ESTIMATION OF A QUADRATIC FUNCTION AND THE p -BANACH-SAKS PROPERTY

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ABSTRACT. Let E be a rearrangement-invariant Banach function space on $[0, 1]$, and let $\Gamma(E)$ denote the set of all $p \geq 1$ such that any sequence $\{x_n\}$ in E converging weakly to 0 has a subsequence $\{y_n\}$ with $\sup_m m^{-1/p} \|\sum_{1 \leq k \leq m} y_n\| < \infty$. The set $\Gamma_i(E)$ is defined similarly, but only sequences $\{x_n\}$ of independent random variables are taken into account. It is proved (under the assumption $\Gamma(E) \neq \{1\}$) that if $\Gamma_i(E) \setminus \Gamma(E) \neq \emptyset$, then $\Gamma_i(E) \setminus \Gamma(E) = \{2\}$.

§1

A classical Banach–Saks theorem (see [B, Chapter 12, Theorem 2]) states that if a sequence $x_n \in L_p[0, 1]$, $1 < p < \infty$, converges weakly to zero, then there exists a sequence $n_k \in \mathbb{N}$ and a number $C > 0$ such that

$$\left\| \sum_{k=1}^m x_{n_k} \right\|_{L_p} \leq C m^{\max(1/2, 1/p)}$$

for all $m \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of positive integers). For $p \in (2, \infty)$ this estimate follows also from [KP]. The exponent $\max(1/2, 1/p)$ is sharp. It suffices to consider the Rademacher system $x_n(t) = \text{sign} \sin 2^n \pi t$, $n \in \mathbb{N}$, for $p \geq 2$ and any sequence of normalized L_p -functions with disjoint supports for $p \leq 2$.

This theorem leads to the following definitions (see [J, Be]). Let E be a Banach space, and let $p \geq 1$. A bounded sequence $\{x_n\}$ in E is called a p -BS-sequence if there exists a subsequence $\{y_n\} \subset \{x_n\}$, such that

$$\sup_{m \in \mathbb{N}} m^{-\frac{1}{p}} \left\| \sum_{k=1}^m y_k \right\|_E < \infty.$$

We shall say that E possesses the p -BS-property and write $E \in BS(p)$ if any sequence converging weakly to zero contains a p -BS-subsequence. Obviously, every Banach space possesses the 1-BS-property. The set $\Gamma(E) = \{p : p \geq 1, E \in BS(p)\}$ is either $[1, \alpha]$ or $[1, \alpha)$, for some $\alpha \in [1, \infty]$. This set will be called the *index set* of the space E , and α is the *Banach–Saks index* of E ; we write $\gamma(E) = \alpha$ if $\Gamma(E) = [1, \alpha]$ and $\gamma(E) = \alpha - 0$ if $\Gamma(E) = [1, \alpha)$. A related notion was introduced in [R], where the coordinate Orlicz spaces with the p -BS-property were described.

The Banach–Saks theorem says that $\gamma(L_p) = \min(p, 2)$ for $1 < p < \infty$. It is known also that $\gamma(L_1) = \gamma(L_\infty) = 1$. In l_p , the sequences that converge weakly to zero have a

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relatively simple structure. By using these results, it is not hard to show that $\gamma(l_p) = p$ for $p \in (1, \infty)$, $\gamma(l_1) = \gamma(c_0) = \infty$, and $\gamma(l_\infty) = 1$.

A Banach space of measurable functions on $[0, 1]$ is said to be *rearrangement invariant* (r.i.) or *symmetric* if

- 1) $|x(t)| \leq |y(t)|$ a.e. and $y \in E \implies x \in E$ and $\|x\|_E \leq \|y\|_E$;
- 2) $x(t)$ and $y(t)$ are equimeasurable and $y \in E \implies x \in E$ and $\|x\|_E = \|y\|_E$.

The theory of r.i. spaces is presented in the books [LT, KPS]. The p -BS-property in r.i. spaces was studied in [SS1, SS2, ASS, DSS]; in particular, there it was shown that the p -BS-property is closely related to other geometric characteristics of r.i. spaces.

The index of any r.i. space does not exceed 2. Every nonseparable r.i. space has the trivial index 1. Therefore, we shall study the p -BS-property for separable r.i. spaces only.

Restricting ourselves, in the definitions of a p -BS-sequence and the index set $\Gamma(E)$, to sequences of independent functions or to sequences of functions with disjoint supports, we arrive at the definitions of the sets $\Gamma_i(E)$ or $\Gamma_d(E)$, respectively. We recall that a system of functions $x_i(t)$ on $[0, 1]$ is *independent* if

$$\begin{aligned} & \text{meas}\{t : t \in [0, 1], x_n(t) \in (\alpha_n, \beta_n), n = 1, 2, \dots, m\} \\ &= \prod_{n=1}^m \text{meas}\{t : t \in [0, 1], x_n(t) \in (\alpha_n, \beta_n)\} \end{aligned}$$

for any $\alpha_n < \beta_n$, $n = 1, 2, \dots, m$ and any $m \in \mathbb{N}$. The simplest and most important example of an independent system of functions is provided by the Rademacher system. Obviously, $\Gamma(E) \subset \Gamma_i(E)$ and $\Gamma(E) \subset \Gamma_d(E)$. In [ASS, L.4.1] it was shown that $\Gamma_i(E) \subset \Gamma_d(E)$ for every separable r.i. space E .

As a rule, determining the index set $\Gamma(E)$ for a specific r.i. space is a fairly complicated task. Therefore, it is of interest to describe the set $\Gamma_i(E)$; this is much simpler, because of the known estimates of sums of independent functions (see [KS, Chapter 2]).

A large part of the paper [SS1] was devoted to the calculation of the index of the spaces $L_{p,q}$.

In the present paper, we aim at proving the following statement: either the set $\Gamma_i(E) \setminus \Gamma(E)$ is empty, or this set consists of one point and this point is 2. For this, we were bound to undertake a special study of a certain quadratic function generated by a system of measure-preserving transformations. Possibly, these auxiliary results are also of independent interest.

§2

Lemma 1. *If $r, j \in \mathbb{N}$ and $r \leq j$, then*

$$(1) \quad \sum_{k=1}^j \frac{\min(kr, j)}{k^{3/2}} \geq \frac{1}{2} \sqrt{jr}.$$

Proof. Simple calculations show that for $r = 1$ and $r = 2$, inequality (1) is true with constant 1 (rather than $\frac{1}{2}$). This implies inequality (1) for $r \leq 8$. If $8 \leq r \leq j$, then

$$\begin{aligned} \sum_{k=1}^j \frac{\min(kr, j)}{k^{3/2}} &> \sum_{\frac{j}{r} \leq k \leq j} \frac{j}{k^{3/2}} \geq j \int_{\frac{j}{r}+1}^j \frac{dt}{t^{3/2}} \\ &= j \left(2 \left(\frac{r}{j+r} \right)^{1/2} - 2 \left(\frac{1}{j} \right)^{1/2} \right) \geq \sqrt{jr} \left(\sqrt{2} - \frac{2}{\sqrt{r}} \right) \\ &\geq \sqrt{jr} \left(\sqrt{2} - \frac{1}{\sqrt{2}} \right) > \frac{1}{2} \sqrt{jr}. \quad \square \end{aligned}$$

Let $k, m \in \mathbb{N}$. We denote by σ_k the dilation operator that acts on R^m as follows: if $x = (x_1, x_2, \dots, x_m)$, then

$$\sigma_k x = (\underbrace{x_1, x_1, \dots, x_1}_k, \underbrace{x_2, x_2, \dots, x_2}_k, \dots).$$

In particular, for $k \geq m$ we have $\sigma_k x = (x_1, x_1, \dots, x_1)$. If $x(t)$ is a function on $[0, 1]$, we put $\sigma_k x(t) = x(t/k)$. It is well known that the operators σ_k are bounded in every r.i. space.

For $x, y \in R^m$, the relation $x \prec y$ will mean that

$$\sum_{i=1}^j x_i^* \leq \sum_{i=1}^j y_i^*$$

for all $j = 1, 2, \dots, m$, where x_i^* is the monotone decreasing rearrangement of the numbers $|x_i|$.

Lemma 2. Suppose $x \in R^m$, $x_1 \geq x_2 \geq \dots \geq x_m \geq 0$. Then

$$\left\{ \left(\frac{1}{k} \sum_{i=1}^k x_i^2 \right)^{1/2} \right\}_{k=1}^m \prec 4 \sum_{k=1}^m \frac{\sigma_k x}{k^{3/2}}.$$

Proof. First, we consider the particular case where

$$(2) \quad x = x^r = (\underbrace{1, 1, \dots, 1}_r, \underbrace{0, 0, \dots, 0}_{m-r}), \quad 1 \leq r \leq m.$$

Then

$$\left(\frac{1}{k} \sum_{i=1}^k x_i^2 \right)^{1/2} = \begin{cases} 1, & k \leq r, \\ \left(\frac{r}{k} \right)^{1/2}, & k \geq r, \end{cases}$$

and

$$\sum_{k=1}^j \left(\frac{1}{k} \sum_{i=1}^k x_i^2 \right)^{1/2} = j$$

for $j \leq r$, while

$$\begin{aligned} \sum_{k=1}^j \left(\frac{1}{k} \sum_{i=1}^k x_i^2 \right)^{1/2} &= r + \sum_{k=r+1}^j \left(\frac{r}{k} \right)^{1/2} \\ &\leq r + \sqrt{r} \int_r^j \frac{dt}{\sqrt{t}} = r + 2\sqrt{r}(\sqrt{j} - \sqrt{r}) < 2\sqrt{jr} \end{aligned}$$

for $j > r$. Therefore, it suffices to show that

$$(3) \quad j \leq 4 \sum_{i=1}^j \sum_{k=1}^m \frac{(\sigma_k x)_i}{k^{3/2}}$$

for $j \leq r$, and

$$(4) \quad \sqrt{jr} \leq 2 \sum_{i=1}^j \sum_{k=1}^m \frac{(\sigma_k x)_i}{k^{3/2}}$$

for $j > r$. The obvious estimate

$$\sum_{i=1}^j \sum_{k=1}^m \frac{(\sigma_k x)_i}{k^{3/2}} > \sum_{i=1}^j \sum_{k=1}^m \frac{1}{k^{3/2}} > \sum_{i=1}^j 1 = j, \quad j \leq r,$$

implies (3). Since

$$\sum_{i=1}^j (\sigma_k x)_i = \min(kr, j),$$

inequality (4) follows from Lemma 1. Thus, the inequality

$$(5) \quad \sum_{k=1}^j \left(\frac{1}{k} \sum_{i=1}^k x_i^2 \right)^{1/2} \leq 4 \sum_{i=1}^j \sum_{k=1}^m \frac{(\sigma_k x)_i}{k^{3/2}}$$

is established for all $j = 1, 2, \dots, m$ and all $x = x^r$, $1 \leq r \leq m$.

Now, let $x \in R^m$, and let $1 \geq x_1 \geq x_2 \geq \dots \geq x_m \geq 0$. The extreme points of this closed convex set are well known: these are the vectors x^r , $0 \leq r \leq m$. There exists a sequence $\lambda_0, \lambda_1, \dots, \lambda_m$, such that $\lambda_r \geq 0$ for all $r = 0, 1, \dots, m$, $\sum_{r=0}^m \lambda_r = 1$, and

$$x = \sum_{r=0}^m \lambda_r x^r.$$

We denote by $P = P_j$ and $Q = Q_j$ the operators occurring on the left and on the right side of (5). Since P is convex and Q is linear, we can use (5) to obtain

$$\begin{aligned} Px &= P \left(\sum_{r=0}^m \lambda_r x^r \right) \leq \sum_{r=0}^m \lambda_r P(x^r) \\ &\leq \sum_{r=0}^m \lambda_r 4Q(x^r) = 4Q \left(\sum_{r=0}^m \lambda_r x^r \right) = 4Qx. \end{aligned}$$

This proves (5) under the additional restriction $x_1 \leq 1$. Using positive homogeneity, we extend this estimate to all nonnegative $x \in R^m$ that are monotone nondecreasing. $\square$

Let $m, n \in \mathbb{N}$. We denote by Π_m the set of all permutations of $\{1, 2, \dots, m\}$ and by $\Pi_{m,n}$ the set of all sequences $(\pi_1, \pi_2, \dots, \pi_n)$, where $\pi_1, \pi_2, \dots, \pi_n \in \Pi_m$. Given $\pi = (\pi_1, \pi_2, \dots, \pi_n) \in \Pi_{m,n}$, we consider the operator

$$(R_{\pi,m} x)_i = \left(\sum_{k=1}^n x_{\pi_k(i)}^2 \right)^{1/2}.$$

Lemma 3. Suppose $m, n \in \mathbb{N}$, $x \in R^m$, $x_1 \geq x_2 \geq \dots \geq x_m \geq 0$, $\varepsilon > 0$, and $\pi \in \Pi_{m,n}$. There exists a vector

$$z \in \text{conv} \left\{ \left(\frac{n}{k} \right)^{1/2+\varepsilon} \sigma_k x, 1 \leq k \leq n \right\}$$

such that $R_{\pi,m} x \prec \frac{4}{\varepsilon} z$.

Proof. There is no loss of generality in assuming that $x_1 > x_2 > \dots > x_m > 0$. The set

$$\{(k, i) : x_{\pi_k(i)} \geq x_j\}$$

consists of jn elements. Therefore,

$$(6) \quad (R_{\pi,m} x)_j^* \leq \left(\frac{1}{j} \sum_{i=1}^j n x_i^2 \right)^{1/2} = \left(\frac{n}{j} \sum_{i=1}^j x_i^2 \right)^{1/2}$$

for $1 \leq j \leq m$.

We have

$$\sum_{k=1}^n \frac{\sigma_k x}{k^{3/2}} = \frac{1}{\varepsilon} \sum_{k=1}^n \frac{\varepsilon}{n^\varepsilon k^{1-\varepsilon}} \frac{n^\varepsilon}{k^{1/2+\varepsilon}} \sigma_k x$$

and

$$\sum_{k=1}^n \frac{\varepsilon}{n^\varepsilon k^{1-\varepsilon}} \leq \frac{\varepsilon}{n^\varepsilon} \int_0^n \frac{dt}{t^{1-\varepsilon}} = 1.$$

Consequently, for some $\gamma \geq \varepsilon$ we have

$$\gamma \sum_{k=1}^n \frac{\sigma_k x}{k^{3/2}} \in \text{conv} \left\{ \frac{n^\varepsilon}{k^{1/2+\varepsilon}} \sigma_k x, 1 \leq k \leq n \right\}.$$

Put

$$z = \gamma \sum_{k=1}^n \frac{\sigma_k x}{k^{3/2}} n^{1/2}.$$

Then

$$z \in \text{conv} \left\{ \left(\frac{n}{k} \right)^{1/2+\varepsilon} \sigma_k x, 1 \leq k \leq n \right\}.$$

Using (6) and Lemma 2, we obtain

$$R_{\pi,m} x \prec \frac{4}{\varepsilon} z. \quad \square$$

Since the norm in any finite-dimensional space F with symmetric basis is monotone relative to the partial ordering $\prec$ (see [LT, 2.a.8]), under the assumptions of Lemma 2 we have

$$\begin{aligned} \|R_{\pi,m} x\|_F &\leq \left\| \frac{4}{\varepsilon} z \right\|_F \leq \frac{4}{\varepsilon} \max_{1 \leq k \leq n} \left\| \left(\frac{n}{k} \right)^{1/2+\varepsilon} \sigma_k x \right\|_F \\ (7) \quad &\leq \frac{4}{\varepsilon} \max_{1 \leq k \leq n} \left(\frac{n}{k} \right)^{1/2+\varepsilon} \|\sigma_k\|_F \|x\|_F. \end{aligned}$$

Let $n \in \mathbb{N}$, and let $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ be a sequence of measure-preserving transformations of the segment $[0, 1]$. The operator

$$S_{n,\omega} x(t) = \left(\sum_{k=1}^n x^2(\omega_k(t)) \right)^{1/2}$$

acts in any r.i. space E . The next lemma is usually viewed as routine. We present the proof for completeness.

Lemma 4. *Suppose $n \in \mathbb{N}$,*

$$x(t) = \sum_{k=1}^n x_k \mathfrak{a}_{\left(\frac{k-1}{n}, \frac{k}{n}\right)}(t)$$

is a step function, ω is a measure-preserving transformation of $[0, 1]$, and $T_\omega x(t) = x(\omega(t))$. Then

$$\lim_{m \rightarrow \infty, \frac{m}{n} \in \mathbb{N}, \pi \in \Pi_m} \min \text{meas} \left\{ t : T_\omega x(t) \neq \sum_{i=1}^m \bar{x}_{\pi(i)} \mathfrak{a}_{\left(\frac{i-1}{m}, \frac{i}{m}\right)}(t) \right\} = 0,$$

where $\bar{x}_i = x_k$ for $\frac{m}{n}(k-1) < i \leq \frac{m}{n}k$, $k = 1, 2, \dots, n$.

Proof. Let e_k denote the image of the segment $\left(\frac{k-1}{n}, \frac{k}{n}\right)$ under ω , i.e., $\mathfrak{a}_{e_k} = T_\omega \mathfrak{a}_{\left(\frac{k-1}{n}, \frac{k}{n}\right)}$, $1 \leq k \leq n$. For a given $\varepsilon > 0$, we find a system of disjoint intervals $(\alpha_{k,i}, \beta_{k,i})$, $i = 1, 2, \dots, s_k$, $k = 1, 2, \dots, n$, such that

$$\sum_{k=1}^n \text{meas}((e_k \setminus g_k) \cup (g_k \setminus e_k)) < \frac{\varepsilon}{2},$$

where

$$g_k = \bigcup_{i=1}^{s_k} (\alpha_{k,i}, \beta_{k,i}).$$

We may assume that $\text{meas } g_k \leq \text{meas } e_k$ for all $k = 1, 2, \dots, n$ and that $g_k \cap g_i = \emptyset$ for $k \neq i$.

We choose $m \in \mathbb{N}$ so that $\frac{m}{n} \in \mathbb{N}$ and $m > \frac{2}{\varepsilon} \sum_{k=1}^n s_k$. Then the measure of the union of all intervals $(\frac{i-1}{m}, \frac{i}{m})$, $1 \leq i \leq m$, that contain at least one point of the set $\{\alpha_{k,i}, \beta_{k,i}, 1 \leq i \leq s_k, k = 1, 2, \dots, n\}$ is less than $\frac{\varepsilon}{2}$. Put

$$A_k = \left\{ i : i \in \mathbb{N}, \left(\frac{i-1}{m}, \frac{i}{m} \right) \subset g_k \right\}, \quad k = 1, 2, \dots, n.$$

Then

$$|A_k| \leq \left| \left\{ i : \left(\frac{i-1}{m}, \frac{i}{m} \right) \subset \left(\frac{k-1}{n}, \frac{k}{n} \right) \right\} \right| = \frac{m}{n}$$

for all $k = 1, 2, \dots, n$. The set $\{i : \frac{m}{n}(k-1) < i \leq \frac{m}{n}k\}$ consists of $\frac{m}{n}$ elements. Therefore, there exists a permutation $\pi \in \Pi_m$ such that $\pi^{-1}(i) \in (\frac{m}{n}(k-1), \frac{m}{n}k]$ for all $i \in A_k$, $k = 1, 2, \dots, n$. For this $\pi \in \Pi_m$, we have

$$\text{meas} \left\{ t : T_\omega x(t) \neq \sum_{i=1}^m \bar{x}_{\pi(i)} \mathfrak{A}_{(\frac{i-1}{m}, \frac{i}{m})}(t) \right\} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

Given an r.i. space E and $m \in \mathbb{N}$, we can construct an m -dimensional space E_m with symmetric basis: if $x = (x_1, x_2, \dots, x_m)$, then

$$\|x\|_{E_m} = \left\| \sum_{k=1}^m x_k \mathfrak{A}_{(\frac{k-1}{m}, \frac{k}{m})} \right\|_E.$$

Lemma 5. *Let E be a separable r.i. space. Then*

$$(8) \quad \|S_{n,\omega}\|_E \leq \sup_{m \in \mathbb{N}, \pi \in \Pi_m} \|R_{\pi,m}\|_{E_m}.$$

Proof. Since E is separable, for any $\varepsilon > 0$ we can find a step function

$$x(t) = \sum_{k=1}^r x_k \mathfrak{A}_{(\frac{k-1}{r}, \frac{k}{r})}(t),$$

such that $\|x\|_E = 1$ and

$$(9) \quad \|S_{n,\omega}x\|_E \geq \|S_{n,\omega}\|_E - \frac{\varepsilon}{2}.$$

The structure of the operator $S_{n,\omega}$ allows us to assume that $x_1 \geq x_2 \geq \dots \geq x_r \geq 0$. In a separable r.i. space, the norm of each function is absolutely continuous, i.e.,

$$\lim_{\delta \rightarrow 0} \sup_{\text{meas } e \leq \delta} \|x \mathfrak{A}_e\|_E = 0.$$

Thus, there exists $\delta > 0$ such that $\|x \mathfrak{A}_e\|_E < \frac{\varepsilon}{4n}$ whenever $\text{meas } e < \delta$. By Lemma 4, there are $m \in \mathbb{N}$ and $\pi = (\pi_1, \pi_2, \dots, \pi_n) \in \Pi_{m,n}$ such that

$$(10) \quad \text{meas} \left\{ t : T_{\omega_k} x(t) \neq \sum_{i=1}^m \bar{x}_{\pi_k(i)} \mathfrak{A}_{(\frac{i-1}{m}, \frac{i}{m})}(t) \right\} < \delta$$

for all $k = 1, 2, \dots, n$. The functions $T_{\omega_k} x$ and

$$\sum_{i=1}^m \bar{x}_{\pi_k(i)} \mathfrak{A}_{(\frac{i-1}{m}, \frac{i}{m})}$$

are equimeasurable with x for all $k = 1, 2, \dots, n$. Therefore,

$$\|T_{\omega_k}x - P_kx\| \leq 2 \sup_{\text{meas } e \leq \delta} \|x\mathfrak{A}_e\| \leq \frac{\varepsilon}{2n}$$

for all $k = 1, 2, \dots, n$, where

$$P_kx(t) = \sum_{i=1}^m \bar{x}_{\pi_k(i)} \mathfrak{A}_{\left(\frac{i-1}{m}, \frac{i}{m}\right)}(t).$$

Then

$$\begin{aligned} (11) \quad & \left\| S_{n,\omega}x - \left(\sum_{k=1}^n (P_kx)^2 \right)^{1/2} \right\|_E \\ &= \left\| \left(\sum_{k=1}^n (T_{\omega_k}x)^2 \right)^{1/2} - \left(\sum_{k=1}^n (P_kx)^2 \right)^{1/2} \right\|_E \\ &\leq \left\| \left(\sum_{k=1}^n (T_{\omega_k}x - P_kx)^2 \right)^{1/2} \right\|_E \\ &\leq \sum_{k=1}^n \|T_{\omega_k}x - P_kx\|_E \leq \sum_{k=1}^n \frac{\varepsilon}{2n} = \frac{\varepsilon}{2}. \end{aligned}$$

By (9) and (11), we have

$$\|S_{n,\omega}\|_E \leq \left\| \left(\sum_{k=1}^n (P_kx)^2 \right)^{1/2} \right\|_E + \varepsilon.$$

If π is the permutation defined in (10), then

$$\left\| \left(\sum_{k=1}^n (P_kx)^2 \right)^{1/2} \right\|_E \leq \|R_{\pi,m}x\|_E \leq \|R_{\pi,m}\|_E.$$

The last two estimates imply (8). $\square$

It is clear that $\|\sigma_k\|_{E_m} \leq \|\sigma_k\|_E$ for all $k, m \in \mathbb{N}$, and that the norm of the operator $S_{n,\omega}$ is attained on the set of nonnegative, monotone nondecreasing functions. Therefore, (7) and Lemma 6 imply the following statement.

Theorem 6. *If E is a separable r.i. space, $n \in \mathbb{N}$ and $\varepsilon > 0$, then*

$$\|S_{n,\omega}\|_E \leq \frac{4}{\varepsilon} \max_{1 \leq k \leq n} \left(\frac{n}{k} \right)^{1/2+\varepsilon} \|\sigma_k\|_E.$$

If E is the Lorentz space $L_{2,1}$, then $\|\sigma_k\|_E = k^{1/2}$ for all $k \in \mathbb{N}$, and

$$\lim_{n \rightarrow \infty} \|S_{n,\omega}\|_E \cdot n^{-1/2} = \infty$$

for some sequence ω . This shows that the estimate

$$\|S_{n,\omega}\|_E \leq C \max_{1 \leq k \leq n} \left(\frac{n}{k} \right)^{1/2} \|\sigma_k\|_E,$$

with C independent of n , cannot be obtained.

§3

Now we pass to the proof of the main result of the paper. If E is an r.i. space, we denote by E'' the set of $x \in L_1[0, 1]$ such that

$$\|x\|_{E''} = \sup_n \|\min(|x(t)|, n)\|_E < \infty.$$

The inclusion $E \subset E''$ is well known to be isometric for any separable r.i. space E . The space E'' either coincides with E or is nonseparable.

Theorem 7. *Suppose $1 < p < 2$, E is a separable r.i. space on $[0, 1]$, $\Gamma(E) \neq \{1\}$, and $p \in \Gamma_i(E)$. Then $p \in \Gamma(E)$.*

Proof. Let $x_n \in E$ be such that $\|x_n\|_E \leq 1$ for all $n \in \mathbb{N}$. Suppose that the x_n converge weakly to zero in E . By [ASS, Lemma 3.6], there is a subsequence $\{y_n\} \subset \{x_n\}$, a function $u \in E''$, and sequences of functions $u_n, v_n, w_n \in E$, such that $y_n = u_n + v_n + w_n$, $u_n^* \leq u$, $\|u\|_E \leq 1$, the v_n are pairwise disjoint, $\|v_n\|_E \leq 2$ for $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} \|w_n\| = 0$.

By [ASS, Lemma 4.1(ii)], we have $p \in \Gamma_d(E)$. Therefore, there is a constant $C_1 > 0$ and a subsequence v_{i_n} such that

$$(12) \quad \left\| \sum_{i=1}^m v_{i_n} \right\|_E \leq C_1 m^{1/p} \sup_n \|v_{i_n}\|_E \leq 2C_1 m^{1/p}$$

for all $m \in \mathbb{N}$. To simplify the notation, we assume that $i_n = n$.

In Theorem 4.3 of [SS1] (which may be applied due to [ASS, Theorem 4.2(i)]), it was proved that there exists a constant $C_2 > 0$ such that

$$\left\| \sum_{n=1}^m u_n \right\|_E \leq C_2 \left\| \left(\sum_{n=1}^m u_n^2 \right)^{1/2} \right\|_E + 1$$

for all $m \in \mathbb{N}$. Since $u_n^* \leq u$, we have $u_n \leq u(\omega_n)$, where ω_n is a measure-preserving transformation of $[0, 1]$, $n \in \mathbb{N}$. Therefore,

$$\left\| \sum_{n=1}^m u_n \right\|_E \leq C_2 \|S_{m,\omega} u\|_{E''} + 1$$

where $(\omega = \omega_1, \omega_2, \dots, \omega_m)$, $m \in \mathbb{N}$. The function u defined in [ASS, Lemma 3.6] belongs to E'' , but may fail to belong to E . However, when estimating $\|\sum_{n=1}^m u_n\|$, we may replace u with the function

$$v(t) = \min \left(u^*(t), \max_{1 \leq n \leq m} u_n^*(t) \right).$$

Since $v \in E$ and the inclusion $E \subset E''$ is isometric, there is no loss of generality in assuming that $u \in E$ and

$$\left\| \sum_{n=1}^m u_n \right\|_E \leq C_2 \|S_{m,\omega} u\|_E + 1.$$

Now we apply Theorem 6, putting $\varepsilon = \frac{1}{p} - \frac{1}{2}$. Then

$$\left\| \sum_{n=1}^m u_n \right\|_E \leq C_2 \frac{2p}{2-p} \cdot 4 \max_{1 \leq k \leq m} \left(\frac{m}{k} \right)^{\frac{1}{p}} \|\sigma_k\|_E + 1.$$

Since $p \in \Gamma_i(E)$, the proof of [ASS, Theorem 4.2(ii)] shows that there exists a constant $C_3 > 0$ such that $\|\sigma_k\|_E \leq C_3 k^{1/p}$ for all $k \in \mathbb{N}$. Consequently,

$$(13) \quad \left\| \sum_{n=1}^m u_n \right\|_E \leq \frac{16C_2}{2-p} \max_{1 \leq k \leq m} \left(\frac{m}{k} \right)^{\frac{1}{p}} C_3 k^{1/p} + 1 = \frac{16C_2 C_3}{2-p} m^{1/p} + 1$$

for all $m \in \mathbb{N}$.

As was mentioned above, $\lim_{n \rightarrow \infty} \|w_n\|_E = 0$. Passing to a subsequence, we may assume that

$$(14) \quad \sum_{n=1}^{\infty} \|w_n\|_E \leq 1.$$

Combining (12), (13), and (14), we see that there exists a subsequence $\{x_{j_n}\} \subset \{x_n\}$ and a constant $C_4 > 0$ such that

$$\left\| \sum_{n=1}^m x_{j_n} \right\|_E \leq C_4 m^{1/p}$$

for all $m \in \mathbb{N}$. We can take $C_4 = 2C_1 + \frac{16C_2 C_3}{2-p} + 2$. This means that $p \in \Gamma(E)$. $\square$

Thus,

$$\Gamma(E) \cap [1, 2) = \Gamma_i(E) \cap [1, 2).$$

In the proof of Proposition 4.12 in [ASS] and in Theorem 5.9 of [SS1] it was shown that $\Gamma_i(L_{2,1}) = [1, 2]$ and $\Gamma(L_{2,1}) = [1, 2)$. This tells us that Theorem 7 does not extend to the case of $p = 2$. However, if $2 \in \Gamma_i(E)$, then $p \in \Gamma_i(E)$ for any $p \in [1, 2)$. We arrive at the following consequence of Theorem 7.

Corollary 8. *If E is a separable r.i. space such that $\Gamma(E) \neq \{1\}$, $2 \in \Gamma_i(E)$, and $1 < p < 2$, then $p \in \Gamma(E)$.*

Lemma 4.1 in [ASS] states that $1 \in \Gamma(E) \subset \Gamma_i(E) \subset [1, 2]$ for any separable r.i. space E . Combining this with Theorem 7 and Corollary 8, we see that the following is true.

Theorem 9. *Let E be a separable r.i. space with $\Gamma(E) \neq \{1\}$. Then either (i) $\Gamma(E) = \Gamma_i(E)$ or (ii) $\Gamma(E) = [1, 2)$ and $\Gamma_i(E) = [1, 2]$.*

We note that, in contrast to $\Gamma_i(E)$, the set $\Gamma_d(E)$ may differ significantly from $\Gamma(E)$. For example, if E is a separable Lorentz space, then $\Gamma_d(E) = [1, \infty)$ and $\Gamma(E) \subset \Gamma_i(E) \subset [1, 2]$.

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