z_i^{(s')} q_{i+2} - y_{i+1} l_{i+2}; \end{cases}$$

$$(2.18) \quad t_i z_i^{(s')} - \rho^{s'-8} y_0 y_1 y_2 p^{s'-1} r_{i+2}, \quad l_i z_i^{(s')} - \rho^{s'-8} y_0 y_1 y_2 p^{s'-1} r_{i+1}.$$

(I.4) Let $s = 2s'$ with $s' \in \mathbb{N} \setminus \{1\}$, and let

$$\mathcal{X}_4 = \{h, h_i, y_i, t_i, z_i^{(m)}, p, p_i\}_{i \in \{0,1,2\}, m \in \{1, \dots, s'-1\}}.$$

On the algebra $K\langle \mathcal{X}_4 \rangle$, we introduce a grading such that

$$\begin{aligned} \deg h = \deg h_i = 0, \quad \deg y_i = 1, \quad \deg t_i = s - 2, \\ \deg z_i^{(m)} = 2m, \quad \deg p = 6, \quad \deg p_i = s \end{aligned}$$

for all $i \in \{0, 1, 2\}$, $m \in \{1, \dots, s' - 1\}$. Put $\mathcal{A}_4 = K\langle \mathcal{X}_4 \rangle / I_4$, where I_4 is the ideal of the algebra $K\langle \mathcal{X}_4 \rangle$ generated by the elements (2.1) (for all $a, b \in \mathcal{X}_4$) and by the following homogeneous elements (for all $i, j \in \{0, 1, 2\}$, $m, m_1, m_2 \in \{1, \dots, s' - 1\}$):

$$(2.19) \quad \begin{cases} h^2, hh_i, hy_i, ht_i, h z_i^{(m)}, hp, hp_i, \\ h_i h_j, h_i t_j, h_i z_j^{(m)}, h_i p, \\ h_i y_j, h_i p_j, y_i z_j^{(m)}, z_i^{(m)} p_j \quad \text{for } i \neq j, \\ y_i t_i, t_i t_j, t_i z_j^{(m)}, t_i p, t_i p_i, \\ z_i^{(m_1)} z_j^{(m_2)}, z_i^{(m)} p; \end{cases}$$

$$(2.20) \quad p^{s'} - p_0 p_1 p_2.$$

(I.5) Consider a countable set

$$\mathcal{X}_5 = \{h, h_i, y_i, z_i^{(m)}, p\}_{i \in \{0,1,2\}, m \in \mathbb{N}}.$$

On the algebra $K\langle \mathcal{X}_5 \rangle$, we introduce a grading such that

$$\begin{aligned} \deg h = \deg h_i = 0, \quad \deg y_i = 1, \\ \deg z_i^{(m)} = 2m, \quad \deg p = 6 \end{aligned}$$

for all $i \in \{0, 1, 2\}$, $m \in \mathbb{N}$. Put $\mathcal{A}_5 = K\langle \mathcal{X}_5 \rangle / I_5$, where I_5 is the ideal of the algebra $K\langle \mathcal{X}_5 \rangle$ generated by the elements (2.1) (for all $a, b \in \mathcal{X}_5$) and by the following homogeneous elements (for all $i, j \in \{0, 1, 2\}$, $m, m_1, m_2 \in \mathbb{N}$):

$$(2.21) \quad \begin{cases} h^2, hh_i, hy_i, h z_i^{(m)}, hp, \\ h_i h_j, h_i z_j^{(m)}, h_i p, \\ h_i y_j, y_i z_j^{(m)} \quad \text{for } i \neq j, \\ z_i^{(m_1)} z_j^{(m_2)}, z_i^{(m)} p. \end{cases}$$

(II) Suppose $\text{char } K = 2$.

(II.1) Let $\mathcal{X}_6 = \{u_i, p_i\}_{i \in \{0,1,2\}}$. On the algebra $K[\mathcal{X}_6]$, we introduce a grading such that $\deg u_i = 0$, $\deg p_i = 1$ for all $i \in \{0,1,2\}$. Consider the commutative K -algebra $\mathcal{A}_6 = K[\mathcal{X}_6]/I_6$, where I_6 is the ideal of the algebra $K[\mathcal{X}_6]$ generated by the elements u_i^2 (for all $i \in \{0,1,2\}$).

(II.2) Let $s \in \mathbb{N} \setminus \{1\}$. Consider a set of generators

$$\mathcal{X}_7 = \{h, h_i, y_i, t_i, z_i^{(m)}, p, p_i\}_{i \in \{0,1,2\}, m \in \{1, \dots, s-1\}}.$$

On the algebra $K[\mathcal{X}_7]$, we introduce a grading such that

$$\begin{aligned} \deg h = \deg h_i = 0, \quad \deg y_i = 1, \quad \deg t_i = 2s - 2, \\ \deg z_i^{(m)} = m, \quad \deg p = 3, \quad \deg p_i = s \end{aligned}$$

for all $i \in \{0,1,2\}$, $m \in \{1, \dots, s-1\}$. By definition, we put $\mathcal{A}_7 = K[\mathcal{X}_7]/I_7$, where I_7 is the ideal of the algebra $K[\mathcal{X}_7]$ generated by the following homogeneous elements (for all $i, j \in \{0,1,2\}$, $m, m_1, m_2 \in \{1, \dots, s-1\}$):

$$(2.22) \quad \left\{ \begin{array}{l} h^2, hh_i, hy_i, ht_i, hz_i^{(m)}, \\ h_i h_j, h_i t_j, h_i z_j^{(m)}, h_i p, \\ h_i y_j, h_i p_j, y_i z_j^{(m)}, t_i z_j^{(m)} \quad \text{for } i \neq j, \\ y_i t_i, t_i^2, z_i^{(m_1)} z_j^{(m_2)}, \\ t_i z_i^{(m)}, z_i^{(m)} p \quad \text{for } m \neq s-1, \\ z_i^{(m)} p_j \quad \text{for } i \neq j, m \neq s-1; \end{array} \right.$$

$$(2.23) \quad p^s - p_0 p_1 p_2, \quad hp - \rho^{-2} y_0 y_1 y_2, \quad hp_i - y_i z_i^{(s-1)};$$

$$(2.24) \quad \left\{ \begin{array}{ll} z_i^{(s-1)} p - \rho^{-2} y_{i+1} y_{i+2} p_i, & t_i z_i^{(s-1)} - \rho^{-4} y_0 y_1 y_2 p^{s-2}, \\ z_{i+1}^{(s-1)} p_i - t_{i+2} y_i, & t_i p - \rho^{-2} y_i p_{i+1} p_{i+2}, \\ z_{i+2}^{(s-1)} p_i - t_{i+1} y_i, & t_i p_i - \rho^{-2} y_i p^{s-1}; \end{array} \right.$$

$$(2.25) \quad t_{i+1} t_{i+2} - \rho^{-4} y_{i+1} y_{i+2} p_i p^{s-2}.$$

(II.3) Consider the countable set

$$\mathcal{X}_8 = \{h, h_i, y_i, z_i^{(m)}, p\}_{i \in \{0,1,2\}, m \in \mathbb{N}}.$$

On the algebra $K[\mathcal{X}_8]$, we introduce a grading such that

$$\deg h = \deg h_i = 0, \quad \deg y_i = 1, \quad \deg z_i^{(m)} = m, \quad \deg p = 3$$

for all $i \in \{0,1,2\}$, $m \in \mathbb{N}$. Put $\mathcal{A}_8 = K[\mathcal{X}_8]/I_8$, where I_8 is the ideal of the algebra $K[\mathcal{X}_8]$ generated by the following elements (for all $i, j \in \{0,1,2\}$, $m, m_1, m_2 \in \mathbb{N}$):

$$(2.26) \quad \left\{ \begin{array}{l} h^2, hh_i, hy_i, hz_i^{(m)}, \\ h_i h_j, h_i z_j^{(m)}, h_i p, \\ h_i y_j, y_i z_j^{(m)} \quad \text{for } i \neq j, \\ z_i^{(m_1)} z_j^{(m_2)}, z_i^{(m)} p; \end{array} \right.$$

$$(2.27) \quad y_0 y_1 y_2 - \rho^2 hp.$$

Since the ideal I_i is homogeneous, we see that the grading on the algebra $K\langle \mathcal{X}_i \rangle$ (or on the algebra $K[\mathcal{X}_i]$, respectively) induces a grading on the algebra $\mathcal{A}_i$ (for all $1 \leq i \leq 8$).

Theorem 2.1. *Let $R = R_\rho$, and let s be the order of ρ in K^* .*

I. Suppose $\text{char } K \neq 2$.

(I.1) *If $s = 1$, then, as a graded K -algebra, the Hochschild cohomology algebra $\text{HH}^*(R)$ is isomorphic to the algebra $\mathcal{A}_1$.*

(I.2) *If $s = 2$, then $\text{HH}^*(R) \simeq \mathcal{A}_2$ as graded K -algebras.*

(I.3) *If s is odd and larger than 1, then $\text{HH}^*(R) \simeq \mathcal{A}_3$ as graded K -algebras.*

(I.4) *If s is even and larger than 2, then $\text{HH}^*(R) \simeq \mathcal{A}_4$ as graded K -algebras.*

(I.5) *If $s = \infty$, then $\text{HH}^*(R) \simeq \mathcal{A}_5$ as graded K -algebras.*

II. Suppose $\text{char } K = 2$.

(II.1) *If $s = 1$, then $\text{HH}^*(R) \simeq \mathcal{A}_6$ as graded K -algebras.*

(II.2) *If $1 < s < \infty$, then $\text{HH}^*(R) \simeq \mathcal{A}_7$ as graded K -algebras.*

(II.3) *If $s = \infty$, then $\text{HH}^*(R) \simeq \mathcal{A}_8$ as graded K -algebras.*

Remark 2.1. This theorem shows that if ρ has finite order in K^* , then $\text{HH}^*(R_\rho)$ is finitely generated. But if ρ is not of finite order (i.e., ρ is not a root of unity), then the algebra $\text{HH}^*(R_\rho)$ is not finitely generated, and in this case we present a countable set of generators of $\text{HH}^*(R_\rho)$. Moreover, we shall prove that if $\text{HH}^*(R_\rho)$ is not finitely generated, then (i) if $\text{char } K \neq 2$, then $\dim_K \text{HH}^n(R) \leq 6$, and (ii) if $\text{char } K = 2$, then $\dim_K \text{HH}^n(R) \leq 9$ (see Proposition 3.1 and Corollary 3.4).

Remark 2.2. In the case (II.1), the algebra R is isomorphic to the group algebra $K[G]$ of the elementary Abelian group G of order 8, and then the description of the algebra $\text{HH}^*(R_\rho)$ can also be obtained from [15, 16].

§3. ADDITIVE STRUCTURE OF THE COHOMOLOGY RING

In this section, we use the minimal projective resolution $Q_\bullet = (Q_n, d_n^Q)$ of the Λ -module R to determine the cohomology groups $\text{HH}^i(R)$, $i \geq 0$. For this, we investigate the complex

$$\text{Hom}_\Lambda(Q_\bullet, R) = \left(\text{Hom}_\Lambda(Q_n, R), \Delta^n := (d_n^Q)^* \right)$$

and calculate its cohomology:

$$H^n(\text{Hom}_\Lambda(Q_\bullet, R)) = \text{Ext}_\Lambda^n(R, R) = \text{HH}^n(R).$$

In [14], the resolution $Q_\bullet$ was constructed as a total complex $\text{Tot}(B_{\bullet\bullet\bullet})$ of the non-negative 3-complex $B_{\bullet\bullet\bullet}$ such that $B_{ijk} = \Lambda$ for all $(i, j, k) \in \mathbb{N}_0^3$ and

$$\begin{aligned} d' &= (\rho^j x_0 \otimes 1' + (-1)^{i+j+k+1} \rho^k 1 \otimes x'_0)^* : B_{i+1jk} \longrightarrow B_{ijk}, \\ d'' &= (\rho^k x_1 \otimes 1' + (-1)^{i+j+k+1} \rho^i 1 \otimes x'_1)^* : B_{ij+1k} \longrightarrow B_{ijk}, \\ d''' &= (\rho^i x_2 \otimes 1' + (-1)^{i+j+k+1} \rho^j 1 \otimes x'_2)^* : B_{ijk+1} \longrightarrow B_{ijk}. \end{aligned}$$

Recall that multiplication on the right by an element $w \in \Lambda$ induces an endomorphism w^* of the left Λ -module Λ .

Next, let $m: \Lambda \rightarrow R$, $m(a \otimes b') = ab$, be the canonical map induced by multiplication in R . Then the map m can be taken for the role of an augmentation map of the resolution $d_{-1}^Q: Q_0 \rightarrow R$.

Therefore, we have $Q_n = \Lambda^{M_n}$, where $M_n = \#\{(i, j, k) \mid 0 \leq i, j, k \leq n, i + j + k = n\} = \binom{n+2}{2}$. Let $e_{i,j,k}$ (for $i + j + k = n$) denote an element of Q_n such that its component with index (i, j, k) (i.e., the (i, j, k) -component) is equal to $1 \otimes 1' \in \Lambda$ and all other components are zero. We identify a homomorphism $f \in \text{Hom}_\Lambda(Q_n, R)$ with the collection of its values $\{f(e_{i,j,k})\}_{i+j+k=n} \in R^{M_n}$. Let $[x]_{i,j,k}$ (for $x \in R$, $i + j + k = n$)

denote the element of R^{M_n} for which only the (i, j, k) -component is nonzero and is equal to x . Since $R = {}_K\langle 1, x_0, x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2 \rangle$, the elements

$$\begin{aligned} & \{[1]_{i,j,k}, [x_0]_{i,j,k}, [x_1]_{i,j,k}, [x_2]_{i,j,k}, [x_0x_1]_{i,j,k}, \\ & [x_1x_2]_{i,j,k}, [x_2x_0]_{i,j,k}, [x_0x_1x_2]_{i,j,k}\}_{i+j+k=n} \end{aligned}$$

form a K -basis of the vector space R^{M_n} ; such a basis is said to be *standard*.

Proposition 3.1. *If $\rho = -1$, then*

$$\mathrm{HH}^0(R) = {}_K\langle 1, x_0, x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2 \rangle = R.$$

In the other cases,

$$\mathrm{HH}^0(R) = {}_K\langle 1, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2 \rangle.$$

Proof. It is easy to prove that a center $Z(R)$ of the algebra R is equal to

$$R = {}_K\langle 1, x_0, x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2 \rangle$$

if $\rho = -1$, and to

$${}_K\langle 1, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2 \rangle$$

otherwise. Since $\mathrm{HH}^0(R) \simeq Z(R)$, after appropriate identifications this basis can be viewed as a K -basis of $\mathrm{HH}^0(R)$. $\square$

Remark 3.1. There are exactly two cases ((I.2) and (II.1)) in Theorem 2.1 in which $\rho = -1$.

We calculate $\mathrm{HH}^n(R) = \mathrm{Ker} \Delta^n / \mathrm{Im} \Delta^{n-1}$ for $n \geq 1$. For every $f \in \mathrm{Hom}_\Lambda(Q_n, R)$, the coefficients in the expansion of $f(e_{i,j,k})$ (for $i + j + k = n$) will be denoted in the following way:

$$\begin{aligned} f(e_{i,j,k}) &= k_{(000)}^{(ijk)} 1 + k_{(100)}^{(ijk)} x_0 + k_{(010)}^{(ijk)} x_1 + k_{(001)}^{(ijk)} x_2 \\ &+ k_{(011)}^{(ijk)} x_1x_2 + k_{(101)}^{(ijk)} x_2x_0 + k_{(110)}^{(ijk)} x_0x_1 + k_{(111)}^{(ijk)} x_0x_1x_2 \end{aligned}$$

(here $k_{(\alpha\beta\gamma)}^{(ijk)} \in K$).

Now, if $i + j + k = n + 1$, then

$$\begin{aligned} (3.1) \quad f \circ d_n(e_{i,j,k}) &= (\rho^j + (-1)^{n+1} \rho^k) k_{(000)}^{(i-1,j,k)} x_0 + (\rho^k + (-1)^{n+1} \rho^i) k_{(000)}^{(i,j-1,k)} x_1 \\ &+ (\rho^i + (-1)^{n+1} \rho^j) k_{(000)}^{(i,j,k-1)} x_2 \\ &+ [(\rho^k + (-1)^n \rho^{i+1}) k_{(001)}^{(i,j-1,k)} - (\rho^{i+1} + (-1)^n \rho^j) k_{(010)}^{(i,j,k-1)}] x_1x_2 \\ &+ [(\rho^i + (-1)^n \rho^{j+1}) k_{(100)}^{(i,j,k-1)} - (\rho^{j+1} + (-1)^n \rho^k) k_{(001)}^{(i-1,j,k)}] x_2x_0 \\ &+ [(\rho^j + (-1)^n \rho^{k+1}) k_{(010)}^{(i-1,j,k)} - (\rho^{k+1} + (-1)^n \rho^i) k_{(100)}^{(i,j-1,k)}] x_0x_1 \\ &+ [(\rho^j + (-1)^{n+1} \rho^k) k_{(011)}^{(i-1,j,k)} + (\rho^k + (-1)^{n+1} \rho^i) k_{(101)}^{(i,j-1,k)} \\ &\quad + (\rho^i + (-1)^{n+1} \rho^j) k_{(110)}^{(i,j,k-1)}] x_0x_1x_2. \end{aligned}$$

Here and in what follows we assume additionally that $k_{(\alpha\beta\gamma)}^{(\lambda\mu\nu)} = 0$ if at least one of λ, μ , or ν is negative.

The condition $f \in \text{Ker } \Delta^n$ is equivalent to the system of equations

$$\begin{aligned}
 (\rho^j + (-1)^{n+1} \rho^k) k_{(000)}^{(i-1,j,k)} &= 0, \\
 (\rho^k + (-1)^{n+1} \rho^i) k_{(000)}^{(i,j-1,k)} &= 0, \\
 (\rho^i + (-1)^{n+1} \rho^j) k_{(000)}^{(i,j,k-1)} &= 0, \\
 (\rho^k + (-1)^n \rho^{i+1}) k_{(001)}^{(i,j-1,k)} - (\rho^{i+1} + (-1)^n \rho^j) k_{(010)}^{(i,j,k-1)} &= 0, \\
 (\rho^i + (-1)^n \rho^{j+1}) k_{(100)}^{(i,j,k-1)} - (\rho^{j+1} + (-1)^n \rho^k) k_{(001)}^{(i-1,j,k)} &= 0, \\
 (\rho^j + (-1)^n \rho^{k+1}) k_{(010)}^{(i-1,j,k)} - (\rho^{k+1} + (-1)^n \rho^i) k_{(100)}^{(i,j-1,k)} &= 0, \\
 (\rho^j + (-1)^{n+1} \rho^k) k_{(011)}^{(i-1,j,k)} + (\rho^k + (-1)^{n+1} \rho^i) k_{(101)}^{(i,j-1,k)} \\
 + (\rho^i + (-1)^{n+1} \rho^j) k_{(110)}^{(i,j,k-1)} &= 0.
 \end{aligned}$$

Analyzing these equations, we find the following generators of $\text{Ker } \Delta^n$:

for $a + b + c = n$:

$$(3.2) \quad [1]_{a,b,c} \quad \text{if } (-1)^n = 1, \rho^a = \rho^b = \rho^c;$$

for $a + b + c = n - 1$:

$$(3.3) \quad [x_0]_{a+1,b,c}, [x_1]_{a,b+1,c}, [x_2]_{a,b,c+1} \quad \text{if } (-1)^{n+1} = 1 \text{ and } \rho^a = \rho^b = \rho^c,$$

$$(3.4) \quad (\rho^b + (-1)^n \rho^c) [x_0]_{a+1,b,c} + (\rho^c + (-1)^n \rho^a) [x_1]_{a,b+1,c} + (\rho^a + (-1)^n \rho^b) [x_2]_{a,b,c+1},$$

otherwise;

for $t + l = n$:

$$(3.5) \quad [x_0]_{0,t,l}, [x_1]_{t,0,l}, [x_2]_{t,l,0} \quad \text{if } \rho^{t+1} = \rho^{l+1} = (-1)^{n+1};$$

for $a + b + c = n - 2$:

$$(3.6) \quad [x_1 x_2]_{a,b+1,c+1}, [x_2 x_0]_{a+1,b,c+1}, [x_0 x_1]_{a+1,b+1,c} \quad \text{if } (-1)^n = 1 \text{ and } \rho^a = \rho^b = \rho^c;$$

$$(3.7) \quad \begin{cases} (\rho^c + (-1)^{n+1} \rho^a) [x_1 x_2]_{a,b+1,c+1} - (\rho^b + (-1)^{n+1} \rho^c) [x_2 x_0]_{a+1,b,c+1}, \\ (\rho^a + (-1)^{n+1} \rho^b) [x_2 x_0]_{a+1,b,c+1} - (\rho^c + (-1)^{n+1} \rho^a) [x_0 x_1]_{a+1,b+1,c}, \\ (\rho^b + (-1)^{n+1} \rho^c) [x_0 x_1]_{a+1,b+1,c} - (\rho^a + (-1)^{n+1} \rho^b) [x_1 x_2]_{a,b+1,c+1} \end{cases} \quad \text{otherwise;}$$

for $t + l = n - 1$:

$$(3.8) \quad \begin{cases} [x_1 x_2]_{t,l+1,0}, [x_1 x_2]_{t,0,l+1}, [x_2 x_0]_{t+1,l,0}, \\ [x_2 x_0]_{0,t,l+1}, [x_0 x_1]_{t+1,0,l}, [x_0 x_1]_{0,t+1,l} \end{cases} \quad \text{if } \rho^{t+1} = \rho^{l+1} = (-1)^n;$$

$$(3.9) \quad \begin{cases} (1 + (-1)^{n+1} \rho^{t+1}) [x_1 x_2]_{t,l+1,0} - (\rho^{l+1} + (-1)^{n+1}) [x_2 x_0]_{t+1,l,0}, \\ (1 + (-1)^{n+1} \rho^{t+1}) [x_2 x_0]_{0,t,l+1} - (\rho^{l+1} + (-1)^{n+1}) [x_0 x_1]_{0,t+1,l}, \\ (1 + (-1)^{n+1} \rho^{l+1}) [x_0 x_1]_{t+1,0,l} - (\rho^{t+1} + (-1)^{n+1}) [x_1 x_2]_{t,0,l+1} \end{cases} \quad \text{otherwise;}$$

$$(3.10) \quad [x_1 x_2]_{n,0,0}, [x_2 x_0]_{0,n,0}, [x_0 x_1]_{0,0,n} \quad \text{if } (-1)^n = 1;$$

for $a + b + c = n$:

$$(3.11) \quad [x_0 x_1 x_2]_{a,b,c}.$$

This set of generators of $\text{Ker } \Delta^n$ is denoted by $\mathcal{K}_n$.

In order to investigate $\text{Im } \Delta^{n-1}$, we consider the analog of (3.1) with $n - 1$ in place of n . Grouping the summands with coefficients $k_{(\alpha\beta\gamma)}^{(abc)}$ (for $a + b + c = n - 1$) we get the following expressions:

at $k_{(000)}^{(abc)}$:

$$(3.12) \quad (\rho^b + (-1)^n \rho^c)[x_0]_{a+1,b,c} + (\rho^c + (-1)^n \rho^a)[x_1]_{a,b+1,c} + (\rho^a + (-1)^n \rho^b)[x_2]_{a,b,c+1};$$

at $k_{(100)}^{(abc)}$:

$$(3.13) \quad (\rho^a + (-1)^{n+1} \rho^{b+1})[x_2 x_0]_{a,b,c+1} - (\rho^{c+1} + (-1)^{n+1} \rho^a)[x_0 x_1]_{a,b+1,c};$$

at $k_{(010)}^{(abc)}$:

$$(3.14) \quad (\rho^b + (-1)^{n+1} \rho^{c+1})[x_0 x_1]_{a+1,b,c} - (\rho^{a+1} + (-1)^{n+1} \rho^b)[x_1 x_2]_{a,b,c+1};$$

at $k_{(001)}^{(abc)}$:

$$(3.15) \quad (\rho^c + (-1)^{n+1} \rho^{a+1})[x_1 x_2]_{a,b+1,c} - (\rho^{b+1} + (-1)^{n+1} \rho^c)[x_2 x_0]_{a+1,b,c};$$

at $k_{(011)}^{(abc)}$:

$$(3.16) \quad (\rho^b + (-1)^n \rho^c)[x_0 x_1 x_2]_{a+1,b,c};$$

at $k_{(101)}^{(abc)}$:

$$(3.17) \quad (\rho^c + (-1)^n \rho^a)[x_0 x_1 x_2]_{a,b+1,c};$$

at $k_{(110)}^{(abc)}$:

$$(3.18) \quad (\rho^a + (-1)^n \rho^b)[x_0 x_1 x_2]_{a,b,c+1}.$$

It is easily seen that the elements (3.12)–(3.18) form a set of generators of $\text{Im } \Delta^{n-1}$.

Note that if $(-1)^{n+1} = 1$ and $\rho^a = \rho^b = \rho^c$, then the elements (3.12) are zero, and otherwise they are precisely the generators of $\text{Ker } \Delta^n$ occurring in (3.4).

We split the generators (3.13) ((3.14) or (3.15)) into two groups where $a > 0$ and $a = 0$ (where $b > 0$ and $b = 0$ or $c > 0$ and $c = 0$, respectively). In the notation for the generators of the first group, we substitute $a + 1$ for a ($b + 1$ for b , or $c + 1$ for c , respectively). After that, we divide these generators by ρ . Now observe that if $(-1)^n = 1$ and $\rho^a = \rho^b = \rho^c$, then the resulting elements are equal to zero, and otherwise they are precisely the generators (3.7). In the second group substitute $0, t, l$ ($t, 0, l$ or $t, l, 0$, respectively) for a, b, c . Note that if $\rho^{t+1} = \rho^{l+1} = (-1)^n$, then the resulting elements are equal to zero, and otherwise they are precisely the generators (3.9).

Finally, consider the generators (3.16) ((3.17) or (3.18), respectively). We split these generators into three groups.

1) Suppose $b, c > 0$ ($a, c > 0$ or $a, b > 0$, respectively). We substitute $b + 1$ for b and $c + 1$ for c in (3.16) ($a + 1$ for a and $c + 1$ for c in (3.17) and $a + 1$ for a and $b + 1$ for b in (3.18)). Then, for every a, b, c such that $a + b + c = n - 3$, the generators of the type (3.16)–(3.18) can be represented in the following form:

$$\begin{aligned} &(\rho^{b+1} + (-1)^n \rho^{c+1})[x_0 x_1 x_2]_{a+1,b+1,c+1}, \\ &(\rho^{c+1} + (-1)^n \rho^{a+1})[x_0 x_1 x_2]_{a+1,b+1,c+1}, \\ &(\rho^{a+1} + (-1)^n \rho^{b+1})[x_0 x_1 x_2]_{a+1,b+1,c+1}. \end{aligned}$$

If $(-1)^{n+1} = 1$ and $\rho^a = \rho^b = \rho^c$, then these elements are equal to zero. On the other hand, there are generators of the form

$$(3.19) \quad [x_0 x_1 x_2]_{a+1,b+1,c+1}, \quad \text{unless } (-1)^{n+1} = 1 \text{ and } \rho^a = \rho^b = \rho^c.$$

2) Suppose $b > 0$, $c = 0$ ($c > 0$, $a = 0$ or $a > 0$, $b = 0$, respectively). We substitute t for a and $l + 1$ for b (t for b and $l + 1$ for c , or $t + 1$ for a and l for c , respectively). If $b = 0$, $c > 0$ ($c = 0$, $a > 0$ or $a = 0$, $b > 0$, respectively), we substitute t for a and $l + 1$

for c ($t+1$ for a and l for b , or $t+1$ for b and l for c , respectively). Thus, for every t, l such that $t+l = n-2$, the generators of the type (3.16)–(3.18) can be represented in the following form:

$$(3.20) \quad \begin{cases} (\rho^{l+1} + (-1)^n)[x_0x_1x_2]_{t+1,l+1,0}, & (1 + (-1)^n\rho^{l+1})[x_0x_1x_2]_{t+1,0,l+1}, \\ (\rho^{l+1} + (-1)^n)[x_0x_1x_2]_{0,t+1,l+1}, & (1 + (-1)^n\rho^{t+1})[x_0x_1x_2]_{t+1,l+1,0}, \\ (\rho^{t+1} + (-1)^n)[x_0x_1x_2]_{t+1,0,l+1}, & (1 + (-1)^n\rho^{t+1})[x_0x_1x_2]_{0,t+1,l+1}. \end{cases}$$

If $\rho^{t+1} = \rho^{l+1} = (-1)^{n+1}$, then the elements (3.20) are equal to zero. On the other hand, there are generators of the form

$$(3.21) \quad \begin{cases} [x_0x_1x_2]_{0,t+1,l+1}, \\ [x_0x_1x_2]_{t+1,0,l+1}, \\ [x_0x_1x_2]_{t+1,l+1,0}, \end{cases} \quad \text{unless } \rho^{t+1} = \rho^{l+1} = (-1)^{n+1}.$$

3) Suppose $b = c = 0$ ($c = a = 0$ or $a = b = 0$, respectively). Then the elements (3.16)–(3.18) have the form

$$(3.22) \quad \begin{cases} (1 + (-1)^n)[x_0x_1x_2]_{n,0,0}, \\ (1 + (-1)^n)[x_0x_1x_2]_{0,n,0}, \\ (1 + (-1)^n)[x_0x_1x_2]_{0,0,n}. \end{cases}$$

If $(-1)^{n+1} = 1$, then the elements (3.22) are equal to zero. On the other hand, there are generators of the form

$$(3.23) \quad \begin{cases} [x_0x_1x_2]_{n,0,0}, \\ [x_0x_1x_2]_{0,n,0}, \\ [x_0x_1x_2]_{0,0,n}, \end{cases} \quad \text{unless } (-1)^{n+1} = 1.$$

The resulting set of generators of $\text{Im } \Delta^{n-1}$, i.e., the set of the elements (3.4), (3.7), (3.9), (3.19), (3.21), and (3.23), will be denoted by $\mathcal{I}_n$. It is easily seen that $\mathcal{I}_n$ is a subset of $\mathcal{K}_n$, and that $\mathcal{K}_n \setminus \mathcal{I}_n$ consists of the following elements:

$$(3.24) \quad [1]_{a,b,c} \quad \text{if } a+b+c = n, \quad (-1)^n = 1, \quad \rho^a = \rho^b = \rho^c;$$

$$(3.25) \quad \begin{cases} [x_0]_{0,t,l}, \\ [x_1]_{t,0,l}, \\ [x_2]_{t,l,0} \end{cases} \quad \text{if } t+l = n, \quad \rho^{t+1} = \rho^{l+1} = (-1)^{n+1};$$

$$(3.26) \quad \begin{cases} [x_0]_{a+1,b,c}, \\ [x_1]_{a,b+1,c}, \\ [x_2]_{a,b,c+1} \end{cases} \quad \text{if } a+b+c = n-1, \quad (-1)^{n+1} = 1, \quad \rho^a = \rho^b = \rho^c;$$

$$(3.27) \quad \begin{cases} [x_1x_2]_{n,0,0}, \\ [x_2x_0]_{0,n,0}, \\ [x_0x_1]_{0,0,n} \end{cases} \quad \text{if } (-1)^n = 1;$$

$$(3.28) \quad \begin{cases} [x_1x_2]_{t,l+1,0}, \\ [x_1x_2]_{t,0,l+1}, \\ [x_2x_0]_{t+1,l,0}, \\ [x_2x_0]_{0,t,l+1}, \\ [x_0x_1]_{t+1,0,l}, \\ [x_0x_1]_{0,t+1,l} \end{cases} \quad \text{if } t+l = n-1, \quad \rho^{t+1} = \rho^{l+1} = (-1)^n;$$

$$(3.29) \quad \begin{cases} [x_1 x_2]_{a,b+1,c+1}, \\ [x_2 x_0]_{a+1,b,c+1}, & \text{if } a+b+c = n-2, (-1)^n = 1, \rho^a = \rho^b = \rho^c; \\ [x_0 x_1]_{a+1,b+1,c} \end{cases}$$

$$(3.30) \quad \begin{cases} [x_0 x_1 x_2]_{n,0,0}, \\ [x_0 x_1 x_2]_{0,n,0}, & \text{if } (-1)^{n+1} = 1; \\ [x_0 x_1 x_2]_{0,0,n} \end{cases}$$

$$(3.31) \quad \begin{cases} [x_0 x_1 x_2]_{0,t+1,l+1}, \\ [x_0 x_1 x_2]_{t+1,0,l+1}, & \text{if } t+l = n-2, \rho^{t+1} = \rho^{l+1} = (-1)^{n+1}; \\ [x_0 x_1 x_2]_{t+1,l+1,0} \end{cases}$$

$$(3.32) \quad [x_0 x_1 x_2]_{a+1,b+1,c+1} \quad \text{if } a+b+c = n-3, (-1)^{n+1} = 1, \rho^a = \rho^b = \rho^c.$$

Proposition 3.2. *The cohomology classes of the elements (3.24)–(3.32) form a K -basis of $\mathrm{HH}^n(R)$.*

Remark 3.2. In the sequel, for an n -cocycle $x \in \mathrm{Ker} \Delta^n$ ($n \geq 1$), its cohomology class in $\mathrm{HH}^n(R)$ will often be denoted also by x .

Proof. All elements (3.24)–(3.32) are elements of the standard basis of R^{M_n} . Moreover, such elements do not appear in the expansions (in the standard basis) of the elements of $\mathcal{I}_n$. Therefore, they are linearly independent. $\square$

Next, we calculate $\dim_K \mathrm{HH}^n(R)$ for $n > 0$. Denote

$$(3.33) \quad A(n) = \#\{(a, b, c) \mid a+b+c = n, (-1)^n = 1, \rho^a = \rho^b = \rho^c\},$$

$$(3.34) \quad B(n) = \#\{(t, l) \mid t+l = n, \rho^{t+1} = \rho^{l+1} = (-1)^{n+1}\},$$

$$(3.35) \quad C = \begin{cases} 3 & \text{if } \mathrm{char} K \neq 2, \\ 6 & \text{if } \mathrm{char} K = 2. \end{cases}$$

Then, using Proposition 3.2, we get

$$(3.36) \quad \begin{aligned} \dim_K \mathrm{HH}^n(R) &= A(n) + 3A(n-1) + 3A(n-2) + A(n-3) \\ &\quad + 3B(n) + 6B(n-1) + 3B(n-2) + C. \end{aligned}$$

Now we consider separately several cases; cf. the items of Theorem 2.1.

Corollary 3.3. *Suppose $R = R_\rho$. For every $n > 0$,*

if $\mathrm{char} K \neq 2$ and $\rho = 1$, then

$$\dim_K \mathrm{HH}^n(R) = 2n^2 + 6n + 4;$$

if $\mathrm{char} K \neq 2$ and $\rho = -1$, then

$$\dim_K \mathrm{HH}^n(R) = \begin{cases} 2m^2 + 11m + 12 & \text{if } n = 2m + 1, \\ 2m^2 + 9m + 7 & \text{if } n = 2m; \end{cases}$$

if $\mathrm{char} K = 2$ and $\rho = 1$, then

$$\dim_K \mathrm{HH}^n(R) = 4n^2 + 12n + 8.$$

Proof. Observe that

$$\begin{aligned} \#\{(a, b, c) \mid a+b+c = n\} &= \frac{1}{2}(n^2 + 3n + 2), \\ \#\{(t, l) \mid t+l = n\} &= n + 1. \end{aligned}$$

Suppose $\text{char } K \neq 2$, $\rho = 1$. The condition $(-1)^{a+b+c} = 1, \rho^a = \rho^b = \rho^c$ is equivalent to the fact that $a + b + c$ is even. Therefore,

$$A(n) = \begin{cases} \frac{1}{2}(n^2 + 3n + 2) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

The condition $\rho^{t+1} = \rho^{l+1} = (-1)^{t+l+1}$ is equivalent to the fact that $t + l$ is odd. Therefore,

$$B(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Suppose $\text{char } K \neq 2$, $\rho = -1$. The condition $(-1)^{a+b+c} = 1, \rho^a = \rho^b = \rho^c$ is equivalent to the fact that a, b , and c are even. Therefore,

$$A(n) = \begin{cases} \frac{1}{2}(m^2 + 3m + 2) & \text{if } n = 2m, \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

The condition $\rho^{t+1} = \rho^{l+1} = (-1)^{t+l+1}$ is equivalent to the fact that t and l are even. Therefore,

$$B(n) = \begin{cases} m + 1 & \text{if } n = 2m, \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Suppose $\text{char } K = 2$, $\rho = 1$. Then the conditions $(-1)^{a+b+c} = 1, \rho^a = \rho^b = \rho^c$, and $\rho^{t+1} = \rho^{l+1} = (-1)^{t+l+1}$ are fulfilled for every $a, b, c, t, l \in \mathbb{N}_0$. Therefore,

$$A(n) = \frac{1}{2}(n^2 + 3n + 2), \quad B(n) = \#\{(t, l) \mid t + l = n\} = n + 1.$$

Substituting this in (3.36), we complete the proof. $\square$

Corollary 3.4. *Suppose $R = R_\rho$ and ρ is of infinite order in the group K^* . For every $n > 0$, if $\text{char } K \neq 2$, then*

$$\dim_K \text{HH}^n(R) = \begin{cases} 6 & \text{if } n \equiv 1 \text{ or } 2 \pmod{6}, \\ 4 & \text{if } n \equiv 3 \text{ or } 0 \pmod{6}, \\ 3 & \text{if } n \equiv 4 \text{ or } 5 \pmod{6}; \end{cases}$$

if $\text{char } K = 2$, then

$$\dim_K \text{HH}^n(R) = \begin{cases} 8 & \text{if } n \equiv 0 \pmod{3}, \\ 9 & \text{otherwise.} \end{cases}$$

Proof. Since ρ is not a root of unity, there are no indices t, l such that $\rho^{t+1} = \rho^{l+1} = (-1)^{t+l+1}$. Therefore, $B(n) = 0$ for every n .

Suppose $\text{char } K \neq 2$. The condition $(-1)^{a+b+c} = 1, \rho^a = \rho^b = \rho^c$ is equivalent to the fact that $a = b = c = 2k$ for some $k \in \mathbb{N}_0$. Therefore,

$$A(n) = \begin{cases} 1 & \text{if } n = 6k, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $\text{char } K = 2$. The condition $(-1)^{a+b+c} = 1, \rho^a = \rho^b = \rho^c$ is equivalent to the fact that $a = b = c = k$ for some $k \in \mathbb{N}_0$. Therefore,

$$A(n) = \begin{cases} 1 & \text{if } n = 3k, \\ 0 & \text{otherwise.} \end{cases}$$

Substituting these values in (3.36), we complete the proof. $\square$

The values of $\dim_K \mathrm{HH}^n(R)$ for the remaining cases are shown in the tables of §7.

§4. YONEDA PRODUCT

For our calculations, the following interpretation of the Yoneda product in

$$\mathrm{HH}^*(R) = \mathrm{Ext}_\Lambda^*(R, R) = \bigoplus_{m \geq 0} \mathrm{Ext}_\Lambda^m(R, R)$$

is convenient (cf., e.g., [13]).

Let $Q_\bullet \rightarrow R$ be a minimal Λ -projective resolution. We denote by $\Omega^n(R)$ the n th syzygy of the module R , i.e., $\Omega^n(R) = \mathrm{Im}(d_{n-1}^Q: Q_n \rightarrow Q_{n-1})$. Let

$$\mathrm{Hom}_\Lambda(Q_\bullet, R) = (\mathrm{Hom}_\Lambda(Q_n, R), \Delta^n);$$

this is the complex studied in §3. Since $\mathrm{Ker} \Delta^n \simeq \mathrm{Hom}_\Lambda(\Omega^n(R), R)$, for any cocycle $f: Q_n \rightarrow R$ there exists a unique Λ -homomorphism $\hat{f}: \Omega^n(R) \rightarrow R$, and $f = \hat{f} \circ \pi_n$, where $\pi_n: Q_n \rightarrow \Omega^n(R)$ is the projective cover. The sequence

$$\cdots \xrightarrow{d_{n+1}^Q} Q_{n+1} \xrightarrow{d_n^Q} Q_n \xrightarrow{\pi_n} \Omega^n(R) \rightarrow 0$$

is a minimal Λ -projective resolution of the module $\Omega^n(R)$. Therefore, the homomorphism $\hat{f}: \Omega^n(R) \rightarrow R$ lifts (uniquely up to homotopy) to a chain map of complexes $\{\varphi_i: Q_{n+i} \rightarrow Q_i\}_{i \geq 0}$. The homomorphism φ_i is called the i th *translate* of the homomorphism $\hat{f}$ or of the corresponding cocycle f and is denoted by $T^i(\hat{f})$ or $T^i(f)$. Moreover, the homomorphism φ_i induces maps $\Omega^k(\hat{f}): \Omega^{n+k}(R) \rightarrow \Omega^k(R)$, $k \geq 0$. Now, for cocycles $f \in \mathrm{Ker} \Delta^n$ and $g \in \mathrm{Ker} \Delta^t$ we have $\mathrm{cl} g \cdot \mathrm{cl} f = \mathrm{cl}(\hat{g} \circ \Omega^t(\hat{f}))$.

On the other hand, the commutative diagram

$$\begin{array}{ccc} Q_{n+t} & \xrightarrow{\pi_{n+t}} & \Omega^{n+t}(R) \\ T^t(\hat{f}) \downarrow & & \downarrow \Omega^t(\hat{f}) \\ Q_t & \xrightarrow{\pi_t} & \Omega^t(R) \\ T^0(\hat{g}) \downarrow & & \downarrow \hat{g} \\ Q_0 & \xrightarrow{d_{-1}^Q} & R \end{array}$$

shows that $(\hat{g} \circ \Omega^t(\hat{f})) \circ \pi_{n+t} = d_{-1}^Q \circ (T^0(\hat{g}) \circ T^t(\hat{f}))$; consequently, for us it suffices to know the composition of the translates $T^0(\hat{g})$ and $T^t(\hat{f})$.

For $i + j + k = n$, $a + b + c = m$, let $e_{(i,j,k) \rightarrow (a,b,c)}$ denote the Λ -homomorphism of Q_m to Q_n that takes an element of the direct summand of Q_n with index (i, j, k) to the corresponding element of the direct summand of Q_m with index (a, b, c) ; all other direct summands of Q_n are mapped to zero. We assume that the indices lie in the Abelian group $\mathbb{Z}^3$, and we can add them accordingly.

Proposition 4.1. *Let $f = [r]_{a,b,c}: Q_n \rightarrow R$ be a cocycle; then for any $t \geq 0$ we have*

$$T^t(f) = \sum_{i+j+k=t} (-1)^{nt} \rho^{aj+bk+ci} (r \otimes 1')^* e_{(a,b,c)+(i,j,k) \rightarrow (i,j,k)}.$$

Proof. Obviously, we may put $T^0(f) = (r \otimes 1')^* e_{(a,b,c) \rightarrow (0,0,0)}$. Direct calculations show easily that the squares

$$\begin{array}{ccc} Q_{n+t} & \xrightarrow{d_{n+t-1}} & Q_{n+t-1} \\ T^t(f) \downarrow & & \downarrow T^{t-1}(f) \\ Q_t & \xrightarrow{d_{t-1}} & Q_{t-1} \end{array}$$

are commutative for all $t \geq 1$. This concludes the proof. $\square$

Therefore, we get the following “multiplication formula”.

Theorem 4.2. *If $f = [r_1]_{a_1, b_1, c_1} \in \text{Ker } \Delta^{n_1}$ and $g = [r_2]_{a_2, b_2, c_2} \in \text{Ker } \Delta^{n_2}$, then $\text{cl } g \cdot \text{cl } f = \text{cl } h$, where*

$$h = [(-1)^{n_1 n_2} \rho^{a_1 b_2 + b_1 c_2 + c_1 a_2} r_1 r_2]_{a_1 + a_2, b_1 + b_2, c_1 + c_2} : Q_{n_1 + n_2} \rightarrow R.$$

Theorem 4.2 allows us to calculate the products of basis elements of $\text{HH}^n(R)$ (see §3).

§5. GENERATORS AND RELATIONS

In §3 we introduced a K -basis of $\text{HH}^n(R)$ (see Proposition 3.2). The conditions $(-1)^{a+b+c} = 1, \rho^a = \rho^b = \rho^c$ (see (3.24), (3.26), (3.29), and (3.32)) and $\rho^{t+1} = \rho^{l+1} = (-1)^{t+l+1}$ (see (3.25), (3.28), and (3.31)) occur on the list of basis elements. Now we reformulate these conditions, using the conditions on $\text{char } K$ and the (multiplicative) order of ρ in K^* .

Lemma 5.1. *Suppose K is a field and $\rho \in K^*$. Let s be the (multiplicative) order of ρ in K^* , and let $a, b, c \in \mathbb{N}_0$. Consider the following condition on a, b, c :*

$$(5.1) \quad (-1)^{a+b+c} = 1, \quad \rho^a = \rho^b = \rho^c.$$

I. Suppose $\text{char } K \neq 2$.

(I.1) *If $s = 1$, then (5.1) is equivalent to the fact that $a + b + c$ is even, which, in its turn, is equivalent to the existence of $a_i \in \mathbb{N}_0$, $i \in \{0, 1, 2\}$, such that*

$$(a, b, c) = a_0(2, 0, 0) + a_1(0, 2, 0) + a_2(0, 0, 2) + \tau,$$

where $\tau \in \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\}$.

(I.2) *If $s = 2$, then (5.1) is equivalent to the fact that a, b , and c are even. In its turn, this is equivalent to the existence of $a_i \in \mathbb{N}_0$, $i \in \{0, 1, 2\}$, such that*

$$(a, b, c) = a_0(2, 0, 0) + a_1(0, 2, 0) + a_2(0, 0, 2).$$

(I.3) *If s is an odd number greater than 1, then (5.1) is equivalent to the fact that $a + b + c$ is even and $a \equiv b \equiv c \pmod{s}$. In its turn, this is equivalent to the existence of $k, a_i \in \mathbb{N}_0$, $i \in \{0, 1, 2\}$, such that*

$$(5.2) \quad (a, b, c) = k(2, 2, 2) + a_0(2s, 0, 0) + a_1(0, 2s, 0) + a_2(0, 0, 2s) + \tau,$$

where

$$\tau \in \{(0, 0, 0), (0, s, s), (s, 0, s), (s, s, 0), (s+1, 1, 1), (1, s+1, 1), (1, 1, s+1)\}$$

and k satisfies the following conditions:

- (1) if $\tau = (0, 0, 0)$, then $k \leq s - 1$;
- (2) if $\tau \in \{(0, s, s), (s, 0, s), (s, s, 0)\}$, then $k \leq s'$;
- (3) if $\tau \in \{(s+1, 1, 1), (1, s+1, 1), (1, 1, s+1)\}$, then $k \leq s' - 1$.

(I.4) If s is an even number greater than 2, then (5.1) is equivalent to the fact that $a + b + c$ is even and $a \equiv b \equiv c \pmod{s}$. In its turn, this is equivalent to the existence of $k, a_i \in \mathbb{N}_0, i \in \{0, 1, 2\}$, such that

$$(a, b, c) = k(2, 2, 2) + a_0(s, 0, 0) + a_1(0, s, 0) + a_2(0, 0, s).$$

(I.5) If $s = \infty$, then (5.1) is equivalent to the fact that $a + b + c$ is even and $a = b = c$. In its turn, this is equivalent to the existence of $k \in \mathbb{N}_0$ such that $(a, b, c) = k(2, 2, 2)$.

II. Suppose $\text{char } K = 2$.

(II.1) If $s = 1$, then (5.1) is equivalent to the fact that a, b , and c are arbitrary nonnegative integers. In its turn, this is equivalent to the existence of $a_i \in \mathbb{N}_0, i \in \{0, 1, 2\}$, such that

$$(a, b, c) = a_0(1, 0, 0) + a_1(0, 1, 0) + a_2(0, 0, 1).$$

(II.2) If $1 < s < \infty$, then (5.1) is equivalent to the fact that $a \equiv b \equiv c \pmod{s}$. In its turn, this is equivalent to the existence of $k, a_i \in \mathbb{N}_0, i \in \{0, 1, 2\}$, such that

$$(a, b, c) = k(1, 1, 1) + a_0(s, 0, 0) + a_1(0, s, 0) + a_2(0, 0, s).$$

(II.3) If $s = \infty$, then (5.1) is equivalent to the fact that $a = b = c$. In its turn, this is equivalent to the existence of $k \in \mathbb{N}_0$ such that $(a, b, c) = k(1, 1, 1)$.

The decompositions of (a, b, c) into a sum as indicated are unique.

Proof. We prove only the second equivalence in (I.3), because the other equivalences are quite simple.

So, suppose that $\text{char } K \neq 2$, s is an odd number exceeding 1, and $a, b, c \in \mathbb{N}_0$ are such that $a + b + c$ is even and $a \equiv b \equiv c \pmod{s}$. We prove (5.2). Let $a = 2sa_0 + w_0$, $b = 2sa_1 + w_1$, and $c = 2sa_2 + w_2$, where $a_i, w_i \in \mathbb{N}_0, 0 \leq w_i \leq 2s - 1$ ($i = 0, 1, 2$). Let k be the integral part of $\min\{w_0, w_1, w_2\}/2$. Observe that $\min\{w_0 - 2k, w_1 - 2k, w_2 - 2k\}$ is equal to zero or to one, these three numbers are congruent modulo s , and their sum is even. Therefore, the triple $(w_0 - 2k, w_1 - 2k, w_2 - 2k)$ is a permutation of one of the following triples: $(0, 0, 0)$, $(0, s, s)$, $(1, 1, s + 1)$. It is easy to realize that the number k as above satisfies conditions (1)–(3).

Now we prove the uniqueness of a decomposition as in (5.2). Suppose we have a decomposition satisfying the conditions indicated. Then $a - 2sa_0$ does not exceed the sum of $2k$ and one of the numbers $0, s$, or $s + 1$; also, k does not exceed $s - 1, s'$, or $s' - 1$, respectively. Therefore, $a - 2sa_0 < 2s$ and similarly $b - 2sa_1 < 2s$ and $c - 2sa_2 < 2s$. Consequently, the a_i are uniquely determined, being the integral parts of the result of division by $2s$. Since $\min\{a - 2sa_0, b - 2sa_1, c - 2sa_2\}$ is equal to $2k$ or to $2k + 1$, the number k is uniquely determined. Therefore, the remainder $(a - 2sa_0 - 2k, b - 2sa_1 - 2k, c - 2sa_2 - 2k)$ is also uniquely determined. $\square$

Lemma 5.2. Suppose K is a field and $\rho \in K^*$. Let s be the (multiplicative) order of ρ in K^* , and let $t, l \in \mathbb{N}_0$. Consider the following condition on t, l :

$$(5.3) \quad \rho^{t+1} = \rho^{l+1} = (-1)^{t+l+1}.$$

I. Suppose $\text{char } K \neq 2$.

(I.1) If $s = 1$, then (5.3) is equivalent to the fact that $t + l$ is odd. In its turn, this is equivalent to the existence of $a_1, a_2 \in \mathbb{N}_0$ such that

$$(t, l) = \theta + a_1(2, 0) + a_2(0, 2),$$

where θ is $(1, 0)$ or $(0, 1)$.

(I.2) If $s = 2$, then (5.3) is equivalent to the fact that t and l are even. In its turn, this is equivalent to the existence of $a_1, a_2 \in \mathbb{N}_0$ such that

$$(t, l) = a_1(2, 0) + a_2(0, 2).$$

(I.3) If s is an odd number greater than 1, then (5.3) is equivalent to the fact that $t + l$ is odd and $t \equiv l \equiv -1 \pmod{s}$. In its turn, this is equivalent to the existence of $a_1, a_2 \in \mathbb{N}_0$ such that

$$(t, l) = \theta + a_1(2s, 0) + a_2(0, 2s),$$

where θ is $(s - 1, 2s - 1)$ or $(2s - 1, s - 1)$.

(I.4) If s is an even number greater than 2, then (5.3) is equivalent to the fact that $t + l$ is even and $t \equiv l \equiv s' - 1 \pmod{s}$, where $s = 2s'$. In its turn, this is equivalent to the existence of $a_1, a_2 \in \mathbb{N}_0$ such that

$$(t, l) = (s' - 1, s' - 1) + a_1(s, 0) + a_2(0, s).$$

(I.5) If $s = \infty$, then no t, l satisfy (5.3).

II. Suppose $\text{char } K = 2$.

(II.1) If $s = 1$, then (5.3) is equivalent to the fact that t and l are arbitrary nonnegative integers. In its turn, this is equivalent to the existence of $a_1, a_2 \in \mathbb{N}_0$ such that

$$(t, l) = a_1(1, 0) + a_2(0, 1).$$

(II.2) If $1 < s < \infty$, then (5.3) is equivalent to the fact that $t \equiv l \equiv -1 \pmod{s}$. In its turn, this is equivalent to the existence of $a_1, a_2 \in \mathbb{N}_0$ such that

$$(t, l) = (s - 1, s - 1) + a_1(s, 0) + a_2(0, s).$$

(II.3) If $s = \infty$, then no t, l satisfy (5.3).

The decompositions of (t, l) into a sum as indicated are unique.

Proof. These statements are obvious. □

Remark 5.1. In fact, we shall need decompositions of the triples $(0, t, l)$, $(t, 0, l)$, and $(t, l, 0)$. However, the decompositions obtained for (t, l) will remain valid for such triples if we add the zero component.

Suppose $R = R_\rho$ and s is the order of ρ in K^* . We proceed to describing generators and relations in the algebra $\text{HH}^*(R)$. We consider several cases separately; see the items in Theorem 2.1. Only cases (I.3) and (I.5) will be treated in detail. The proofs for the others cases are similar; we leave them to the reader.

(I.3) Suppose $\text{char } K \neq 2$ and $s = 2s' + 1$, where $s' \in \mathbb{N}$.

Consider the elements of $\text{HH}^0(R)$ given by

$$\begin{aligned} h_0 &= [x_1 x_2]_{0,0,0}, & h_1 &= [x_2 x_0]_{0,0,0}, & h_2 &= [x_0 x_1]_{0,0,0}, \\ h &= [x_0 x_1 x_2]_{0,0,0}; \end{aligned}$$

the elements of $\text{HH}^1(R)$ given by

$$y_0 = [x_0]_{1,0,0}, \quad y_1 = [x_1]_{0,1,0}, \quad y_2 = [x_2]_{0,0,1};$$

the elements of $\text{HH}^{3s-2}(R)$ given by

$$\begin{aligned} t_0 &= [x_0]_{0,s-1,2s-1}, & t_1 &= [x_1]_{2s-1,0,s-1}, & t_2 &= [x_2]_{s-1,2s-1,0}, \\ l_0 &= [x_0]_{0,2s-1,s-1}, & l_1 &= [x_1]_{s-1,0,2s-1}, & l_2 &= [x_2]_{2s-1,s-1,0}; \end{aligned}$$

the elements of $\text{HH}^{2m}(R)$ ($m = 1, \dots, s - 1$) given by

$$z_0^{(m)} = [x_1 x_2]_{2m,0,0}, \quad z_1^{(m)} = [x_2 x_0]_{0,2m,0}, \quad z_2^{(m)} = [x_0 x_1]_{0,0,2m};$$

the element of $\text{HH}^6(R)$ given by

$$p = [1]_{2,2,2};$$

the elements of $\mathrm{HH}^{2s}(R)$ given by

$$\begin{aligned} p_0 &= [1]_{2s,0,0}, & p_1 &= [1]_{0,2s,0}, & p_2 &= [1]_{0,0,2s}, \\ q_0 &= [1]_{0,s,s}, & q_1 &= [1]_{s,0,s}, & q_2 &= [1]_{s,s,0}; \end{aligned}$$

and, finally, the elements of $\mathrm{HH}^{s+3}(R)$ given by

$$r_0 = [1]_{s+1,1,1}, \quad r_1 = [1]_{1,s+1,1}, \quad r_2 = [1]_{1,1,s+1}.$$

Theorem 5.3. *Suppose $\mathrm{char} K \neq 2$ and $s = 2s' + 1$, where $s' \in \mathbb{N}$. By definition, put*

$$(5.4) \quad \mathcal{Y}_3 = \{h, h_i, y_i, t_i, l_i, z_i^{(m)}, p, p_i, q_i, r_i\}_{i \in \{0,1,2\}, m \in \{1, \dots, s-1\}}.$$

Then, as a K -algebra, $\mathrm{HH}^(R)$ is generated by the set $\mathcal{Y}_3$.*

Proof. By Proposition 3.1, we have $\mathrm{HH}^0(R) = {}_K \langle 1, h_i, h \rangle_{i \in \{0,1,2\}}$. We fix $n > 0$ and show how the elements of the K -basis $\mathrm{HH}^n(R)$ (see (3.24)–(3.32)) are expressed, by using multiplication only, in terms of the elements of the set $\mathcal{Y}_3$.

By Lemma 5.1(I.3), there exist $k, a_j \in \mathbb{N}_0$, $j = 0, 1, 2$, and, possibly, some $i \in \{0, 1, 2\}$ such that

$$[1]_{a,b,c} = \begin{cases} \rho^{-6k(k-1)} p_0^{a_0} p_1^{a_1} p_2^{a_2} p^k, \\ \rho^{-6k(k-1)} p_0^{a_0} p_1^{a_1} p_2^{a_2} p^k q_i, \\ \rho^{-6k^2} p_0^{a_0} p_1^{a_1} p_2^{a_2} p^k r_i. \end{cases}$$

The following generators also come from elements of $\mathcal{Y}_3$:

$$(5.5) \quad \left\{ \begin{aligned} [x_0]_{a+1,b,c} &= \rho^{-c} y_0 [1]_{a,b,c}, \\ [x_1]_{a,b+1,c} &= \rho^{-a} y_1 [1]_{a,b,c}, \\ [x_2]_{a,b,c+1} &= \rho^{-b} y_2 [1]_{a,b,c}, \\ [x_1 x_2]_{a,b+1,c+1} &= \rho^{-(a+1)} y_1 [x_2]_{a,b,c+1}, \\ [x_2 x_0]_{a+1,b,c+1} &= \rho^{-(b+1)} y_2 [x_0]_{a+1,b,c}, \\ [x_0 x_1]_{a+1,b+1,c} &= \rho^{-(c+1)} y_0 [x_1]_{a,b+1,c}, \\ [x_0 x_1 x_2]_{a+1,b+1,c+1} &= \rho^{-(c+1)} y_0 [x_1 x_2]_{a,b+1,c+1}. \end{aligned} \right.$$

By Lemma 5.2(I.3) (see also Remark 5.1), there exist $a_j \in \mathbb{N}_0$, $j = 0, 1, 2$, such that

$$[x_0]_{0,t,l} = \begin{bmatrix} t_0 p_1^{a_1} p_2^{a_2}, \\ l_0 p_1^{a_1} p_2^{a_2}, \end{bmatrix} \quad [x_1]_{t,0,l} = \begin{bmatrix} t_1 p_0^{a_0} p_2^{a_2}, \\ l_1 p_0^{a_0} p_2^{a_2}, \end{bmatrix} \quad [x_2]_{t,l,0} = \begin{bmatrix} t_2 p_0^{a_0} p_1^{a_1}, \\ l_2 p_0^{a_0} p_1^{a_1}. \end{bmatrix}$$

Furthermore,

$$\begin{aligned} [x_1 x_2]_{t,l+1,0} &= y_1 [x_2]_{t,l,0}, & [x_1 x_2]_{t,0,l+1} &= [x_1]_{t,0,l} y_2, \\ [x_2 x_0]_{0,t,l+1} &= y_2 [x_0]_{0,t,l}, & [x_2 x_0]_{t+1,l,0} &= [x_2]_{t,l,0} y_0, \\ [x_0 x_1]_{t+1,0,l} &= y_0 [x_1]_{t,0,l}, & [x_0 x_1]_{0,t+1,l} &= [x_0]_{0,t,l} y_1, \\ [x_0 x_1 x_2]_{0,t+1,l+1} &= [x_2 x_0]_{0,t,l+1} y_1, \\ [x_0 x_1 x_2]_{t+1,0,l+1} &= [x_0 x_1]_{t+1,0,l} y_2, \\ [x_0 x_1 x_2]_{t+1,l+1,0} &= [x_1 x_2]_{t,l+1,0} y_0. \end{aligned}$$

Suppose that n is even, and let $a_j \in \mathbb{N}_0$, $j = 0, 1, 2$, and m with $1 \leq m \leq s-1$ be such that $n = 2(a_j s + m)$. Then

$$[x_1 x_2]_{n,0,0} = z_0^{(m)} p_0^{a_0}, \quad [x_2 x_0]_{0,n,0} = z_1^{(m)} p_1^{a_1}, \quad [x_0 x_1]_{0,0,n} = z_2^{(m)} p_2^{a_2}.$$

On the other hand, if $n = 2a_js$, then

$$[x_1x_2]_{n,0,0} = h_0p_0^{a_0}, \quad [x_2x_0]_{0,n,0} = h_1p_1^{a_1}, \quad [x_0x_1]_{0,0,n} = h_2p_2^{a_2}.$$

Finally, for odd n we have

$$(5.6) \quad \begin{cases} [x_0x_1x_2]_{n,0,0} = y_0[x_1x_2]_{n-1,0,0}, \\ [x_0x_1x_2]_{0,n,0} = y_1[x_2x_0]_{0,n-1,0}, \\ [x_0x_1x_2]_{0,0,n} = y_2[x_0x_1]_{0,0,n-1}, \\ [x_0x_1x_2]_{1,0,0} = y_0h_0, \quad [x_0x_1x_2]_{0,1,0} = y_1h_1, \quad [x_0x_1x_2]_{0,0,1} = y_2h_2. \quad \square \end{cases}$$

Proposition 5.4. *Suppose $\text{char } K \neq 2$ and $s = 2s' + 1$, where $s' \in \mathbb{N}$. Then the elements of $\mathcal{Y}_3$ (see (5.4)) satisfy the following relations, in which $i, j \in \{0, 1, 2\}$ and $m, m_1, m_2 \in \{1, \dots, s-1\}$:*

$$(5.7) \quad \begin{cases} ha = 0 \quad \text{for all } a \in \mathcal{Y}_3, \\ h_ih_j = h_it_j = h_il_j = h_iz_j^{(m)} = h_ip = h_iq_j = h_ir_j = 0, \\ h_iy_j = h_ip_j = y_iz_j^{(m)} = t_iz_j^{(m)} = l_iz_j^{(m)} = 0 \quad \text{for } i \neq j, \\ y_it_i = y_il_i = t_il_i = z_i^{(m_1)}z_j^{(m_2)} = 0, \\ t_iz_i^{(m)} = l_iz_i^{(m)} = z_i^{(m)}p = z_i^{(m)}q_j = z_i^{(m)}r_j = 0 \quad \text{for } m \neq s', \\ z_i^{(m)}p_j = 0 \quad \text{for } i \neq j, m \neq s'; \end{cases}$$

(5.8)

$$p^s = p_0p_1p_2;$$

$$(5.9) \quad \begin{cases} q_i^2 = p_{i+1}p_{i+2}, \quad r_i^2 = \rho^3pp_i, \\ q_{i+1}q_{i+2} = p_iq_i, \quad r_{i+1}r_{i+2} = \rho^3pq_i, \\ q_ir_i = \rho^{s'+2}p^{s'+1}, \\ q_{i+1}r_i = p_ir_{i+2}, \quad q_{i+2}r_i = p_ir_{i+1}, \\ p^{s'+1}q_i = \rho^{s'-1}p_{i+1}p_{i+2}r_i, \quad p^{s'}r_i = \rho^{s'+2}p_iq_i; \end{cases}$$

$$(5.10) \quad \begin{cases} t_ip_i = \rho^{s'-3}y_ip^{s'}q_{i+1}, \quad l_ip_i = \rho^{s'-3}y_ip^{s'}q_{i+2}, \\ t_ip = \rho^{-5}y_ip_{i+2}r_{i+1}, \quad l_ip = \rho^{-5}y_ip_{i+1}r_{i+2}, \\ t_iq_{i+1} = \rho^{s'-3}y_ip^{s'}p_{i+2}, \quad l_iq_{i+1} = \rho^{s'-3}y_ip^{s'}q_i, \\ t_iq_{i+2} = \rho^{s'-3}y_ip^{s'}q_i, \quad l_iq_{i+2} = \rho^{s'-3}y_ip^{s'}p_{i+1}, \\ t_ir_i = \rho^{-2}y_ip_{i+2}q_{i+2}, \quad l_ir_i = \rho^{-2}y_ip_{i+1}q_{i+1}, \\ t_ir_{i+1} = \rho^{-2}y_ip_{i+1}p_{i+2}, \quad l_ir_{i+1} = \rho^{-2}y_ip_{i+1}q_i, \\ t_ir_{i+2} = \rho^{-2}y_ip_{i+2}q_i, \quad l_ir_{i+2} = \rho^{-2}y_ip_{i+1}p_{i+2}; \end{cases}$$

(5.11)

$$\begin{cases} t_iq_i = l_ip_{i+2}, \\ l_iq_i = t_ip_{i+1}; \end{cases}$$

(5.12)

$$\begin{cases} t_{i+1}t_{i+2} = \rho^{s'-8}y_{i+1}y_{i+2}p^{s'-1}p_ir_{i+1}, \\ l_{i+1}l_{i+2} = \rho^{s'-8}y_{i+1}y_{i+2}p^{s'-1}p_ir_{i+2}, \\ t_{i+1}l_{i+2} = \rho^{s'-8}y_{i+1}y_{i+2}p^{s'-1}p_ir_i, \\ l_{i+1}t_{i+2} = \rho^{s'-8}y_{i+1}y_{i+2}p^{s'-1}p_iq_i; \end{cases}$$

$$(5.13) \quad \begin{cases} z_i^{(s')} p = \rho^{-5} y_{i+1} y_{i+2} r_i, \\ z_i^{(s')} q_i = \rho^{s'-3} y_{i+1} y_{i+2} p^{s'}, \\ z_i^{(s')} r_i = \rho^{-2} y_{i+1} y_{i+2} p_i, \\ z_i^{(s')} r_{i+1} = \rho^{-2} y_{i+1} y_{i+2} q_{i+2}, \\ z_i^{(s')} r_{i+2} = \rho^{-2} y_{i+1} y_{i+2} q_{i+1}; \end{cases}$$

$$(5.14) \quad \begin{cases} z_i^{(s')} p_{i+1} = y_{i+1} t_{i+2}, \\ z_i^{(s')} p_{i+2} = l_{i+1} y_{i+2}, \\ z_i^{(s')} q_{i+1} = t_{i+1} y_{i+2}, \\ z_i^{(s')} q_{i+2} = y_{i+1} l_{i+2}; \end{cases}$$

$$(5.15) \quad \begin{cases} t_i z_i^{(s')} = \rho^{s'-8} y_0 y_1 y_2 p^{s'-1} r_{i+2}, \\ l_i z_i^{(s')} = \rho^{s'-8} y_0 y_1 y_2 p^{s'-1} r_{i+1}. \end{cases}$$

Proof. The proof is straightforward, with the use of Theorem 4.2. $\square$

(I.1) Suppose $\text{char } K \neq 2$ and $s = 1$.

Consider the elements of $\text{HH}^0(R)$ given by

$$\begin{aligned} h_0 &= [x_1 x_2]_{0,0,0}, & h_1 &= [x_2 x_0]_{0,0,0}, & h_2 &= [x_0 x_1]_{0,0,0}, \\ h &= [x_0 x_1 x_2]_{0,0,0}; \end{aligned}$$

the elements of $\text{HH}^1(R)$ given by

$$\begin{aligned} y_0 &= [x_0]_{1,0,0}, & l_0 &= [x_0]_{0,1,0}, & t_0 &= [x_0]_{0,0,1}, \\ y_1 &= [x_1]_{0,1,0}, & l_1 &= [x_1]_{0,0,1}, & t_1 &= [x_1]_{1,0,0}, \\ y_2 &= [x_2]_{0,0,1}, & l_2 &= [x_2]_{1,0,0}, & t_2 &= [x_2]_{0,1,0}; \end{aligned}$$

and the elements of $\text{HH}^2(R)$ given by

$$\begin{aligned} p_0 &= [1]_{2,0,0}, & p_1 &= [1]_{0,2,0}, & p_2 &= [1]_{0,0,2}, \\ q_0 &= [1]_{0,1,1}, & q_1 &= [1]_{1,0,1}, & q_2 &= [1]_{1,1,0}. \end{aligned}$$

Theorem 5.5. Suppose $\text{char } K \neq 2$ and $s = 1$. By definition, put

$$(5.16) \quad \mathcal{Y}_1 = \{h, h_i, y_i, t_i, l_i, p_i, q_i\}_{i \in \{0,1,2\}}.$$

Then, as a K -algebra, $\text{HH}^*(R)$ is generated by the set $\mathcal{Y}_1$.

Remark 5.2. In the same way as in Proposition 5.4, it can be proved that the generators belonging to $\mathcal{Y}_1$ satisfy the relations that correspond to the generators (2.2)–(2.7) of the ideal I_1 .

(I.2) Suppose $\text{char } K \neq 2$ and $s = 2$.

Consider the elements of $\text{HH}^0(R)$ given by

$$u_0 = [x_0]_{0,0,0}, \quad u_1 = [x_1]_{0,0,0}, \quad u_2 = [x_2]_{0,0,0};$$

the elements of $\text{HH}^1(R)$ given by

$$y_0 = [x_0]_{1,0,0}, \quad y_1 = [x_1]_{0,1,0}, \quad y_2 = [x_2]_{0,0,1};$$

and the elements of $\mathrm{HH}^2(R)$ given by

$$p_0 = [1]_{2,0,0}, \quad p_1 = [1]_{0,2,0}, \quad p_2 = [1]_{0,0,2}.$$

Theorem 5.6. *Suppose $\mathrm{char} K \neq 2$ and $s = 2$. By definition, put*

$$(5.17) \quad \mathcal{Y}_2 = \{u_i, y_i, p_i\}_{i \in \{0,1,2\}}.$$

Then, as a K -algebra, $\mathrm{HH}^(R)$ is generated by the set $\mathcal{Y}_2$.*

Remark 5.3. In the same way as in Proposition 5.4, it can be proved that the generators belonging to $\mathcal{Y}_2$ satisfy the relations that correspond to the generators (2.8) of the ideal I_2 .

(I.4) Suppose $\mathrm{char} K \neq 2$ and $s = 2s'$, where $s' \in \mathbb{N} \setminus \{1\}$.

Consider the elements of $\mathrm{HH}^0(R)$ given by

$$\begin{aligned} h_0 &= [x_1x_2]_{0,0,0}, & h_1 &= [x_2x_0]_{0,0,0}, & h_2 &= [x_0x_1]_{0,0,0}, \\ h &= [x_0x_1x_2]_{0,0,0}; \end{aligned}$$

the elements of $\mathrm{HH}^1(R)$ given by

$$y_0 = [x_0]_{1,0,0}, \quad y_1 = [x_1]_{0,1,0}, \quad y_2 = [x_2]_{0,0,1};$$

the elements of $\mathrm{HH}^{s-2}(R)$ given by

$$t_0 = [x_0]_{0,s'-1,s'-1}, \quad t_1 = [x_1]_{s'-1,0,s'-1}, \quad t_2 = [x_2]_{s'-1,s'-1,0};$$

the elements of $\mathrm{HH}^{2m}(R)$ ($m \in \{1, \dots, s' - 1\}$) given by

$$z_0^{(m)} = [x_1x_2]_{2m,0,0}, \quad z_1^{(m)} = [x_2x_0]_{0,2m,0}, \quad z_2^{(m)} = [x_0x_1]_{0,0,2m};$$

the element of $\mathrm{HH}^6(R)$ given by

$$p = [1]_{2,2,2};$$

and, finally, the elements of $\mathrm{HH}^s(R)$ given by

$$p_0 = [1]_{s,0,0}, \quad p_1 = [1]_{0,s,0}, \quad p_2 = [1]_{0,0,s}.$$

Theorem 5.7. *Suppose $\mathrm{char} K \neq 2$ and $s = 2s'$, where $s' \in \mathbb{N} \setminus \{1\}$. By definition, put*

$$(5.18) \quad \mathcal{Y}_4 = \{h, h_i, y_i, t_i, z_i^{(m)}, p, p_i\}_{i \in \{0,1,2\}, m \in \{1, \dots, s'-1\}}.$$

Then, as a K -algebra, $\mathrm{HH}^(R)$ is generated by the set $\mathcal{Y}_4$.*

Remark 5.4. In the same way as in Proposition 5.4, it can be proved that the generators belonging to $\mathcal{Y}_4$ satisfy the relations that correspond to the generators (2.19), (2.20) of the ideal I_4 .

(I.5) Suppose $\mathrm{char} K \neq 2$ and $s = \infty$.

Consider the elements of $\mathrm{HH}^0(R)$ given by

$$\begin{aligned} h_0 &= [x_1x_2]_{0,0,0}, & h_1 &= [x_2x_0]_{0,0,0}, & h_2 &= [x_0x_1]_{0,0,0}, \\ h &= [x_0x_1x_2]_{0,0,0}; \end{aligned}$$

the elements of $\mathrm{HH}^1(R)$ given by

$$y_0 = [x_0]_{1,0,0}, \quad y_1 = [x_1]_{0,1,0}, \quad y_2 = [x_2]_{0,0,1};$$

the elements of $\mathrm{HH}^{2m}(R)$ ($m \in \mathbb{N}$) given by

$$z_0^{(m)} = [x_1x_2]_{2m,0,0}, \quad z_1^{(m)} = [x_2x_0]_{0,2m,0}, \quad z_2^{(m)} = [x_0x_1]_{0,0,2m};$$

and, finally, the element of $\mathrm{HH}^6(R)$ given by

$$p = [1]_{2,2,2}.$$

Theorem 5.8. Suppose $\text{char } K \neq 2$ and $s = \infty$. By definition, put

$$(5.19) \quad \mathcal{Y}_5 = \{h, h_i, y_i, z_i^{(m)}, p\}_{i \in \{0,1,2\}, m \in \mathbb{N}}.$$

Then, as a K -algebra, $\text{HH}^*(R)$ is generated by the set $\mathcal{Y}_5$.

Proof. By Proposition 3.1, we have $\text{HH}^0(R) = {}_K \langle 1, h_i, h \rangle_{i \in \{0,1,2\}}$. We fix $n > 0$ and show how the elements of the K -basis $\text{HH}^n(R)$ (see (3.24)–(3.32)) are expressed in terms of the elements of the set $\mathcal{Y}_5$.

By Lemma 5.1(I.5), there exists $k \in \mathbb{N}_0$ such that $(a, b, c) = k(2, 2, 2)$. Therefore, $[1]_{a,b,c} = \rho^{-6k(k-1)} p^k$. By Lemma 5.2 (I.5), no t and l satisfy (5.3). If $n = 2m$, $m > 0$, then

$$[x_1 x_2]_{n,0,0} = z_0^{(m)}, \quad [x_2 x_0]_{0,n,0} = z_1^{(m)}, \quad [x_0 x_1]_{0,0,n} = z_2^{(m)}.$$

The other generators satisfy (5.5) and (5.6). Hence, they can also be expressed (by using multiplication only) in terms of the elements of $\mathcal{Y}_5$. $\square$

Proposition 5.9. Suppose $\text{char } K \neq 2$ and $s = \infty$. Then the elements of $\mathcal{Y}_5$ (see (5.19)) satisfy the following relations (where $i, j \in \{0, 1, 2\}$ and $m, m_1, m_2 \in \mathbb{N}$):

$$(5.20) \quad \begin{cases} h^2 = hh_i = hy_i = hz_i^{(m)} = hp = 0, \\ h_i h_j = h_i z_j^{(m)} = h_i p = 0, \\ h_i y_j = y_i z_j^{(m)} = 0 \quad \text{for } i \neq j, \\ z_i^{(m_1)} z_j^{(m_2)} = z_i^{(m)} p = 0. \end{cases}$$

Proof. The proof is straightforward, with the use of Theorem 4.2. $\square$

Corollary 5.10. If $\text{char } K \neq 2$ and $s = \infty$, then the algebra $\text{HH}^*(R)$ is not finitely generated.

Proof. Suppose there exists a finite set $\mathcal{Y}$ that generates $\text{HH}^*(R)$. Let M denote the maximal degree of the elements of $\mathcal{Y}$. Consider the set $\mathcal{Y}' = \{h, h_i, y_i, z_i^{(m)}, p\}_{i \in \{0,1,2\}, 2m \leq M}$, which is also finite. All elements of $\mathcal{Y}$ are generated by $\mathcal{Y}'$, because all elements of the K -basis of $\text{HH}^n(R)$ for $n \leq M$ are generated by elements of $\mathcal{Y}'$. (This is similar to the proof of Theorem 5.8.) However, every nonzero product of elements of $\mathcal{Y}'$ is equal (up to a multiplicative constant) to one of the following products: p^k , $y_i p^k$, $y_{i+1} y_{i+2} p^k$, $y_0 y_1 y_2 p^k$, h_i , $z_i^{(m)}$, $y_i h_i$, and $y_i z_i^{(m)}$; here $k \in \mathbb{N}_0$, $i \in \{0, 1, 2\}$, and $m \in \mathbb{N}_0$ is such that $2m \leq M$. Clearly, among such products there are no elements of the form (3.27) or (3.30) for $n > M$. $\square$

(II.1) Suppose $\text{char } K = 2$ and $s = 1$.

Consider the elements of $\text{HH}^0(R)$ given by

$$u_0 = [x_0]_{0,0,0}, \quad u_1 = [x_1]_{0,0,0}, \quad u_2 = [x_2]_{0,0,0}$$

and the elements of $\text{HH}^1(R)$ given by

$$p_0 = [1]_{1,0,0}, \quad p_1 = [1]_{0,1,0}, \quad p_2 = [1]_{0,0,1}.$$

Theorem 5.11. Suppose $\text{char } K = 2$ and $s = 1$. By definition, put

$$(5.21) \quad \mathcal{Y}_6 = \{u_i, p_i\}_{i \in \{0,1,2\}}.$$

Then, as a K -algebra, $\text{HH}^*(R)$ is generated by the set $\mathcal{Y}_6$.

Remark 5.5. It is easy to check that, in $\text{HH}^*(R)$, we have $u_i^2 = 0$ for all $i \in \{0, 1, 2\}$.

(II.2) Suppose $\text{char } K = 2$ and $s \in \mathbb{N} \setminus \{1\}$.

Consider the elements of $\text{HH}^0(R)$ given by

$$\begin{aligned} h_0 &= [x_1x_2]_{0,0,0}, & h_1 &= [x_2x_0]_{0,0,0}, & h_2 &= [x_0x_1]_{0,0,0}, \\ & & h &= [x_0x_1x_2]_{0,0,0}; \end{aligned}$$

the elements of $\text{HH}^1(R)$ given by

$$y_0 = [x_0]_{1,0,0}, \quad y_1 = [x_1]_{0,1,0}, \quad y_2 = [x_2]_{0,0,1};$$

the elements of $\text{HH}^{2s-2}(R)$ given by

$$t_0 = [x_0]_{0,s-1,s-1}, \quad t_1 = [x_1]_{s-1,0,s-1}, \quad t_2 = [x_2]_{s-1,s-1,0};$$

the elements of $\text{HH}^m(R)$ ($m \in \{1, \dots, s-1\}$) given by

$$z_0^{(m)} = [x_1x_2]_{m,0,0}, \quad z_1^{(m)} = [x_2x_0]_{0,m,0}, \quad z_2^{(m)} = [x_0x_1]_{0,0,m};$$

the element of $\text{HH}^3(R)$ given by

$$p = [1]_{1,1,1};$$

and, finally, the elements of $\text{HH}^s(R)$ given by

$$p_0 = [1]_{s,0,0}, \quad p_1 = [1]_{0,s,0}, \quad p_2 = [1]_{0,0,s}.$$

Theorem 5.12. Suppose $\text{char } K = 2$ and $1 < s < \infty$. By definition, put

$$(5.22) \quad \mathcal{Y}_7 = \{h, h_i, y_i, t_i, z_i^{(m)}, p, p_i\}_{i \in \{0,1,2\}, m \in \{1, \dots, s-1\}}.$$

Then, as a K -algebra, $\text{HH}^*(R)$ is generated by the set $\mathcal{Y}_7$.

Remark 5.6. In the same way as in Proposition 5.4, it can be proved that the generators belonging to $\mathcal{Y}_7$ satisfy the relations that correspond to the generators (2.22)–(2.25) of the ideal I_7 .

(II.3) Suppose $\text{char } K = 2$ and $s = \infty$.

Consider the elements of $\text{HH}^0(R)$ given by

$$\begin{aligned} h_0 &= [x_1x_2]_{0,0,0}, & h_1 &= [x_2x_0]_{0,0,0}, & h_2 &= [x_0x_1]_{0,0,0}, \\ & & h &= [x_0x_1x_2]_{0,0,0}; \end{aligned}$$

the elements of $\text{HH}^1(R)$ given by

$$y_0 = [x_0]_{1,0,0}, \quad y_1 = [x_1]_{0,1,0}, \quad y_2 = [x_2]_{0,0,1};$$

the elements of $\text{HH}^m(R)$ ($m \in \mathbb{N}$) given by

$$z_0^{(m)} = [x_1x_2]_{m,0,0}, \quad z_1^{(m)} = [x_2x_0]_{0,m,0}, \quad z_2^{(m)} = [x_0x_1]_{0,0,m};$$

and, finally, the element of $\text{HH}^3(R)$ given by

$$p = [1]_{1,1,1}.$$

Theorem 5.13. Suppose $\text{char } K = 2$ and $s = \infty$. By definition, put

$$(5.23) \quad \mathcal{Y}_8 = \{h, h_i, y_i, z_i^{(m)}, p\}_{i \in \{0,1,2\}, m \in \mathbb{N}}.$$

Then, as a K -algebra, $\text{HH}^*(R)$ is generated by the set $\mathcal{Y}_8$.

Remark 5.7. In the same way as in Proposition 5.4, it can be proved that the generators belonging to $\mathcal{Y}_8$ satisfy the relations that correspond to the generators (2.26), (2.27) of the ideal I_8 .

Remark 5.8. As in the case of (I.5), if $\text{char } K = 2$ and $s = \infty$, then the algebra $\text{HH}^*(R)$ is not finitely generated (see Corollary 5.10).

§6. ISOMORPHISM OF ALGEBRAS

As usual, s is the order of ρ in K^* .

Suppose $\text{char } K \neq 2$. If s is an odd number greater than 1, then, by Theorem 5.3 and Proposition 5.4, there exists a surjective homomorphism $\varphi: \mathcal{A}_3 \rightarrow \text{HH}^*(R_\rho)$ of graded K -algebras that takes the generators in the set $\mathcal{X}_3$ (see (2.9)) to the corresponding generators in $\mathcal{Y}_3$ (see (5.4)). Not hesitating about ambiguity, we use the same letters to denote elements of both sets that correspond to each other. In the same way, we construct surjective homomorphisms of graded algebras $\mathcal{A}_1 \rightarrow \text{HH}^*(R_1)$, $\mathcal{A}_2 \rightarrow \text{HH}^*(R_{-1})$, $\mathcal{A}_4 \rightarrow \text{HH}^*(R_\rho)$ if s is an even number greater than 2, and $\mathcal{A}_5 \rightarrow \text{HH}^*(R_\rho)$ if $s = \infty$.

If $\text{char } K = 2$, then, similarly, we can construct surjective homomorphisms of graded algebras $\mathcal{A}_6 \rightarrow \text{HH}^*(R_1)$, $\mathcal{A}_7 \rightarrow \text{HH}^*(R_\rho)$ if $1 < s < \infty$, and $\mathcal{A}_8 \rightarrow \text{HH}^*(R_\rho)$ if $s = \infty$. Thus, Theorem 2.1 follows from the next proposition.

Proposition 6.1. *Suppose $\varphi: \mathcal{A} \rightarrow \text{HH}^*(R_\rho)$ is one of the homomorphisms constructed above, where $\mathcal{A}$ is $\mathcal{A}_1, \mathcal{A}_2, \dots$, or $\mathcal{A}_8$, respectively. Let $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}^n$ be a direct decomposition of $\mathcal{A}$ into homogenous direct summands. Then*

$$(6.1) \quad \dim_K \mathcal{A}^n = \dim_K \text{HH}^n(R).$$

Proof. We prove (6.1) only in the case where $\text{char } K \neq 2$ and $s = 2s' + 1$ for $s' \in \mathbb{N}$, i.e., in the case of $\mathcal{A} = \mathcal{A}_3$. The remaining cases are treated similarly.

By Proposition 3.1, we have $\text{HH}^0(R) = {}_K\langle 1, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2 \rangle$. On the other hand, $\mathcal{A}^0 = {}_K\langle 1, h_0, h_1, h_2, h \rangle$. In what follows we fix $n \geq 1$.

The (nonzero) images of the monomials in $K\langle \mathcal{X}_3 \rangle$ under the canonical epimorphism $K\langle \mathcal{X}_3 \rangle \rightarrow \mathcal{A}_3$ will also be called monomials. Any element of $\mathcal{A}_3$ can be represented as a linear combination of monomials (with coefficients in K).

We say that the following monomials in $\mathcal{A}_3$ are *canonical* (here $i, j \in \{0, 1, 2\}$ and $a_j \in \mathbb{N}_0, 1 \leq m \leq s - 1$):

$$(6.2) \quad \begin{cases} (w :=) p_0^{a_0} p_1^{a_1} p_2^{a_2} p^k & \text{for } 0 \leq k \leq s - 1 = 2s', \\ (u_j :=) p_0^{a_0} p_1^{a_1} p_2^{a_2} p^k q_j & \text{for } 0 \leq k \leq s', \\ (v_j :=) p_0^{a_0} p_1^{a_1} p_2^{a_2} p^k r_j & \text{for } 0 \leq k \leq s' - 1; \end{cases}$$

$$(6.3) \quad t_i p_{i+1}^{a_{i+1}} p_{i+2}^{a_{i+2}}, \quad l_i p_{i+1}^{a_{i+1}} p_{i+2}^{a_{i+2}};$$

$$(6.4) \quad y_i w, \quad y_i u_j, \quad y_i v_j;$$

$$(6.5) \quad h_i p_i^{a_i} \ (a_i \neq 0), \quad z_i^{(m)} p_i^{a_i};$$

$$(6.6) \quad \begin{cases} y_{i+2} t_i p_{i+1}^{a_{i+1}} p_{i+2}^{a_{i+2}}, & y_{i+2} l_i p_{i+1}^{a_{i+1}} p_{i+2}^{a_{i+2}}, \\ t_i y_{i+1} p_{i+1}^{a_{i+1}} p_{i+2}^{a_{i+2}}, & l_i y_{i+1} p_{i+1}^{a_{i+1}} p_{i+2}^{a_{i+2}}; \end{cases}$$

$$(6.7) \quad y_{i+1} y_{i+2} w, \quad y_{i+1} y_{i+2} u_j, \quad y_{i+1} y_{i+2} v_j;$$

$$(6.8) \quad y_i z_i^{(m)} p_i^{a_i}, \quad y_i h_i p_i^{a_i};$$

$$(6.9) \quad t_i y_{i+1} y_{i+2} p_{i+1}^{a_{i+1}} p_{i+2}^{a_{i+2}}, \quad l_i y_{i+1} y_{i+2} p_{i+1}^{a_{i+1}} p_{i+2}^{a_{i+2}};$$

$$(6.10) \quad y_0 y_1 y_2 w, \quad y_0 y_1 y_2 u_j, \quad y_0 y_1 y_2 v_j.$$

Let d_k denote the number of canonical monomials in $\mathcal{A}^k$, $k \geq 1$. We shall prove that any (nonzero) monomial $a \in \mathcal{A}_3$ is equal to a canonical monomial up to a multiplicative constant (in fact, this constant is a power of ρ up to a sign).

By definition, the *reduction* of a monomial a in $\mathcal{A}$ is the process, and also the result, of replacement of some submonomials of a by other elements of $\mathcal{A}$ in accordance with the rules (6.11)–(6.18). These rules correspond to the generators (2.11)–(2.18) of the ideal I_3 , so that any monomial a is equal (up to a multiplicative constant) to the reduction monomial. A rule $b \mapsto c$ means the replacement of every occurrence of the monomial b by the element c . By the graded commutative property, we may apply a reduction $b \mapsto c$ to a monomial a if a contains some factors whose product is b .

$$(6.11) \quad \left\{ \begin{array}{l} q_i^2 \mapsto p_{i+1}p_{i+2}, \quad r_i^2 \mapsto pp_i, \\ q_{i+1}q_{i+2} \mapsto p_iq_i, \quad r_{i+1}r_{i+2} \mapsto pq_i, \\ q_i r_i \mapsto p^{s'+1}, \\ q_{i+1}r_i \mapsto p_i r_{i+2}, \quad q_{i+2}r_i \mapsto p_i r_{i+1}; \end{array} \right.$$

$$(6.12) \quad \left\{ \begin{array}{l} p^s \mapsto p_0 p_1 p_2, \\ p^{s'+1} q_i \mapsto p_{i+1} p_{i+2} r_i, \quad p^{s'} r_i \mapsto p_i q_i; \end{array} \right.$$

$$(6.13) \quad \left\{ \begin{array}{ll} t_i p_i \mapsto y_i p^{s'} q_{i+1}, & l_i p_i \mapsto y_i p^{s'} q_{i+2}, \\ t_i p \mapsto y_i p_{i+2} r_{i+1}, & l_i p \mapsto y_i p_{i+1} r_{i+2}, \\ t_i q_{i+1} \mapsto y_i p^{s'} p_{i+2}, & l_i q_{i+1} \mapsto y_i p^{s'} q_i, \\ t_i q_{i+2} \mapsto y_i p^{s'} q_i, & l_i q_{i+2} \mapsto y_i p^{s'} p_{i+1}, \\ t_i r_i \mapsto y_i p_{i+2} q_{i+2}, & l_i r_i \mapsto y_i p_{i+1} q_{i+1}, \\ t_i r_{i+1} \mapsto y_i p_{i+1} p_{i+2}, & l_i r_{i+1} \mapsto y_i p_{i+1} q_i, \\ t_i r_{i+2} \mapsto y_i p_{i+2} q_i, & l_i r_{i+2} \mapsto y_i p_{i+1} p_{i+2}; \end{array} \right.$$

$$(6.14) \quad t_i q_i \mapsto l_i p_{i+2}, \quad l_i q_i \mapsto t_i p_{i+1};$$

$$(6.15) \quad \left\{ \begin{array}{l} t_{i+1} t_{i+2} \mapsto y_{i+1} y_{i+2} p^{s'-1} p_i r_{i+1}, \\ l_{i+1} l_{i+2} \mapsto y_{i+1} y_{i+2} p^{s'-1} p_i r_{i+2}, \\ t_{i+1} l_{i+2} \mapsto y_{i+1} y_{i+2} p^{s'-1} p_i r_i, \\ l_{i+1} t_{i+2} \mapsto y_{i+1} y_{i+2} p^{s'-1} p_i q_i; \end{array} \right.$$

$$(6.16) \quad \left\{ \begin{array}{l} z_i^{(s')} p \mapsto y_{i+1} y_{i+2} r_i, \\ z_i^{(s')} q_i \mapsto y_{i+1} y_{i+2} p^{s'}, \\ z_i^{(s')} r_i \mapsto y_{i+1} y_{i+2} p_i, \\ z_i^{(s')} r_{i+1} \mapsto y_{i+1} y_{i+2} q_{i+2}, \\ z_i^{(s')} r_{i+2} \mapsto y_{i+1} y_{i+2} q_{i+1}; \end{array} \right.$$

$$(6.17) \quad \left\{ \begin{array}{ll} z_i^{(s')} p_{i+1} \mapsto y_{i+1} t_{i+2}, & z_i^{(s')} q_{i+1} \mapsto y_{i+2} t_{i+1}, \\ z_i^{(s')} p_{i+2} \mapsto y_{i+2} l_{i+1}, & z_i^{(s')} q_{i+2} \mapsto y_{i+1} l_{i+2}; \end{array} \right.$$

$$(6.18) \quad t_i z_i^{(s')} \mapsto y_0 y_1 y_2 p^{s'-1} r_{i+2}, \quad l_i z_i^{(s')} \mapsto y_0 y_1 y_2 p^{s'-1} r_{i+1}.$$

By (2.10), it follows that our nonzero monomial a does not contain h ($a \neq h$ because $n \geq 1$) and contains at most one element of h_i and $z_i^{(m)}$. If a contains h_i for some $i \in \{0, 1, 2\}$ or $z_i^{(m)}$ for some $i \in \{0, 1, 2\}$ and $1 \leq m \leq s-1$, $m \neq s'$, then a can only contain y_i and powers of p_i . Therefore, up to a multiplicative constant, a is equal to a canonical monomial of the form (6.5) or (6.8). Now suppose that a contains neither h_i , $i = 0, 1, 2$, nor $z_i^{(m)}$, $i = 0, 1, 2$, $1 \leq m \leq s-1$, $m \neq s'$. Using several steps of the type (6.11), we reduce the sum of the exponents of q_i and r_i to 0 or 1. Then we use (6.12) to reduce the exponent of p to a quantity not exceeding $s-1$; if a contains q_i for some $i \in \{0, 1, 2\}$, then we reduce the exponent of p to a quantity not exceeding s' ; if a contains r_i , then we reduce the exponent of p to a quantity not exceeding $s'-1$. The resulting monomial a contains at most one of the q_i or r_i , and the exponent of p is not greater than $s-1$; moreover, this exponent is at most s' if a q_i occurs, and this exponent is at most $s'-1$ if an r_i occurs. If a contains p , p_i , q_i , or r_i only, then we obtain a canonical monomial of the form (6.2).

Now consider the following cases.

1) The resulting monomial a contains $z_i^{(s')}$ for some $i \in \{0, 1, 2\}$. By (2.10), a contains at most one element among y_j, t_j, l_j , $j \in \{0, 1, 2\}$. If it contains t_i or l_i , then, using (6.18), we obtain a canonical monomial of the form (6.10). Now, if a contains at least one of the elements p, q_i, r_i, r_{i+1} , or r_{i+2} ($p_{i+1}, p_{i+2}, q_{i+1}$, or q_{i+2} , respectively), then we can use (6.16) (or (6.17), respectively) to obtain a monomial free of $z_i^{(m)}$. Such monomials will be considered in item 2) below. If a contains none of the elements p, p_j, q_j, r_j , $j \in \{0, 1, 2\}$, then we arrive at a canonical monomial of the form (6.5) or (6.8) with $m = s'$.

2) Suppose a contains no $z_i^{(m)}$ for $i = 0, 1, 2$ and $1 \leq m \leq s-1$. If, moreover, no elements t_i and l_i , $i = 0, 1, 2$, occur, then we already have a canonical monomial of the form (6.4), (6.7), or (6.10). On the other hand, if a contains at least two t 's or l 's (with some indices), then, using the reduction (6.15) (and additionally (6.13) if there are three such elements, e.g., $t_0 l_1 t_2$), we obtain a monomial of the form (6.7) or (6.10). If a contains only one element of the type t_i or l_i , then either we already have a canonical monomial of the form (6.3), (6.6), or (6.9); or, if a contains q_i , we obtain a canonical monomial of the same form (6.3), (6.6), or (6.9) after using (6.14); or, if we have at least one element among $p, p_i, q_{i+1}, q_{i+2}, r_i, r_{i+1}$, or r_{i+2} , we can use (6.13) to obtain a monomial of the form (6.4), (6.7), or (6.10).

Thus we have shown how an arbitrary monomial can be reduced to a canonical one. Therefore, $d_n \geq \dim_K \mathcal{A}^n$.

Since there is an epimorphism $\mathcal{A}^n \rightarrow \mathrm{HH}^n(R)$, we have $\dim_K \mathcal{A}^n \geq \dim_K \mathrm{HH}^n(R)$. Therefore it suffices to prove that

$$(6.19) \quad d_n = \dim_K \mathrm{HH}^n(R).$$

Let $d_k^{(1)}, \dots, d_k^{(9)}$ denote the number of the canonical monomials in $\mathcal{A}^k$ that have the form (6.2), $\dots$, (6.10), respectively, and let $h_k^{(1)}, \dots, h_k^{(9)}$ be the number of the additive generators of $\mathrm{HH}^k(R)$ that have the form (3.24), $\dots$, (3.32), respectively.

The decomposition (5.2) (see Lemma 5.1(I.3)) induces a one-to-one correspondence between the monomials of the form (6.2) of degree n and the elements of $\mathrm{HH}^n(R)$ of the form (3.24). Thus, $d_m^{(1)} = h_m^{(1)}$. By Lemma 5.2(I.3) (and Remark 5.1), we have $d_m^{(2)} = h_m^{(2)}$. Obviously, $d_n^{(3)} = 3d_{n-1}^{(1)} = 3h_{n-1}^{(1)} = h_n^{(3)}$ (we assume that $d_0^{(1)} = h_0^{(1)} = 1$).

Also,

$$d_n^{(4)} = 3\#\{(m, a) \mid m \in \{0, 1, \dots, s-1\}, a \in \mathbb{N}_0, 2m + a2s = n\},$$

$$h_n^{(4)} = \begin{cases} 3 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Therefore, $d_n^{(4)} = h_n^{(4)}$. It is clear that

$$\begin{aligned} d_n^{(5)} &= 2d_{n-1}^{(2)} = 2h_{n-1}^{(2)} = h_n^{(5)}, \\ d_n^{(6)} &= 3d_{n-2}^{(1)} = 3h_{n-2}^{(1)} = h_n^{(6)}, \\ d_n^{(7)} &= d_{n-1}^{(4)} = h_{n-1}^{(4)} = h_n^{(7)}, \\ d_n^{(8)} &= d_{n-2}^{(2)} = h_{n-2}^{(2)} = h_n^{(8)}, \\ d_n^{(9)} &= d_{n-3}^{(1)} = h_{n-3}^{(1)} = h_n^{(9)}, \end{aligned}$$

provided we assume that $d_0^{(4)} = h_0^{(4)} = 3$. For small n , obviously, $d_1^{(5)} = h_1^{(5)} = 0$, $d_1^{(6)} = h_1^{(6)} = 0$, $d_1^{(8)} = h_1^{(8)} = d_2^{(8)} = h_2^{(8)} = 0$, $d_1^{(9)} = h_1^{(9)} = d_2^{(9)} = h_2^{(9)} = 0$.

Since $d_n = \sum_{i=1}^9 d_n^{(i)}$ and $\dim_K \mathrm{HH}^n(R) = \sum_{i=1}^9 h_n^{(i)}$, we arrive at (6.19). $\square$

§7. TABLES: DIMENSION OF $\mathrm{HH}^n(R)$

In this section we present the results of calculating $\dim_K \mathrm{HH}^n(R_\rho)$ for $n > 0$ in the cases where ρ has a finite order greater than 2 if $\mathrm{char} K \neq 2$, or ρ has a finite order greater than 1 if $\mathrm{char} K = 2$.

Let $n \in \mathbb{N}$. We introduce and calculate the following auxiliary quantities:

$$K_n^{(s)} = \#\{(a, b, c) \mid a + b + c = n, a \equiv b \equiv c \pmod{s}\};$$

for odd s ,

$$T_n^{(s)} = \#\{(t, l) \mid t + l = n, t \equiv l \equiv s-1 \pmod{s}\};$$

for even $s = 2s'$,

$$L_n^{(s)} = \#\{(t, l) \mid t + l = n, t \equiv l \equiv s'-1 \pmod{s}\}.$$

Lemma 7.1. *If $s = 3s_1$, $s_1 \in \mathbb{N}$, then*

$$K_n^{(s)} = \begin{cases} \frac{1}{2}(3a^2 + 3a + 2) & \text{if } n = 3n_1 \text{ and } n_1 = as_1 + k_1, 0 \leq k_1 \leq s_1 - 1, \\ 0 & \text{if } 3 \text{ does not divide } n. \end{cases}$$

Suppose 3 does not divide s . Then:

1) *if $n \equiv 0 \pmod{3}$, then*

$$K_n^{(s)} = \binom{3a+2}{2}, \quad \text{where } n = 3m \text{ and } m = as + k, 0 \leq k \leq s-1;$$

2) *if $n \equiv s \pmod{3}$, then*

$$K_n^{(s)} = \begin{cases} 0 & \text{if } n < s, \\ \binom{3a+3}{2} & \text{otherwise,} \end{cases}$$

where $n - s = 3m$ and $m = as + k$, $0 \leq k \leq s-1$;

3) *if $n \equiv 2s \pmod{3}$, then*

$$K_n^{(s)} = \begin{cases} 0 & \text{if } n < 2s, \\ \binom{3a+4}{2} & \text{otherwise,} \end{cases}$$

where $n - 2s = 3m$ and $m = as + k$, $0 \leq k \leq s-1$.

For odd s ,

$$T_n^{(s)} = \begin{cases} 0 & \text{if } s \text{ does not divide } n+2 \text{ or } n+2 < 2s, \\ \alpha - 1 & \text{if } n+2 = \alpha s; \end{cases}$$

and for even s ,

$$L_n^{(s)} = \begin{cases} 0 & \text{if } s \text{ does not divide } n+2, \\ \alpha & \text{if } n+2 = \alpha s. \end{cases}$$

Now, analyzing several cases separately and using (3.36) and the values of $K_n^{(s)}$, $T_n^{(s)}$, and $L_n^{(s)}$ obtained above, we arrive at the following formulas for $\dim_K \mathrm{HH}^n(R)$. The proof is somewhat bulky, and we omit it.

Proposition 7.2. *Let $R = R_\rho$, and let s be the order of ρ in K^* . Suppose $\mathrm{char} K \neq 2$ and s is an odd number greater than 1.*

- 1) *Suppose $s = 3s_1$, $s_1 \in \mathbb{N}$. The values of $\dim_K \mathrm{HH}^n(R)$, $n > 0$, are listed in Table 1, where $2m = 2[\frac{n}{6}] = as_1 + k_1$, $a, k_1 \in \mathbb{N}_0$, $0 \leq k_1 \leq s_1 - 1$.*
- 2) *Suppose $s = 3s_1 + 1$, $s_1 \in \mathbb{N}$. The values of $\dim_K \mathrm{HH}^n(R)$, $n > 0$, are listed in Table 2, where $2m = 2[\frac{n}{6}] = as + k$, $0 \leq k \leq s - 1$, and $x = 3a$.*
- 3) *Suppose $s = 3s_1 + 2$, $s_1 \in \mathbb{N}$. The values of $\dim_K \mathrm{HH}^n(R)$, $n > 0$, are listed in Table 3, where $2m = 2[\frac{n}{6}] = as + k$, $0 \leq k \leq s - 1$, and $x = 3a$.*

TABLE 1.

n	$\dim_K \mathrm{HH}^n(R)$
$6m$	$3\frac{a(a+1)}{2} + 4$
$6m+1$	$9\frac{a(a+1)}{2} + 6 + \begin{cases} 3a & \text{if } k_1 = s_1 - 1, \\ 0 & \text{otherwise} \end{cases}$
$6m+2$	$9\frac{a(a+1)}{2} + 6 + \begin{cases} 6a & \text{if } k_1 = s_1 - 1, \\ 0 & \text{otherwise} \end{cases}$
$6m+3$	$3\frac{a(a+1)}{2} + 4 + \begin{cases} 3a & \text{if } k_1 = s_1 - 1, \\ 0 & \text{otherwise} \end{cases}$
$6m+4$	3
$6m+5$	3

Proposition 7.3. *Let $R = R_\rho$, and let s be the order of ρ in K^* . Suppose $\mathrm{char} K \neq 2$ and s is an even number greater than 2.*

- 1) *Suppose $s = 3s_1$, $s_1 \in \mathbb{N}$. The values of $\dim_K \mathrm{HH}^n(R)$, $n > 0$, are listed in Table 4, where $2m = 2[\frac{n}{6}] = as_1 + k_1$, $0 \leq k_1 \leq s_1 - 1$.*
- 2) *Suppose $s = 3s_1 + 1$, $s_1 \in \mathbb{N}$. The values of $\dim_K \mathrm{HH}^n(R)$, $n > 0$, are listed in Table 5, where $2m = 2[\frac{n}{6}] = as + k$, $0 \leq k \leq s - 1$, and $x = 3a$.*
- 3) *Suppose $s = 3s_1 + 2$, $s_1 \in \mathbb{N}$. The values of $\dim_K \mathrm{HH}^n(R)$, $n > 0$, are listed in Table 6, where $2m = 2[\frac{n}{6}] = as + k$, $0 \leq k \leq s - 1$, and $x = 3a$.*

TABLE 2.

n	$\dim_K \mathrm{HH}^n(R)$
$6m$	$2x^2 + 4 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 1, \\ 6x & \text{if } k = s_1, \\ 9x + 9 & \text{if } s_1 + 1 \leq k \leq 3s_1 \end{cases}$
$6m + 1$	$2x^2 + 4x + 6 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 1, \\ 3x & \text{if } k = s_1, \\ 3x + 3 & \text{if } s_1 + 1 \leq k \leq 3s_1 - 1, \\ 6x + 9 & \text{if } k = 3s_1 \end{cases}$
$6m + 2$	$2x^2 + 5x + 6 + \begin{cases} 0 & \text{if } 0 \leq k \leq 2s_1 - 1, \\ 3x + 6 & \text{if } 2s_1 \leq k \leq 3s_1 - 1, \\ 9x + 18 & \text{if } k = 3s_1 \end{cases}$
$6m + 3$	$2x^2 + 3x + 4 + \begin{cases} 0 & \text{if } 0 \leq k \leq 2s_1 - 2, \\ 3x + 3 & \text{if } k = 2s_1 - 1, \\ 9x + 18 & \text{if } 2s_1 \leq k \leq 3s_1 - 1, \\ 12x + 24 & \text{if } k = 3s_1 \end{cases}$
$6m + 4$	$2x^2 + x + 3 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 2, \\ 3x + 3 & \text{if } s_1 - 1 \leq k \leq 2s_1 - 2, \\ 9x + 9 & \text{if } k = 2s_1 - 1, \\ 12x + 21 & \text{if } 2s_1 \leq k \leq 3s_1 \end{cases}$
$6m + 5$	$2x^2 - x + 3 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 3, \\ 3x & \text{if } k = s_1 - 2, \\ 9x + 9 & \text{if } s_1 - 2 \leq k \leq 2s_1 - 2, \\ 12x + 12 & \text{if } k = 2s_1 - 2, \\ 12x + 15 & \text{if } 2s_1 \leq k \leq 3s_1 \end{cases}$

Proposition 7.4. *Let $R = R_\rho$, and let s be the order of ρ in K^* . Suppose $\mathrm{char} K = 2$ and $s \in \mathbb{N} \setminus \{1\}$.*

1) *Suppose $s = 3s_1$, $s_1 \in \mathbb{N}$. The values of $\dim_K \mathrm{HH}^n(R)$, $n > 0$, are listed in Table 7, where $m = [\frac{n}{3}] = as_1 + k_1$, $0 \leq k_1 \leq s_1 - 1$.*

2) *Suppose $s = 3s_1 + 1$, $s_1 \in \mathbb{N}$. The values of $\dim_K \mathrm{HH}^n(R)$, $n > 0$, are listed in Table 8, where $m = [\frac{n}{3}] = as + k$, $0 \leq k \leq s - 1$, and $x = 3a$.*

3) *Suppose $s = 3s_1 + 2$, $s_1 \in \mathbb{N}_0$. The values of $\dim_K \mathrm{HH}^n(R)$, $n > 0$, are listed in Table 9, where $m = [\frac{n}{3}] = as + k$, $0 \leq k \leq s - 1$, and $x = 3a$.*

TABLE 3.

n	$\dim_K \mathrm{HH}^n(R)$
$6m$	$2x^2 + 3x + 4 + \begin{cases} 0 & \text{if } 0 \leq k \leq 2s_1, \\ 15x + 24 & \text{if } k = 2s_1 + 1, \\ 9x + 18 & \text{if } 2s_1 + 2 \leq k \leq 3s_1 + 1 \end{cases}$
$6m + 1$	$2x^2 + 5x + 6 + \begin{cases} 0 & \text{if } 0 \leq k \leq 2s_1, \\ 6x + 9 & \text{if } k = 2s_1 + 1, \\ 3x + 6 & \text{if } 2s_1 + 2 \leq k \leq 3s_1, \\ 6x + 12 & \text{if } k = 3s_1 + 1 \end{cases}$
$6m + 2$	$2x^2 + 4x + 6 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 1, \\ 3x + 3 & \text{if } s_1 \leq k \leq 3s_1, \\ 9x + 15 & \text{if } k = 3s_1 + 1 \end{cases}$
$6m + 3$	$2x^2 + 4 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 2, \\ 3x & \text{if } k = s_1 - 1, \\ 9x + 9 & \text{if } s_1 \leq k \leq 3s_1, \\ 12x + 15 & \text{if } k = 3s_1 + 1 \end{cases}$
$6m + 4$	$2x^2 - x + 3 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 2, \\ 6x & \text{if } k = s_1 - 1, \\ 9x + 9 & \text{if } s_1 \leq k \leq 2s_1 - 2, \\ 12x + 15 & \text{if } 2s_1 - 1 \leq k \leq 3s_1 + 1 \end{cases}$
$6m + 5$	$2x^2 + x + 3 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 2, \\ 3x & \text{if } k = s_1 - 1, \\ 3x + 3 & \text{if } s_1 \leq k \leq 2s_1 - 2, \\ 15x + 24 & \text{if } k = 2s_1 - 1, \\ 12x + 21 & \text{if } 2s_1 \leq k \leq 3s_1 + 1 \end{cases}$

TABLE 4.

n	$\dim_K \mathrm{HH}^n(R)$
$6m$	$3 \frac{a(a+1)}{2} + 4 + \begin{cases} 3a & \text{if } k_1 = 0, \\ 0 & \text{otherwise} \end{cases}$
$6m + 1$	$9 \frac{a(a+1)}{2} + 6$
$6m + 2$	$9 \frac{a(a+1)}{2} + 6$
$6m + 3$	$3 \frac{a(a+1)}{2} + 4$
$6m + 4$	$3 + \begin{cases} 3a + 3 & \text{if } k_1 = s_1 - 2, \\ 0 & \text{otherwise} \end{cases}$
$6m + 5$	$3 + \begin{cases} 6a + 6 & \text{if } k_1 = s_1 - 2, \\ 0 & \text{otherwise} \end{cases}$

TABLE 5.

n	$\dim_K \mathrm{HH}^n(R)$
$6m$	$2x^2 + 4 + \begin{cases} 3x & \text{if } k = 0, \\ 0 & \text{if } 1 \leq k \leq s_1, \\ 9x + 9 & \text{if } s_1 + 1 \leq k \leq 2s_1 - 1, \\ 12x + 15 & \text{if } k = 2s_1, \\ 9x + 9 & \text{if } 2s_1 + 1 \leq k \leq 3s_1 \end{cases}$
$6m + 1$	$2x^2 + 4x + 6 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1, \\ 3x + 3 & \text{if } s_1 + 1 \leq k \leq 2s_1 - 1, \\ 9x + 15 & \text{if } k = 2s_1, \\ 3x + 3 & \text{if } 2s_1 + 1 \leq k \leq 3s_1 \end{cases}$
$6m + 2$	$2x^2 + 5x + 6 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 2, \\ 3x + 3 & \text{if } k = s_1 - 1, \\ 0 & \text{if } s_1 \leq k \leq 2s_1 - 1, \\ 6x + 12 & \text{if } k = 2s_1, \\ 3x + 6 & \text{if } 2s_1 + 1 \leq k \leq 3s_1 \end{cases}$
$6m + 3$	$2x^2 + 3x + 4 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 2, \\ 6x + 6 & \text{if } k = s_1 - 1, \\ 0 & \text{if } s_1 \leq k \leq 2s_1 - 1, \\ 9x + 18 & \text{if } 2s_1 \leq k \leq 3s_1 \end{cases}$
$6m + 4$	$2x^2 + x + 3 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 2, \\ 6x + 6 & \text{if } k = s_1 - 1, \\ 3x + 3 & \text{if } s_1 \leq k \leq 2s_1 - 1, \\ 12x + 21 & \text{if } 2s_1 \leq k \leq 3s_1 - 2, \\ 15x + 30 & \text{if } k = 3s_1 - 1, \\ 12x + 21 & \text{if } k = 3s_1 \end{cases}$
$6m + 5$	$2x^2 - x + 3 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 2, \\ 9x + 9 & \text{if } s_1 - 1 \leq k \leq 2s_1 - 1, \\ 12x + 15 & \text{if } 2s_1 \leq k \leq 3s_1 - 2, \\ 18x + 33 & \text{if } k = 3s_1 - 1, \\ 12x + 15, & \text{if } k = 3s_1 \end{cases}$

TABLE 6.

n	$\dim_K \mathrm{HH}^n(R)$
$6m$	$2x^2 + 3x + 4 + \begin{cases} 3x & \text{if } k = 0, \\ 0 & \text{if } 1 \leq k \leq s_1 - 1, \\ 3x + 3 & \text{if } k = s_1, \\ 0 & \text{if } s_1 + 1 \leq k \leq 2s_1, \\ 9x + 18 & \text{if } 2s_1 + 1 \leq k \leq 3s_1 + 1 \end{cases}$
$6m + 1$	$2x^2 + 5x + 6 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 1, \\ 6x + 6 & \text{if } k = s_1, \\ 0 & \text{if } s_1 + 1 \leq k \leq 2s_1, \\ 3x + 6 & \text{if } 2s_1 + 1 \leq k \leq 3s_1 + 1 \end{cases}$
$6m + 2$	$2x^2 + 4x + 6 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 1, \\ 6x + 6 & \text{if } k = s_1, \\ 3x + 3 & \text{if } s_1 + 1 \leq k \leq 2s_1 - 1, \\ 6x + 9 & \text{if } k = 2s_1, \\ 3x + 3 & \text{if } 2s_1 + 1 \leq k \leq 3s_1 + 1 \end{cases}$
$6m + 3$	$2x^2 + 4 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 1, \\ 9x + 9 & \text{if } s_1 \leq k \leq 2s_1 - 1, \\ 15x + 21 & \text{if } k = 2s_1, \\ 9x + 9 & \text{if } 2s_1 + 1 \leq k \leq 3s_1 + 1 \end{cases}$
$6m + 4$	$2x^2 - x + 3 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 1, \\ 9x + 9 & \text{if } s_1 \leq k \leq 2s_1 - 2, \\ 12x + 15 & \text{if } k = 2s_1 - 1, \\ 15x + 21 & \text{if } k = 2s_1, \\ 12x + 15 & \text{if } 2s_1 + 1 \leq k \leq 3s_1 - 1, \\ 15x + 24 & \text{if } k = 3s_1, \\ 12x + 15 & \text{if } k = 3s_1 + 1 \end{cases}$
$6m + 5$	$2x^2 + x + 3 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 1, \\ 3x + 3 & \text{if } s_1 \leq k \leq 2s_1 - 2, \\ 12x + 21 & \text{if } 2s_1 - 1 \leq k \leq 3s_1 - 1, \\ 18x + 39 & \text{if } k = 3s_1, \\ 12x + 21 & \text{if } k = 3s_1 + 1 \end{cases}$

TABLE 7.

n	$\dim_K \mathrm{HH}^n(R)$
$3m+1$	$9\frac{a(a+1)}{2} + 9 + \begin{cases} 3a & \text{if } k_1 = s_1 - 1, \\ 0 & \text{otherwise} \end{cases}$
$3m+2$	$9\frac{a(a+1)}{2} + 9 + \begin{cases} 6a & \text{if } k_1 = s_1 - 1, \\ 0 & \text{otherwise} \end{cases}$
$3m+3$	$3a(a+1) + 8 + \begin{cases} 6a+3 & \text{if } k_1 = s_1 - 1, \\ 0 & \text{otherwise} \end{cases}$

TABLE 8.

n	$\dim_K \mathrm{HH}^n(R)$
$3m+1$	$4x^2 + 5x + 9 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 1, \\ 6x + 3 & \text{if } k = s_1, \\ 6x + 6 & \text{if } s_1 + 1 \leq k \leq 2s_1 - 1, \\ 12x + 12 & \text{if } k = 2s_1, \\ 15x + 24 & \text{if } 2s_1 + 1 \leq k \leq 3s_1 - 1, \\ 18x + 30 & \text{if } k = 3s_1 \end{cases}$
$3m+2$	$4x^2 + 4x + 9 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 2, \\ 3x & \text{if } k = s_1 - 1, \\ 9x + 9 & \text{if } s_1 \leq k \leq 2s_1 - 1, \\ 15x + 18 & \text{if } k = 2s_1, \\ 15x + 21 & \text{if } 2s_1 + 1 \leq k \leq 3s_1 - 1, \\ 21x + 33 & \text{if } k = 3s_1 \end{cases}$
$3m+3$	$4x^2 + 3x + 8 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 2, \\ 6x & \text{if } k = s_1 - 1, \\ 9x + 9 & \text{if } s_1 \leq k \leq 2s_1 - 2, \\ 12x + 12 & \text{if } k = 2s_1 - 1, \\ 18x + 27 & \text{if } 2s_1 \leq k \leq 3s_1 - 1, \\ 24x + 42 & \text{if } k = 3s_1 \end{cases}$

TABLE 9.

n	$\dim_K \mathrm{HH}^n(R)$
$3m+1$	$4x^2 + 4x + 9 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 1, \\ 6x & \text{if } k = s_1, \\ 9x + 9 & \text{if } s_1 + 1 \leq k \leq 2s_1, \\ 15x + 18 & \text{if } k = 2s_1 + 1, \\ 15x + 21 & \text{if } 2s_1 + 2 \leq k \leq 3s_1, \\ 18x + 27 & \text{if } k = 3s_1 + 1 \end{cases}$
$3m+2$	$4x^2 + 5x + 9 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 1, \\ 6x + 3 & \text{if } k = s_1, \\ 6x + 6 & \text{if } s_1 + 1 \leq k \leq 2s_1 - 1, \\ 9x + 9 & \text{if } k = 2s_1, \\ 15x + 24 & \text{if } 2s_1 + 1 \leq k \leq 3s_1, \\ 21x + 36 & \text{if } k = 3s_1 + 1 \end{cases}$
$3m+3$	$4x^2 + 3x + 8 + \begin{cases} 0 & \text{if } 0 \leq k \leq s_1 - 2, \\ 3x & \text{if } k = s_1 - 1, \\ 9x + 9 & \text{if } s_1 \leq k \leq 2s_1 - 1, \\ 15x + 15 & \text{if } k = 2s_1, \\ 18x + 27 & \text{if } 2s_1 + 1 \leq k \leq 3s_1, \\ 24x + 42 & \text{if } k = 3s_1 + 1 \end{cases}$

REFERENCES

- [1] S. Eilenberg and S. MacLane, *Cohomology theory in abstract groups*. I, Ann. of Math. (2) **48** (1947), 51–78. MR0019092 (8:367f)
- [2] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, NJ, 1999. MR1731415 (2000h:18022)
- [3] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2) **78** (1963), 267–288. MR0161898 (28:5102)
- [4] M. Gerstenhaber and S. D. Schack, *Algebraic cohomology and deformation theory*, Deformation Theory of Algebras and Structures and Applications (Il Ciocco, 1986) (M. Hazewinkel, M. Gerstenhaber, eds.), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 247, Kluwer Acad. Publ., Dordrecht, 1988, pp. 11–264. MR981619 (90c:16016)
- [5] C. M. Ringel, *The Liu–Schulz example*, Representation Theory of Algebras (Cocoyoc, 1994), CMS Conf. Proc., vol. 18, Amer. Math. Soc., Providence, RI, 1996, pp. 587–600. MR1388076 (97d:16017)
- [6] Sh. Liu and R. Schulz, *The existence of bounded infinite DTr-orbits*, Proc. Amer. Math. Soc. **122** (1994), no. 4, 1003–1005. MR1223516 (95b:16007)
- [7] K. Erdmann, *Blocks of tame representation type and related algebras*, Lecture Notes in Math., vol. 1428, Springer-Verlag, Berlin, 1990. MR1064107 (91c:20016)
- [8] R. Schulz, *Boundedness and periodicity of modules over QF rings*, J. Algebra **101** (1986), no. 2, 450–469. MR847170 (87i:16023)
- [9] E. S. Golod, *The cohomology ring of a finite p-group*, Dokl. Akad. Nauk SSSR **125** (1959), no. 4, 703–706. (Russian) MR0104720 (21:3473)
- [10] B. B. Venkov, *Cohomology algebras for some classifying spaces*, Dokl. Akad. Nauk SSSR **127** (1959), no. 5, 943–944. (Russian) MR0108788 (21:7500)
- [11] L. Evens, *The cohomology ring of a finite group*, Trans. Amer. Math. Soc. **101** (1961), 224–239. MR0137742 (25:1191)
- [12] A. I. Generalov and N. Yu. Fëdorova, *The Yoneda algebra for the “Liu–Schulz example”*, Algebra i Analiz **14** (2002), no. 4, 19–35; English transl., St. Petersburg Math. J. **14** (2003), no. 4, 549–562. MR1935915 (2004b:16010)

- [13] A. I. Generalov, *Hochschild cohomology of algebras of dihedral type. I. The family $D(3K)$ in characteristic 2*, Algebra i Analiz **16** (2004), no. 6, 53–122; English transl., St. Petersburg Math. J. **16** (2005), no. 6, 961–1012. MR2117449 (2005i:16013)
- [14] N. Yu. Kosovskaya, *Bimodule resolution for Liu–Schulz algebras*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. **321** (2005), 213–223; English transl., J. Math. Sci. (N.Y.) **136** (2006), no. 3, 3951–3956. MR2138419 (2006e:16012)
- [15] Th. Holm, *The Hochschild cohomology ring of a modular group algebra: The commutative case*, Comm. Algebra **24** (1996), 1957–1969. MR1386022 (97c:13012)
- [16] C. Cibils and A. Solotar, *Hochschild cohomology algebra of abelian groups*, Arch. Math. (Basel) **68** (1997), 17–21. MR1421841 (97k:13018)

DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, UNIVERSITETSKIĬ PROSPECT 28, STARYĬ PETERHOF, ST. PETERSBURG 198504, RUSSIA

DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, UNIVERSITETSKIĬ PROSPECT 28, STARYĬ PETERHOF, ST. PETERSBURG 198504, RUSSIA

E-mail address: nadyakosovsk@mail.ru

Received 3/OCT/2005

Translated by N. YU. KOSOVSKAYA