DIMENSIONS OF PRODUCTS OF HYPERBOLIC SPACES

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To dear Viktor Abramovich Zalgaller on the occasion of his 85th birthday

ABSTRACT. Estimates on asymptotic dimension are given for products of general hyperbolic spaces, with applications to hyperbolic groups. Examples are presented where strict inequality occurs in the product theorem for the asymptotic dimension in the class of hyperbolic groups and in the product theorem for the hyperbolic dimension. It is proved that $\mathbb{R}$ is dimensionally full for the asymptotic dimension in the class of hyperbolic groups.

§1. INTRODUCTION

Our purpose in this paper is to answer some questions concerning the theory of asymptotic dimension (in particular, those posed in [Dr3]). For this, we give various estimates on the asymptotic dimension of general hyperbolic spaces and products of hyperbolic spaces and apply these estimates to hyperbolic groups.

To formulate our results we need some definitions. We say that the metric spaces $\{X_\alpha : \alpha \in \mathcal{A}\}$ have linearly controlled dimension ℓ -dim not exceeding k uniformly if there is a constant $\delta \in (0, 1)$ such that for every sufficiently small $\tau > 0$ and every $\alpha \in \mathcal{A}$ there exists a $(k + 1)$ -colored open covering $\mathcal{U}_\alpha$ of X_α with $\text{mesh}(\mathcal{U}_\alpha) \leq \tau$ and with Lebesgue number $L(\mathcal{U}_\alpha) \geq \delta\tau$.

For a metric space X and $a > 0$, we denote by aX the metric space obtained from X by multiplying all distances by a .

In Subsection 2.3, we recall the definitions of the asymptotic dimension asdim and the linearly controlled asymptotic dimension ℓ - asdim , used in the following theorem; these are some quasi-isometry invariants of metric spaces.

Theorem 1.1. *Let X_1, X_2 be visual Gromov hyperbolic spaces. Assume that for the metric spaces of the family $\{a\partial_\infty X_1 \times b\partial_\infty X_2 : a \geq 1, b \geq 1\}$ we have ℓ -dim $\leq k$ uniformly, where the boundaries at infinity $\partial_\infty X_1, \partial_\infty X_2$ are taken with some visual metrics. Then*

$$\text{asdim}(X_1 \times X_2) \leq k + 2.$$

The property of a Gromov hyperbolic space X to be visual is a rough version of the property that every point $x \in X$ lies on a geodesic ray emanating from a fixed point $x_0 \in X$; for the precise definition see §2.

Remark 1.2. A similar result is true in the case of an arbitrary finite number (more than two) of factors, and moreover, the estimate remains valid if we replace asdim by ℓ - asdim . In other words, the following is true.

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Let $X_1, \dots, X_n$ be visual Gromov hyperbolic spaces. Assume that for the metric spaces of the family $\{a_1 \partial_\infty X_1 \times \dots \times a_n \partial_\infty X_n : a_i \geq 1, i \in \{1, \dots, n\}\}$ we have $\ell\text{-dim} \leq k$ uniformly. Then

$$\text{asdim}(X_1 \times \dots \times X_n) \leq k + n$$

and

$$\ell\text{-asdim}(X_1 \times \dots \times X_n) \leq k + n.$$

For simplicity of notation, we shall consider the case of two spaces and the asymptotic dimension asdim .

Theorem 1.3. *Let $X_1, \dots, X_n$ be geodesic Gromov hyperbolic spaces. Then*

$$\text{asdim}(X_1 \times \dots \times X_n) \geq \ell\text{-dim}(\partial_\infty X_1 \times \dots \times \partial_\infty X_n \times [0, 1]^n).$$

Corollary 1.4. *Let Γ_1, Γ_2 be Gromov hyperbolic groups. Then*

$$\text{asdim}(\Gamma_1 \times \Gamma_2) = \dim(\partial_\infty \Gamma_1 \times \partial_\infty \Gamma_2) + 2.$$

As an application of this corollary, we

1) give examples of strict inequality in the product theorem for the asymptotic dimension in the class of hyperbolic groups (cf. [Dr3, Problem 23]; the corresponding example in the class of general metric spaces was given in [BL]);

2) give examples of strict inequality in the product theorem for the hyperbolic dimension;

3) prove that $\text{asdim}(\Gamma \times \mathbb{R}) = \text{asdim}(\Gamma) + 1$ for any hyperbolic group Γ .

The above identity fails for general metric spaces. An example of a metric space X of bounded geometry with finite asymptotic dimension for which $\text{asdim}(X \times \mathbb{R}) = \text{asdim} X$ was constructed in [Dr4], and the question is whether the identity in 3) is true for groups ([Dr3, Problem 21]).

Theorem 1.5. *Let X be a visual Gromov hyperbolic space. Then*

$$\ell\text{-asdim} X \leq \ell\text{-dim}(\partial_\infty X \times [0, 1]).$$

This refines the estimate $\text{asdim} X \leq \ell\text{-dim} \partial_\infty X + 1$ proved in [Bu1, Bu2]. Unfortunately, we do not know whether the space $[0, 1]$ is dimensionally full for the linearly controlled dimension, i.e., whether $\ell\text{-dim}(Y \times [0, 1]) = \ell\text{-dim} Y + 1$ for any (compact or proper) metric space Y .

Corollary 1.6. *Let X be a visual, geodesic Gromov hyperbolic space. Then*

$$\ell\text{-asdim} X = \text{asdim} X = \ell\text{-dim}(\partial_\infty X \times [0, 1]).$$

The first identity fails in general, and the question is whether it holds for all finitely presented groups ([Dr3, Problem 41]).

§2. PRELIMINARIES

Here, we recall some notions and facts necessary for the paper.

2.1. Coverings. Let Z be a metric space. For $U, U' \subset Z$, we denote by $\text{dist}(U, U')$ the distance between U and U' , $\text{dist}(U, U') = \inf\{|uu'| : u \in U, u' \in U'\}$, where $|uu'|$ is the distance between u, u' . For $r > 0$ we denote by $B_r(U)$ the open r -neighborhood of U , $B_r(U) = \{z \in Z : \text{dist}(z, U) < r\}$, and by $\overline{B}_r(U)$ the closed r -neighborhood of U , $\overline{B}_r(U) = \{z \in Z : \text{dist}(z, U) \leq r\}$. We extend this notation to all real r by putting $B_r(U) = U$ for $r = 0$ and by defining $B_r(U)$ for $r < 0$ as the complement to the closed $|r|$ -neighborhood of $Z \setminus U$, $B_r(U) = Z \setminus \overline{B}_{|r|}(Z \setminus U)$.

Given a family $\mathcal{U}$ of subsets in a metric space Z , we define $\text{mesh}(\mathcal{U}) = \sup\{\text{diam } U : U \in \mathcal{U}\}$. The *multiplicity* $m(\mathcal{U})$ of $\mathcal{U}$ is the maximal number of the members of $\mathcal{U}$ with nonempty intersection. We say that a family $\mathcal{U}$ is *disjoint* if $m(\mathcal{U}) = 1$.

A family $\mathcal{U}$ is called a *covering* of Z if $\bigcup\{U : U \in \mathcal{U}\} = Z$. A covering $\mathcal{U}$ is said to be *colored* if it is the union of $m \geq 1$ disjoint families, $\mathcal{U} = \bigcup_{a \in A} \mathcal{U}^a$, $|A| = m$. In this case we also say that $\mathcal{U}$ is *m-colored*. Clearly, the multiplicity of an *m-colored* covering is at most m .

Let $\mathcal{U}$ be a family of open subsets in a metric space Z that cover $A \subset Z$. Given $z \in A$, we let

$$L(\mathcal{U}, z) = \sup\{\text{dist}(z, Z \setminus U) : U \in \mathcal{U}\}$$

be the Lebesgue number of $\mathcal{U}$ at z , and $L(\mathcal{U}) = \inf_{z \in A} L(\mathcal{U}, z)$ the Lebesgue number of the covering $\mathcal{U}$ of A . For every $z \in A$, the open ball $B_r(z) \subset Z$ of radius $r = L(\mathcal{U})$ centered at z is contained in some member of the covering $\mathcal{U}$.

We shall use the following obvious fact (see, e.g., [Bu1]).

Lemma 2.1. *Let $\mathcal{U}$ be an open covering of $A \subset Z$ with $L(\mathcal{U}) > 0$. Then for every $s \in (0, L(\mathcal{U}))$ the family $\mathcal{U}_{-s} = B_{-s}(\mathcal{U})$ is still an open covering of A .* $\square$

2.2. Hyperbolic spaces. Here we recall the necessary facts from the theory of hyperbolic spaces. For more details, we refer the reader, e.g., to [BoS]. We also assume that the reader is familiar with notions such as a geodesic metric space, a geodesic ray, etc.

Let X be a metric space. We use the notation $|xx'|$ for the distance between $x, x' \in X$. For $o \in X$ and $x, x' \in X$, put $(x|x')_o = \frac{1}{2}(|xo| + |x'o| - |xx'|)$. The number $(x|x')_o$ is nonnegative by the triangle inequality; it is called the Gromov product of x, x' with respect to o .

A metric space X is said to be δ -hyperbolic, $\delta \geq 0$, if the inequality

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta$$

is fulfilled for every base point $o \in X$ and all $x, y, z \in X$.

Let X be a hyperbolic space, i.e., X is δ -hyperbolic for some $\delta \geq 0$. We denote by $\partial_\infty X$ the (Gromov) boundary of X at infinity. The Gromov product based at $o \in X$ extends naturally to $\partial_\infty X$.

If a hyperbolic space X is geodesic, then every geodesic ray in X represents a point at infinity. Conversely, if a geodesic hyperbolic space X is proper (i.e., the closed balls in X are compact), then every point at infinity is represented by a geodesic ray.

A metric d on the boundary at infinity $\partial_\infty X$ of X is said to be *visual* if there are $o \in X$, $a > 1$, and positive constants c_1, c_2 , such that

$$c_1 a^{-(\xi|\xi')_o} \leq d(\xi, \xi') \leq c_2 a^{-(\xi|\xi')_o}$$

for all $\xi, \xi' \in \partial_\infty X$. In this case, we say that d is visual with respect to the base point o and the parameter a . The boundary at infinity is bounded and complete with respect to any visual metric; moreover, if X is proper, then $\partial_\infty X$ is compact. If $a > 1$ is sufficiently close to 1, then a visual metric with respect to a does exist.

A hyperbolic space X is *visual* if for some base point $x_0 \in X$ there is a positive constant D such that for every $x \in X$ there is $\xi \in \partial_\infty X$ with $|xx_0| \leq (x|\xi)_{x_0} + D$ (it is easily seen that this property is independent of the choice of x_0).

2.3. Definitions of the linearly controlled and the asymptotic dimension. We shall say that a covering $\mathcal{U}$ is an (L, M, n) -covering if it is n -colored with $\text{mesh}(\mathcal{U}) \leq M$ and $L(\mathcal{U}) \geq L$.

The notion of the linearly controlled metric dimension of a metric space was introduced in [Bu1], where it was called the “capacity dimension”. One of many equivalent definitions of this notion is the following.

The linearly controlled dimension, or ℓ -dimension, of a metric space Z , $\ell\text{-dim}(Z)$, is the minimal integer $m \geq 0$ with the following property: there is a constant $\delta \in (0, 1)$ such that for every sufficiently small $\tau > 0$ there exists an open $(\delta\tau, \tau, m+1)$ -covering of Z . Here (and in all similar definitions of a dimension below) we assume that $\ell\text{-dim } Z = \infty$ if there is no m with the property indicated.

The asymptotic dimension is a quasi-isometry invariant of a metric space; it was introduced in [Gr]. There are also several equivalent definitions; see [Gr], [BD]. We use the following one. The asymptotic dimension of a metric space X is the minimal integer $\text{asdim } X = m$ such that for every positive d there is an open covering $\mathcal{U}$ of X with $m(\mathcal{U}) \leq m+1$, $\text{mesh}(\mathcal{U}) < \infty$, and $L(\mathcal{U}) \geq d$.

The linearly controlled asymptotic dimension is also a quasi-isometry invariant of a metric space. We use the following definition. The linearly controlled asymptotic dimension of a metric space X is the minimal integer $\ell\text{-asdim } X = n$ with the following property: there is a constant $C > 1$ such that for every sufficiently large d there is an open $(d, Cd, n+1)$ -covering $\mathcal{U}$ of X .

2.4. Coverings of the product. Throughout the paper, we assume that the product of metric spaces $X \times Y$ is endowed with the l_∞ -metric; i.e., $|(x, y)(x', y')| = \max\{|xx'|, |yy'|\}$ for each $(x, y), (x', y') \in X \times Y$.

Let $\mathcal{U}$ be an open covering of the product $X \times Y$ of metric spaces. We say that a pair (p, q) , $p, q \geq 0$, dominates $\text{mesh}^\times(\mathcal{U})$ and write $\text{mesh}^\times(\mathcal{U}) \leq (p, q)$ or $(p, q) = \text{mesh}^\boxtimes(\mathcal{U})$ if for each member of $\mathcal{U}$ there exists a point $(x, y) \in X \times Y$ such that this member is contained in the product $\overline{B}_p(x) \times \overline{B}_q(y)$.

We also say that a function $f(z) \in [0, \infty) \times [0, \infty)$, $z \in X \times Y$, dominates the pointwise $\text{mesh}^\times(\mathcal{U}, z)$ and write $\text{mesh}^\times(\mathcal{U}, z) \leq f(z)$ or $f(z) = \text{mesh}^\boxtimes(\mathcal{U}, z)$ if for each member of $\mathcal{U}$ that contains z there exists a point $(x, y) \in X \times Y$ such that this member is contained in $\overline{B}_p(x) \times \overline{B}_q(y)$, where $(p, q) = f(z)$. In this sense, the $\text{mesh}^\times$ (the pointwise $\text{mesh}^\times$) is determined basically by the meshes (the meshes at the projected points) of the coverings projected to the factors.

Similarly, we say that the *Lebesgue number* $L^\times$ at a point $z = (x, y) \in X \times Y$ of the covering $\mathcal{U}$ dominates $f(z) = (p, q) \in [0, \infty) \times [0, \infty)$, $L^\times(\mathcal{U}, z) \geq f(z)$, if the product of the balls $\overline{B}_p(x) \times \overline{B}_q(y)$ is contained in some member of the covering. In this case, we also write $f(z) = L^\boxtimes(\mathcal{U}, z)$.

We write $L^\times(\mathcal{U}) \geq (p, q)$ or $(p, q) = L^\boxtimes(\mathcal{U})$ if $L^\times(\mathcal{U}, z) \geq (p, q)$ for every $z \in X \times Y$. In general, there is no way to recover $L^\times$ by the Lebesgue numbers of the projected coverings.

For the product $X \times Y$, it is straightforward to check that the following two properties are equivalent.

- (1) The spaces of the family $\{aX \times bY : a \geq 1, b \geq 1\}$ have $\ell\text{-dim} \leq k$ uniformly.
- (2) There is a constant $\delta \in (0, 1)$ such that for every sufficiently small $\tau > 0$ and every $\alpha, \beta \in (0, 1]$ there exists a $(k+1)$ -colored open covering $\mathcal{U}$ of $X \times Y$ with $\text{mesh}^\times(\mathcal{U}) \leq (\alpha\tau, \beta\tau)$ and $L^\times(\mathcal{U}) \geq (\alpha\delta\tau, \beta\delta\tau)$.

Let $\mathcal{U}, \mathcal{V}$ be open coverings of $X \times Y$. We write $\text{mesh}^\times(\mathcal{U}) \leq (p, q)L^\times(\mathcal{V})$ for some $p, q \geq 0$ if $\text{mesh}^\times(\mathcal{U}) \leq (pr_1, qr_2)$ for some $(r_1, r_2) = L^\boxtimes(\mathcal{V})$. If $p = q$, we write $\text{mesh}^\times(\mathcal{U}) \leq pL^\times(\mathcal{V})$. Observe that if $\text{mesh}^\times(\mathcal{U}) \leq L^\times(\mathcal{V})$, then the covering $\mathcal{U}$ is *inscribed* in the covering $\mathcal{V}$; that is, every member of $\mathcal{U}$ is contained in some member of $\mathcal{V}$.

We write $\text{mesh}^\times(\mathcal{U}, z) \leq \text{mesh}^\times(\mathcal{V}, z)$ if every pair $\text{mesh}^\boxtimes(\mathcal{V}, z)$ is a pair $\text{mesh}^\boxtimes(\mathcal{U}, z)$. We write $L^\times(\mathcal{U}, z) \geq sL^\times(\mathcal{V}, z)$ for $s \geq 0$ if for any $p, q \geq 0$ with $L^\times(\mathcal{V}, z) \geq (p, q)$ we have $L^\times(\mathcal{U}, z) \geq (sp, sq)$.

For $A \subset Z = X \times Y$ and $r_1, r_2 > 0$, we denote by $B_{(r_1, r_2)}(A)$ the union of all products $B_{r_1}(x) \times B_{r_2}(y)$ with $(x, y) \in A$, $B_{(r_1, r_2)}(A) = \bigcup \{B_{r_1}(x) \times B_{r_2}(y) : (x, y) \in A\}$, and put $B_{(-r_1, -r_2)}(A) = Z \setminus \overline{B_{(r_1, r_2)}(Z \setminus A)}$.

We also write $(p, q) \geq (p', q')$ for reals p, q, p', q' if and only if $p \geq p'$ and $q \geq q'$.

§3. AUXILIARY FACTS

3.1. Separate and qualified families of coverings. In this section we give a generalization of the following lemma, the proof of which can be found, e.g., in [BL].

Lemma 3.1. *Suppose that Z is a metric space and $A, B \subset Z$. Let $\mathcal{U}$ be an open covering of A , $\mathcal{V}$ an open covering of B , both of multiplicity at most m . If $\text{mesh}(\mathcal{V}) \leq L(\mathcal{U})/2$, then there exists an open covering $\mathcal{W}$ of $A \cup B$ with multiplicity at most m and $\text{mesh}(\mathcal{W}) \leq \max\{\text{mesh}(\mathcal{V}), \text{mesh}(\mathcal{U})\}$, $L(\mathcal{W}) \geq \min\{L(\mathcal{U})/2, L(\mathcal{V})\}$. $\square$*

Let $\mathcal{U}_1, \mathcal{U}_2$ be families of subsets in a metric space Z . We say that $\mathcal{U}_1$ and $\mathcal{U}_2$ are separate if every member of $\mathcal{U}_1$ intersects no member of $\mathcal{U}_2$.

Let $Z = X \times Y$ be the product of metric spaces, and let $Z_\alpha \subset Z$, $\alpha \in \mathcal{A}$. Let $\mathcal{U}_\alpha$ be a covering of Z_α . Let $S = \{0, \dots, N\}$ be the set of “scales” and $i : \mathcal{A} \rightarrow S$ a *scale function*.

We say that the family $\mathcal{U}_\alpha$, $\alpha \in \mathcal{A}$, is *separate with the scale function i* if for every $U \in \mathcal{U}_\alpha$, $V \in \mathcal{U}_{\alpha'}$ with $\alpha \neq \alpha'$ and $i(\alpha) = i(\alpha')$, we have $U \cap V = \emptyset$.

We say that the family $\mathcal{U}_\alpha$ is *qualified by the scale function i* if the condition $U \cap V \neq \emptyset$ for some $U \in \mathcal{U}_\alpha$, $V \in \mathcal{U}_{\alpha'}$ with $i(\alpha) < i(\alpha')$ implies $L^\boxtimes(\mathcal{U}_{\alpha'}) \geq 4 \text{mesh}^\boxtimes(\mathcal{U}_\alpha)$ for appropriate dominating numbers fixed for each covering $\mathcal{U}_\alpha$.

Proposition 3.2. *Suppose that $Z = X \times Y$ is the product of metric spaces and $Z_\alpha \subset Z$, $\alpha \in \mathcal{A}$. Let $S = \{0, \dots, N\}$, and let $i : \mathcal{A} \rightarrow S$ be a scale function. Let $\mathcal{U}_\alpha$ be an open m -colored covering of Z_α , $\alpha \in \mathcal{A}$, such that the family of coverings $\mathcal{U}_\alpha$, $\alpha \in \mathcal{A}$, is separate with and qualified by the scale function i . Then there exists an open m -colored covering $\mathcal{W}$ of $\bigcup_\alpha Z_\alpha$ with $\text{mesh}^\times(\mathcal{W}, z) \leq \text{mesh}^\times(\bigcup_\alpha \mathcal{U}_\alpha, z)$ and $L^\times(\mathcal{W}, z) \geq L^\times(\bigcup_\alpha \mathcal{U}_\alpha, z)/2$ for every $z \in \bigcup_\alpha Z_\alpha$.*

Proof. Let A be the set of colors, $|A| = m$, which may be assumed to be common for all coverings $\mathcal{U}_\alpha$, so that $\mathcal{U}_\alpha$ is the union of disjoint families, $\mathcal{U}_\alpha = \bigcup_{a \in A} \mathcal{U}_\alpha^a$. For $s \in S$, we denote $\mathcal{A}_s := \{\alpha \in \mathcal{A} : i(\alpha) = s\}$.

For every $\alpha \in \mathcal{A}$ and $a \in A$, we consider the family $\mathcal{V}_\alpha^a = B_{-L^\boxtimes(\mathcal{U}_\alpha)/2}(\mathcal{U}_\alpha^a)$, where the pair $L^\boxtimes(\mathcal{U}_\alpha)$ is dominated by $L^\times(\mathcal{U}_\alpha)$ as in the definition of coverings qualified by a scale function.

Assume that $U \in \mathcal{U}_\alpha$ and $V = B_{-L^\boxtimes(\mathcal{U}_\alpha)/2}(V')$ for $V' \in \mathcal{U}_{\alpha'}^a$. Then

(*) for $i(\alpha) < i(\alpha')$ and $U \cap V \neq \emptyset$, we have $U \cup V \subset V'$.

This easily follows from the fact that $L^\boxtimes(\mathcal{U}_{\alpha'}) \geq 4 \text{mesh}^\boxtimes(\mathcal{U}_\alpha)$ in this case. In particular,

(**) for every $U \in \mathcal{U}_\alpha$, every $\alpha' \in \mathcal{A}$ with $i(\alpha) < i(\alpha')$, and every $a \in A$ there exists at most one $V \in \mathcal{V}_{\alpha'}^a$ with $U \cap V \neq \emptyset$,

because the family $\mathcal{U}_{\alpha'}^a$ is disjoint.

We construct an m -colored covering $\mathcal{W}$ by induction on $j \in S$, independently for each color $a \in A$.

Fix a color $a \in A$ and set $\mathcal{W}_0^a := \bigcup_{\alpha \in \mathcal{A}_0} \mathcal{U}_\alpha^a$. Then the family $\mathcal{W}_0^a$ is disjoint; moreover, it possesses properties (2), (3) below. Assume that for some $j \in S$ we have already constructed a family $\mathcal{W}_j^a$ with the following properties:

- (1) $\mathcal{W}_j^a$ is disjoint;
- (2) for every $U \in \mathcal{V}_\alpha^a$ with $i(\alpha) \leq j$ there exists $W \in \mathcal{W}_j^a$ such that $U \subset W$;
- (3) for every $W \in \mathcal{W}_j^a$ there exists $U \in \mathcal{U}_\alpha^a$ with $i(\alpha) \leq j$ such that $W \subset U$.

In view of (**), property (3) implies that for every $W \in \mathcal{W}_j^a$ there exists at most one member $V \in \bigcup_{\mathcal{A}_{j+1}} \mathcal{V}_\alpha^a$ such that $V \cap W \neq \emptyset$. We set $I(W) = V$ if such a V exists and $I(W) = \emptyset$ otherwise; this yields a function $I : \mathcal{W}_j^a \rightarrow \bigcup_{\mathcal{A}_{j+1}} \mathcal{V}_\alpha^a \cup \{\emptyset\}$.

We put

$$\mathcal{W}_{j+1}^a = \{W \in \mathcal{W}_j^a : I(W) = \emptyset\} \cup \{(V \cup I^{-1}(V)) : V \in \bigcup_{\mathcal{A}_{j+1}} \mathcal{V}_\alpha^a\}.$$

The family $\bigcup_{\mathcal{A}_{j+1}} \mathcal{V}_\alpha^a$ is disjoint. Thus, by (*) and (1), the family $\mathcal{W}_{j+1}^a$ is also disjoint. Furthermore, $\mathcal{W}_{j+1}^a$ possesses property (2) (with j replaced with $j+1$) by construction. Property (3) is also fulfilled for $\mathcal{W}_{j+1}^a$ by (*) and the inductive assumption.

Proceeding by induction, we obtain a family $\mathcal{W}_N^a$ for each $a \in A$. We set $\mathcal{W}^a = \mathcal{W}_N^a$ and $\mathcal{W} = \bigcup_a \mathcal{W}^a$. Property (2) for $\mathcal{W}_N^a$ implies that $\mathcal{W}$ is a covering of $\bigcup_\alpha Z_\alpha$. By property (1) for $\mathcal{W}_N^a$, $\mathcal{W}$ is m -colored. Property (3) implies that $\text{mesh}^\times(\mathcal{W}, z) \leq \text{mesh}^\times(\bigcup_\alpha \mathcal{U}_\alpha, z)$ and property (2) implies that $L^\times(\mathcal{W}, z) \geq L^\times(\bigcup_\alpha \mathcal{U}_\alpha, z)/2$ for every $z \in \bigcup_\alpha Z_\alpha$. That is, $\mathcal{W}$ possesses all the required properties. $\square$

3.2. Barycentric triangulation of products. We recall some standard constructions related to simplicial polyhedra. Given an index set J , we let R^J be the Euclidean space of functions $J \rightarrow \mathbb{R}$ with finite support. For $x, x' \in \mathbb{R}^J$, the distance $|xx'|$ is well defined by

$$|xx'|^2 = \sum_{j \in J} (x_j - x'_j)^2.$$

Let $\Delta^J \subset R^J$ be the standard simplex; i.e., $x \in \Delta^J$ if and only if $x_j \geq 0$ for all $j \in J$ and $\sum_{j \in J} x_j = 1$. The metric of R^J induces a metric on Δ^J and on every subcomplex $K \subset \Delta^J$. If J is finite, then $|J| - 1 = \dim \Delta^J$ is the (combinatorial) dimension of Δ^J .

For every simplicial polyhedron K , there is a canonical embedding $u : K \rightarrow \Delta^J$, where J is the vertex set of K , which is affine on every simplex. Its image $K' = u(K)$ is called the *uniformization* of K . The (combinatorial) dimension of K is the maximal dimension of its simplexes.

For a vertex $v \in K$, its *star* $\overline{\text{st}}_v \subset K$ consists of all simplexes of K containing v . The *open star* st_v of v is the star without the faces that miss v . If a simplicial polyhedron K is uniform, then the open star st_v of any vertex $v \in K$ is an open neighborhood of v .

There are several ways to triangulate the product of simplicial complexes. For that, one needs to choose some ordering of simplexes. Since the barycentric triangulation is ordered canonically, we prefer to use the following construction, which can be found, e.g., in the forthcoming book [BS3]. Given an index set J , we denote by $\overline{\text{ba}}\Delta^J$ the barycentric subdivision of Δ^J , which is a simplicial complex isometric to Δ^J . The vertices of this complex are barycenters of all simplexes (including the 0-dimensional ones). The simplexes of this complex are the convex hulls of all sets S of vertices with the following property: if $s, s' \in S$, then they are the barycenters of two simplexes, one of which is contained in the other. If J is finite, then the covering of Δ^J by the open stars of the vertices of its barycentric subdivision is $|J|$ -colored: as the color of a star st_v , we can take the dimension of the face for which v is the barycenter.

Now, we describe the barycentric triangulation of the product of two simplexes $\Delta^{J_1}, \Delta^{J_2}$. We regard this product $\Delta^{J_1} \times \Delta^{J_2}$ as a complex with faces that are products of standard simplexes. Let b_1, b_2 be barycenters of simplexes $S_1 \subset \Delta^{J_1}, S_2 \subset \Delta^{J_2}$. We call the point $(b_1, b_2) \in \Delta^{J_1} \times \Delta^{J_2}$ the barycenter of the face $S_1 \times S_2$. We define the *barycentric subdivision* of $\Delta^{J_1} \times \Delta^{J_2}$ as a simplicial complex $\Delta^{J_1} \overline{\text{ba}}_s \Delta^{J_2}$, isometric to $\Delta^{J_1} \times \Delta^{J_2}$, in the following way. The vertices of this complex are the barycenters of all faces (including the 0-dimensional ones). The simplexes of this complex are the convex

hulls of all sets S of vertices with the following property: if $s, s' \in S$, then they are the barycenters of two faces, one of which is contained in the other.

We denote the uniformization of $\Delta^{J_1} \times_s \Delta^{J_2}$ by $\Delta^{J_1} \times_s \Delta^{J_2}$ and call it the *barycentric triangulation* of the product $\Delta^{J_1} \times \Delta^{J_2}$.

The canonical bijection $\phi : \Delta^{J_1} \times \Delta^{J_2} \rightarrow \Delta^{J_1} \times_s \Delta^{J_2}$ is called the *barycentric triangulation map*.

Now, let K_1, K_2 be uniform simplicial polyhedra, which we can identify with subpolyhedra of the standard simplexes $\Delta^{J_1}, \Delta^{J_2}$; we define the *barycentric triangulation* of $K_1 \times K_2$ as $\phi(K_1 \times K_2)$.

Note that the covering of $K_1 \times_s K_2$ by open stars of its vertices is $(n_1 + n_2 + 1)$ -colored, where $n_1 = \dim K_1$, $n_2 = \dim K_2$: as the color of a star st_v , we can take the dimension of the minimal simplex of $\Delta^{J_1 \cup J_2}$ containing v .

We give the following two technical lemmas without proof (for the proof, see [BS3]).

Lemma 3.3. *Let K_1, K_2 be uniform simplicial polyhedra both of finite dimension. The barycentric triangulation map*

$$\phi : K_1 \times K_2 \rightarrow K_1 \times_s K_2$$

is bi-Lipschitz, with bi-Lipschitz constant depending only on the dimensions of K_1 and K_2 . $\square$

Lemma 3.4. *Let K_1, K_2 be uniform simplicial polyhedra, and let $K = K_1 \times_s K_2$. For every vertex $v \in K$, there are vertices $v_1 \in K_1$ and $v_2 \in K_2$ such that $\phi^{-1}(st_v) \subset st_{v_1} \times st_{v_2}$.* $\square$

Recall that the *nerve* $\mathcal{N}(\mathcal{U})$ of a covering $\mathcal{U} = \{U_j\}_{j \in J}$ is the simplicial complex whose vertices are the members of the covering and a collection of vertices spans a simplex if and only if the corresponding members of $\mathcal{U}$ have nonempty intersection. We assume that the nerves we consider are uniform polyhedra, unless the opposite is stated explicitly; moreover, we can regard $\mathcal{N}(\mathcal{U})$ as a subcomplex of Δ^J .

Let $\mathcal{U}$ be an open locally finite covering of a metric space Z such that no member of $\mathcal{U}$ coincides with Z . The barycentric map $p : Z \rightarrow \mathcal{N}(\mathcal{U})$ is defined as follows. For every $j \in J$ we put $q_j(z) = \text{dist}(z, Z \setminus U_j)$. Since the covering is open, $\sum_{j \in J} q_j(z) > 0$. Since no member of $\mathcal{U}$ coincides with Z and the covering is locally finite, $\sum_{j \in J} q_j(z) < \infty$ for every point $z \in Z$. Then we define the coordinate functions of the map $p : Z \rightarrow \mathbb{R}^J$ by $p_j(z) = q_j(z) / \sum_{j \in J} q_j(z)$. Clearly, $p(Z) \subset \mathcal{N}(\mathcal{U})$. For every vertex $v \in \mathcal{N}(\mathcal{U})$, the preimage $p^{-1}(st_v)$ coincides with the corresponding element of the covering.

Suppose, moreover, that $m(\mathcal{U}) \leq m + 1$ and $L(\mathcal{U}) \geq d > 0$. Then, it is not difficult to prove (see, e.g., [BS1]) that the map p is Lipschitz with Lipschitz constant

$$\text{Lip}(p) \leq (m + 2)^2 / d.$$

Lemma 3.5. *Let X_1, X_2 be metric spaces. Let $\mathcal{U}_1$ be an open covering of X_1 and $\mathcal{U}_2$ an open covering of X_2 , of multiplicities at most $n_1 + 1$ and $n_2 + 1$, respectively. Then there exists an $(n_1 + n_2 + 1)$ -colored open covering $\mathcal{W}$ of $X_1 \times X_2$ with*

$$L^\times(\mathcal{W}) \geq q(L(\mathcal{U}_1), L(\mathcal{U}_2)) \quad \text{and} \quad \text{mesh}^\times(\mathcal{W}) \leq (\text{mesh}(\mathcal{U}_1), \text{mesh}(\mathcal{U}_2)),$$

where the constant q depends only on n_1, n_2 .

Proof. In accordance with the definitions of $L^\times$ and $\text{mesh}^\times$, rescaling the metric of $X_1 \times X_2$ in one of the factors does not change the problem; i.e., if we construct a required covering for, say, $aX_1 \times X_2$, then rescaling the metric back gives a required covering for $X_1 \times X_2$.

So, by rescaling of the metric, we may assume that $L(\mathcal{U}_2) = \frac{(n_2+2)^2}{(n_1+2)^2} L(\mathcal{U}_1)$. Denote

$$\lambda := \frac{(n_1+2)^2}{L(\mathcal{U}_1)} = \frac{(n_2+2)^2}{L(\mathcal{U}_2)}.$$

Let K_i be the nerve of $\mathcal{U}_i$ and $p_i : X_i \rightarrow K_i$ the corresponding barycentric map, $i = 1, 2$. We have

$$\text{Lip}(p_i) \leq \lambda.$$

Then for the map $p : X_1 \times X_2 \rightarrow K_1 \times K_2$ defined by $p(x_1, x_2) = (p_1(x_1), p_2(x_2))$, we have $\text{Lip}(p) \leq \lambda$.

By Lemma 3.3, the barycentric triangulation map

$$\phi : K_1 \times K_2 \rightarrow K_1 \times_s K_2$$

is Lipschitz with Lipschitz constant c depending only on n_1, n_2 . Consider the covering $\mathcal{V}$ of $K_1 \times_s K_2$ by open stars of its vertices, which is $(n_1 + n_2 + 1)$ -colored. Since the polyhedron $K_1 \times_s K_2$ is uniform, the Lebesgue number of this covering is bounded from below by a positive constant l depending only on $\dim(K_1 \times_s K_2)$.

Now, let $\mathcal{W}$ be the open covering of $X_1 \times X_2$ by the preimages of the elements of $\mathcal{V}$ under the map $\phi \circ p$. Then $\mathcal{W}$ is $(n_1 + n_2 + 1)$ -colored and

$$L(\mathcal{W}) \geq \lambda^{-1} c^{-1} l = \frac{l}{c(n_1+2)^2} L(\mathcal{U}_1) = \frac{l}{c(n_2+2)^2} L(\mathcal{U}_2);$$

in particular, $L^\times(\mathcal{W}) \geq q(L(\mathcal{U}_1), L(\mathcal{U}_2))$ with $q = \frac{l}{c \max\{(n_1+2)^2, (n_2+2)^2\}}$.

By Lemma 3.4, for every $W \in \mathcal{W}$ there are $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$ with $W \subset U_1 \times U_2$, so that $\text{mesh}^\times(\mathcal{W}) \leq (\text{mesh } \mathcal{U}_1, \text{mesh } \mathcal{U}_2)$. The lemma follows. $\square$

We denote the covering described in this lemma by $\mathcal{W} = \mathcal{U} * \mathcal{V}$. In the following corollary, we use the natural extension of our notation $\text{mesh}^\times$ and $L^\times$ to the case of more than two factors.

Corollary 3.6. *Let $\mathcal{U}_i$ be an open covering with multiplicity at most $n_i + 1$ of a metric space X_i , $i \in \{1, \dots, k\}$. We let $\mathcal{W}_i = \mathcal{W}_{i-1} * \mathcal{U}_i$ be the covering of $X_1 \times \dots \times X_i$ for $i = 1, \dots, k$, where $\mathcal{W}_1 = \mathcal{U}_1$. Then $\mathcal{W} = \mathcal{W}_k$ is an open m -colored covering of $X_1 \times \dots \times X_k$ with $m = n_1 + \dots + n_k + 1$,*

$$L^\times(\mathcal{W}) \geq q(L(\mathcal{U}_1), \dots, L(\mathcal{U}_k))$$

and

$$\text{mesh}^\times(\mathcal{W}) \leq (\text{mesh}(\mathcal{U}_1), \dots, \text{mesh}(\mathcal{U}_k)),$$

where the constant q depends only on $n_1, \dots, n_k$.

Proof. Arguing as at the beginning of the above proof and rescaling the metrics of the factors, we may assume that $L(\mathcal{U}_1) = \dots = L(\mathcal{U}_k) = r$.

Note that, for a covering $\mathcal{U}$ of a product, the property $L^\times(\mathcal{U}) \geq (r, \dots, r)$ and the property $L(\mathcal{U}) \geq r$ are equivalent. Using this and proceeding by induction, we obtain $L^\times(\mathcal{W}) \geq q(r, \dots, r)$ for some constant q depending only on $n_1, \dots, n_k$.

By Lemma 3.4, for every $W \in \mathcal{W}_i$ there are $W' \in \mathcal{W}_{i-1}$ and $U \in \mathcal{U}_i$ such that $W \subset W' \times U$, for every $i = 2, \dots, k$. Consequently,

$$\text{mesh}^\times(\mathcal{W}) \leq (\text{mesh } \mathcal{U}_1, \dots, \text{mesh } \mathcal{U}_k). \quad \square$$

We shall need the following fact.

Lemma 3.7. *Let X, X_1 and X_2 be metric spaces.*

- (1) Assume that $\ell\text{-dim}(X \times [0, 1]) \leq k$. Then there exist $\tau_0 > 0$ and $\sigma > 1$ such that for every $0 < \tau < \tau_0$ and $\tau' > 0$ there exists a $(k+1)$ -colored open covering $\mathcal{U}$ of $X \times \mathbb{R}$ with $\text{mesh}^\times(\mathcal{U}) \leq (\sigma\tau, \sigma\tau')$ and $L^\times(\mathcal{U}) \geq (\tau, \tau')$.
- (2) Assume that for the spaces of the family $aX_1 \times bX_2$ we have $\ell\text{-dim} \leq k$ uniformly in $a, b \geq 1$. Then there exist $\tau_0 > 0$ and $\sigma > 1$ such that for every $0 < \tau_1, \tau_2 < \tau_0$ and $\tau'_1, \tau'_2 > 0$ there exists a $(k+3)$ -colored open covering $\mathcal{W}$ of $Z = X_1 \times X_2 \times \mathbb{R} \times \mathbb{R}$ with $\text{mesh}^\times(\mathcal{W}) \leq \sigma\tau$, $L^\times(\mathcal{W}) \geq \tau$, where $\tau = (\tau_1, \tau_2, \tau'_1, \tau'_2)$.

Proof. (1) First, we show that $\ell\text{-dim}(X \times \mathbb{R}) \leq k$. For this, we represent $X \times \mathbb{R} = Z_1 \cup Z_2$, where $Z_1 = X \times \bigcup_{k \in \mathbb{Z}} (k, k + 3/4)$, $Z_2 = X \times \bigcup_{k \in \mathbb{Z}} (k - 1/2, k + 1/4)$. Taking two appropriate qualified coverings of these sets and applying Proposition 3.2, we see that $\ell\text{-dim}(X \times \mathbb{R}) \leq k$. Then taking an appropriate covering of $X \times \mathbb{R}$ and applying a homothety in the $\mathbb{R}$ -factor, we find a desired covering.

(2) Consider the 2-colored covering $\mathcal{U} = \mathcal{U}^1 \cup \mathcal{U}^2$ of $\mathbb{R}$, where $\mathcal{U}^1 = \{(2k, 2k + 2) : k \in \mathbb{Z}\}$, $\mathcal{U}^2 = \{(2k + 1, 2k + 3) : k \in \mathbb{Z}\}$. Clearly, $L(\mathcal{U}) = 1/2$, $\text{mesh}(\mathcal{U}) = 2$.

There is a constant $\delta \in (0, 1)$ such that for every sufficiently small $\tau_1, \tau_2 > 0$ there exists a $(k+1)$ -colored open covering $\mathcal{V}$ of $X \times Y$ with $\text{mesh}^\times(\mathcal{V}) \leq (\tau_1, \tau_2)$ and $L^\times(\mathcal{V}) \geq \delta(\tau_1, \tau_2)$.

By Corollary 3.6, for the product covering $\mathcal{W} = (\mathcal{V} * \mathcal{U}) * \mathcal{U}$ of Z we have $L^\times(\mathcal{W}) \geq q(\tau_1, \tau_2, 1/2, 1/2)$ and $\text{mesh}^\times(\mathcal{W}) \leq (\tau_1, \tau_2, 2, 2)$, with some constant $q > 0$ depending only on k, δ . Applying homotheties independently in two $\mathbb{R}$ -factors and changing the constants, we obtain a desired covering. $\square$

3.3. Locally self-similar spaces. Let $\lambda \geq 1$ and $R > 0$ be given. A map $f : Z \rightarrow Z'$ between metric spaces is λ -quasihomothetic with coefficient R if for all $z, z' \in Z$ we have

$$R|zz'|/\lambda \leq |f(z)f(z')| \leq \lambda R|zz'|.$$

A metric space Z is *locally self-similar* if there is $\lambda \geq 1$ such that for every sufficiently large $R > 1$ and every $A \subset Z$ with $\text{diam } A \leq 1/R$, there is a λ -quasihomothetic map $f : A \rightarrow Z$ with coefficient R .

In [BL] it was proved that the linearly controlled dimension $\ell\text{-dim } Z$ of every compact locally self-similar metric space Z is finite and coincides with $\dim Z$, $\ell\text{-dim } Z = \dim Z$.

We shall use the following facts, obviously implied by the definition of a quasihomothetic map.

Lemma 3.8. Let $h : Z \rightarrow Z'$ be a λ -quasihomothetic map with coefficient R . Let $V \subset Z$ and let $\tilde{\mathcal{U}}$ be an open covering of $h(V)$, $\mathcal{U} = h^{-1}(\tilde{\mathcal{U}})$. Then:

- (1) $R \text{mesh}(\mathcal{U})/\lambda \leq \text{mesh}(\tilde{\mathcal{U}}) \leq \lambda R \text{mesh}(\mathcal{U})$;
- (2) $\lambda R \cdot L(\mathcal{U}) \geq L(\tilde{\mathcal{U}}) \geq R \cdot L(\mathcal{U})/\lambda$, where $L(\mathcal{U})$ is the Lebesgue number of $\mathcal{U}$ as a covering of V . $\square$

Proposition 3.9. If compact metric spaces X, Y are locally self-similar, then for the spaces of the family $aX \times bY$ we have $\ell\text{-dim} \leq n$, $n = \dim(X \times Y)$, uniformly in $a, b \geq 1$.

Proof. By the remark above, $\ell\text{-dim } X = N$ is finite. Furthermore, the space $Z = X \times Y$ is compact and locally self-similar. Thus, $\ell\text{-dim } Z = n$ is finite.

It suffices to prove that there is a constant $\delta \in (0, 1)$ such that for every sufficiently small $\tau > 0$ and every $\alpha, \beta \in (0, 1]$ there exist $(n+1)$ -colored open coverings $\mathcal{U}, \mathcal{U}'$ of Z with $\text{mesh}^\times(\mathcal{U}) \leq (\alpha\tau, \tau)$, $L^\times(\mathcal{U}) \geq \delta(\alpha\tau, \tau)$, $\text{mesh}^\times(\mathcal{U}') \leq (\tau, \beta\tau)$, and $L^\times(\mathcal{U}') \geq \delta(\tau, \beta\tau)$.

Let $\delta_1, \delta_2, \delta_3$ be the constants occurring in the definition of $\ell\text{-dim}$ for X, Y, Z , respectively, and let $\delta' = \min\{\delta_1, \delta_2, \delta_3\}$. We may assume that the constant λ in the definition of self-similarity serves both X and Y .

We explain how to construct the covering $\mathcal{U}$; the covering $\mathcal{U}'$ is constructed similarly. Fix a positive $\tau < \min\{\delta'/\lambda, \text{diam } Y\}$. For every $\alpha \in (0, 1]$, we construct a covering with $\text{mesh}^\times \leq (\alpha\tau, \tau)$ and $L^\times \geq \delta(\alpha\tau, \tau)$, where $\delta = (\delta'/4\lambda^2)^{N+1}/2$.

The argument is similar to that in [BL, Theorem 1.1]. We fix an $(N+1)$ -colored open covering $\mathcal{V}''$ of X with $\text{mesh}(\mathcal{V}'') \leq \alpha/\lambda$, $L(\mathcal{V}'') \geq \delta'\alpha/\lambda$ and put $\mathcal{V} = \{V \times Y : V \in \mathcal{V}''\}$. Then $\mathcal{V}$ is an $(N+1)$ -colored open covering of Z with $L(\mathcal{V}) \geq \delta'\alpha/\lambda$, and we assume that $\mathcal{V}$ is colored by the set $S = \{0, \dots, N\}$, $\mathcal{V} = \bigcup_{a \in S} \mathcal{V}^a$.

For every $V \in \mathcal{V}$, consider the slightly smaller subset $V' = B_{-\delta'\alpha/2\lambda}(V)$. Then, the sets $Z_a = \bigcup_{V \in \mathcal{V}^a} V' \subset Z$, $a \in S$, cover Z , $Z = \bigcup_{a \in S} Z_a$, because $L(\mathcal{V}) \geq \delta'\alpha/\lambda$. The idea is to construct a family of open $(n+1)$ -colored coverings $\mathcal{U}_a$ of Z_a , $a \in S$, which is separated with and qualified by the scale function $i : S \rightarrow \{0, \dots, N\}$, $i(a) = N - a$, and then to construct the desired $\mathcal{U}$ with the help of Proposition 3.2. The construction of $\mathcal{U}_a$ is based on the self-similarity of X .

Using the fact that $\ell\text{-dim}(X \times Y) = n$ and assuming that τ is sufficiently small, for every $a \in S$ we can find an $(n+1)$ -colored covering $\tilde{\mathcal{U}}_a$ of $X \times Y$ with $\text{mesh}(\tilde{\mathcal{U}}_a) \leq (\delta'/4\lambda^2)^a \tau/4$ and $L(\tilde{\mathcal{U}}_a) \geq \delta'(\delta'/4\lambda^2)^a \tau/4$.

Given $V \in \mathcal{V}$, we fix a map $h_V = (h_V^1, \text{id}) : V \rightarrow X \times Y$, where h_V^1 is λ -quasihomothetic with coefficient $R = \lambda/\alpha$, and put $\tilde{V} = h_V(V')$. Now, for every $a \in S$, $V \in \mathcal{V}^a$, consider the family $\tilde{\mathcal{U}}_{a,V} = \{\tilde{U} \in \tilde{\mathcal{U}}_a : \tilde{V} \cap \tilde{U} \neq \emptyset\}$, which is obviously an $(n+1)$ -colored covering of $\tilde{V}$. Then

$$\mathcal{U}_{a,V} = \{h_V^{-1}(\tilde{U}) : \tilde{U} \in \tilde{\mathcal{U}}_{a,V}\}$$

is an open $(n+1)$ -colored covering of V' .

Note that $U = h_V^{-1}(\tilde{U}) \subset V$ for every $\tilde{U} \in \tilde{\mathcal{U}}_{a,V}$, because $\text{mesh}^\times \tilde{U} \leq (\tau/4, \tau/4)$. Therefore, $\text{mesh}^\times U \leq (\alpha\tau/4, \tau/4)$ by Lemma 3.8, whence $U \subset B_{(\alpha\tau/2, \text{diam } Y)}(V') \subset V$ by the choice of τ . Thus, the family $\mathcal{U}_{a,V}$ is contained in V . Now, the family $\mathcal{U}_a = \bigcup_{V \in \mathcal{V}^a} \mathcal{U}_{a,V}$ covers the set Z_a of the color a , and it has the following properties:

- (1) for every $a \in S$, the covering $\mathcal{U}_a$ is $(n+1)$ -colored;
- (2) $L(\mathcal{U}_a)^\boxtimes \geq 4 \text{mesh}^\boxtimes \mathcal{U}_{a+1}$ for every $a \in S$, $a \leq N-1$;
- (3) $\text{mesh}^\times(\bigcup_{a \in S} \mathcal{U}_a) \leq (\alpha\tau, \tau)$, $L^\times(\bigcup_{a \in S} \mathcal{U}_a) \geq (\delta'/4\lambda^2)^{N+1}(\alpha\tau, \tau)$.

Indeed, since distinct $V_1, V_2 \in \mathcal{V}^a$ are disjoint, any $U_1 \in \mathcal{U}_{a,V_1}$, $U_2 \in \mathcal{U}_{a,V_2}$ are disjoint because $U_1 \subset V_1$, $U_2 \subset V_2$. This proves (1).

Lemma 3.8 shows that $\text{mesh}^\times(\mathcal{U}_a) \leq \text{mesh}(\tilde{\mathcal{U}}_a)(\alpha, 1)$ and $L^\times(\mathcal{U}_a) \geq L(\tilde{\mathcal{U}}_a)(\alpha/\lambda^2, 1/\lambda^2)$ for every $a \in S$. These estimates together with the estimates on $\text{mesh}(\tilde{\mathcal{U}}_a)$ and $L(\tilde{\mathcal{U}}_a)$ yield (3), and together with the inequalities

$$4\lambda^2 \text{mesh}(\tilde{\mathcal{U}}_{a+1}) \leq \delta'(\delta'/4\lambda^2)^a \tau/4 \leq L(\tilde{\mathcal{U}}_a)$$

for every $a \in S$, $a \leq N-1$, yield (2).

In view of (1) and (2), the family of coverings $\mathcal{U}_a$, $a \in S$, satisfies the assumption of Proposition 3.2. Applying that proposition and using (3), we obtain an open $(n+1)$ -colored covering $\mathcal{U}$ of $X \times Y$ with $\text{mesh}^\times(\mathcal{U}) \leq (\alpha\tau, \tau)$ and $L^\times(\mathcal{U}) \geq \delta(\alpha\tau, \tau)$, where $\delta = (\delta'/4\lambda^2)^{N+1}/2$. $\square$

3.4. “Hyperbolic cone” map. Let Z be a bounded metric space. Assuming that $\text{diam } Z > 0$, we put $\mu = \pi/\text{diam } Z$ and note that $\mu|zz'| \in [0, \pi]$ for every $z, z' \in Z$. Recall that the hyperbolic cone $\text{Co}(Z)$ over Z is the space $Z \times [0, \infty)/Z \times \{0\}$ with metric defined as follows. Given $x = (z, t)$, $x' = (z', t') \in \text{Co}(Z)$, we consider a triangle $\overline{ox}\overline{x'} \subset \mathbb{H}^2$ with $|\overline{ox}| = t$, $|\overline{ox'}| = t'$, and the angle $\angle_{\overline{ox}}(\overline{ox}, \overline{ox'}) = \mu|zz'|$. Now, we put $|xx'| := |\overline{xx'}|$. In the degenerate case $Z = \{\text{pt}\}$, we define $\text{Co}(Z) = \{\text{pt}\} \times [0, \infty)$ as the metric product. The point $o = Z \times \{0\} \in \text{Co}(Z)$ is called the *vertex* of $\text{Co}(Z)$.

We let $h : Z \times [0, \infty) \rightarrow \text{Co}(Z)$ be the canonical projection, and $\delta = \delta_{H^2}$ the hyperbolicity constant of H^2 .

It is well known that the hyperbolic cone $\text{Co}(Z)$ is a hyperbolic space satisfying the δ -inequality with respect to the vertex o . Furthermore, there is a canonical inclusion $Z \subset \partial_\infty \text{Co}(Z)$, and the metric of Z is visual with respect to the base point o and the parameter e ,

$$e^{-(\xi|\xi')_o - c_0} \leq d(\xi, \xi') \leq e^{-(\xi|\xi')_o + c_0}$$

for some $c_0 \geq 0$ and all $\xi, \xi' \in Z$. In general, $\text{Co}(Z)$ is not geodesic; however, for every point $z \in \text{Co}(Z)$ there is a geodesic segment $oz \subset \text{Co}(Z)$, and every $\xi \in Z$ is represented by a geodesic ray $o\xi \subset \text{Co}(Z)$.

The following lemma is similar to [BoS, Lemma 5.1].

Lemma 3.10. *Let X be a δ -hyperbolic space. Given $o, z_1, z_2 \in X$ and $x_1 \in oz_1, x_2 \in oz_2$, we have*

$$\min\{|ox_1|, |ox_2|, (z_1|z_2)_o\} - 2\delta \leq (x_1|x_2)_o \leq \min\{|ox_1|, |ox_2|, (z_1|z_2)_o\} + 2\delta.$$

Proof. Applying the δ -inequality twice, we obtain

$$\begin{aligned} (z_1|z_2)_o &\geq \min\{(z_1|x_1)_o, (x_1|x_2)_o, (x_2|z_2)_o\} - 2\delta \\ &= \min\{|ox_1|, (x_1|x_2)_o, |ox_2|\} - 2\delta = (x_1|x_2)_o - 2\delta. \end{aligned}$$

Similarly, $(x_1|x_2)_o \geq \min\{|ox_1|, (z_1|z_2)_o, |ox_2|\} - 2\delta$. The lemma follows. $\square$

Corollary 3.11. *Let X be a δ -hyperbolic space. Suppose that points $x_1 \in o\xi_1, x_2 \in o\xi_2$ satisfy $|x_1o| = |x_2o|$ and $|x_1x_2| > 4\delta$, where $\xi_1, \xi_2 \in \partial_\infty X$. Then*

$$(x_1|x_2)_o - 2\delta \leq (\xi_1|\xi_2)_o \leq (x_1|x_2)_o + 2\delta.$$

Proof. In this case $(x_1|x_2)_o < |ox_1| - 2\delta = |ox_2| - 2\delta$. Hence, by the preceding lemma, $\min\{|ox_1|, |ox_2|, (z_1|z_2)_o\} = (z_1|z_2)_o$ for each $z_1 \in (x_1, \xi_1), z_2 \in (x_2, \xi_2)$. Now, the statement follows from that lemma. $\square$

We put $c = c_0 + 2\delta$.

Claim 1. For every $R > 0$, the map $h|_{Z \times \{R\}} : Z \times \{R\} \rightarrow \text{Co}(Z)$ possesses the following properties:

- (1) $|h(z, R)h(z', R)| \geq 2D$ for every $D > 0$ and every $z, z' \in Z$ with $|zz'| \geq e^{-R+D+c}$;
- (2) $|h(z, R)h(z', R)| \leq 2D$ for every $D > 2\delta$ and every $z, z' \in Z$ with $|zz'| \leq e^{-R+D-c}$.

Proof. (1) We identify $z = \xi, z' = \xi' \in \partial_\infty \text{Co}(Z)$ and denote $h(z, R) = x, h(z', R) = x'$. Thus,

$$e^{-R+D+c} \leq |\xi\xi'| \leq e^{-(\xi|\xi')_o + c_0},$$

and we obtain $(\xi|\xi')_o \leq R - D - 2\delta$. By Lemma 3.10,

$$(x|x')_o \leq \min\{R, (\xi|\xi)_o\} + 2\delta \leq R - D.$$

Consequently, $|xx'| = 2(R - (x|x')_o) \geq 2D$.

(2) Using the notation as above, we obtain $e^{-(\xi|\xi')_o - c_0} \leq |\xi\xi'| \leq e^{-R+D-c}$. Hence, $(\xi|\xi')_o \geq R - D + 2\delta$. In the case where $|xx'| \leq 4\delta$, we see that $|xx'| < 2D$ by the assumption $D > 2\delta$. If $|xx'| > 4\delta$, we apply Corollary 3.11, obtaining $(x|x')_o \geq (\xi|\xi')_o - 2\delta \geq R - D$. Thus, $|xx'| = 2(R - (x|x')_o) \leq 2D$. $\square$

The proof of the following claim is similar, and we leave it to the reader as an exercise. We only mention that at some point one should use the monotonicity of the Gromov product, $(x|x')_o \leq (\xi|\xi')_o$ for every $x \in o\xi, x' \in o\xi'$.

Claim 2. For every $R > 0$, the map $h|_{Z \times \{R\}} : Z \times \{R\} \rightarrow \text{Co}(Z)$ possesses the following properties:

- (1) $|zz'| \geq e^{-R+D-c}$ for every $D > 2\delta$ and every $x = h(z, R)$, $x' = h(z', R)$ with $|xx'| \geq 2D$;
- (2) $|zz'| \leq e^{-R+D+c}$ for every $D > 0$ and every $x = h(z, R)$, $x' = h(z', R)$ with $|xx'| \leq 2D$.

□

Claim 3. (1) Given $l > 0$, $R \geq R_1 \geq l$, put $r = e^{-R_1+3l/2+c}$. Then

$$h(B_r(x) \times [R-l, R+l]) \supset B_l(h(x, R)).$$

(2) Given $l > 2\delta$, $t > 0$, $R_2 \geq R \geq t$, put $r = e^{-R_2+l-c}$. Then

$$h(B_r(x) \times [R-t, R+t]) \subset B_{3t+2l}(h(x, R)).$$

Proof. (1) Given $y' \in \text{Co}(Z)$ with $|y'h(x, R)| < l$, we represent $y' = h(y, R_0)$. Using the definition of the distances in $\text{Co}(Z)$, we see that $R_0 \in [R-l, R+l]$ and $|h(y, R')h(x, R')| < l$, where $R' = \min\{R_0, R\}$. Then, by Claim 2(2), we have

$$|yx| \leq e^{-R'+l/2+c} \leq e^{-R+l/2+c} \leq e^{-R_1+3l/2+c}.$$

Claim 3(1) follows.

(2) Assume that $(y, R_0) \in B_r(x) \times [R-t, R+t]$. Then $R_0 \in [R-t, R+t]$ and $|yx| < e^{-R_2+l-c} \leq e^{-R_0+t+l-c}$. By Claim 1(2), we have $|h(y, R_0)h(x, R_0)| < 2t + 2l$. By the triangle inequality,

$$|h(x, R)h(y, R_0)| < 3t + 2l.$$

Claim 3(2) follows. □

Claim 4. (1) Given $R_1 \geq R > 0$, $l > 8\delta$, put $r_1 = e^{-R_1-l/4-c}$. Then $h^{-1}(B_l(h(x, R))) \supset B_{r_1}(x) \times [R-l/2, R+l/2]$.

(2) Given $l > 0$, $R \geq R_2 \geq l$, put $r_2 = e^{-R_2+3l/2+c}$. Then $h^{-1}(B_l(h(x, R))) \subset B_{r_2}(x) \times [R-l, R+l]$.

Proof. (1) Assume that $(y, R_0) \in B_{r_1}(x) \times [R-l/2, R+l/2]$. Then $R_0 \in [R-l/2, R+l/2]$ and $|xy| < e^{-R_1-l/4-c} \leq e^{-R_0+l/4-c}$. Using Claim 1(2), we obtain $|h(y, R_0)h(x, R_0)| < l/2$. Then $|h(y, R_0)h(x, R)| < l$ by the triangle inequality. Claim 4(1) follows.

(2) Claim 4(2) follows from Claim 3(1). □

§4. PROOFS

Proposition 4.1. *Let Z_1, Z_2 be bounded metric spaces such that for the spaces of the family $\{aZ_1 \times bZ_2 : a \geq 1, b \geq 1\}$ we have $\ell\text{-dim} \leq k$ uniformly. Then*

$$\text{asdim}(\text{Co}(Z_1) \times \text{Co}(Z_2)) \leq k + 2.$$

We need some preparation for the proof. We denote

$$\begin{aligned} \overline{P_0}(T) &= Z_1 \times [0, T] \times Z_2 \times [0, T], \\ \overline{P_1}(T) &= Z_1 \times [T, \infty] \times Z_2 \times [0, T], \\ \overline{P_2}(T) &= Z_1 \times [0, T] \times Z_2 \times [T, \infty], \\ \overline{P_3}(T) &= Z_1 \times [T, \infty] \times Z_2 \times [T, \infty], \end{aligned}$$

and $P_i = (h_1, h_2)(\overline{P_i})$, where $h_i : Z_i \times [0, \infty) \rightarrow \text{Co } Z_i$ is the canonical projection, $i = 1, 2$.

Lemma 4.2. *For every $L > 0$ there exists T_3 and $M > 0$ such that for every $T > T_3$ there exists an $(L, M, k+3)$ -covering of $P_3(T)$.*

Proof. Let $\sigma > 1$ be the constant in Lemma 3.7(2). We fix a constant $L > 0$, put $H = 512\sigma^4 L$, and for any integers $m, n \geq 0$ consider the product

$$A_{m,n} = Z_1 \times Z_2 \times [Hm, Hm + H] \times [Hn, Hn + H].$$

Fixing some $T > 0$, first we construct a $(k+3)$ -colored covering $\mathcal{U}$ of $\overline{P_3}(T)$ such that for all sufficiently large integers m, n we have

$$L^\times(\mathcal{U})|_{A_{m,n}} \geq (e^{-Hm+3L/2+c}, e^{-Hn+3L/2+c}, L, L),$$

$$\text{mesh}^\times(\mathcal{U})|_{A_{m,n}} \leq (e^D e^{-Hm}, e^D e^{-Hn}, H/4, H/4)$$

for some $D > 0$, where $c > 0$ is the constant defined before Claim 1.

It is convenient to denote $c_1 = 2e^{3L/2+c}$, $b = e^{-H}$. Using Lemma 3.7(2), we fix N so large that for all integers $m, n \geq N$ and every $i = 0, 1, 2, 3$ there exists a $(k+3)$ -colored covering $\tilde{\mathcal{U}}_{m,n}^i$ of $Z_1 \times Z_2 \times \mathbb{R} \times \mathbb{R}$ with $L^\times(\tilde{\mathcal{U}}_{m,n}^i) \geq (4\sigma)^i (c_1 b^{m-i}, c_1 b^{n-i}, 2L, 2L)$ and $\text{mesh}^\times(\tilde{\mathcal{U}}_{m,n}^i) \leq \sigma(4\sigma)^i (c_1 b^{m-i}, c_1 b^{n-i}, 2L, 2L)$. For $m, n \geq N$, we consider the subfamily $\mathcal{U}_{m,n}^i$ of $\tilde{\mathcal{U}}_{m,n}^i$ consisting of all members that intersect $A_{m,n}$.

Next, we define a function $i : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1, 2, 3\}$ by $i(m, n) := m \bmod 2 + 2(n \bmod 2)$ and put $\mathcal{U}_{m,n} := \mathcal{U}_{m,n}^{i(m,n)}$. We show that the family of coverings $\mathcal{U}_{m,n}$ of $A_{m,n}$, $m, n \geq N$, is separated with and qualified by the scale function i .

We have $\sigma(4\sigma)^3 2L = H/4 < H/2$ by the choice of the constants. The estimate on $\text{mesh}^\times(\tilde{\mathcal{U}}_{m,n}^i)$ implies that any member $U \in \mathcal{U}_{m,n}$ is disjoint with any $U' \in \mathcal{U}_{m',n'}$ if $i(m, n) = i(m', n')$ and $(m, n) \neq (m', n')$. That is, the family $\mathcal{U}_{m,n}$, $m, n \geq N$, is separated with the scale function i . In particular, if $U \cap U' \neq \emptyset$ for some $U \in \mathcal{U}_{m,n}$, $U' \in \mathcal{U}_{m',n'}$, then the pairs (m, n) and (m', n') are adjacent; i.e., $|m' - m| \leq 1$ and $|n' - n| \leq 1$.

Assume that the pairs (m', n') , (m, n) are adjacent and $i(m', n') > i(m, n)$. We have $(4\sigma)^{i+1} 2L = 4\sigma(4\sigma)^i 2L$ and $(4\sigma)^{i+1} c_1 b^{m-(i+1)} > 4\sigma(4\sigma)^i c_1 b^{m-i}$, because $b < 1$. These inequalities together with the estimates on $L^\times(\tilde{\mathcal{U}}_{m,n}^i)$ and $\text{mesh}^\times(\tilde{\mathcal{U}}_{m,n}^i)$ show that

$$L^\boxtimes(\mathcal{U}_{m',n'}) \geq 4 \text{mesh}^\boxtimes(\mathcal{U}_{m,n}).$$

It follows that these coverings are qualified by the scale function i . Applying Proposition 3.2 to this family of coverings, we obtain a $(k+3)$ -colored covering $\mathcal{U}$ of $\overline{P_3}(T)$, $T = HN$, for which

$$\begin{aligned} L^\times(\mathcal{U})|_{A_{m,n}} &\geq \frac{1}{2}(c_1 e^{-Hm}, c_1 e^{-Hn}, 2L, 2L) \\ &= (e^{-Hm+3L/2+c}, e^{-Hn+3L/2+c}, L, L), \\ \text{mesh}^\times(\mathcal{U})|_{A_{m,n}} &\leq \sigma(4\sigma)^3 (c_1 e^{-H(m-i(m,n))}, c_1 e^{-H(n-i(m,n))}, 2L, 2L) \\ &\leq (e^D e^{-Hm}, e^D e^{-Hn}, H/4, H/4) \end{aligned}$$

for some $D > 0$.

Now, we estimate the Lebesgue number and the mesh of the covering $\mathcal{U}' = (h_1, h_2)(\mathcal{U})$. Applying Claim 3(1) with $l = L$, $R_1 = Hm$, $R \in [Hm, Hm + H]$ (respectively, $R_1 = Hn$, $R \in [Hn, Hn + H]$) to the map h_1 (respectively, h_2), we obtain $L(\mathcal{U}') \geq L$.

Next, we represent the upper estimate for $\text{mesh}^\times(\mathcal{U})|_{A_{m,n}}$ as follows:

$$\text{mesh}^\times(\mathcal{U})|_{A_{m,n}} \leq (e^{-(Hm+H)+H+D+c-c}, e^{-(Hn+H)+H+D+c-c}, H/4, H/4).$$

Now, applying Claim 3(2) with $l = H + D + c$ (note that $l > 2\delta = 2\delta_{H^2}$), $t = H/4$, $R_2 = Hm + H$, $R \in [Hm, Hm + H]$ (respectively, $R_2 = Hn + H$, $R \in [Hn, Hn + H]$) to the map h_1 (respectively, h_2), we obtain $\text{mesh}(\mathcal{U}') \leq M = 2(3H/4 + 2(H + D + c))$.

It follows that $\mathcal{U}'$ is an $(L, M, k+3)$ -covering of $P_3(T)$. $\square$

Lemma 4.3. *Given $L > 0$, there is T_2 such that for every $T > T_2$ there exist $(L, M, k+2)$ -coverings of $P_1(T)$, $P_2(T)$ with $M > 0$ depending on T , L .*

Proof. The proof is similar to that of Lemma 4.2. For simplicity, we construct a desired covering of $P_1(T)$. For $P_2(T)$ the construction is the same with obvious modifications.

We fix $L > 0$ and put $H = 40\sigma^2 L$. Let σ be the constant in Lemma 3.7(1). For every integer $m \geq 0$, consider the product

$$A_m = Z_1 \times Z_2 \times [Hm, Hm + H] \times [0, T].$$

For some $T > 0$, first we construct a $(k+2)$ -colored covering $\mathcal{U}$ of $\overline{P_1}(T)$ such that for all $m \geq N$ we have

$$\begin{aligned} L(\mathcal{U})|_{A_m} &\geq (e^{-Hm+3L/2+c}, \text{diam } Z_2, L, L), \\ \text{mesh}(\mathcal{U})|_{A_m} &\leq (e^D e^{-Hm}, \text{diam } Z_2, H/4, T + L) \end{aligned}$$

for some $D > 0$, where $c = c_0 + 2\delta$ is the constant introduced before Claim 1.

For convenience, we put $c_1 = 2e^{3L/2+c}$, $b = e^{-H}$. Since Z_1 is isometrically embedded into $Z_1 \times Z_2$, we have $\ell\text{-dim } Z_1 \leq \ell\text{-dim}(Z_1 \times Z_2) \leq k$, whence $\ell\text{-dim}(Z_1 \times [0, 1]) \leq k+1$. Then, using Lemma 3.7(1), we can find N so large that for every integer $m \geq N$ and every $i = 0, 1$ there exists a $(k+2)$ -colored covering $\tilde{\mathcal{U}}_m^i$ of $Z_1 \times \mathbb{R}$ with

$$\begin{aligned} L^\times(\tilde{\mathcal{U}}_m^i) &\geq (4\sigma)^i (c_1 b^{m-i}, 2L), \\ \text{mesh}^\times(\tilde{\mathcal{U}}_m^i) &\leq \sigma(4\sigma)^i (c_1 b^{m-i}, 2L). \end{aligned}$$

We take its subfamily $\mathcal{U}_m^i$ consisting of all members that intersect $\hat{A}_m = Z_1 \times [Hm, Hm + h]$.

We define a function $i : \mathbb{N} \rightarrow \{0, 1\}$ by $i(m) := m \bmod 2$ and put $\mathcal{U}_m := \mathcal{U}_m^{i(m)}$. Now, we show that the family of the coverings $\mathcal{U}_m$ of $\hat{A}_m$, $m \geq N$, is separated with and qualified by the scale function i .

We have $\sigma(4\sigma)2L < H/4$ by the choice of the constants. The estimate on $\text{mesh}^\times(\tilde{\mathcal{U}}_m^i)$ implies that any member $U \in \mathcal{U}_m$ is disjoint with any $U' \in \mathcal{U}_{m'}$ if $i(m) = i(m')$ and $m \neq m'$. That is, the family $\mathcal{U}_m$, $m \geq N$, is separated with the scale function i . In particular, if $U \cap U' \neq \emptyset$ for some $U \in \mathcal{U}_m$, $U' \in \mathcal{U}_{m'}$, then $|m' - m| \leq 1$.

Assume that $|m' - m| = 1$ and $i(m') > i(m)$; i.e., $i(m') = 1, i(m) = 0$. Using the estimates on $L^\times(\tilde{\mathcal{U}}_m^i)$ and $\text{mesh}^\times(\tilde{\mathcal{U}}_m^i)$, and recalling that $b < 1$, we obtain

$$L^\boxtimes(\mathcal{U}_{m'}) \geq 4 \text{mesh}^\boxtimes(\mathcal{U}_m).$$

Thus, these coverings are qualified by the scale function i . So, we apply Proposition 3.2 to the family of the coverings $\mathcal{U}_m$, $m \geq N$, and obtain a $(k+2)$ -colored covering $\mathcal{U}'$ of $Z_1 \times [T, \infty)$, $T = HN$, with

$$\begin{aligned} L^\times(\mathcal{U}')|_{A_m} &\geq \frac{1}{2}(c_1 e^{-Hm}, 2L) = (e^{-Hm+3L/2+c}, L), \\ \text{mesh}^\times(\mathcal{U}')|_{A_m} &\leq 4\sigma^2(c_1 e^{-H(m-i(m))}, 2L) \leq (e^D e^{-Hm}, H/4) \end{aligned}$$

for some $D > 0$.

Now, the family $\mathcal{U} = \{U \times Z_2 \times [0, T + L] : U \in \mathcal{U}'\}$ is a $(k+2)$ -colored covering of $\overline{P_1}(T)$ that satisfies the required estimates on $L^\times$ - and $\text{mesh}^\times$ -numbers.

We estimate the Lebesgue number and the mesh of the covering $\mathcal{U}'' = (h_1, h_2)(\mathcal{U})$. Applying Claim 3(1) with $l = L$, $R_1 = Hm$, and $R \in [Hm, Hm + H]$ to the map h_1 , we obtain $L(\mathcal{U}'') \geq L$.

Next, we represent the upper estimate for $\text{mesh}^\times(\mathcal{U})|_{A_m}$ as follows:

$$\text{mesh}^\times(\mathcal{U})|_{A_m} \leq (e^{-(Hm+H)+H+D+c-c}, \text{diam } Z_2, H/4, T + L).$$

Applying Claim 3(2) with $l = H + D + c$ (note that $l > 2\delta = 2\delta_{H^2}$), $t = H/4$, $R_2 = Hm + 2H$, and $R \in [Hm, Hm + H]$ to the map h_1 and using the fact that the diameter of $h_2(Z_2 \times [0, T + L]) \subset \text{Co}(Z_2)$ is at most $2(T + L)$, we obtain $\text{mesh}(\mathcal{U}'') \leq M = 2 \max\{3H/4 + 2(2H + D + c), T + L\}$.

It follows that $\mathcal{U}''$ is an $(L, M, k + 2)$ -covering of $P_1(T)$. $\square$

Lemma 4.4. *For every $T, L > 0$ there exists an $(L, M, 1)$ -covering of $P_0(T)$ with $M > 0$ depending only on T and L .*

Proof. The covering of $P_0(T)$ consisting of a unique element $U = h_1(Z_1 \times [0, T + 2L]) \times h_2(Z_2 \times [0, T + 2L])$ has the required properties. $\square$

Proof of Proposition 4.1. Given $L_0 > 0$, we construct a uniformly bounded covering $\mathcal{U}$ of $\text{Co}(Z_1) \times \text{Co}(Z_2)$ with $L(\mathcal{U}) \geq L_0$.

We apply Lemma 4.2 with $L = L_0$, find the corresponding constants T_3, M_3 , and let $\mathcal{U}_3$ be an $(L_0, M_3, k + 3)$ -covering of $P_3(T_3)$. We may assume that $M_3 \geq L_0$.

Next, we apply Lemma 4.3 with $L = 2M_3$ and denote by T_2 the corresponding constant. Thus, for $T_0 = \max\{T_2, T_3\}$ there is a $(2M_3, M_2, k + 2)$ -covering $\mathcal{U}_2$ of $P_2(T_0)$, where the constant $M_2 \geq 2M_3$ depends on T_0, M_3 .

Then we again apply Lemma 4.3, this time with $L = 2M_2$, and denote by T_1 the corresponding constant. Thus, for $T^* = \max\{T_0, T_1\}$ there is a $(2M_2, M_1, k + 2)$ -covering $\mathcal{U}_1$ of $P_1(T^*)$, where the constant $M_1 \geq 2M_2$ depends on T^* and M_2 .

Finally, by Lemma 4.4, there is a $(2M_1, M, 1)$ -covering $\mathcal{U}_0$ of $P_0(T^*)$, where $M > 0$ depends on M_1 and T^* .

We have

$$\text{Co}(Z_1) \times \text{Co}(Z_2) = P_0(T^*) \cup P_1(T^*) \cup P_2(T_0) \cup P_3(T_3).$$

Now we apply Lemma 3.1 consecutively to the sequence of the coverings $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$, obtaining a bounded $(k + 3)$ -colored covering $\mathcal{U}$ of $\text{Co}(Z_1) \times \text{Co}(Z_2)$ with $L(\mathcal{U}) \geq L_0$. This shows that $\text{asdim}(\text{Co}(Z_1) \times \text{Co}(Z_2)) \leq k + 2$. $\square$

Proposition 4.5. *Let Z be a bounded metric space. Then*

$$\ell\text{-asdim } \text{Co}(Z) \leq \ell\text{-dim}(Z \times [0, 1]).$$

Proof. We may assume that $\ell\text{-dim}(Z \times [0, 1]) = k$ is finite. We fix some $L > 4\delta$, $\delta = \delta_{H^2}$. Let $\tau_0 > 0$, $\sigma > 1$ be the constants in Lemma 3.7(1). We put $H = 40\sigma^2 L$ and denote

$$A_m = Z \times [Hm, Hm + H].$$

There exists $N > 0$ such that for $m \geq N$ we have $e^{-H(m-1)+3L/2+c} < \tau_0$ uniformly in $L > 4\delta$, where, as above, $c = c_0 + 2\delta$ is the constant introduced before Claim 1.

Precisely as in the proof of Lemma 4.3, we construct a $(k + 1)$ -colored covering $\mathcal{U}$ of $Z \times [HN, \infty)$ such that

$$\begin{aligned} L^\times(\mathcal{U})|_{A_m} &\geq (e^{-Hm+3L/2+c}, L), \\ \text{mesh}^\times(\mathcal{U})|_{A_m} &\leq (e^D e^{-Hm}, H/4) \end{aligned}$$

for all $m \geq N$, for some constant $D > 0$ linearly depending on L .

We have

$$\text{mesh}^\times(\mathcal{U})|_{A_m} \leq (e^{-(Hm+H)+(H+3L/2+D+c)-c}, H/4).$$

Applying Claim 3 to the canonical map $h : Z \times [0, \infty) \rightarrow \text{Co}(Z)$, we see that the covering $h(\mathcal{U}) = \mathcal{U}'$ satisfies $L(\mathcal{U}') \geq L$ and $\text{mesh}(\mathcal{U}') \leq (3H/4 + 2(H + D + 3L/2 + c))$.

Now for every $L > 4\delta$ we have the $(k + 1)$ -colored covering $\mathcal{U}'$ of $h(Z \times [CL, \infty))$ with $L(\mathcal{U}') \geq L$ and $\text{mesh}(\mathcal{U}') \leq CL$ for some constant $C \geq 1$ independent of L . Then we cover the ball $B_{CL}(o) \subset \text{Co } Z$ by the ball $\mathcal{U}'' = B_{3CL}(o)$ and apply Lemma 3.1 to $\mathcal{U}'$,

$\mathcal{U}''$ to obtain a covering $\mathcal{W}$ of $\text{Co } Z$ with multiplicity $m(\mathcal{W}) \leq k+1$ and with $L(\mathcal{W}) \geq L$ and $\text{mesh}(\mathcal{W}) \leq 6CL$. This shows that $\text{asdim } \text{Co}(Z) \leq \ell\text{-dim}(Z \times [0, 1])$. $\square$

Proposition 4.6. *Let $Z_1, \dots, Z_n$ be bounded metric spaces. Then*

$$\ell\text{-dim}(Z_1 \times \dots \times Z_n \times [0, 1]^n) \leq \text{asdim}(\text{Co } Z_1 \times \dots \times \text{Co } Z_n).$$

We reduce the proof to the following two lemmas.

Lemma 4.7. *Assume that $\text{asdim}(\text{Co } Z_1 \times \dots \times \text{Co } Z_n) \leq k$. Then for some $\bar{\varepsilon} > 0$ there is a function $\delta : [0, \bar{\varepsilon}] \rightarrow [0, 1]$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$, $\tau \in (0, 1)$ there exists a $(\delta(\varepsilon)\tau, \varepsilon\tau, k+1)$ -covering of $Z_1 \times \dots \times Z_n \times [0, \tau]^n$ by open subsets of $Z_1 \times \dots \times Z_n \times \mathbb{R}^n$.*

Proof. We put $Z = Z_1 \times \dots \times Z_n$, $X = \text{Co } Z_1 \times \dots \times \text{Co } Z_n$; let $h : Z \times \mathbb{R}^n \rightarrow X$ be the product of canonical projections. By assumption, for every $L > 0$ there exists an $(L, M, k+1)$ -covering $\mathcal{U}$ of X with some $M < \infty$. We fix such a covering for $L > 8\delta_{\text{H}^2}$, together with $\varepsilon \in (0, \bar{\varepsilon})$ and $\tau \in (0, 1)$, and put $r = -\ln(\varepsilon\tau) + 5M/2 + c$, where $c = c_0 + 2\delta_{\text{H}^2}$ is the constant defined before Claim 1.

The annulus $\text{An}_{[r, r+M/\varepsilon]} \subset X$ that consists of all $(x_1, \dots, x_n) \in X$ with $r \leq |x_i| \leq r + M/\varepsilon$, $i = 1, \dots, n$, is covered by $\mathcal{U}(\varepsilon, \tau) = \{U \in \mathcal{U} : U \cap \text{An}_{[r, r+M/\varepsilon]} \neq \emptyset\}$. We lift this covering to the covering $\mathcal{U}'(\varepsilon, \tau) = h^{-1}(\mathcal{U}(\varepsilon, \tau))$ of $Z \times [r, r + M/\varepsilon]^n$ (the latter will be identified with $Z \times [0, M/\varepsilon]^n$).

We have $\text{mesh}(\mathcal{U}(\varepsilon, \tau)) \leq M$. Applying Claim 4(2) with $l = M$, $R_2 = r - M$ (note that $R_2 \geq l$), and $R \geq R_2$, we obtain $\text{mesh}^\times \mathcal{U}'(\varepsilon, \tau) \leq (\varepsilon\tau, M)$, due to our choices. Applying Claim 4(1) with $R_1 = r + M/\varepsilon$, $l = L$, and $R \leq R_1$, we obtain

$$L^\times \mathcal{U}'(\varepsilon, \tau) \geq (e^{-(r+M/\varepsilon)-L/4-c}, L/2) = (e^{-M/\varepsilon-L/4-5M/2-2c} \varepsilon\tau, L/2),$$

the last identity being ensured by the choice of r .

Consider the homothety “along the second factor”, $Z \times [0, M/\varepsilon]^n \rightarrow Z \times [0, \tau]^n$, with coefficient $\varepsilon\tau/M$, and the image $\mathcal{V}$ of $\mathcal{U}'$ under this homothety. Then $\text{mesh}^\times(\mathcal{V}) \leq (\varepsilon\tau, \varepsilon\tau)$, $L^\times(\mathcal{V}) \geq (\delta_0(\varepsilon)\tau, \varepsilon\tau L/(2M))$ for $\delta_0(\varepsilon) = c_1 \varepsilon e^{-M/\varepsilon}$, $c_1 = e^{-L/4-5M/2-2c}$, and we obtain a covering with the required properties taking $\delta(\varepsilon) = \min\{\delta_0(\varepsilon), L\varepsilon/(2M)\}$. $\square$

Lemma 4.8. *Let Z be a metric space. Suppose that for some $\bar{\varepsilon} > 0$ there exists a function $\delta : [0, \bar{\varepsilon}] \rightarrow [0, 1]$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$, $\tau \in (0, 1)$ there is a $(\delta(\varepsilon)\tau, \varepsilon\tau, k+1)$ -covering of $Z \times [0, \tau]^n$ by open subsets of $Z \times [0, \infty)^n$. Then $\ell\text{-dim}(Z \times [0, 1]^n) \leq k$.*

Proof. It suffices to find, for sufficiently large $m \in \mathbb{N}$, a $(c/m, C/m, k+1)$ -covering of $Z \times [0, 1]^n$ with c, C depending only on Z .

We represent

$$Z \times [0, 1]^n = \bigcup_{(l_1, \dots, l_n) \in A} Z \times \left[\frac{l_1}{m}, \frac{l_1+1}{m} \right] \times \dots \times \left[\frac{l_n}{m}, \frac{l_n+1}{m} \right],$$

where $A = \{0, \dots, m-1\}^n$. We define $i : A \rightarrow S = \{0, \dots, 2^n - 1\}$ by

$$i(l_1, \dots, l_n) = (l_1 \bmod 2)2^0 + \dots + (l_n \bmod 2)2^{n-1}.$$

Let $\varepsilon_0 = \min\{1/4, \bar{\varepsilon}\}$, and let $\varepsilon_{i+1} = \delta(\varepsilon_i)/2$, $i \in S$. We fix a $(\delta(\varepsilon_i)/m, \varepsilon_i/m, k+1)$ -covering $\mathcal{U}_{(l_1, \dots, l_n)}$ of $Z \times \left[\frac{l_1}{m}, \frac{l_1+1}{m} \right] \times \dots \times \left[\frac{l_n}{m}, \frac{l_n+1}{m} \right]$, where $i = i(l_1, \dots, l_n)$. For $s \in S$, we put

$$\mathcal{U}_s = \bigcup_{(l_1, \dots, l_n) \in i^{-1}(s)} \mathcal{U}_{(l_1, \dots, l_n)}.$$

Then $\text{mesh}(\mathcal{U}_{s+1}) \leq L(\mathcal{U}_s)/2$. Proceeding by induction on $s \in S$ and applying Lemma 3.1, we obtain a $(c/m, C/m, k+1)$ -covering of $Z \times [0, 1]^n$ with $c = \delta(\varepsilon_{2^n-1})$, $C = \varepsilon_0$. $\square$

Proof of Proposition 4.6. This follows immediately from the two lemmas above. $\square$

4.1. Proof of the main results. It is known that any visual hyperbolic space X can be quasi-isometrically embedded in the hyperbolic cone over its boundary at infinity $\partial_\infty X$, and that for a geodesic hyperbolic space Y the hyperbolic cone over its boundary at infinity can be quasi-isometrically embedded in the space itself (see, e.g., [BS3]).

It follows that in these cases,

$$\begin{aligned} \operatorname{asdim} X &\leq \operatorname{asdim} \operatorname{Co}(\partial_\infty X), \\ \operatorname{asdim} Y &\geq \operatorname{asdim} \operatorname{Co}(\partial_\infty Y), \end{aligned}$$

and similar estimates are true for products of such spaces.

Now, Theorem 1.1 follows immediately from Proposition 4.1, Theorem 1.3 from Proposition 4.6, and Theorem 1.5 from Proposition 4.5.

For the proof of Corollary 1.4, we note that any Gromov hyperbolic group is a visual hyperbolic space with a compact, locally self-similar boundary at infinity; see [BL]. By Proposition 3.9, for the spaces of the family $\{a\partial_\infty \Gamma_1 \times b\partial_\infty \Gamma_2 : a, b \geq 1\}$ we have $\ell\text{-dim} \leq n$ uniformly, $n = \dim(\partial_\infty \Gamma_1 \times \partial_\infty \Gamma_2)$, for any hyperbolic groups Γ_1, Γ_2 . By Theorem 1.1,

$$\begin{aligned} \operatorname{asdim}(\Gamma_1 \times \Gamma_2) &\leq \ell\text{-dim}(\partial_\infty \Gamma_1 \times \partial_\infty \Gamma_2) + 2 \\ &= \dim(\partial_\infty \Gamma_1 \times \partial_\infty \Gamma_2) + 2, \end{aligned}$$

where the last identity follows from [BL] and the fact that $\partial_\infty \Gamma_1 \times \partial_\infty \Gamma_2$ is a compact, locally self-similar space. The reverse inequality

$$\operatorname{asdim}(\Gamma_1 \times \Gamma_2) \geq \dim(\partial_\infty \Gamma_1 \times \partial_\infty \Gamma_2) + 2$$

follows from Theorem 1.3 and the inequalities

$$\ell\text{-dim}(Z \times [0, 1]) \geq \dim(Z \times [0, 1]) = \dim Z + 1,$$

which are valid for any metric space Z .

Corollary 1.6 follows from Theorem 1.3 and Theorem 1.5.

§5. APPLICATIONS

As applications of Corollary 1.4, we have:

1) Examples of a strict inequality in the product theorem for the asymptotic dimension in the class of hyperbolic groups. Namely, in [Dr1, Dr2] it was proved that for every prime p , there is a hyperbolic Coxeter group Γ_p with a Pontryagin surface Π_p as the boundary at infinity. Then, by Corollary 1.4,

$$\begin{aligned} \operatorname{asdim}(\Gamma_p \times \Gamma_q) &= \dim(\Pi_p \times \Pi_q) + 2 \\ &< \dim \Pi_p + \dim \Pi_q + 2 \\ &= \operatorname{asdim} \Gamma_p + \operatorname{asdim} \Gamma_q \end{aligned}$$

for prime $p \neq q$ (the second identity follows from the main result of [BL]).

2) Examples of a strict inequality in the product theorem for the hyperbolic dimension (the hyperbolic dimension is a quasi-isometry invariant of metric space introduced in [BS2]). Indeed, let Γ_p and Γ_q for $p \neq q$ be as in 1). Then

$$\begin{aligned} \operatorname{hypdim}(\Gamma_p \times \Gamma_q) &\leq \operatorname{asdim}(\Gamma_p \times \Gamma_q) \\ &< \dim \Pi_p + \dim \Pi_q + 2 \\ &= \operatorname{hypdim}(\Gamma_p) + \operatorname{hypdim}(\Gamma_q) \end{aligned}$$

(the above identity also follows from the main result of [BL] and [BS2]).

3) The formula $\operatorname{asdim}(\Gamma \times \mathbb{R}) = \operatorname{asdim}(\Gamma) + 1$ for any hyperbolic group Γ .

It is known that for a visual, proper, geodesic hyperbolic space X we have

$$(*) \quad \dim(\partial_\infty X) + 1 \leq \operatorname{asdim} X \leq \ell\text{-dim}(\partial_\infty X) + 1.$$

The lower estimate is simple; the ideas of the proof are contained, e.g., in [Gr, 1.E'₁]. The upper estimate was proved in [Bu1, Bu2]. Corollary 1.6 gives a better estimate from below, making it possible to show that the first inequality in (*) may be strict. More precisely, we give an example of a hyperbolic space with asymptotic dimension arbitrarily greater than the topological dimension of its boundary at infinity. Let $Z = \{0\} \cup \{1/m : m \in \mathbb{N}\}$, and let $X = \text{Co } Z^k$. It is known that $\ell\text{-dim}(Z^k \times [0, 1]) = k + 1$; see [BL]. Then $\text{asdim } X = k + 1$, while $\dim(\partial_\infty X) = 0$. The natural conjecture is that for every compact metric space Y we have $\ell\text{-dim}(Y \times [0, 1]) = \ell\text{-dim}(Y) + 1$. If this is true, then both inequalities in (*) become identities.

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