HOCHSCHILD COHOMOLOGY OF THE INTEGRAL GROUP RING OF THE DIHEDRAL GROUP. I: EVEN CASE

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Dedicated to the centenary of D. K. Faddeev's birth

ABSTRACT. A free bimodule resolution is constructed for the integral group ring of the dihedral group of order $4m$. This resolution is applied for a description, in terms of generators and defining relations, of the Hochschild cohomology algebra of this group ring.

INTRODUCTION

Let K be a commutative ring with unity, let R be an associative K -algebra that is a finitely generated projective K -module, let $\Lambda = R^e = R \otimes_K R^{\text{op}}$ be its enveloping algebra, and let $\text{HH}^n(R) = \text{Ext}_{\Lambda}^n(R, R)$ be the n th Hochschild cohomology group of the algebra R (with coefficients in the R -bimodule R). On the Abelian group

$$\text{HH}^*(R) = \bigoplus_{n \geq 0} \text{HH}^n(R) = \bigoplus_{n \geq 0} \text{Ext}_{\Lambda}^n(R, R),$$

we introduce a structure of an associative algebra with the $\smile$ -product (see [1, §5], [2, Chapter XI], and [3]); this algebra is called the *Hochschild cohomology algebra*. The algebra $\text{HH}^*(R)$ is a graded commutative algebra, see [3]; moreover, it is well known (see, e.g., [4, p. 120]) that the $\smile$ -product on $\text{HH}^*(R)$ coincides with the Yoneda product.

In recent years, interest has grown in the investigation of the multiplicative structure of the Hochschild cohomology algebra, and appreciable success has been made in the solution of this problem for finite-dimensional algebras. In [5], a description was obtained of the Hochschild cohomology algebra for the symmetric group S_3 over the field $\mathbb{F}_3$ and for the alternating group A_4 and the dihedral 2-groups over the field $\mathbb{F}_2$. In [6], the algebra $\text{HH}^*(R)$ was described in the case where R is a self-injective Nakayama algebra. In the author's papers [7, 8], a description of the Hochschild cohomology algebra was given for algebras of dihedral type in the family $D(3\mathcal{K})$ over an algebraically closed ground field of characteristic two and for a family of local algebras of quaternion type, respectively. We recall that algebras of dihedral, semidihedral, and quaternion types appeared in the work of K. Erdmann on classification of group blocks of tame representation type (see [9]). Some partial results were obtained in the description of the algebra $\text{HH}^*(R)$ for the so-called Möbius algebra (see [10, 11]) and for group blocks of tame representation type that have one or three simple modules (see [12]).

On the other hand, the Hochschild cohomology algebra was calculated recently in [13] for the integral group ring of the generalized quaternion group (see also [14]). It should be noted that a ring isomorphism $\text{HH}^*(K[G]) \simeq H^*(G) \otimes_K K[G]$, where K is a commutative

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ring, was established in [15, 16]. But results in [13] and [7] show that the relationship between ordinary cohomology and Hochschild cohomology for noncommutative groups may turn out to be much more complicated.

In the present paper, a similar problem of description of the algebra $\mathrm{HH}^*(\mathbb{Z}[D_{4m}])$ is solved for the group ring $\mathbb{Z}[D_{4m}]$ of the “even” dihedral group of order $4m$. Our approach to the calculation of products in $\mathrm{HH}^*(R)$ differs from that of the papers [5, 13, 14] where, in calculating the $\smile$ -product in $\mathrm{HH}^*(R)$, the authors used reduction to the $\smile$ -product in the usual cohomology of groups. Namely, we apply an approach that was used earlier in [7, 8]. Its specific feature is that, on the basis of some empirical observations, a conjecture is stated about the structure of the bimodule projective resolution of the algebra R , and then, upon establishing this conjecture, the resolution is used in the calculation of the Hochschild cohomology algebra; in this way, we interpret the multiplication in $\mathrm{HH}^*(R)$ as the Yoneda product. We note that, for the first time, such a technique of calculating cohomology was used in [17], where the Yoneda algebra was described for two families of algebras of semidihedral type. The resulting description of the algebra $\mathrm{HH}^*(\mathbb{Z}[D_{4m}])$ can be used in the further investigation of the relationship between the Hochschild cohomology and the usual cohomology. Recall that the (usual) cohomology ring $H^*(\mathbb{Z}[D_{2t}])$ for the dihedral group D_{2t} was calculated in [18].

We outline the structure of the paper. In §1, we state the main result of the paper, Theorem 1.1: the algebra $\mathrm{HH}^*(\mathbb{Z}[D_{4m}])$ is determined in terms of generators and relations. In §2, we build some bicomplex and use the spectral sequence of the bicomplex to prove that the total complex of the bicomplex is a Λ -free resolution of the Λ -module $R = \mathbb{Z}[D_{4m}]$; here $\Lambda = R^e$. With the help of this resolution, in §3 we compute the groups $\mathrm{HH}^n(\mathbb{Z}[D_{4m}])$. Finally, in §4 we pick a (finite) set of generators for the algebra $\mathrm{HH}^*(\mathbb{Z}[D_{4m}])$ and find relations satisfied by these generators.

The author has found also a Λ -free bimodule resolution for the algebra $\mathbb{Z}[D_{4m+2}]$ (the “odd case”). This resolution differs from that in the “even case”. Because of this, the cohomology algebra $\mathrm{HH}^*(\mathbb{Z}[D_{4m+2}])$ is computed independently of the results of the present paper. These computations will be presented in the second part of this article.

§1. STATEMENT OF THE MAIN RESULT

The dihedral group D_{2t} of order $2t$ can be defined as a semidirect product of the cyclic group C_t of order $t > 2$ by the cyclic group C_2 :

$$(1.1) \quad D_{2t} = C_t \rtimes C_2 = \langle a, b \mid a^t = b^2 = 1, bab = a^{-1} \rangle.$$

In the sequel, we study only the case where t is even: $t = 2m, m \geq 2$ (the “even case”). Then the dihedral group D_{4m} admits the following description (see, e.g., [19]):

$$(1.2) \quad D_{4m} = \langle b, c \mid b^2 = c^2 = 1, (bc)^m = (cb)^m \rangle$$

(to deduce (1.2) from (1.1), it suffices to put $c = ab$). Any element of D_{4m} different from 1 and $(bc)^m$ is represented uniquely (in the “alphabet” $\{b, c\}$) by one of the following “words”:

$$(bc)^i, (cb)^i \quad (1 \leq i \leq m-1); \quad b(cb)^i, c(bc)^i \quad (0 \leq i \leq m-1).$$

Such a representation of elements of D_{4m} is said to be *standard*. The empty word 1 will be called the standard representation of the unit element of the group; and the standard representation for the element $(bc)^m = (cb)^m$ is the word $(bc)^m$. For an element x of the group algebra $\mathbb{Z}[G]$ with $G = D_{4m}$, a presentation in the form $x = \sum_{g \in G} x_g g$, where $x_g \in \mathbb{Z}$ and all elements $g \in G$ have standard representations, is called the *canonical expansion*. The canonical expansion of an element of the envelope algebra, $\lambda = \sum_{g, h \in G} \lambda(g, h) g \otimes h \in \Lambda = R^e$, is defined similarly (g and h have the standard representations and $\lambda(g, h) \in \mathbb{Z}$).

Consider the polynomials defined by the following formulas:

$$(1.3) \quad f_i(T) = \left(\frac{T + \sqrt{T^2 - 4}}{2} \right)^i + \left(\frac{T - \sqrt{T^2 - 4}}{2} \right)^i \in \mathbb{Z}[T] \quad (i \in \mathbb{N}).$$

This family of polynomials can also be introduced with the help of the recurrence formulas

$$f_0(T) := 2, \quad f_1(T) := T, \quad f_{i+1}(T) := f_i(T) \cdot T - f_{i-1}(T) \quad \text{for } i \geq 1.$$

Observe that, for $i \geq 1$, the f_i are unital polynomials with $\deg f_i = i$, so that we can write

$$(1.4) \quad f_i(T) = T^i + g_i(T)$$

for some $g_i \in \mathbb{Z}[T]$, $i \geq 1$, with $\deg g_i < i$.

Put $R_m = \mathbb{Z}[D_{4m}]$. To describe the Hochschild cohomology algebra $\mathrm{HH}^*(R_m)$, we construct auxiliary graded $\mathbb{Z}$ -algebras. Set

$$(1.5) \quad \mathcal{X} = \{p_1, p'_1, p_2, p_3, u_0, u_1, u'_1, u_2, u'_2, v, w\}.$$

On the algebra $\mathbb{Z}[\mathcal{X}]$, we introduce a grading such that

$$\begin{aligned} \deg p_1 &= \deg p'_1 = \deg p_2 = \deg p_3 = 0, \\ \deg u_0 &= \deg u_1 = \deg u'_1 = \deg u_2 = \deg u'_2 = 2, \\ \deg v &= 3, \quad \deg w = 4. \end{aligned}$$

For brevity of the further statements, we denote

$$q_i := f_i(p_2) \in \mathbb{Z}[\mathcal{X}], \quad 1 \leq i \leq m.$$

Then we define a graded $\mathbb{Z}$ -algebra $\mathcal{A}_m = \mathbb{Z}[\mathcal{X}]/I_m$, where I_m is the ideal of the algebra $\mathbb{Z}[\mathcal{X}]$ generated by the following elements:

— of degree 0:

$$(1.6) \quad p_1^2 - (p'_1)^2, \quad p_1^2 - \varkappa_1$$

with

$$(1.7) \quad \varkappa_1 = \begin{cases} m \left(1 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \right) & \text{if } m \text{ is odd,} \\ m \left(1 + p_3 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \right) & \text{if } m \text{ is even;} \end{cases}$$

$$p_1 p'_1 - \varkappa_2$$

with

$$(1.8) \quad \varkappa_2 = \begin{cases} m \left(p_3 + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} q_{2i-1} \right) & \text{if } m \text{ is odd,} \\ m \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} q_{2i-1} & \text{if } m \text{ is even;} \end{cases}$$

$$p_1 p_3 - \varkappa_3, \quad p'_1 p_3 - \varkappa'_3$$

with

$$\varkappa_3 = \begin{cases} p'_1 & \text{if } m \text{ is odd,} \\ p_1 & \text{if } m \text{ is even,} \end{cases} \quad \varkappa'_3 = \begin{cases} p_1 & \text{if } m \text{ is odd,} \\ p'_1 & \text{if } m \text{ is even;} \end{cases}$$

$$(1.9) \quad p_1 p_2 - 2p'_1, p'_1 p_2 - 2p_1, q_m - 2p_3, p_2 p_3 - q_{m-1}, p_3^2 - 1;$$

— of degree 2:

$$(1.10) \quad p_1 u_0, p'_1 u_0, 2m u_0; 2u_i, 2u'_i \text{ for } i \in \{1, 2\};$$

$$(1.11) \quad \left(1 + \sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} q_{m-1-2i}\right) u_0 \quad \text{if } m \text{ is odd,}$$

$$(1.12) \quad \sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} q_{m-1-2i} \cdot u_0 \quad \text{if } m \text{ is even;}$$

$$(1.13) \quad p_3 u_0 + \varkappa_4$$

with

$$(1.14) \quad \varkappa_4 = \begin{cases} \left(\sum_{i=0}^{\lfloor \frac{m-3}{2} \rfloor} q_{m-2-2i}\right) u_0 & \text{if } m \text{ is odd,} \\ \left(1 + \sum_{i=0}^{\lfloor \frac{m-3}{2} \rfloor} q_{m-2-2i}\right) u_0 & \text{if } m \text{ is even;} \end{cases}$$

$$p_1 u_1 - \varkappa_5, \quad p'_1 u'_1 - \varkappa'_5$$

with

$$(1.15) \quad \varkappa_5 = \begin{cases} u_2 & \text{if } m \text{ is odd,} \\ u_2 + p_3 u_2 & \text{if } m \text{ is even,} \end{cases} \quad \varkappa'_5 = \begin{cases} u'_2 & \text{if } m \text{ is odd,} \\ u'_2 + p_3 u'_2 & \text{if } m \text{ is even;} \end{cases}$$

$$p'_1 u_1 - \varkappa_6, \quad p_1 u'_1 - \varkappa'_6$$

with

$$(1.16) \quad \varkappa_6 = \begin{cases} p_3 u_2 & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even;} \end{cases} \quad \varkappa'_6 = \begin{cases} p_3 u'_2 & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even;} \end{cases}$$

$$p_2 u_1 - p_2 u'_1, p_2 u_1 - m p_3 u_0, p_2 u_2, p_2 u'_2;$$

$$(1.17) \quad p_1 u_2 - \varkappa_7, \quad p'_1 u'_2 - \varkappa'_7$$

with

$$(1.18) \quad \varkappa_7 = \begin{cases} u_1 + m \left(\left\lfloor \frac{m}{2} \right\rfloor \cdot 1 + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i} \right) u_0 & \text{if } m \text{ is odd,} \\ u_1 + p_3 u_1 + m \left(\sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i-1} \right) u_0 & \text{if } m \text{ is even;} \end{cases}$$

$$\varkappa'_7 = \begin{cases} u'_1 + m \left(\left\lfloor \frac{m}{2} \right\rfloor \cdot 1 + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i} \right) u_0 & \text{if } m \text{ is odd,} \\ u'_1 + p_3 u'_1 + m \left(\sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i-1} \right) u_0 & \text{if } m \text{ is even;} \end{cases}$$

$$p'_1 u_2 - \varkappa_8, \quad p_1 u'_2 - \varkappa'_8$$

with

$$\begin{aligned}\varkappa_8 &= \begin{cases} p_3 u_1 + m \left(\sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} \left(\lfloor \frac{m}{2} \rfloor - i \right) q_{2i+1} \right) u_0 & \text{if } m \text{ is odd,} \\ m \left(\lfloor \frac{m}{2} \rfloor \cdot 1 + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\lfloor \frac{m}{2} \rfloor - i \right) q_{2i} \right) u_0 & \text{if } m \text{ is even;} \end{cases} \\ \varkappa'_8 &= \begin{cases} p_3 u'_1 + m \left(\sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} \left(\lfloor \frac{m}{2} \rfloor - i \right) q_{2i+1} \right) u_0 & \text{if } m \text{ is odd,} \\ m \left(\lfloor \frac{m}{2} \rfloor \cdot 1 + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\lfloor \frac{m}{2} \rfloor - i \right) q_{2i} \right) u_0 & \text{if } m \text{ is even;} \end{cases}\end{aligned}$$

— of degree 3:

$$(1.19) \quad 2v, \ p_2 v;$$

— of degree 4:

$$(1.20) \quad 2mw, \ 2p_1 w, \ 2p'_1 w, \ u_0 u_2, \ u_0 u'_2, \ u_0^2 - (2 - q_2)w;$$

$$(1.21) \quad u_0 u_1 - u_0 u'_1, \ u_0 u_1 - m q_{m-1} w, \ u_1 u'_1 - \varkappa_9$$

with

$$\begin{aligned}\varkappa_9 &= \begin{cases} mw & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even;} \end{cases} \\ (1.22) \quad &u_1 u'_2 - \varkappa_{10}, \quad u'_1 u_2 - \varkappa'_{10}\end{aligned}$$

with

$$\begin{aligned}\varkappa_{10} &= \begin{cases} p'_1 w & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even,} \end{cases} \quad \varkappa'_{10} = \begin{cases} p_1 w & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even;} \end{cases} \\ (1.23) \quad &u_2^2 - u_1^2 - (u'_2)^2 + (u'_1)^2, \ u_2^2 - u_1^2 - \varkappa_{11}\end{aligned}$$

with

$$\begin{aligned}\varkappa_{11} &= \begin{cases} m \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \cdot w & \text{if } m \text{ is odd,} \\ m \left(1 + p_3 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \right) w & \text{if } m \text{ is even;} \end{cases} \\ (1.24) \quad &u_2 u'_2 - \varkappa_{12}\end{aligned}$$

with

$$\varkappa_{12} = \begin{cases} m \left(p_3 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i-1} \right) w & \text{if } m \text{ is odd,} \\ m \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i+1} \cdot w & \text{if } m \text{ is even;} \end{cases}$$

— of degree 5:

$$(1.25) \quad u_0 v;$$

— of degree 6:

$$(1.26) \quad v^2 - (u_1 + u'_1)w.$$

Moreover, we induce a grading on the algebra $\mathcal{A}_m$ from the grading on $\mathbb{Z}[\mathcal{X}]$.

Theorem 1.1. *Let $R = \mathbb{Z}[D_{4m}]$, where $m \in \mathbb{N} \setminus \{1\}$. The Hochschild cohomology algebra $\mathrm{HH}^*(R)$ is isomorphic, as a graded algebra, to the algebra $\mathcal{A}_m$.*

Remark 1.2. We observe that the above description of the algebra $\mathrm{HH}^*(\mathbb{Z}[D_{4m}])$ implies immediately that this algebra is commutative.

Remark 1.3. The defining relations (1.6)–(1.26) of the algebra $\mathcal{A}_m$ imply that its description can be simplified with regard to the parity of m (for example, if m is odd, then in the generating set $\mathcal{X}$ we can omit the elements p'_1, u_2, u'_2). The above statement of Theorem 1.1 is convenient for unifying the arguments in the proofs. But at the end of the paper, for the reader's convenience, we state Corollaries 4.11 and 4.12 in which we take into account the possibilities of simplifying the description of the cohomology algebra $\mathrm{HH}^*(\mathbb{Z}[D_{4m}])$.

In the course of the proof of Theorem 1.1 we compute the groups $\mathrm{HH}^n(R)$. This is of independent interest, and we collect the results in the following statement.

Proposition 1.4. *Let $R = \mathbb{Z}[D_{4m}]$. Then*

$$\mathrm{HH}^n(R) = \begin{cases} \mathbb{Z}^{m+3} & \text{if } n = 0, \\ \mathbb{Z}_{2m}^{m+1} \oplus \mathbb{Z}_2^{8k+2} & \text{if } n = 4k > 0, \\ \mathbb{Z}_2^{8k} & \text{if } n = 4k + 1, \\ \mathbb{Z}_{2m}^{m-1} \oplus \mathbb{Z}_2^{8(k+1)} & \text{if } n = 4k + 2, \\ \mathbb{Z}_2^{8k+4} & \text{if } n = 4k + 3. \end{cases}$$

§2. RESOLUTION

Put $G = D_{4m}$ and $R = R_m = \mathbb{Z}[G]$. We consider the following elements of the algebra $\Lambda = R^e$:

$$(2.1) \quad B^+ = b \otimes 1 + 1 \otimes b, \quad C^+ = c \otimes 1 + 1 \otimes c,$$

$$(2.2) \quad B^- = b \otimes 1 - 1 \otimes b, \quad C^- = c \otimes 1 - 1 \otimes c,$$

$$\zeta^+ = \sum_{i=0}^{m-1} c(bc)^i \otimes (cb)^{m-1-i} + \sum_{i=0}^{m-1} (bc)^i \otimes (cb)^{m-1-i} c,$$

$$\eta^+ = \sum_{i=0}^{m-1} b(cb)^i \otimes (bc)^{m-1-i} + \sum_{i=0}^{m-1} (cb)^i \otimes (bc)^{m-1-i} b,$$

$$\zeta^- = \sum_{i=0}^{m-1} c(bc)^i \otimes (cb)^{m-1-i} - \sum_{i=0}^{m-1} (bc)^i \otimes (cb)^{m-1-i} c,$$

$$\eta^- = \sum_{i=0}^{m-1} b(cb)^i \otimes (bc)^{m-1-i} - \sum_{i=0}^{m-1} (cb)^i \otimes (bc)^{m-1-i} b,$$

$$\rho^+ = c(bc)^{m-1} \otimes 1 + 1 \otimes (cb)^{m-1} c, \quad \omega^+ = b(cb)^{m-1} \otimes 1 + 1 \otimes (bc)^{m-1} b,$$

$$\rho^- = c(bc)^{m-1} \otimes 1 - 1 \otimes (cb)^{m-1} c, \quad \omega^- = b(cb)^{m-1} \otimes 1 - 1 \otimes (bc)^{m-1} b.$$

In the category of (left) Λ -modules, we construct the following bicomplex, lying in the first quadrant of the plane (i.e., its rows and columns are enumerated by $0, 1, 2, \dots$):

$$\begin{array}{ccccccc}
 & \dots & & \dots & & \dots & & \dots \\
 & D^- \downarrow & & \downarrow D^- & & \downarrow D^- & & \downarrow D^- \\
 & \Lambda^2 & \xleftarrow{\sigma^-} & \Lambda^2 & \xleftarrow{\sigma^+} & \Lambda^2 & \xleftarrow{\sigma^-} & \Lambda^2 & \xleftarrow{\tau^+} & \dots \\
 & D^+ \downarrow & & \downarrow D^+ & & \downarrow D^+ & & \downarrow (B^+ \ C^+) \\
 & \Lambda^2 & \xleftarrow{-\sigma^-} & \Lambda^2 & \xleftarrow{-\sigma^+} & \Lambda^2 & \xleftarrow{-\tau^-} & \Lambda \\
 (2.3) \quad \mathcal{B}_{\bullet\bullet} : & D^- \downarrow & & \downarrow D^- & & \downarrow (B^- \ C^-) \\
 & \Lambda^2 & \xleftarrow{\sigma^-} & \Lambda^2 & \xleftarrow{\tau^+} & \Lambda \\
 & D^+ \downarrow & & \downarrow (B^+ \ C^+) \\
 & \Lambda^2 & \xleftarrow{-\tau^-} & \Lambda \\
 & \downarrow (B^- \ C^-) \\
 & \Lambda
 \end{array}$$

where

$$(2.4) \quad D^+ = \begin{pmatrix} B^+ & 0 \\ 0 & C^+ \end{pmatrix}, \quad D^- = \begin{pmatrix} B^- & 0 \\ 0 & C^- \end{pmatrix}, \quad \tau^+ = \begin{pmatrix} \zeta^+ \\ -\eta^+ \end{pmatrix}, \quad \tau^- = \begin{pmatrix} \zeta^- \\ -\eta^- \end{pmatrix},$$

$$(2.5) \quad \sigma^+ = \begin{pmatrix} \rho^+ & 0 \\ 0 & -\omega^+ \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} \rho^- & 0 \\ 0 & -\omega^- \end{pmatrix}.$$

Theorem 2.1. *The total complex $Q_\bullet = \text{Tot}(\mathcal{B}_{\bullet\bullet})$ of the bicomplex (2.3) is a free resolution of the Λ -module R .*

Remark 2.2. A direct verification shows that diagram (2.3) indeed represents a bicomplex; i.e., its rows and columns are differential sequences and all squares are anticommutative.

Remark 2.3. As usual, as an augmentation map $\mu: Q_0 = \Lambda \rightarrow R$ we take the canonical map induced by multiplication on R : $\mu(x \otimes y) = xy$. Then the exactness of the corresponding sequence at the member Q_0 , i.e., the fact that $\text{Im } d_0^Q = \text{Ker } \mu$, is well known.

Lemma 2.4. *Let $B^\pm, C^\pm: \Lambda \rightarrow \Lambda$ be homomorphisms of multiplication (on the right) by the corresponding elements (see (2.1), (2.2)). Then*

$$\begin{aligned}
 (2.6) \quad \text{Ker } B^- &= \text{Im } B^+, & \text{Ker } B^+ &= \text{Im } B^-, \\
 \text{Ker } C^- &= \text{Im } C^+, & \text{Ker } C^+ &= \text{Im } C^-.
 \end{aligned}$$

Proof. We only prove the first relation in (2.6). The remaining relations can be proved similarly (or can be derived from the first by symmetry).

Let $\lambda \in \text{Ker } B^-$, and let $\lambda = \sum_{u,v \in G} \lambda(u,v)u \otimes v$ be the canonical expansion of this element. Since

$$\lambda \cdot b \otimes 1 = \sum_{u,v \in G} \lambda(ub^{-1}, v)u \otimes v, \quad \lambda \cdot 1 \otimes b = \sum_{u,v \in G} \lambda(u, b^{-1}v)u \otimes v,$$

we have $\lambda(ub, v) = \lambda(u, bv)$ for all $u, v \in G$. Consequently,

$$\lambda = \sum_{u,v} \lambda(ub, v)(ub \otimes v + u \otimes bv) = \sum_{u,v} \lambda(ub, v)u \otimes v \cdot (b \otimes 1 + 1 \otimes b) \in \text{Im } B^+.$$

□

Lemma 2.5. $\text{Ker } d_0^Q = \text{Im } d_1^Q$.

Proof. For brevity, we denote $d_i = d_i^Q$. It remains to prove the inclusion $\text{Ker } d_0 \subset \text{Im } d_1$. Let h_1, h_2, h_3 denote the columns of the matrix of the differential d_1 ,

$$(2.7) \quad d_1 = \begin{pmatrix} -\zeta^- & B^+ & 0 \\ \eta^- & 0 & C^+ \end{pmatrix},$$

which are also viewed as elements in $\text{Im } d_1 \subset \Lambda^2$. Suppose that $q = (\lambda, \mu) \in \text{Ker } d_0$ with $\lambda, \mu \in \Lambda$. Let $\lambda = \sum_{u,v \in G} \lambda(u, v)u \otimes v$ and $\mu = \sum_{u,v \in G} \mu(u, v)u \otimes v$ be canonical expansions, where $\lambda(u, v), \mu(u, v) \in \mathbb{Z}$. Since

$$\lambda \cdot B^- + \mu \cdot C^- = 0,$$

we have

$$(2.8) \quad \lambda(ub, v) - \lambda(u, bv) + \mu(uc, v) - \mu(u, cv) = 0$$

for all $u, v \in G$. Indeed, the left-hand side of equation (2.8) is the coefficient of $u \otimes v$ in the expansion of the element $\lambda \cdot B^- + \mu \cdot C^-$ in the canonical basis of the algebra Λ . If the standard representation of an element $v \in G$ starts with the letter c or if $v = 1$, then we may assume that $\lambda(u, bv) = 0$ for all u . Indeed, if $\lambda(u, bv) \neq 0$, then we replace the original $q \in \text{Ker } d_0$ by the element $q' = q - \lambda(u, bv)u \otimes v \cdot h_2$. By symmetry, we may also assume that $\mu(u, cv) = 0$ if the standard representation for an element $v \in G$ starts with b (or if $v = 1$). We fix $u \in G$ and put $\mu_u := \mu(uc, 1)$. From (2.8), it follows that

$$\begin{aligned} \mu(u(cb)^i c, (bc)^i) &= \lambda(u(cb)^{i+1}, c(bc)^i) \text{ for } 0 \leq i \leq m-1, \\ \lambda(u(cb)^i, c(bc)^{i-1}) &= \mu(u(cb)^i c, (bc)^i) \text{ for } 1 \leq i \leq m-1, \end{aligned}$$

whence

$$\begin{aligned} \mu(u(cb)^i c, (bc)^i) &= \mu_u \text{ for } 1 \leq i \leq m-1, \\ \lambda(u(cb)^i, c(bc)^{i-1}) &= \mu_u \text{ for } 1 \leq i \leq m. \end{aligned}$$

Furthermore, putting $v = 1$ and $v = (bc)^m$ in (2.8), we obtain two additional relations:

$$\lambda(ub, 1) + \mu(uc, 1) = 0, \quad \lambda(u(bc)^m, c(bc)^{m-1}) + \mu(u(bc)^m, b(cb)^{m-1}) = 0.$$

By symmetry, we have

$$\begin{aligned} \lambda(u(bc)^i b, (cb)^i) &= \mu(u(bc)^{i+1}, b(cb)^i) \text{ for } 0 \leq i \leq m-1, \\ \mu(u(bc)^i, b(cb)^{i-1}) &= \lambda(u(bc)^i b, (cb)^i) \text{ for } 1 \leq i \leq m-1, \end{aligned}$$

whence

$$\begin{aligned} \lambda(u(bc)^i b, (cb)^i) &= -\mu_u \text{ for } 0 \leq i \leq m-1, \\ \mu(u(bc)^i, b(cb)^{i-1}) &= -\mu_u \text{ for } 1 \leq i \leq m-1. \end{aligned}$$

These relations imply easily that

$$q = \left(\sum_u \mu_u u \right) (bc)^m \otimes 1 \cdot h_1 \in \text{Im } d_1.$$

□

Lemma 2.6. a) $\text{Ker}(\zeta^- B^-)^* = \text{Im}(B^+)^* + \text{Im}(C^+)^*$;

b) $\text{Ker}(\zeta^+ B^+)^* = \text{Im}(B^-)^* + \text{Im}(C^-)^*$.

Proof. a) Using Remark 2.2, we deduce the inclusion $\text{Im}(B^+)^* + \text{Im}(C^+)^* \subset \text{Ker}(\zeta^- B^-)^*$ from the fact that

$$B^+ \cdot \zeta^- B^- = B^+ \eta^- C^- = 0, \quad C^+ \cdot \zeta^- B^- = 0$$

in the algebra Λ . Now we prove the reverse inclusion. Let $\lambda = \sum_{u,v \in G} \lambda(u,v) u \otimes v \in \text{Ker}(\zeta^- B^-)^*$ be a nonzero element, where $\lambda(u,v) \in \mathbb{Z}$. Consider the set

$$V_\lambda = \{v \in G \mid \lambda(u,v) \neq 0 \text{ for some } u\}$$

and put $l(\lambda) := \max\{l(v) \mid v \in V_\lambda\}$. Also, we consider the set $M_\lambda = \{v \in V_\lambda \mid l(v) = l(\lambda)\}$. Since

$$\begin{aligned} \zeta^- B^- = \sum_{i=0}^{m-1} & \left((cb)^{i+1} \otimes (cb)^{m-1-i} - (bc)^i b \otimes (cb)^{m-1-i} c \right. \\ & \left. - c(bc)^i \otimes b(cb)^{m-1-i} + (bc)^i \otimes (bc)^{m-i} \right), \end{aligned}$$

we have

$$\begin{aligned} (2.9) \quad \sum_{i=0}^{m-1} & \left(\lambda(u \cdot (bc)^{i+1}, (bc)^{m-1-i} \cdot v) - \lambda(u \cdot (bc)^i b, (cb)^{m-1-i} c \cdot v) \right. \\ & \left. - \lambda(u \cdot c(bc)^i, (bc)^{m-1-i} b \cdot v) + \lambda(u \cdot (cb)^i, (cb)^{m-i} \cdot v) \right) = 0 \end{aligned}$$

for all $u, v \in G$. Fix $v \in M_\lambda$. First, we assume that the standard representation of v starts with b and write $v = bv_1$, where either $v_1 = 1$, or the standard representation of v_1 starts with c . Replacing the initial element λ by the element $\lambda' := \lambda - \lambda(u,v) u \otimes v_1 \cdot B^+$, we obtain a new element with zero coefficient of $u \otimes v$ in its canonical expansion. Now, if the standard representation of v starts with the letter c and $v = cv_2$, where either $v_2 = 1$ or the standard representation of v_2 starts with b , then we replace λ by $\lambda'' := \lambda - \lambda(u,v) u \otimes v_2 \cdot C^+$. Applying a similar argument for each $u \in G$ with $\lambda(u,v) \neq 0$, we obtain an element $\lambda_1 \in \text{Ker}(\zeta^- B^-)^*$ for which the cardinality of the set M_{λ_1} is less than that of M_λ . Repeating this procedure, we obtain an element $\tilde{\lambda}$ with $l(\tilde{\lambda}) < l(\lambda)$.

The above process terminates at an element of $\text{Ker}(\zeta^- B^-)^*$ whose canonical expansion can have nonzero coefficients only at basis elements of the form $u \otimes 1$, $u \in G$. Hence, we may assume that the initial element $\lambda \in \text{Ker}(\zeta^- B^-)^*$ has the property that $\lambda(u,v) = 0$ for all $v \in G \setminus \{1\}$ and $u \in G$.

Observe that, for a fixed $v \in G$, the set G is a disjoint union of the sets

$$\{(bc)^{m-1-i} \cdot v\}_{i=0}^{m-1}, \quad \{(cb)^{m-1-i} c \cdot v\}_{i=0}^{m-1}, \quad \{(bc)^{m-1-i} b \cdot v\}_{i=0}^{m-1}, \quad \{(cb)^{m-i} \cdot v\}_{i=0}^{m-1}.$$

By the above assumptions, only one summand remains on the left-hand side of (2.9), and this summand must also be zero. Since G is a disjoint union of the sets

$$\{u \cdot (bc)^{i+1}\}_{i=0}^{m-1}, \quad \{u \cdot (bc)^i b\}_{i=0}^{m-1}, \quad \{u \cdot (cb)^i c\}_{i=0}^{m-1}, \quad \{u \cdot (cb)^i\}_{i=0}^{m-1},$$

we see that $\lambda(u,v) = 0$ for all $u, v \in G$. This completes the proof of the inclusion $\text{Ker}(\zeta^- B^-)^* \subset \text{Im}(B^+)^* + \text{Im}(C^+)^*$.

b) The proof is similar to that of part a). $\square$

Proof of Theorem 2.1. We consider the spectral sequence of the bicomplex $B_{\bullet\bullet}$ (see (2.3)),

$$(2.10) \quad H_p^h H_q^v(B_{\bullet\bullet}) \Rightarrow H_{p+q}(\text{Tot}(B_{\bullet\bullet})),$$

and prove that the second sheet of this spectral sequence degenerates.

By Lemma 2.4, the (nonzero) columns in $\mathcal{B}_{\bullet\bullet}$ are exact at all terms except the two bottom terms Λ^2 and Λ . Consequently, every row with number $i > 0$ of the first sheet of the spectral sequence (2.10) contains two nonzero terms, namely,

$$\mathrm{Ker}(B^-, C^-) / \mathrm{Im} D^+ \xleftarrow{\overline{\tau^-}} \mathrm{Coker}(B^+, C^+)$$

for odd i and

$$\mathrm{Ker}(B^+, C^+) / \mathrm{Im} D^- \xleftarrow{\overline{\tau^+}} \mathrm{Coker}(B^-, C^-)$$

for even i ; here $\overline{\tau^-}$ and $\overline{\tau^+}$ denote the homomorphisms induced by the homomorphisms τ^- and τ^+ , respectively. We shall prove that $\overline{\tau^-}$ is an isomorphism; the fact that $\overline{\tau^+}$ is also an isomorphism is established similarly.

The surjectivity of $\overline{\tau^-}$ follows from Lemma 2.5. Next, we assume that $\overline{\tau^-}(\bar{\lambda}) = 0$ for some $\bar{\lambda} = \lambda + \mathrm{Im}(B^+, C^+) \in \mathrm{Coker}(B^+, C^+)$, i.e., $\tau^-(\lambda) = D^+(\lambda', \lambda'')$ for some $\lambda', \lambda'' \in \Lambda$. The latter relation is equivalent to the system

$$\begin{cases} -\lambda \cdot \zeta^- = \lambda' \cdot B^+, \\ \lambda \cdot \eta^- = \lambda'' \cdot C^+. \end{cases}$$

Consequently, we have $\lambda \cdot \zeta^- B^- = 0$. By Lemma 2.6 a), there exist $\mu', \mu'' \in \Lambda$ such that $\lambda = \mu' \cdot B^+ + \mu'' \cdot C^+$, i.e., $\lambda \in \mathrm{Im}(B^+, C^+)$, whence $\bar{\lambda} = 0$.

Finally, in the 0th row of the first sheet of the spectral sequence, there is only one nonzero term: $\mathrm{Coker}(B^-, C^-) \simeq R$.

Thus, we have proved that $Q_\bullet = \mathrm{Tot}(\mathcal{B}_{\bullet\bullet})$ is a Λ -free resolution of the module R . $\square$

Consider the bicomplex $\mathcal{A}_{\bullet\bullet}$ that consists of the first two columns of the bicomplex $\mathcal{B}_{\bullet\bullet}$ with numbers 0 and 1 (the remaining columns in $\mathcal{A}_{\bullet\bullet}$ are zero). Set $X_\bullet = \mathrm{Tot}(\mathcal{A}_{\bullet\bullet})$.

Proposition 2.7. *There is a short exact sequence of complexes*

$$0 \rightarrow X_\bullet \xrightarrow{i} Q_\bullet \xrightarrow{\pi} Q_\bullet[-4] \rightarrow 0$$

that splits in each degree.

Proof. This follows immediately from the construction of the bicomplex $\mathcal{B}_{\bullet\bullet}$. $\square$

§3. ADDITIVE STRUCTURE OF THE COHOMOLOGY ALGEBRA

We keep the notation R for the group algebra $\mathbb{Z}[G]$ with $G = D_{4m}$. In this section, we use the Λ -free resolution $Q_\bullet = (Q_n, d_n^Q)$ of the Λ -module R (see Theorem 2.1) to determine the cohomology groups $\mathrm{HH}^i(R)$, $i \geq 0$. For this, we investigate the complex

$$(3.1) \quad \mathrm{Hom}_\Lambda(Q_\bullet, R) = \left(\mathrm{Hom}_\Lambda(Q_n, R), \delta^n := (d_n^Q)^* \right)$$

and calculate its cohomology:

$$\mathrm{H}^n(\mathrm{Hom}_\Lambda(Q_\bullet, R)) = \mathrm{Ext}_\Lambda^n(R, R) = \mathrm{HH}^n(R).$$

To describe the homomorphisms in $\mathrm{Hom}_\Lambda(Q_n, R)$, we fix the decomposition

$$Q_n = \sum_{i+j=n} B_{ij}$$

ordered in accordance with the increase of the second index j .

Since $Q_n \simeq \Lambda^{n+1}$, we have $\mathrm{Hom}_\Lambda(Q_n, R) \simeq R^{n+1}$, and any Λ -homomorphism $f: Q_n \rightarrow R$ is identified with the collection $(f(e_1), f(e_2), \dots, f(e_{n+1}))$ of its values, where $e_1 = (1 \otimes 1, 0, \dots, 0)$, $\dots$, $e_{n+1} = (0, \dots, 0, 1 \otimes 1)$ is the canonical basis of the free Λ -module Λ^{n+1} . In the sequel, if in this collection there is a subsequence containing only zeros, say r zeros, we denote such a subsequence by O_r . Similarly, the zero $(r \times s)$ -matrix will be

denoted by $O_{r,s}$; moreover, we omit the indices indicating the size of such a matrix if no confusion is possible.

We observe that, for the homomorphism $f = w^*: \Lambda \rightarrow \Lambda$ of multiplication on the right by $w \in \Lambda$, the induced homomorphism (of Abelian groups)

$$\text{Hom}_\Lambda(f, R): \text{Hom}_\Lambda(\Lambda, R) \simeq R \rightarrow \text{Hom}_\Lambda(\Lambda, R) \simeq R$$

acts, in accordance with the identification mentioned above, as follows: $r \in R$ is mapped to $w * r$; here $*$ corresponds to the Λ -module structure on R .

Then the differential $\delta^0: R \rightarrow R^2$ can be described by the formula

$$(3.2) \quad \delta^0(r) = (B^- * r, C^- * r) = (br - rb, cr - rc)$$

for any $r \in R$.

We consider the following elements of the algebra R :

$$(3.3) \quad \begin{aligned} \theta_b &:= \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} b(cb)^{2i} + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} c(bc)^{2i-1}, \\ \theta_c &:= \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} c(bc)^{2i} + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} b(cb)^{2i-1}. \end{aligned}$$

It is clear that θ_b (respectively, θ_c) is the sum of all elements of the group G that are conjugate to b (respectively, c).

Denote by $Z(R)$ the centre of the algebra R . The following statement is a consequence of well-known properties of group algebras (also it can be deduced directly from the analysis of the relation $\delta^0(r) = 0$ with the help of (3.2)).

Proposition 3.1. *The group $\text{HH}^0(R) \simeq Z(R)$ is a free $\mathbb{Z}$ -module with the basis*

$$(3.4) \quad \{1, \theta_b, \theta_c, (bc)^m\} \cup \{(bc)^i + (cb)^i\}_{i=1}^{m-1}.$$

Corollary 3.2. *The group $\text{Im } \delta^0$ is a free $\mathbb{Z}$ -module of rank $3m - 3$.*

Proof. Since $\text{Im } \delta^0 \subset R^2$, $\text{Im } \delta^0$ is a free $\mathbb{Z}$ -module, and we obtain

$$\text{rk Im } \delta^0 = \text{rk } R - \text{rk Ker } \delta^0 = 4m - (m + 3) = 3m - 3. \quad \square$$

Remark 3.3. It is easy to verify that the following set is a basis of the $\mathbb{Z}$ -module $\text{Im } \delta^0$:

$$(3.5) \quad \begin{aligned} &\left\{ \left((bc)^i - (cb)^i, 0 \right) \right\}_{i=1}^{m-1} \cup \left\{ \left(0, (cb)^i - (bc)^i \right) \right\}_{i=1}^{m-1} \\ &\cup \left\{ \left(c(bc)^{i-1} - b(cb)^i, c(bc)^i - b(cb)^{i-1} \right) \right\}_{i=1}^{m-1}. \end{aligned}$$

The following properties of the elements θ_b and θ_c will be useful in the sequel.

Proposition 3.4. $\theta_b = \sum_{i=0}^{m-1} (bc)^{2i} b$, $\theta_c = \sum_{i=0}^{m-1} (cb)^{2i} c$.

Proof. We establish only the first relation; the second follows by symmetry. Replacing the index in the second sum in (3.3) by $j = m - i$, we obtain

$$\sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} c(bc)^{2i-1} = \sum_{j=\lfloor \frac{m+1}{2} \rfloor}^{m-1} c(bc)^{2m-2j-1} = \sum_{j=\lfloor \frac{m+1}{2} \rfloor}^{m-1} c(cb)^{2j+1} = \sum_{j=\lfloor \frac{m+1}{2} \rfloor}^{m-1} b(cb)^{2j},$$

whence

$$\theta_b = \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} b(cb)^{2i} + \sum_{i=\lfloor \frac{m+1}{2} \rfloor}^{m-1} b(cb)^{2i} = \sum_{i=0}^{m-1} b(cb)^{2i}. \quad \square$$

Corollary 3.5. $bc \cdot \theta_b = \theta_c = cb \cdot \theta_b$, $bc \cdot \theta_c = \theta_b = cb \cdot \theta_c$.

Proof. Proposition 3.4 yields

$$bc\theta_b = b\left(\sum_{i=0}^{m-1} b(cb)^{2i}\right)c = \sum_{i=0}^{m-1} (cb)^{2i}c = \theta_c.$$

The remaining relations are proved similarly. $\square$

Corollary 3.6. *For any natural j , we have*

$$(3.6) \quad \sum_{i=0}^{m-1} (bc)^{2i+2j} = \sum_{i=0}^{m-1} (bc)^{2i}.$$

Proof. Multiplying both sides of the identity $\theta_c \cdot (bc)^{2j} = \theta_c$ on the left by c , we arrive at (3.6). $\square$

Using the form of the differential d_1^Q (see (2.7)), we see that the membership relation $(r_1, r_2) \in \text{Ker } \delta^1$ is equivalent to the fact that the following system of three equations (over R) is satisfied:

$$\begin{aligned} -\zeta^- * r_1 + \eta^- * r_2 &= 0, \\ B^+ * r_1 &= 0, \\ C^+ * r_2 &= 0, \end{aligned}$$

or, in detail,

$$\begin{aligned} (3.7) \quad & -\sum_{i=0}^{m-1} c(bc)^i \cdot r_1 \cdot (cb)^{m-1-i} + \sum_{i=0}^{m-1} (bc)^i \cdot r_1 \cdot (cb)^{m-1-i}c \\ & + \sum_{i=0}^{m-1} b(cb)^i \cdot r_2 \cdot (bc)^{m-1-i} - \sum_{i=0}^{m-1} (cb)^i \cdot r_2 \cdot (bc)^{m-1-i}b = 0, \\ & b \cdot r_1 + r_1 \cdot b = 0, \\ & c \cdot r_2 + r_2 \cdot c = 0. \end{aligned}$$

Proposition 3.7. $\text{HH}^1(R) = 0$.

Proof. Assume that $(r_1, r_2) \in \text{Ker } \delta^1$. Let

$$r_1 = \sum_{g \in G} \lambda_g g, \quad r_2 = \sum_{g \in G} \mu_g g$$

be canonical expansions (where $\lambda_g, \mu_g \in \mathbb{Z}$). The second equation in (3.7) implies

$$\begin{aligned} \lambda_{(bc)^i} + \lambda_{(cb)^i} &= 0, \quad \lambda_{b(cb)^i} + \lambda_{(cb)^{i-1}c} = 0 \quad \text{for } 1 \leq i \leq m-1, \\ \lambda_1 &= \lambda_b = \lambda_{(cb)^{m-1}c} = \lambda_{(bc)^m} = 0. \end{aligned}$$

By symmetry, the third equation in (3.7) yields

$$\begin{aligned} \mu_{(bc)^i} + \mu_{(cb)^i} &= 0, \quad \mu_{c(bc)^i} + \mu_{(bc)^{i-1}b} = 0 \quad \text{for } 1 \leq i \leq m-1, \\ \mu_1 &= \mu_c = \mu_{(bc)^{m-1}b} = \mu_{(bc)^m} = 0. \end{aligned}$$

Consequently, r_1 and r_2 are presented in the form

$$\begin{aligned} (3.8) \quad r_1 &= \sum_{j=1}^{m-1} \lambda_{(bc)^j} \left((bc)^j - (cb)^j \right) + \sum_{j=1}^{m-1} \lambda_{b(cb)^j} \left(b(cb)^j - (cb)^{j-1}c \right), \\ r_2 &= \sum_{j=1}^{m-1} \mu_{(bc)^j} \left((bc)^j - (cb)^j \right) + \sum_{j=1}^{m-1} \mu_{c(bc)^j} \left(c(bc)^j - (bc)^{j-1}b \right). \end{aligned}$$

Since the elements $\left((bc)^j - (cb)^j, 0\right), \left(0, (bc)^j - (cb)^j\right), 1 \leq j \leq m-1$, lie in $\text{Im } \delta^0 \subset \text{Ker } \delta^1$, subtracting a suitable linear combination of these elements from (r_1, r_2) , we may assume that in the initial elements r_1, r_2 we have

$$(3.9) \quad \lambda_{(bc)^j} = 0 = \mu_{(bc)^j} \quad \text{for } 1 \leq j \leq m-1.$$

Observe that

$$\begin{aligned} \sum_{i=0}^{m-1} c(bc)^i \cdot b(cb)^j \cdot (cb)^{m-1-i} &= \sum_{i=0}^{m-1} (cb)^{m+j} = m(bc)^{m-j}, \\ \sum_{i=0}^{m-1} c(bc)^i \cdot (cb)^{j-1} c \cdot (cb)^{m-1-i} &= \sum_{i=0}^{m-1} (cb)^i (bc)^{j-1} (cb)^{m-1-i} \\ &= \sum_{i=0}^{m-1} (cb)^{m-j} = m(cb)^{m-j} \end{aligned}$$

for $1 \leq j \leq m-1$. Similarly,

$$\begin{aligned} \sum_{i=0}^{m-1} (bc)^i \cdot b(cb)^j \cdot (cb)^{m-1-i} c &= m(cb)^{m-j}, \\ \sum_{i=0}^{m-1} (bc)^i \cdot (cb)^{j-1} c \cdot (cb)^{m-1-i} c &= m(bc)^{m-j}. \end{aligned}$$

Substituting (3.8) in the first equation in (3.7) and keeping relations (3.9) in mind, we obtain

$$2m \sum_{j=1}^{m-1} (\lambda_{b(cb)^j} + \mu_{c(bc)^j}) \left((cb)^{m-j} - (bc)^{m-j} \right) = 0,$$

whence $\lambda_{b(cb)^j} + \mu_{c(bc)^j} = 0$ for all $1 \leq j \leq m-1$. This shows that $\text{Ker } \delta^1$ is generated by the set (3.5), so that $\text{Ker } \delta^1 = \text{Im } \delta^0$. $\square$

Proposition 3.8. *The free $\mathbb{Z}$ -module $\text{Im } \delta^1$ admits a basis formed by the following elements:*

$$(3.10) \quad (0, 2, 0), (0, 2b, 0), (0, 0, 2), (0, 0, 2c);$$

$$(3.11) \quad (0, 2c(bc)^{m-1}, 0), (0, 2(bc)^m, 0), (0, 0, 2b(cb)^{m-1}), (0, 0, 2(bc)^m);$$

$$(3.12) \quad (2m((bc)^j - (cb)^j), 0, 0), \quad 1 \leq j \leq m-1;$$

$$(3.13) \quad (m((cb)^{m-j} - (bc)^{m-j}), (bc)^j + (cb)^j, 0), \quad 1 \leq j \leq m-1;$$

$$(3.14) \quad (0, c(bc)^{j-1} + b(cb)^j, 0), \quad 1 \leq j \leq m-1;$$

$$(3.15) \quad (m((cb)^{m-j} - (bc)^{m-j}), 0, (bc)^j + (cb)^j), \quad 1 \leq j \leq m-1;$$

$$(3.16) \quad (0, 0, b(cb)^{j-1} + c(bc)^j), \quad 1 \leq j \leq m-1.$$

Proof. First, observe that for $0 \leq j \leq m-1$ we have

$$\delta^1((bc)^j, 0) = (\Xi_j, c(bc)^{j-1} + (bc)^j b, 0),$$

where

$$\Xi_j := - \sum_{i=0}^{m-1} c(cb)^{m-1-2i-j} + \sum_{i=0}^{m-1} (cb)^{m-1-2i-j} c.$$

Then we obtain

$$\begin{aligned} c \cdot \Xi_j &= - \sum_{i=0}^{m-1} (cb)^{m-1-2i-j} + \sum_{i=0}^{m-1} (bc)^{m-1-2i-j} \\ &= - \sum_{i=0}^{m-1} (bc)^{-m+1+2i+j} + \sum_{l=0}^{m-1} (bc)^{-m+1+2l-j}, \end{aligned}$$

where, in the second sum, we have changed the index of summation: $l := m - 1 - i$. Consequently,

$$\begin{aligned} c \cdot \Xi_j &= - \sum_{i=0}^{m-1} (bc)^{m+1+2i+j} + \sum_{l=0}^{m-1} (bc)^{m+1+2l-j} \\ &= (bc)^{m+1-j} \left(- \sum_{i=0}^{m-1} (bc)^{2i+2j} + \sum_{l=0}^{m-1} (bc)^{2l} \right) = 0 \end{aligned}$$

(the latter identity is obtained with the help of (3.6)), whence $\Xi_j = 0$. Consequently, $\delta^1((bc)^j, 0) = (0, c(bc)^{j-1} + (bc)^j b, 0)$ ($1 \leq j \leq m-1$) and $\delta^1(1, 0) = (0, 2b, 0)$. By symmetry, $\delta^1(0, (cb)^j) = (0, 0, b(cb)^{j-1} + (cb)^j c)$, $1 \leq j \leq m-1$, and $\delta^1(0, 1) = (0, 0, 2c)$. Furthermore, it is easily seen that

$$\begin{aligned} \delta^1(b, 0) &= (0, 2, 0), \quad \delta^1(c(bc)^{m-1}, 0) = (0, 2(bc)^m, 0), \\ \delta^1((bc)^m, 0) &= (0, 2c(bc)^{m-1}, 0); \\ \delta^1(b(cb)^j, 0) &= (m((cb)^{m-j} - (bc)^{m-j}), (bc)^j + (cb)^j, 0), \quad 1 \leq j \leq m-1; \\ \delta^1(b(cb)^j - c(bc)^{j-1}, 0) &= (2m((cb)^{m-j} - (bc)^{m-j}), 0, 0), \quad 1 \leq j \leq m-1. \end{aligned}$$

By symmetry, we also have

$$\begin{aligned} \delta^1(0, c) &= (0, 0, 2), \quad \delta^1(0, b(cb)^{m-1}) = (0, 0, 2(bc)^m), \\ \delta^1(0, (bc)^m) &= (0, 0, 2b(cb)^{m-1}); \\ \delta^1(0, c(bc)^j) &= (m((cb)^{m-j} - (bc)^{m-j}), 0, (bc)^j + (cb)^j), \quad 1 \leq j \leq m-1; \\ \delta^1(0, c(bc)^j - b(cb)^{j-1}, 0) &= (2m((cb)^{m-j} - (bc)^{m-j}), 0, 0), \quad 1 \leq j \leq m-1. \end{aligned}$$

It follows that the set of elements in (3.10)–(3.16) generates $\text{Im } \delta^1$. Since

$$(3.17) \quad \text{rk Im } \delta^1 = \text{rk } R^2 - \text{rk Ker } \delta^1 = 8m - (3m - 3) = 5m + 3$$

and the cardinality of the set of generators indicated is also equal to $5m + 3$, this set is a basis of $\text{Im } \delta^1$. $\square$

Since

$$d_2^Q = \begin{pmatrix} B^+ & C^+ & 0 & 0 \\ \rho^- & 0 & B^- & 0 \\ 0 & -\omega^- & 0 & C^- \end{pmatrix},$$

the condition $(r_1, r_2, r_3) \in \text{Ker } \delta^2$ is equivalent to the following system of equations:

$$(3.18) \quad \begin{aligned} B^+ * r_1 + \rho^- * r_2 &= 0, \\ C^+ * r_1 - \omega^- * r_3 &= 0, \\ B^- * r_2 &= 0, \\ C^- * r_3 &= 0. \end{aligned}$$

Proposition 3.9. *The free $\mathbb{Z}$ -module $\text{Ker } \delta^2$ admits a basis formed by the elements (3.13)–(3.16) and by the following elements:*

$$(3.19) \quad (0, 1, 0), (0, b, 0), (0, c(bc)^{m-1}, 0), (0, (bc)^m, 0);$$

$$(3.20) \quad (0, 0, 1), (0, 0, c), (0, 0, b(cb)^{m-1}), (0, 0, (bc)^m);$$

$$(3.21) \quad ((bc)^j - (cb)^j, 0, 0), \quad 1 \leq j \leq m-1.$$

The following fact, needed in the proof of this proposition, can be established by direct calculation.

Lemma 3.10. a) *For any $r \in R$, we have $\rho^- * r = (B^- * r)(bc)^m$.*

b) *For any $r \in R$, we have $\omega^- * r = (C^- * r)(bc)^m$.*

Proof of Proposition 3.9. Lemma 3.10 implies that system (3.18) is equivalent to

$$(3.22) \quad \begin{aligned} B^+ * r_1 &= 0, \\ C^+ * r_1 &= 0, \\ B^- * r_2 &= 0, \\ C^- * r_3 &= 0. \end{aligned}$$

Assume that $x = (r_1, r_2, r_3) \in \text{Ker } \delta^2$. Put $r_2 = \sum_{g \in G} \mu_g g$ (with $\mu_g \in \mathbb{Z}$) and substitute this in the third equation in (3.22). It is easily seen that this equation is equivalent to the relations

$$\mu_{(bc)^i} = \mu_{(cb)^i}, \quad \mu_{b(cb)^i} = \mu_{(cb)^{i-1}c} \quad \text{for } 1 \leq i \leq m-1.$$

Consequently, the subgroup of $\text{Ker } \delta^2$ that consists of elements of the form $(0, r_2, 0)$ with $r_2 \in R$ is generated by the elements in (3.19), (3.14) and by the elements

$$(3.23) \quad (0, (bc)^i + (cb)^i, 0) \quad \text{for } 1 \leq i \leq m-1.$$

A similar analysis of the fourth equation in (3.22) shows that the subgroup of $\text{Ker } \delta^2$ that consists of elements of the form $(0, 0, r_3)$ with $r_3 \in R$ is generated by the elements in (3.20), (3.16) and the elements

$$(3.24) \quad (0, 0, (bc)^i + (cb)^i) \quad \text{for } 1 \leq i \leq m-1.$$

Substituting the canonical expansion $r_1 = \sum_{g \in G} \lambda_g g$ ($\lambda_g \in \mathbb{Z}$) in the first equation in (3.22), we obtain

$$(3.25) \quad \lambda_1 = \lambda_b = \lambda_{c(bc)^{m-1}} = \lambda_{(bc)^m} = 0;$$

$$(3.26) \quad \lambda_{(bc)^i} + \lambda_{(cb)^i} = 0 \quad \text{for } 1 \leq i \leq m-1;$$

$$(3.27) \quad \lambda_{b(cb)^i} + \lambda_{(cb)^{i-1}c} = 0 \quad \text{for } 1 \leq i \leq m-1.$$

Similarly, the second equation in (3.22) implies that

$$(3.28) \quad \lambda_c = \lambda_{b(cb)^{m-1}} = 0;$$

$$\lambda_{c(bc)^i} + \lambda_{(bc)^{i-1}b} = 0 \quad \text{for } 1 \leq i \leq m-1.$$

Then (using the relation $\lambda_b = \lambda_c = 0$), from (3.27) and (3.28) we deduce that $\lambda_{b(cb)^i} = \lambda_{c(bc)^i} = 0$ for $0 \leq i \leq m-1$. Hence, the subgroup of $\text{Ker } \delta^2$ that consists of elements of the form $(r_1, 0, 0)$ with $r_1 \in R$ is generated by the elements in (3.21). Consequently, $\text{Ker } \delta^2$ is generated by the set of elements in (3.19), (3.20), (3.21), (3.14), (3.16) and the elements in (3.23) and (3.24). But it is clear that the set of elements in (3.23) and (3.24) can be replaced by (3.13) and (3.15), respectively. It remains to observe that the cardinality of the resulting set of generators of the $\mathbb{Z}$ -module $\text{Ker } \delta^2$ is equal to $5m + 3 = \text{rk Im } \delta^1$ (see (3.17)); hence, this set is linearly independent (over $\mathbb{Z}$). $\square$

Proposition 3.11. *The group $\text{HH}^2(R)$ is generated by the set of cohomology classes of the following elements:*

$$\begin{aligned} u_0^{(i)} &:= \left((bc)^i - (cb)^i, 0, 0 \right) \text{ for } 1 \leq i \leq m-1; \\ u_1 &:= (0, 1, 0), \quad u'_1 := (0, 0, 1), \quad u_2 := (0, b, 0), \quad u'_2 := (0, 0, c), \\ u_3 &:= (0, c(bc)^{m-1}, 0), \quad u'_3 := (0, 0, b(cb)^{m-1}), \\ u_4 &:= (0, (bc)^m, 0), \quad u'_4 := (0, 0, (bc)^m), \end{aligned}$$

which satisfy the relations

$$\begin{aligned} 2mu_0^{(i)} &= 0 \text{ for } 1 \leq i \leq m-1; \\ 2u_j &= 2u'_j = 0 \text{ for } j \in \{1, 2, 3, 4\}. \end{aligned}$$

In particular, we have

$$\text{HH}^2(R) \simeq \mathbb{Z}_{2m}^{m-1} \oplus \mathbb{Z}_2^8.$$

Proof. This follows from the fact that the bases of the groups $\text{Ker } \delta^2$ and $\text{Im } \delta^1$ presented in Propositions 3.9 and 3.8 (respectively) are the so-called stacked bases. $\square$

Remark 3.12. In the sequel, if $f \in \text{Ker } \delta^n$ is an n -cocycle, its cohomology class $\text{cl } f \in \text{HH}^n(R)$ is often still denoted by f .

Proposition 3.13. *The free $\mathbb{Z}$ -module $\text{Im } \delta^2$ admits a basis formed by the following elements:*

$$(3.29) \quad (2b, 2c, 0, 0), \quad (2c(bc)^{m-1}, 2b(cb)^{m-1}, 0, 0);$$

$$(3.30) \quad ((bc)^i b + c(bc)^{i-1}, (cb)^i c + b(cb)^{i-1}, 0, 0), \quad 1 \leq i \leq m-1;$$

$$(3.31) \quad ((bc)^i + (cb)^i, (bc)^{i+1} + (cb)^{i+1}, 0, 0), \quad 0 \leq i \leq m-1;$$

$$(3.32) \quad ((bc)^{i+1} + (cb)^{i+1}, (bc)^i + (cb)^i, 0, 0), \quad 0 \leq i \leq m-1;$$

$$(3.33) \quad (b(cb)^{m-i} - (cb)^{m-1-i}c, 0, c(bc)^{i-1} - (bc)^i b, 0), \quad 1 \leq i \leq m-1;$$

$$(3.34) \quad ((bc)^{m-i} - (cb)^{m-i}, 0, (cb)^i - (bc)^i, 0), \quad 1 \leq i \leq m-1;$$

$$(3.35) \quad (0, c(bc)^{m-i} - (bc)^{m-1-i}b, 0, c(bc)^i - (bc)^{i-1}b), \quad 1 \leq i \leq m-1;$$

$$(3.36) \quad (0, (bc)^{m-i} - (cb)^{m-i}, 0, (bc)^i - (cb)^i), \quad 1 \leq i \leq m-1.$$

Proof. It is verified directly that the elements on the list (3.29)–(3.36) coincide with the following values of the differential δ^2 :

$$\begin{aligned} &\delta^2(1, 0, 0), \quad \delta^2((bc)^m, 0, 0), \quad \delta^2((bc)^i, 0, 0) \quad (1 \leq i \leq m-1), \\ &\delta^2(b(cb)^i, 0, 0) \quad (0 \leq i \leq m-1), \quad \delta^2(c(bc)^i, 0, 0) \quad (0 \leq i \leq m-1), \\ &\delta^2(0, (bc)^i, 0) \quad (1 \leq i \leq m-1), \quad \delta^2(0, b(cb)^i, 0) \quad (1 \leq i \leq m-1), \\ &\delta^2(0, 0, (bc)^i) \quad (1 \leq i \leq m-1), \quad \delta^2(0, 0, c(bc)^i) \quad (1 \leq i \leq m-1). \end{aligned}$$

Also, it is easily seen that this set of elements generates $\text{Im } \delta^2$ and has cardinality equal to $7m - 3$. Since $\text{rk Im } \delta^2 = \text{rk } R^3 - \text{rk Ker } \delta^2 = 12m - (5m + 3) = 7m - 3$, this set is linearly independent. $\square$

Since

$$d_3^Q = \begin{pmatrix} \zeta^+ & B^- & 0 & 0 & 0 \\ -\eta^+ & 0 & C^- & 0 & 0 \\ 0 & -\rho^- & 0 & B^+ & 0 \\ 0 & 0 & \omega^- & 0 & C^+ \end{pmatrix},$$

the condition $(r_1, r_2, r_3, r_4) \in \text{Ker } \delta^3$ is equivalent to the following system of equations:

$$\begin{aligned} &\zeta^+ * r_1 - \eta^+ * r_2 = 0, \\ &B^- * r_1 - \rho^- * r_3 = 0, \\ &C^- * r_2 + \omega^- * r_4 = 0, \\ &B^+ * r_3 = 0, \\ &C^+ * r_4 = 0. \end{aligned} \tag{3.37}$$

Proposition 3.14. a) If m is even, then the free $\mathbb{Z}$ -module $\text{Ker } \delta^3$ admits a basis formed by the elements in (3.33)–(3.36) and by the following elements:

$$(3.38) \quad (b, c, 0, 0), \quad (c(bc)^{m-1}, b(cb)^{m-1}, 0, 0);$$

$$(3.39) \quad (b(cb)^i + c(bc)^{i-1}, c(bc)^i + b(cb)^{i-1}, 0, 0), \quad 1 \leq i \leq m-1;$$

$$(3.40) \quad (1 - (bc)^m, 0, 0, 0), \quad (0, 1 - (bc)^m, 0, 0);$$

$$(3.41) \quad ((bc)^i + (cb)^i, (bc)^{i+1} + (cb)^{i+1}, 0, 0), \quad 0 \leq i \leq m-2;$$

$$(3.42) \quad ((bc)^{i+1} + (cb)^{i+1}, (bc)^i + (cb)^i, 0, 0), \quad 0 \leq i \leq m-2.$$

b) If m is odd, then in order to get a basis of the $\mathbb{Z}$ -module $\text{Ker } \delta^3$, it suffices to replace the elements (3.40) in the set described in part a) by the elements

$$(3.43) \quad (1, (bc)^m, 0, 0), \quad ((bc)^m, 1, 0, 0).$$

Lemma 3.15. a) For the homomorphisms (of Abelian groups)

$$\begin{aligned} f_b^+ : R &\rightarrow R, & f_b^+(r) &= B^+ * r, \\ f_b^- : R &\rightarrow R, & f_b^-(r) &= B^- * r, \end{aligned}$$

we have $\text{Ker } f_b^+ = \text{Im } f_b^-$.

b) For the homomorphisms

$$\begin{aligned} f_c^+ : R &\rightarrow R, & f_c^+(r) &= C^+ * r, \\ f_c^- : R &\rightarrow R, & f_c^-(r) &= C^- * r, \end{aligned}$$

we have $\text{Ker } f_c^+ = \text{Im } f_c^-$.

Proof. a) The inclusion $\text{Im } f_b^- \subset \text{Ker } f_b^+$ follows from the fact that f_b^+ and f_b^- are induced by the Λ -homomorphisms $(B^+)^*, (B^-)^*: \Lambda \rightarrow \Lambda$, for which we have $(B^-)^* \circ (B^+)^* = 0$. Next, if $f_b^+(r) = 0$ with $r = \sum_{g \in G} \lambda_g g \in R$, then relations (3.25)–(3.27) are fulfilled, whence we have

$$\begin{aligned} r &= \sum_{i=1}^{m-1} \lambda_{(bc)^i} ((bc)^i - (cb)^i) + \sum_{i=1}^{m-1} \lambda_{b(cb)^i} (b(cb)^i - c(bc)^{i-1}) \\ &= f_b^- \left(- \sum_{i=1}^{m-1} \lambda_{(bc)^i} b(cb)^i + \sum_{i=1}^{m-1} \lambda_{b(cb)^i} (cb)^i \right). \end{aligned}$$

Part b) of the lemma is proved similarly. $\square$

Proof of Proposition 3.14. Assume that $x = (r_1, r_2, r_3, r_4) \in \text{Ker } \delta^3$. Using Lemma 3.15, from the fourth and fifth equations in (3.37) we deduce the relations $r_3 = B^- * u$, $r_3 = C^- * v$ for some $u, v \in R$. The proof of Proposition 3.13 shows that the element $\delta^2(0, u, v) = (\rho^- * u, -\omega^- * v, r_3, r_4)$ can be represented as a linear combination of elements in (3.33)–(3.36). Replacing x by $x - \delta^2(0, u, v) = (r_1 - \rho^- * u, r_2 + \omega^- * v, 0, 0)$, we may assume that in the initial element $x \in \text{Ker } \delta^3$ we have $r_3 = r_4 = 0$.

We consider the canonical expansions

$$(3.44) \quad r_1 = \sum_{g \in G} \lambda_g g, \quad r_2 = \sum_{g \in G} \mu_g g \quad (\lambda_g, \mu_g \in \mathbb{Z})$$

and substitute formulas (3.44) in the first equation in (3.37). It is easily verified that $\zeta^+ * r, \eta^+ * r \in Z(R)$ for any $r \in R$. Hence, the element on the left-hand side of this equation can be expanded in a basis of $Z(R)$ (see Proposition 3.1).

We introduce an auxiliary notation:

$$(3.45) \quad \Phi_1 := (bc)^m \theta_b, \quad \Phi_2 := (bc)^m \theta_c.$$

By Corollary 3.5, if m is even, then $\Phi_1 = \theta_b$ and $\Phi_2 = \theta_c$, and if m is odd, then $\Phi_1 = \theta_c$ and $\Phi_2 = \theta_b$. A direct calculation shows that

$$\begin{aligned} (3.46) \quad & \zeta^+ * r_1 - \eta^+ * r_2 = 2m \left(\lambda_{c(bc)^{m-1}} - \mu_{b(cb)^{m-1}} \right) \cdot 1 + 2m \left(\lambda_b - \mu_c \right) \cdot (bc)^m \\ & + 2 \left(\sum_{j \text{ is even}} \lambda_{(bc)^j} + \sum_{j \text{ is even}} \lambda_{(cb)^j} - \sum_{j \text{ is odd}} \mu_{(bc)^j} - \sum_{j \text{ is odd}} \mu_{(cb)^j} \right) \cdot \Phi_1 \\ & + 2 \left(\sum_{j \text{ is odd}} \lambda_{(bc)^j} + \sum_{j \text{ is odd}} \lambda_{(cb)^j} - \sum_{j \text{ is even}} \mu_{(bc)^j} - \sum_{j \text{ is even}} \mu_{(cb)^j} \right) \cdot \Phi_2 \\ & + m \sum_{j=1}^{m-1} \left(\lambda_{b(cb)^{m-j}} + \lambda_{c(bc)^{m-1-j}} - \mu_{b(cb)^{m-1-j}} - \mu_{c(bc)^{m-j}} \right) \cdot \left((bc)^j + (cb)^j \right). \end{aligned}$$

Consequently, the first equation in (3.37) is equivalent to the system of equations (over K) for λ_g, μ_g ($g \in G$) that is obtained by equating the coefficients of the decomposition (3.46) to zero.

Now, the second and third equations in (3.37) imply the following relations (cf. the proof of Proposition 3.9):

$$(3.47) \quad \lambda_{(bc)^j} = \lambda_{(cb)^j}, \quad \lambda_{b(cb)^j} = \lambda_{c(bc)^{j-1}} \quad \text{for } 1 \leq j \leq m-1;$$

$$(3.48) \quad \mu_{(bc)^j} = \mu_{(cb)^j}, \quad \mu_{c(bc)^j} = \mu_{b(cb)^{j-1}} \quad \text{for } 1 \leq j \leq m-1.$$

A direct verification shows that the elements in (3.38) and (3.39) lie in $\text{Ker } \delta^3$. Consequently, subtracting suitable multiples of these elements, we may assume that for the

initial element $x \in \text{Ker } \delta^3$ we have

$$\begin{aligned}\lambda_b &= \lambda_{c(bc)^{m-1}} = \mu_c = \mu_{b(cb)^{m-1}} = 0; \\ \lambda_{b(cb)^j} &= \lambda_{c(bc)^{j-1}} = \mu_{c(bc)^j} = \mu_{b(cb)^{j-1}} = 0 \quad \text{for } 1 \leq j \leq m-1.\end{aligned}$$

Assume additionally that m is even: $m = 2k$. In this case,

$$\left(1 - (bc)^m, 0, 0, 0\right), \left(0, 1 - (bc)^m, 0, 0\right) \in \text{Ker } \delta^3.$$

Adding to x suitable multiples of these elements, we may assume additionally that $\lambda_{(bc)^m} = \mu_{(bc)^m} = 0$. Then the first equation in (3.37), combined with (3.47) and (3.48), implies the following relations for the remaining coefficients of the canonical expansions (3.44):

$$(3.49) \quad \lambda_1 + 2 \sum_{t=1}^{k-1} \lambda_{(bc)^{2t}} = 2 \sum_{t=1}^k \mu_{(bc)^{2t-1}},$$

$$(3.50) \quad \mu_1 + 2 \sum_{t=1}^{k-1} \mu_{(bc)^{2t}} = 2 \sum_{t=1}^k \lambda_{(bc)^{2t-1}}.$$

It is easily seen that the set of (integral) solutions of equation (3.49) is generated by the elements

$$(3.51) \quad \left((bc)^{2i} + (cb)^{2i}, (bc)^{2i+1} + (cb)^{2i+1}, 0, 0\right), \quad 0 \leq i \leq \frac{m-2}{2};$$

$$(3.52) \quad \left((bc)^{2i} + (cb)^{2i}, (bc)^{2i-1} + (cb)^{2i-1}, 0, 0\right), \quad 1 \leq i \leq \frac{m-2}{2}.$$

By symmetry, the set of solutions of equation (3.50) is generated by the elements

$$(3.53) \quad \left((bc)^{2i+1} + (cb)^{2i+1}, (bc)^{2i} + (cb)^{2i}, 0, 0\right), \quad 0 \leq i \leq \frac{m-2}{2};$$

$$(3.54) \quad \left((bc)^{2i-1} + (cb)^{2i-1}, (bc)^{2i} + (cb)^{2i}, 0, 0\right), \quad 1 \leq i \leq \frac{m-2}{2}.$$

Clearly, the union of the sets of elements in (3.51)–(3.54) coincides with the union of the sets (3.41) and (3.42). Consequently, we have proved that the set of elements described in Proposition 3.14 a) generates $\text{Ker } \delta^3$. Since its cardinality is equal to $\text{rk Im } \delta^2 = 7m-3$, this set is linearly independent.

Now assume that m is odd. Then

$$\left(1, (bc)^m, 0, 0\right), \left((bc)^m, 1, 0, 0\right) \in \text{Ker } \delta^3.$$

Adding to x suitable multiples of these elements, again we may assume that $\lambda_{(bc)^m} = \mu_{(bc)^m} = 0$. Then we can argue much as in the preceding case to obtain the desired statement. $\square$

Proposition 3.16. a) *If m is even, then the group $\text{HH}^3(R)$ is generated by the set of cohomology classes of the following elements:*

$$(3.55) \quad \begin{aligned} v_1 &:= (b, c, 0, 0), & v'_1 &:= (c(bc)^{m-1}, b(cb)^{m-1}, 0, 0), \\ v_2 &:= (1 - (bc)^m, 0, 0, 0), & v'_2 &:= (0, 1 - (bc)^m, 0, 0), \end{aligned}$$

which satisfy the relations

$$2v_j = 2v'_j = 0 \quad \text{for } j \in \{1, 2\}.$$

b) If m is odd, then the group $\mathrm{HH}^3(R)$ is generated by the set of cohomology classes of the elements

$$\tilde{v}_2 := (1, (bc)^m, 0, 0), \quad \tilde{v}_2' := ((bc)^m, 1, 0, 0),$$

together with the cohomology classes of the elements v_1, v_1' in (3.55), which satisfy the relations

$$2v_1 = 2v_1' = 2\tilde{v}_2 = 2\tilde{v}_2' = 0.$$

In particular, for any $m \geq 2$ we have

$$\mathrm{HH}^3(R) \simeq \mathbb{Z}_2^4.$$

Proof. a) Assume that m is even: $m = 2k$. Since

$$\begin{aligned} 2\left(1 - (bc)^m, 0, 0, 0\right) &= \sum_{t=0}^{k-1} \left((bc)^{2t} + (cb)^{2t}, (bc)^{2t+1} + (cb)^{2t+1}, 0, 0\right) \\ &\quad - \sum_{t=1}^k \left((bc)^{2t} + (cb)^{2t}, (bc)^{2t-1} + (cb)^{2t-1}, 0, 0\right), \end{aligned}$$

we can replace the element $(2(bc)^m, (bc)^{m-1} + (cb)^{m-1}, 0, 0)$ in the basis of the $\mathbb{Z}$ -module $\mathrm{Im} \delta^2$ indicated in Proposition 3.13 (see (3.32)) by the element $2(1 - (bc)^m, 0, 0, 0)$. By symmetry, we can replace the element $((bc)^{m-1} + (cb)^{m-1}, 2(bc)^m, 0, 0)$ in this basis by the element $2(0, 1 - (bc)^m, 0, 0)$. The desired statement follows from the fact that the new basis for $\mathrm{Im} \delta^2$ and the basis of $\mathrm{Ker} \delta^3$ described in Proposition 3.14 are stacked bases.

b) Assume that m is odd: $m = 2k + 1$. Then

$$\begin{aligned} 2\left(1, (bc)^m, 0, 0\right) &= \sum_{t=0}^k \left((bc)^{2t} + (cb)^{2t}, (bc)^{2t+1} + (cb)^{2t+1}, 0, 0\right) \\ &\quad - \sum_{t=1}^k \left((bc)^{2t} + (cb)^{2t}, (bc)^{2t-1} + (cb)^{2t-1}, 0, 0\right). \end{aligned}$$

Hence, we can replace the element $((bc)^{m-1} + (cb)^{m-1}, 2(bc)^m, (0, 0))$ in the basis of $\mathrm{Im} \delta^2$ by the element $2(1, (bc)^m, 0, 0)$. By symmetry, the element $(2(bc)^m, (bc)^{m-1} + (cb)^{m-1}, 0, 0)$ in the basis of $\mathrm{Im} \delta^2$ can be replaced by $2((bc)^m, 1, 0, 0)$. As in the preceding case, we obtain stacked bases of the modules $\mathrm{Im} \delta^2 \subset \mathrm{Ker} \delta^3$, and the desired statement follows. $\square$

Proposition 3.17. *The free $\mathbb{Z}$ -module $\mathrm{Im} \delta^3$ admits a basis formed by the following elements:*

$$(3.56) \quad (2\Phi_1, O_4), \quad (2m(bc)^m, O_4), \quad (2m, O_4);$$

$$(3.57) \quad (m((bc)^{m-j} + (cb)^{m-j}), (cb)^j - (bc)^j, O_3), \quad 1 \leq j \leq m-1;$$

$$(3.58) \quad (2\Phi_1(bc)^j, c(bc)^{j-1} - b(cb)^j, O_3), \quad 1 \leq j \leq m-1;$$

$$(3.59) \quad (2m((bc)^{m-j} + (cb)^{m-j}), O_4), \quad 1 \leq j \leq m-1;$$

$$(3.60) \quad (2\Phi_2, O_4), \quad (O_3, 2b, 0), \quad (O_3, 2, 0);$$

$$(3.61) \quad \left(m((bc)^{m-j} + (cb)^{m-j}), 0, (bc)^j - (cb)^j, O_2 \right), \quad 1 \leq j \leq m-1;$$

$$(3.62) \quad \left(2\Phi_2(bc)^j, 0, (bc)^{j-1}b - c(bc)^j, O_2 \right), \quad 1 \leq j \leq m-1;$$

$$(3.63) \quad \left(0, (cb)^{m-j} - (bc)^{m-j}, 0, (bc)^j + (cb)^j, 0 \right), \quad 1 \leq j \leq m-1;$$

$$(3.64) \quad \left(0, c(bc)^{m-j-1} - b(cb)^{m-j}, 0, c(bc)^{j-1} + b(cb)^j, 0 \right), \quad 1 \leq j \leq m-1;$$

$$(3.65) \quad \left(O_3, 2(bc)^m, 0 \right), \quad \left(O_3, 2c(bc)^{m-1}, 0 \right);$$

$$(3.66) \quad \left(O_4, 2c \right), \quad \left(O_4, 2 \right), \quad \left(O_4, 2(bc)^m \right), \quad \left(O_4, 2b(cb)^{m-1} \right);$$

$$(3.67) \quad \left(O_2, (cb)^{m-j} - (bc)^{m-j}, 0, (bc)^j + (cb)^j \right), \quad 1 \leq j \leq m-1;$$

$$(3.68) \quad \left(O_2, c(bc)^{m-j} - b(cb)^{m-j-1}, 0, c(bc)^j + b(cb)^{j-1} \right), \quad 1 \leq j \leq m-1.$$

Remark 3.18. Recall that Φ_1 and Φ_2 were defined in (3.45).

Proof of Proposition 3.17. It is verified directly that the elements on the list (3.56)–(3.68) coincide with the following values of the differential δ^3 :

$$\begin{aligned} & \delta^3(1, O_3), \quad \delta^3(b, O_3), \quad \delta^3(c(bc)^{m-1}, O_3); \\ & \delta^3(b(cb)^j, O_3) \quad (1 \leq j \leq m-1); \quad \delta^3((bc)^j, O_3) \quad (1 \leq j \leq m-1); \\ & \delta^3(b(cb)^j + c(bc)^{j-1}, O_3) \quad (1 \leq j \leq m-1); \\ & \delta^3(0, -1, O_2), \quad \delta^3(O_2, 1, 0), \quad \delta^3(O_2, b, 0); \\ & \delta^3(0, -b(cb)^{j-1}, O_2) \quad (1 \leq j \leq m-1); \quad \delta^3(0, -(bc)^j, O_2) \quad (1 \leq j \leq m-1); \\ & \delta^3(O_2, b(cb)^j, 0) \quad (1 \leq j \leq m-1); \quad \delta^3(O_2, (bc)^j, 0) \quad (1 \leq j \leq m-1); \\ & \delta^3(O_2, c(bc)^{m-1}, 0), \quad \delta^3(O_2, (bc)^m, 0), \quad \delta^3(O_3, 1), \quad \delta^3(O_3, c); \\ & \delta^3(O_3, b(cb)^{m-1}), \quad \delta^3(O_3, (bc)^m); \\ & \delta^3(O_3, c(bc)^j, 0) \quad (1 \leq j \leq m-1); \quad \delta^3(O_3, (bc)^j) \quad (1 \leq j \leq m-1). \end{aligned}$$

Using Proposition 3.14, we easily see that these elements generate $\text{Im } \delta^3$; moreover, the cardinality of the set of elements in (3.56)–(3.68) is equal to $9m + 3$. Since $\text{rk Im } \delta^3 = \text{rk } R^4 - \text{rk Ker } \delta^2 = 16m - (7m - 3) = 9m + 3$, this set is linearly independent. $\square$

Remark 3.19. For what follows, we describe another basis of $\text{Im } \delta^3$; namely, if in the basis presented in Proposition 3.17 we replace the subset of elements of the form (3.58), (3.62), (3.64), and (3.68) by the family

$$\begin{aligned} & \left(0, c(bc)^{j-1} - b(cb)^j, O_3 \right), \quad \left(O_2, (bc)^{j-1}b - c(bc)^j, O_2 \right), \\ & \left(O_3, c(bc)^{j-1} + b(cb)^j, 0 \right), \quad \left(O_4, c(bc)^j + b(cb)^{j-1} \right) \end{aligned}$$

for all $1 \leq j \leq m-1$, then we obtain a basis of $\text{Im } \delta^3$.

Proposition 3.20. *The free $\mathbb{Z}$ -module $\text{Ker } \delta^4$ admits a basis formed by the following elements:*

$$\begin{aligned} & (1, O_4), \quad (\theta_b, O_4), \quad (\theta_c, O_4), \quad ((bc)^m, O_4); \\ & ((bc)^i + (cb)^i, O_4), \quad 1 \leq i \leq m-1; \\ (3.69) \quad & (0, (bc)^i - (cb)^i, O_3), \quad 1 \leq i \leq m-1; \end{aligned}$$

$$\begin{aligned}
(3.70) \quad & \left(0, b(cb)^i - c(bc)^{i-1}, O_3\right), 1 \leq i \leq m-1; \\
& \left(O_2, (bc)^i - (cb)^i, O_2\right), 1 \leq i \leq m-1; \\
& \left(O_2, c(bc)^i - b(cb)^{i-1}b, O_2\right), 1 \leq i \leq m-1; \\
& \left(O_3, 1, 0\right), \left(O_3, b, 0\right), \left(O_3, c(bc)^{m-1}, 0\right), \left(O_3, (bc)^m, 0\right); \\
(3.71) \quad & \left(O_3, (bc)^i + (cb)^i, 0\right), 1 \leq i \leq m-1; \\
& \left(O_3, b(cb)^i + c(bc)^{i-1}, 0\right), 1 \leq i \leq m-1; \\
& \left(O_4, 1\right), \left(O_4, c\right), \left(O_4, b(cb)^{m-1}\right), \left(O_4, (bc)^m\right); \\
(3.72) \quad & \left(O_4, (bc)^i + (cb)^i\right), 1 \leq i \leq m-1; \\
& \left(O_4, c(bc)^i + b(cb)^{i-1}\right), 1 \leq i \leq m-1.
\end{aligned}$$

Proof. Since

$$d_4^Q = \begin{pmatrix} B^- & C^- & 0 & 0 & 0 & 0 \\ -\rho^+ & 0 & B^+ & 0 & 0 & 0 \\ 0 & \omega^+ & 0 & C^+ & 0 & 0 \\ 0 & 0 & \rho^- & 0 & B^- & 0 \\ 0 & 0 & 0 & -\omega^- & 0 & C^- \end{pmatrix},$$

the condition $(r_1, r_2, r_3, r_4, r_5) \in \text{Ker } \delta^4$ is equivalent to the following system of equations:

$$\begin{aligned}
(3.73) \quad & B^- * r_1 - \rho^+ * r_2 = 0, \\
& C^- * r_1 + \omega^+ * r_3 = 0, \\
& B^+ * r_2 + \rho^- * r_4 = 0, \\
& C^+ * r_3 - \omega^- * r_5 = 0, \\
& B^- * r_4 = 0, \\
& C^- * r_5 = 0.
\end{aligned}$$

Using Lemma 3.10, we observe that this system is equivalent to the following relations:

$$\begin{aligned}
B^- * r_1 &= 0; & B^+ * r_2 &= 0; & B^- * r_4 &= 0; \\
C^- * r_1 &= 0; & C^+ * r_3 &= 0; & C^- * r_5 &= 0
\end{aligned}$$

(to see this, we simplify system (3.73) “from bottom to top”). Next, using Proposition 3.1 and arguments in the proof of Proposition 3.9, we arrive at the above basis of $\text{Ker } \delta^4$. $\square$

Proposition 3.21. *The group $\text{HH}^4(R)$ is generated by the set of cohomology classes of the following elements:*

$$\begin{aligned}
w_0^{(i)} &:= \left((bc)^i + (cb)^i, O_4\right) \text{ for } 1 \leq i \leq m-1; \\
w_1 &:= \left(1, O_4\right), w'_1 := \left((bc)^m, O_4\right), w_2 := \left(\theta_b, O_4\right), w'_2 := \left(\theta_c, O_4\right), \\
w_3 &:= \left(O_3, 1, 0\right), w'_3 := \left(O_4, 1\right), w_4 := \left(O_3, b, 0\right), w'_4 := \left(O_4, c\right), \\
w_5 &:= \left(O_3, c(bc)^{m-1}, 0\right), w'_5 := \left(O_4, b(cb)^{m-1}\right), \\
w_6 &:= \left(O_3, (bc)^m, 0\right), w'_6 := \left(O_4, (bc)^m\right),
\end{aligned}$$

which satisfy the relations

$$\begin{aligned} 2mw_0^{(i)} &= 0 \text{ for } 1 \leq i \leq m-1; \quad 2mw_1 = 2mw'_1 = 0; \\ 2w_j &= 2w'_j = 0 \text{ for } 2 \leq j \leq 6. \end{aligned}$$

In particular, we have

$$\mathrm{HH}^4(R) \simeq \mathbb{Z}_{2m}^{m+1} \oplus \mathbb{Z}_2^{10}.$$

Proof. In the basis of the group $\mathrm{Ker} \delta^4$ described in Proposition 3.20, we replace the subset of elements of the form (3.69), (3.70), (3.71), (3.72) by the set of elements of the form (3.57), (3.61), (3.63), and (3.67). It is clear that the resulting collection is a basis for $\mathrm{Ker} \delta^4$. Moreover, this new basis and the basis of the group $\mathrm{Im} \delta^3$ presented in Remark 3.19 are stacked bases, whence the desired statement follows. $\square$

Proposition 3.22. *For any $n \geq 5$, we have*

$$\mathrm{HH}^n(R) \simeq \mathrm{HH}^{n-4}(R) \oplus \mathbb{Z}_2^8.$$

Proof. By Proposition 2.7, we have the following short exact sequences of complexes:

$$0 \rightarrow \mathrm{Hom}_\Lambda(Q_\bullet[-4], R) \xrightarrow{\pi^*} \mathrm{Hom}_\Lambda(Q_\bullet, R) \xrightarrow{i^*} \mathcal{X}^\bullet \rightarrow 0,$$

where $\mathcal{X}^\bullet = \mathrm{Hom}_\Lambda(X_\bullet, R)$. In its turn, this leads to the long exact cohomology sequence

$$(3.74) \quad \dots \xrightarrow{\Delta^{n-1}} \mathrm{HH}^{n-4}(R) \xrightarrow{\pi^*} \mathrm{HH}^n(R) \xrightarrow{i^*} \mathrm{H}^n(\mathcal{X}^\bullet) \xrightarrow{\Delta^n} \mathrm{HH}^{n-3}(R) \xrightarrow{\pi^*} \dots$$

Now, we need to compute the cohomology for the auxiliary complex $\mathcal{X}^\bullet$.

Lemma 3.23. *For any $n \geq 4$, we have $\mathrm{H}^n(\mathcal{X}^\bullet) \simeq \mathbb{Z}_2^8$.*

Proof. Observe that $\mathcal{X}^n \simeq R^4$ for $n \geq 3$. Hence, as at the beginning of this section, we may identify a cochain in $\mathcal{X}^n$ with the collection of its values at the corresponding generators. Also, we note that the complex $\mathcal{X}^\bullet$ is 2-periodic starting with the term $\mathcal{X}^3$.

a) If n is odd, then the condition $(r_1, r_2, r_3, r_4) \in \mathrm{Ker} \delta_\mathcal{X}^n$ is equivalent to the system of equations obtained from (3.37) by omitting the first equation. Arguing as at the beginning of the proof of Proposition 3.14, we may assume that $r_3 = 0 = r_4$. Then $B^- * r_1 = 0$ and $C^- * r_2 = 0$. As in the proof of Proposition 3.9, we see that $\mathrm{Ker} \delta_\mathcal{X}^n$ has a basis formed by the elements

$$(3.75) \quad (1, O_3), (b, O_3), (c(bc)^{m-1}, O_3), ((bc)^m, O_3);$$

$$(3.76) \quad \begin{aligned} &(0, 1, O_2), (0, c, O_2), (0, b(cb)^{m-1}, O_2), (0, (bc)^m, O_2); \\ &((bc)^i + (cb)^i, O_3), (c(bc)^{i-1} + b(cb)^i, O_3), \quad 1 \leq i \leq m-1; \\ &(0, (bc)^i + (cb)^i, O_2), (0, b(cb)^{i-1} + c(bc)^i, O_2), \quad 1 \leq i \leq m-1. \end{aligned}$$

We can verify directly (cf. the proof of Proposition 3.8) that $\mathrm{Im} \delta_\mathcal{X}^{n-1}$ admits the following basis:

$$\begin{aligned} &(2, O_3), (2, O_3), (2c(bc)^{m-1}, O_3), (2(bc)^m, O_3); \\ &(0, 2, O_2), (0, 2c, O_2), (0, 2b(cb)^{m-1}, O_2), (0, 2(bc)^m, O_2); \\ &((bc)^i + (cb)^i, O_3), (c(bc)^{i-1} + b(cb)^i, O_3), \quad 1 \leq i \leq m-1; \\ &(0, (bc)^i + (cb)^i, O_2), (0, b(cb)^{i-1} + c(bc)^i, O_2), \quad 1 \leq i \leq m-1. \end{aligned}$$

Since these bases of the $\mathbb{Z}$ -modules $\text{Im } \delta_{\mathcal{X}}^{n-1}$ and $\text{Ker } \delta_{\mathcal{X}}^n$ are stacked, the desired statement follows. Moreover, the cohomology classes of the elements in (3.75)–(3.76) form a set of generators for $H^n(\mathcal{X}^\bullet)$, and the order of every element of this set in the group $H^n(\mathcal{X}^\bullet)$ is equal to 2.

b) In the case where n is even, the proof is similar. In this case we can take a generating set for $H^n(\mathcal{X}^\bullet)$ formed by the cohomology classes of the elements

$$\begin{aligned} & \left(O_2, 1, 0 \right), \left(O_2, b, 0 \right), \left(O_2, c(bc)^{m-1}, 0 \right), \left(O_2, (bc)^m, 0 \right); \\ & \left(O_3, 1 \right), \left(O_3, c \right), \left(O_3, b(cb)^{m-1} \right), \left(O_3, (bc)^m \right). \end{aligned}$$

The order of each class in this set is equal to 2. $\square$

We continue the proof of Proposition 3.22. First, we prove that the connecting homomorphisms Δ^n in the sequence (3.74) are zero if $n \geq 4$.

Case 1: Assume that $n = 2k + 1$ with $k \in \mathbb{N}$, $k \geq 2$. We consider a $(2k + 1)$ -cocycle $f = (r_1, r_2, r_3, r_4) \in \mathcal{X}^{2k+1}$, i.e. $\delta_{\mathcal{X}}^{2k+1}(f) = 0$. By the description of the generating set for $H^{2k+1}(\mathcal{X}^\bullet)$ given in the proof of Lemma 3.23 (see (3.75) and (3.76)), we may assume that $r_3 = 0 = r_4$. Then we also have $B^- * r_1 = 0 = C^- * r_2$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_\Lambda(Q_{2k-3}, R) & \xrightarrow{\pi^*} & \text{Hom}_\Lambda(Q_{2k+1}, R) & \xrightarrow{i^*} & \mathcal{X}^{2k+1} \longrightarrow 0 \\ & & \delta^{2k-3} \downarrow & & \downarrow \delta^{2k+1} & & \downarrow \delta_{\mathcal{X}}^{2k+1} \\ 0 & \longrightarrow & \text{Hom}_\Lambda(Q_{2k-2}, R) & \xrightarrow{\pi^*} & \text{Hom}_\Lambda(Q_{2k+2}, R) & \xrightarrow{i^*} & \mathcal{X}^{2k+2} \longrightarrow 0. \end{array}$$

Define $\tilde{f} = f \circ \nu$, where $\nu: Q_{2k+1} \rightarrow X_{2k+1}$ is the projection onto the corresponding direct summand. In other words, the map $\tilde{f}$ is obtained from f via extension by zero to the direct summand complementary to X_{2k+1} in Q_{2k+1} ; thus, $\tilde{f} = (O_{2k-2}, r_1, r_2, 0, 0)$. Consequently, $i^*(\tilde{f}) = \tilde{f} \circ i = f \circ \nu \circ i = f$. By the construction of the connecting homomorphism, we have $\Delta^{2k+1}(\text{cl } f) = \text{cl } g$, where $g: Q_{2k-2} \rightarrow R$ is a cocycle for which $\pi^*(g) = \delta^{2k+1}(\tilde{f})$.

Consider the cochain $\varphi = (O_{2k-4}, r_1 \cdot (bc)^m, -r_2 \cdot (bc)^m) \in \text{Hom}_\Lambda(Q_{2k-3}, R)$. If $k \geq 3$, then by Lemma 3.10,

$$\begin{aligned} \delta^{2k-3}(\varphi) &= (O_{2k-5}, -\rho^- * r_1(bc)^m, -\omega^- * \rho_2(bc)^m, B^+ * r_1(bc)^m, -C^+ * r_2(bc)^m) \\ &= (O_{2k-3}, \rho^+ * r_1, -\omega^+ * r_2). \end{aligned}$$

Consequently, $\pi^*(\delta^{2k-3}(\varphi)) = \delta^{2k+1}(\tilde{f})$. Putting $g = \delta^{2k-3}(\varphi)$ we obtain $\Delta^{2l+1}(\text{cl } f) = \text{cl } g = 0$.

Before studying the case where $k = 2$, we prove the following auxiliary statement.

Lemma 3.24. *Let $f_{\zeta^-}, f_{\eta^-}: R \rightarrow R$ be homomorphisms such that $f_{\zeta^-}(r) = \zeta^- * r$ and $f_{\eta^-}(r) = \eta^- * r$, and let $f_b^-, f_c^-: R \rightarrow R$ be the homomorphisms occurring in Lemma 3.15. Then*

a) $\text{Ker } f_b^- \subset \text{Ker } f_{\zeta^-}$; b) $\text{Ker } f_c^- \subset \text{Ker } f_{\eta^-}$.

Proof. a) For any $r \in \text{Ker } f_b^-$, we have

$$\begin{aligned} \sum_{i=0}^{m-1} c(bc)^i \cdot r \cdot (cb)^{m-1-i} &= \sum_{i=0}^{m-1} c(bc)^i (cb)^m \cdot r \cdot (cb)^{-1-i} = \sum_{i=0}^{m-1} (bc)^{m-1-i} b \cdot r \cdot (bc)^{1+i} \\ &= \sum_{i=0}^{m-1} (bc)^{m-1-i} \cdot r \cdot b(bc)^{1+i} = \sum_{j=0}^{m-1} (bc)^j \cdot r \cdot (cb)^{m-1-j} c \end{aligned}$$

(in the latter identity we have changed the summation: $j = m - 1 - i$). Consequently, $\zeta^- * r = 0$.

Part b) follows from a) by symmetry. $\square$

We continue the proof of Proposition 3.22. Assume that $k = 2$, i.e., $n = 5$. For the 1-cochain $\varphi = (r_1 \cdot (bc)^m, -r_2 \cdot (bc)^m)$, which has already been used, Lemma 3.24 shows that

$$\begin{aligned} \delta^1(\varphi) &= (-\zeta^- * r_1(bc)^m - \eta^- * r_2(bc)^m, B^+ * r_1(bc)^m, -C^+ * r_2(bc)^m) \\ &= (0, \rho^+ * r_1, -\omega^+ * r_2). \end{aligned}$$

Putting again $g = \delta^1(\varphi)$, we obtain $\Delta^5(\text{cl } f) = \text{cl } g = 0$.

Case 2: Assume that $n = 2k$ with $k \in \mathbb{N}$, $k \geq 2$.

Lemma 3.25. a) If $B^+ * r + \rho^- * s = 0$ with $r, s \in R$, then $B^+ * r = 0 = \rho^- * s$.

b) If $C^+ * r + \omega^- * s = 0$ with $r, s \in R$, then $C^+ * r = 0 = \omega^- * s$.

Proof. a) Since

$$2(r + brb) = B^+ * (B^+ * r) = -B^+ * (\rho^- * r) = -(B^+ \rho^-) * r = 0,$$

we obtain $br + rb = b(r + brb) = 0$.

Part b) follows from a) by symmetry. $\square$

We continue the proof of Proposition 3.22. Consider a $2k$ -cocycle

$$f \in \mathcal{X}^{2k} = (r_1, r_2, r_3, r_4) \in \text{Hom}_\Lambda(X_{2k}, R),$$

i.e., $\delta_{\mathcal{X}}^{2k}(f) = 0$. Using Lemma 3.25 we see, in particular, that $B^+ * r_1 = 0 = C^+ * r_2$. Set $\tilde{f} = f \circ \nu$, where $\nu: Q_{2k} \rightarrow X_{2k}$ is the projection onto the corresponding direct summand. Then $\delta^{2k}(\tilde{f}) = (O_{2k-4}, -\rho^+ * r_1, \omega^+ * r_1, O_4) = 0$, whence $\Delta^{2k}(\text{cl } f) = 0$.

Consequently, for any $n \geq 5$ we have the short exact sequence

$$(3.77) \quad 0 \rightarrow \text{HH}^{n-4}(R) \xrightarrow{\pi^*} \text{HH}^n(R) \xrightarrow{i^*} \text{H}^n(\mathcal{X}^\bullet) \rightarrow 0.$$

Since $\text{HH}^1(R) = 0$ (see Proposition 3.7), the sequence (3.77) and Lemma 3.23 show that $\text{HH}^5(R) \simeq \mathbb{Z}_2^8$. Moreover, (3.77) yields the following identity:

$$(3.78) \quad |\text{HH}^n(R)| = |\text{HH}^{n-4}(R)| + 2^8.$$

Now we assume that n is odd: $n = 2l + 1$ with $l \geq 3$. Observe that the matrix of the differential d_{2l+1}^Q has the following block structure:

$$(3.79) \quad d_{2l+1}^Q = \left(\begin{array}{cc|cc} d_{2l-3}^Q & & O_{2l-2,4} & \\ \hline O_{2,2l-3} & \sigma^+ & D^- & O \\ O_{2,2l-3} & O & -\sigma^- & D^+ \end{array} \right);$$

we have used the abbreviated notation (2.4) and (2.5). It is easily seen that the following cochains in $\text{Hom}_\Lambda(Q_{2l+1}, R)$ are cocycles:

$$\begin{aligned} f_1 &:= (O_{2l-4}, -(bc)^m, 0, 1, O_3), & f_2 &:= (O_{2l-4}, -c(bc)^{m-1}, 0, b, O_3), \\ f_3 &:= (O_{2l-4}, -b, 0, c(bc)^{m-1}, O_3), & f_4 &:= (O_{2l-4}, -1, 0, (bc)^m, O_3), \\ f_5 &:= (O_{2l-3}, (bc)^m, 0, 1, O_2), & f_6 &:= (O_{2l-3}, b(cb)^{m-1}, 0, c, O_2), \\ f_7 &:= (O_{2l-3}, c, 0, b(cb)^{m-1}, O_2), & f_8 &:= (O_{2l-3}, 1, 0, (bc)^m, O_2). \end{aligned}$$

Let L be a subgroup of $\text{HH}^{2l+1}(R)$ generated by the set $\{\text{cl } f_i\}_{i=1}^8$. We shall prove that

$$(3.80) \quad \text{HH}^{2l+1}(R) = \pi^*(\text{HH}^{2l-3}(R)) + L.$$

We write any cocycle $x \in \text{Ker } \delta^{2l+1}$ in the form $x = (x', x'')$, where $x' \in \text{Hom}_\Lambda(Q_{2l-3}, R)$ and $x'' \in \text{Hom}_\Lambda(X_{2l+1}, R)$. By (3.79), we have $x'' = (r_1, r_2, r_3, r_4) \in \text{Ker } \delta_{\mathcal{X}}^{2l+1}$. Since $D^+(r_3, r_4) = 0$, Lemma 3.15 shows that $r_3 = B^- * u$ and $r_4 = C^- * u'$ for some $u, u' \in R$.

Replacing x by $x - \delta^{2l}(O_{2l-1}, u, u')$, we may assume that $r_3 = r_4 = 0$ for the initial x . Subtracting from x a suitable linear combination of elements of the set $\{f_i\}_{i=1}^8$ and using the arguments of part a) of the proof of Lemma 3.23, we get an element of the form $y = (y', O_4) \in \text{Ker } \delta^{2l+1}$. Using (3.79), we see that $y' \in \text{Ker } \delta^{2l-3}$. Hence, we have $y \in \text{Im } \pi^*$, so that $x \in \text{Im } \pi^* + L$. This completes the proof of (3.80).

Now, we observe that the matrix of the differential d_{2l}^Q has the following block structure:

$$(3.81) \quad d_{2l}^Q = \left(\begin{array}{cc|cc} d_{2l-4}^Q & & O_{2l-3,4} & \\ \hline O_{2,2l-4} & -\sigma^+ & D^+ & O \\ O_{2,2l-4} & O & \sigma^- & D^- \end{array} \right).$$

Consider the following cochains in $\text{Hom}_\Lambda(Q_{2l}, R)$:

$$\begin{aligned} g_1 &:= (O_{2l-3}, b, O_3), & g_2 &:= (O_{2l-3}, 1, O_3), \\ g_3 &:= (O_{2l-3}, (bc)^m, O_3), & g_4 &:= (O_{2l-3}, c(bc)^{m-1}, O_3), \\ g_5 &:= (O_{2l-2}, c, O_2), & g_6 &:= (O_{2l-2}, 1, O_2), \\ g_7 &:= (O_{2l-2}, (bc)^m, O_2), & g_8 &:= (O_{2l-2}, b(cb)^{m-1}, O_2). \end{aligned}$$

Using (3.81), we easily check that $\delta^{2l}(g_i) = 2f_i$ for all $i = 1, \dots, 8$. In particular, $|L| \leq 2^8$. Moreover, (3.80) implies the existence of an epimorphism $\text{HH}^{2l-3}(R) \oplus L \rightarrow \text{HH}^{2l+1}(R)$. Then, by (3.78), this epimorphism is an isomorphism and $L \simeq \mathbb{Z}_2^8$.

Finally, we study the groups $\text{HH}^n(R)$ for even n : let $n = 2l$ with $l \geq 3$. It is easily verified that the following cochains in $\text{Hom}_\Lambda(Q_{2l}, R)$ are cocycles:

$$\begin{aligned} f_1 &:= (O_{2l-1}, 1, 0), & f_2 &:= (O_{2l-1}, b, 0), \\ f_3 &:= (O_{2l-1}, c(bc)^{m-1}, 0), & f_4 &:= (O_{2l-1}, (bc)^m, 0), \\ f_5 &:= (O_{2l}, 1), & f_6 &:= (O_{2l}, c), \\ f_7 &:= (O_{2l}, b(cb)^{m-1}), & f_8 &:= (O_{2l}, (bc)^m). \end{aligned}$$

Let L be the subgroup of $\mathrm{HH}^{2l}(R)$ generated by the set $\{\mathrm{cl} f_i\}_{i=1}^8$. Arguing as above, we check that $\mathrm{HH}^{2l}(R) = \pi^*(\mathrm{HH}^{2l-4}(R)) + L$. The cochains

$$\begin{aligned} g_1 &:= (O_{2l-1}, b, 0), & g_2 &:= (O_{2l-1}, 1, 0), \\ g_3 &:= (O_{2l-1}, (bc)^m, 0), & g_4 &:= (O_{2l-1}, c(bc)^{m-1}, 0), \\ g_5 &:= (O_{2l}, c), & g_6 &:= (O_{2l}, 1), \\ g_7 &:= (O_{2l}, (bc)^m), & g_8 &:= (O_{2l}, b(cb)^{m-1}) \end{aligned}$$

satisfy the relations $\delta^{2l-1}(g_i) = 2f_i$ for all $i = 1, \dots, 8$. As above, we conclude that $\mathrm{HH}^{2l}(R) = \mathrm{HH}^{2l-4}(R) \oplus L$ and $L \simeq \mathbb{Z}_2^8$. $\square$

Corollary 3.26. *The group $\mathrm{HH}^5(R)$ is generated by the cohomology classes of the following elements:*

$$(3.82) \quad t_1 := (- (bc)^m, 0, 1, O_3), \quad t_2 := (- c(bc)^{m-1}, 0, b, O_3),$$

$$(3.83) \quad t_3 := (- b, 0, c(bc)^{m-1}, O_3), \quad t_4 := (- 1, 0, (bc)^m, O_3),$$

$$(3.84) \quad t'_1 := (0, (bc)^m, 0, 1, O_2), \quad t'_2 := (0, b(cb)^{m-1}, 0, c, O_2),$$

$$(3.85) \quad t'_3 := (0, c, 0, b(cb)^{m-1}, O_2), \quad t'_4 := (0, 1, 0, (bc)^m, O_2),$$

which satisfy the relations

$$2t_j = 2t'_j = 0 \text{ for } 1 \leq j \leq 4.$$

Proof. Clearly, the images of the cohomology classes of the elements (3.82)–(3.85) under the map $i^*: \mathrm{HH}^5(R) \rightarrow \mathrm{HH}^5(\mathcal{X})$ generate the group $\mathrm{HH}^5(\mathcal{X})$ (see the elements (3.75) and (3.76) indicated in the proof of Lemma 3.23). Since i^* is an isomorphism, the corollary follows. $\square$

Proof of Proposition 1.4. The statement follows by induction from Proposition 3.22 with the help of Propositions 3.1, 3.7, 3.11, 3.16, and 3.21. $\square$

§4. GENERATORS AND RELATIONS

We recall briefly an interpretation of the Yoneda product in the algebra $\mathrm{HH}^*(R) = \bigoplus_{n \geq 0} \mathrm{Ext}_{\Lambda}^n(R, R)$, which was used earlier in [7, 8]. Let $\mu: Q_{\bullet} \rightarrow R$ be the Λ -free resolution constructed in §2. Let $\mathrm{Hom}_{\Lambda}(Q_{\bullet}, R) = (\mathrm{Hom}_{\Lambda}(Q_n, R), \delta^n)$ be the complex (3.1). Any n -cocycle $f \in \mathrm{Ker} \delta^n$ is lifted (uniquely up to homotopy) to a chain map of complexes $\{\varphi_i: Q_{n+i} \rightarrow Q_i\}_{i \geq 0}$. The homomorphism φ_i is called the *i th translate* of the cocycle f and will be denoted by $T^i(f)$. For cocycles $f \in \mathrm{Ker} \delta^n$ and $g \in \mathrm{Ker} \delta^t$ we have $\mathrm{cl} g \cdot \mathrm{cl} f = \mathrm{cl}(\mu T^0(g) T^t(f))$. Consequently, it suffices to know the composition of the translates $T^0(g)$ and $T^t(f)$. In the sequel, using direct decompositions of the modules $Q_i \simeq \Lambda^{i+1}$, we shall describe translates of cocycles and their products in terms of matrices that correspond to such decompositions (cf. the description of the differentials d_n^Q in §3). Observe that, by the graded commutativity of the algebra $\mathrm{HH}^*(R)$, in order to compute the product of a pair of different homogeneous elements in $\mathrm{HH}^*(R)$, it suffices to know, for an element of degree i , its translates of order not exceeding i .

We denote by $\mathcal{Y}$ the set formed by the following homogeneous elements of $\mathrm{HH}^*(R)$ (these elements occurred in the description of the groups $\mathrm{HH}^i(R)$ with $i \leq 4$; see §3):

- of degree 0 : $p_1 := \theta_b, p'_1 := \theta_c, p_2 := bc + cb, p_3 := (bc)^m;$
- of degree 2 : $u_0 := u_0^{(1)} := (bc - cb, O_2), u_1 := (0, 1, 0), u'_1 := (O_2, 1),$
 $u_2 := (0, b, 0), u'_2 := (O_2, c);$
- of degree 3 : $v := v_1 := (b, c, O_2);$
- of degree 4 : $w := w_1 := (1, O_4).$

Remark 4.1. By Proposition 2.7, there is a chain map $\pi: Q_\bullet \rightarrow Q_\bullet[-4]$ for which $\pi_n: Q_n = Q_{n-4} \oplus X_n \rightarrow Q_{n-4}$ is the projection onto the corresponding direct summand. Consequently, w coincides with $\mu\pi_4$ (here, $\mu: Q_0 = \Lambda \rightarrow R$ is the augmentation map of the resolution; see Remark 2.3); moreover, we can take the following translates of the cocycle $w: T^i(w) = \pi_{i+4}$ ($i \geq 0$).

Proposition 4.2. *For the role of translates of the elements in $\mathcal{Y} \setminus \{w\}$ that have positive degree, we can take the homomorphisms determined by the following matrices:*

$$T^0(u_0) = (b \otimes c - c \otimes b, 0, 0), \quad T^1(u_0) = \begin{pmatrix} 0 & 1 \otimes 1 + c \otimes c & 0 & 0 \\ -1 \otimes 1 - b \otimes b & 0 & 0 & 0 \end{pmatrix},$$

$$T^2(u_0) = \begin{pmatrix} cb \otimes 1 - bc \otimes 1 & 0 & 0 & 0 & 0 \\ \star & 0 & 0 & 0 & 0 \\ \star & 0 & 0 & 0 & 0 \end{pmatrix},$$

with

$$T^2(u_0)_{2,1} = - \sum_{i=0}^{m-2} b(cb)^i \otimes c(bc)^{m-2-i} - \sum_{i=0}^{m-1} (cb)^i \otimes (cb)^{m-1-i} - c \otimes c(bc)^{m-1},$$

$$T^2(u_0)_{3,1} = - \sum_{i=0}^{m-2} c(bc)^i \otimes b(cb)^{m-2-i} - \sum_{i=0}^{m-1} (bc)^i \otimes (bc)^{m-1-i} - b \otimes b(cb)^{m-1}$$

at the positions (2, 1) and (3, 1) respectively;

$$T^0(u_1) = (0, 1 \otimes 1, 0), \quad T^1(u_1) = \begin{pmatrix} \sum_{i=0}^{m-2} c(bc)^i \otimes c(bc)^{m-2-i} & 0 & 1 \otimes 1 & 0 \\ \sum_{i=0}^{m-1} (cb)^i \otimes (bc)^{m-1-i} & 0 & 0 & 0 \end{pmatrix},$$

$$T^2(u_1) = \begin{pmatrix} 0 & 1 \otimes 1 & 0 & 0 & 0 \\ \star & 0 & 0 & 1 \otimes 1 & 0 \\ \star & 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$T^2(u_1)_{2,1} = \sum_{i=0}^{m-2} (m-1-i)(bc)^i \otimes (cb)^i + \sum_{i=0}^{m-2} (m-1-i)c(bc)^i \otimes c(bc)^i,$$

$$T^2(u_1)_{3,1} = \sum_{i=0}^{m-1} (m-i)(cb)^i \otimes (bc)^i + \sum_{i=1}^{m-1} (m-i)b(cb)^{i-1} \otimes b(cb)^{i-1};$$

$$T^0(u'_1) = (0, 0, 1 \otimes 1), \quad T^1(u'_1) = \begin{pmatrix} 0 & -\sum_{i=0}^{m-1} (bc)^i \otimes (cb)^{m-1-i} & 0 & 0 \\ 0 & -\sum_{i=0}^{m-2} b(cb)^i \otimes b(cb)^{m-2-i} & 0 & 1 \otimes 1 \end{pmatrix},$$

$$T^2(u'_1) = \begin{pmatrix} 0 & 0 & 1 \otimes 1 & 0 & 0 \\ \star & 0 & 0 & 0 & 0 \\ \star & 0 & 0 & 0 & 1 \otimes 1 \end{pmatrix}$$

with

$$T^2(u'_1)_{2,1} = \sum_{i=0}^{m-1} (m-i)(bc)^i \otimes (cb)^i + \sum_{i=1}^{m-1} (m-i)c(bc)^{i-1} \otimes c(bc)^{i-1},$$

$$T^2(u'_1)_{3,1} = \sum_{i=0}^{m-2} (m-1-i)(cb)^i \otimes (bc)^i + \sum_{i=0}^{m-2} (m-1-i)b(cb)^i \otimes b(cb)^i;$$

$$T^0(u_2) = (0, b \otimes 1, 0), \quad T^1(u_2) = \begin{pmatrix} \sum_{i=1}^{m-1} (bc)^i \otimes c(bc)^{m-1-i} & 0 & b \otimes 1 & 0 \\ \sum_{i=0}^{m-1} b(cb)^i \otimes (bc)^{m-1-i} & 0 & 0 & 0 \end{pmatrix},$$

$$T^2(u_2) = \begin{pmatrix} 0 & b \otimes 1 & 0 & 0 & 0 \\ \star & 0 & 0 & b \otimes 1 & 0 \\ \star & 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$T^2(u_2)_{2,1} = \sum_{i=0}^{m-2} \sum_{j=0}^{m-1-i} (bc)^{i+2j} \otimes b(cb)^i + \sum_{i=1}^{m-1} \sum_{j=0}^{m-1-i} c(bc)^{i+1+2j} \otimes (bc)^i,$$

$$T^2(u_2)_{3,1} = \sum_{i=1}^{m-1} \sum_{j=0}^{m-1-i} b(cb)^{i+2j} \otimes (cb)^i + \sum_{i=0}^{m-1} \sum_{j=0}^{m-1-i} (cb)^{i+1+2j} \otimes c(bc)^i;$$

$$T^0(u'_2) = (0, 0, c \otimes 1), \quad T^1(u'_2) = \begin{pmatrix} 0 & -\sum_{i=1}^{m-1} c(bc)^i \otimes (cb)^{m-1-i} & 0 & 0 \\ 0 & -\sum_{i=0}^{m-1} (cb)^i \otimes b(cb)^{m-1-i} & 0 & c \otimes 1 \end{pmatrix},$$

$$T^2(u'_2) = \begin{pmatrix} 0 & 0 & c \otimes 1 & 0 & 0 \\ \star & 0 & 0 & 0 & 0 \\ \star & 0 & 0 & 0 & c \otimes 1 \end{pmatrix}$$

with

$$T^2(u'_2)_{2,1} = \sum_{i=1}^{m-1} \sum_{j=0}^{m-1-i} c(bc)^{i+2j} \otimes (bc)^i + \sum_{i=0}^{m-1} \sum_{j=0}^{m-1-i} (bc)^{i+1+2j} \otimes b(cb)^i,$$

$$T^2(u'_2)_{3,1} = \sum_{i=0}^{m-2} \sum_{j=1}^{m-1-i} (cb)^{i+2j} \otimes c(bc)^i + \sum_{i=1}^{m-1} \sum_{j=0}^{m-1-i} b(cb)^{i+1+2j} \otimes (cb)^i;$$

$$\begin{aligned}
T^0(v) &= (b \otimes 1, c \otimes 1, 0, 0), \quad T^1(v) = \begin{pmatrix} \sum_{i=0}^{m-1} c(bc)^i \otimes (cb)^{m-1-i} & b \otimes 1 & 0 & 0 & 0 \\ -\sum_{i=0}^{m-1} b(cb)^i \otimes (bc)^{m-1-i} & 0 & c \otimes 1 & 0 & 0 \end{pmatrix}, \\
T^2(v) &= \begin{pmatrix} -b \otimes 1 & -c \otimes 1 & 0 & 0 & 0 & 0 \\ -c(bc)^{m-1} \otimes 1 & 0 & b \otimes 1 & 0 & 0 & 0 \\ 0 & b(cb)^{m-1} \otimes 1 & 0 & c \otimes 1 & 0 & 0 \end{pmatrix}, \\
T^3(v) &= t \begin{pmatrix} -\sum_{i=0}^{m-1} (bc)^i \otimes c(bc)^{m-1-i} & -b \otimes 1 & 0 & 0 & 0 & 0 \\ \sum_{i=0}^{m-1} (cb)^i \otimes b(cb)^{m-1-i} & 0 & -c \otimes 1 & 0 & 0 & 0 \\ 0 & c(bc)^{m-1} \otimes 1 & 0 & b \otimes 1 & 0 & 0 \\ 0 & 0 & -b(cb)^{m-1} \otimes 1 & 0 & c \otimes 1 & 0 \end{pmatrix}.
\end{aligned}$$

Proof. This proposition is proved by direct verification of the relations $\mu T^0(x) = x$, $d_{i-1} T^i(x) = T^{i-1}(x) d_{i+\deg x}$ ($0 < i \leq \deg x$), where $x \in \mathcal{Y} \setminus \{w\}$ with $\deg x > 0$, and we leave this to the reader. $\square$

We introduce an auxiliary notation:

$$q_i := (bc)^i + (cb)^i \in Z(R) \text{ for } 1 \leq i \leq m$$

(clearly, $q_1 = p_2$, $q_m = 2p_3$).

The following statement is deduced immediately from the definitions.

Lemma 4.3. *Let f_i with $i \in \mathbb{N}$ be the polynomials defined by (1.3).*

- a) *For any $1 \leq i \leq m$, we have $q_i = f_i(p_2)$.*
- b) *For any $1 \leq i \leq m$, we have*

$$p_2^i = \begin{cases} C_i^{\frac{i}{2}} \cdot 1 + \sum_{t=0}^{\lfloor \frac{i-1}{2} \rfloor} C_i^t q_{i-2t} & \text{if } i \text{ is even,} \\ \sum_{t=0}^{\lfloor \frac{i-1}{2} \rfloor} C_i^t q_{i-2t} & \text{if } i \text{ is odd.} \end{cases}$$

Remark 4.4. Lemma 4.3 implies that the subset $\{q_i\}_{i=1}^{m-1}$ of the basis of the group $\text{HH}^0(R) \simeq Z(R)$ that was indicated in (3.4) can be replaced by the set $\{p_2^i\}_{i=1}^{m-1}$. Furthermore, we have $q_i = p_2^i + g_i(p_2)$ ($1 \leq i \leq m$).

Proposition 4.5. *In the algebra $\text{HH}^*(R)$, the elements of the set*

$$(4.1) \quad \mathcal{Y} = \{p_1, p'_1, p_2, p_3, u_0, u_1, u'_1, u_2, u'_2, v, w\}$$

satisfy the following relations:

$$(4.2) \quad p_1^2 = (p'_1)^2 = \begin{cases} m \left(1 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \right) & \text{if } m \text{ is odd,} \\ m \left(1 + p_3 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \right) & \text{if } m \text{ is even;} \end{cases}$$

$$(4.3) \quad p_1 p'_1 = \begin{cases} m \left(p_3 + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} q_{2i-1} \right) & \text{if } m \text{ is odd,} \\ m \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} q_{2i-1} & \text{if } m \text{ is even;} \end{cases}$$

$$(4.4) \quad p_1 p_3 = \begin{cases} p'_1 & \text{if } m \text{ is odd,} \\ p_1 & \text{if } m \text{ is even;} \end{cases} \quad p'_1 p_3 = \begin{cases} p_1 & \text{if } m \text{ is odd,} \\ p'_1 & \text{if } m \text{ is even;} \end{cases}$$

$$(4.5) \quad p_1 p_2 = 2p'_1, \quad p'_1 p_2 = 2p_1, \quad p_2 p_3 = q_{m-1}, \quad p_3^2 = 1;$$

$$(4.6) \quad p_1 u_0 = p'_1 u_0 = 0;$$

$$(4.7) \quad \left(1 + \sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} q_{m-1-2i}\right) u_0 = 0 \quad \text{if } m \text{ is odd,}$$

$$(4.8) \quad \sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} q_{m-1-2i} \cdot u_0 = 0 \quad \text{if } m \text{ is even;}$$

$$(4.9) \quad p_3 u_0 = \begin{cases} -\left(\sum_{i=0}^{\lfloor \frac{m-3}{2} \rfloor} q_{m-2-2i}\right) u_0 & \text{if } m \text{ is odd,} \\ -\left(1 + \sum_{i=0}^{\lfloor \frac{m-3}{2} \rfloor} q_{m-2-2i}\right) u_0 & \text{if } m \text{ is even;} \end{cases}$$

$$(4.10) \quad p_1 u_1 = \begin{cases} u_2 & \text{if } m \text{ is odd,} \\ u_2 + p_3 u_2 & \text{if } m \text{ is even;} \end{cases} \quad p'_1 u_1 = \begin{cases} p_3 u_2 & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even;} \end{cases}$$

$$(4.11) \quad p_1 u'_1 = \begin{cases} p_3 u'_2 & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even;} \end{cases} \quad p'_1 u'_1 = \begin{cases} u'_2 & \text{if } m \text{ is odd,} \\ u'_2 + p_3 u'_2 & \text{if } m \text{ is even;} \end{cases}$$

$$(4.12) \quad p_2 u_1 = p_2 u'_1 = m p_3 u_0, \quad p_2 u_2 = p_2 u'_2 = 0;$$

$$(4.13) \quad p_1 u_2 = \begin{cases} u_1 + m \left(\left\lfloor \frac{m}{2} \right\rfloor \cdot 1 + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i} \right) u_0 & \text{if } m \text{ is odd,} \\ u_1 + p_3 u_1 + m \left(\sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i-1} \right) u_0 & \text{if } m \text{ is even;} \end{cases}$$

$$(4.14) \quad p'_1 u'_2 = \begin{cases} u'_1 + m \left(\left\lfloor \frac{m}{2} \right\rfloor \cdot 1 + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i} \right) u_0 & \text{if } m \text{ is odd,} \\ u'_1 + p_3 u'_1 + m \left(\sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i-1} \right) u_0 & \text{if } m \text{ is even;} \end{cases}$$

$$(4.15) \quad p'_1 u_2 = \begin{cases} p_3 u_1 + m \left(\sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i+1} \right) u_0 & \text{if } m \text{ is odd,} \\ m \left(\left\lfloor \frac{m}{2} \right\rfloor \cdot 1 + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i} \right) u_0 & \text{if } m \text{ is even;} \end{cases}$$

$$(4.16) \quad p_1 u'_2 = \begin{cases} p_3 u'_1 + m \left(\sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i+1} \right) u_0 & \text{if } m \text{ is odd,} \\ m \left(\left\lfloor \frac{m}{2} \right\rfloor \cdot 1 + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i} \right) u_0 & \text{if } m \text{ is even;} \end{cases}$$

$$(4.17) \quad p_2 v = 0, \quad 2p_1 w = 2p'_1 w = 0,$$

$$u_0 u_2 = u_0 u'_2 = 0, \quad u_0^2 = (2 - q_2)w, \quad u_0 u_1 = u_0 u'_1 = m q_{m-1} w,$$

$$u_1 u'_1 = \begin{cases} mw & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even;} \end{cases}$$

$$\begin{aligned}
u_1 u'_2 &= \begin{cases} p'_1 w & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even;} \end{cases} & u'_1 u_2 &= \begin{cases} p_1 w & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even;} \end{cases} \\
u_2^2 - u_1^2 &= (u'_2)^2 - (u'_1)^2 = \begin{cases} m \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \cdot w & \text{if } m \text{ is odd,} \\ m \left(1 + p_3 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \right) w & \text{if } m \text{ is even;} \end{cases} \\
u_2 u'_2 &= \begin{cases} m \left(p_3 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i-1} \right) w & \text{if } m \text{ is odd,} \\ m \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i+1} \cdot w & \text{if } m \text{ is even;} \end{cases} \\
(4.18) \quad u_0 v &= 0, \quad v^2 = (u_1 + u'_1)w.
\end{aligned}$$

Proof. Relations (4.2)–(4.17) are verified directly. For the remaining relations we use the translates of the corresponding elements that are described in Proposition 4.2. To illustrate the technique, we only verify relations (4.18). The remaining relations are established similarly, and we leave this to the reader. (Our choice of relations that we prove is partly explained by the fact that the description of the bases for $\text{Im } \delta^4$ and $\text{Im } \delta^5$ was not given, contrary to the description of those for the modules $\text{Im } \delta^i$ with $i \leq 3$.)

A direct calculation, which uses the formulas for the translates $T^0(u_0)$ and $T^2(v)$ in Proposition 4.2, yields

$$T^0(u_0) \circ T^2(v) = \left(-1 \otimes c + bc \otimes b, -cb \otimes c + 1 \otimes b, O_3 \right).$$

Hence,

$$\mu \circ T^0(u_0) \circ T^2(v) = \left(-c + bcb, -cbc + b, O_3 \right) = -\delta^4(bc, O_4),$$

and consequently, in $\text{HH}^*(R)$ we have $u_0 v = 0$.

Now, using Remark 4.1, we easily obtain

$$\mu \circ \left(T^0(u_1) + T^0(u'_1) \right) \circ T^2(w) = \left(0, 1, 1, O_4 \right).$$

On the other hand, Proposition 4.2 shows that

$$T^0(v) \circ T^3(v) = \left(-\sum_{i=0}^{m-1} (bc)^i b \otimes c(bc)^{m-1-i} + \sum_{i=0}^{m-1} (bc)^i b \otimes c(bc)^{m-1-i}, -1 \otimes 1, -1 \otimes 1, O_4 \right),$$

whence

$$\mu T^0(v) \circ T^3(v) = \left(0, -1, -1, O_4 \right).$$

Since $\delta^5(b, c, O_4) = (0, 2, 2, O_4)$, in $\text{HH}^*(R)$ we have $v^2 = (u_1 + u'_1)w$. $\square$

Proposition 4.6. *The set $\mathcal{Y}$ (see (4.1)) generates $\text{HH}^*(R)$ as a K -algebra.*

Proof. Let $\mathcal{H}$ denote the K -subalgebra of $\text{HH}^*(R)$ generated by the set $\mathcal{Y} \cup \{1\}$ (here 1 denotes the unity of the algebra $\text{HH}^*(R)$). By Remark 4.4, we have $\text{HH}^0(R) \subset \mathcal{H}$. The fact that the groups $\text{HH}^i(R)$ with $2 \leq i \leq 5$ are contained in $\mathcal{H}$ is established with the help of the following formulas (see Propositions 3.11, 3.16, 3.21 and Corollary 3.26).

Lemma 4.7. *The following relations are valid:*

a) *for any $2 \leq i \leq m-1$, we have*

$$u_0^{(i)} = \begin{cases} \left(1 + \sum_{t=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{i+1-2t}\right) u_0 & \text{if } m \text{ is odd,} \\ \sum_{t=0}^{\lfloor \frac{m-1}{2} \rfloor} q_{i-1-2t} \cdot u_0 & \text{if } m \text{ is even;} \end{cases}$$

b) *if m is even, then $v_2 = p_1 v$, $v'_2 = p'_1 v$; if m is odd, then $\tilde{v}_2 = p_1 v$, $\tilde{v}'_2 = p'_1 v$; moreover, for any m we have $v'_1 = p_3 v$;*

c) *for any $2 \leq i \leq m-1$ we have $w_0^{(i)} = q_i w$; moreover, $w'_1 = p_3 w$, $w_2 = p_1 w$, $w'_2 = p'_1 w$;*

$$\begin{aligned} w_3 &= \begin{cases} u_1^2 & \text{if } m \text{ is odd,} \\ u_1^2 - mw & \text{if } m \text{ is even;} \end{cases} & w'_3 &= \begin{cases} (u'_1)^2 & \text{if } m \text{ is odd,} \\ (u'_1)^2 - mw & \text{if } m \text{ is even;} \end{cases} \\ w_4 &= \begin{cases} u_1 u_2 & \text{if } m \text{ is odd,} \\ u_1 u_2 - p_1 w & \text{if } m \text{ is even;} \end{cases} & w'_4 &= \begin{cases} u'_1 u'_2 & \text{if } m \text{ is odd,} \\ u'_1 u'_2 - p'_1 w & \text{if } m \text{ is even;} \end{cases} \\ w_5 &= p_3 w_4, \quad w_6 = p_3 w_3, \quad w'_5 = p_3 w'_4, \quad w'_6 = p_3 w'_3; \end{aligned}$$

d)

$$\begin{aligned} t_1 &= u_2 v, \quad t_2 = u_1 v, \quad t_3 = p_3 u_1 v, \quad t_4 = p_3 u_2 v, \\ t'_1 &= u'_2 v, \quad t'_2 = u'_1 v, \quad t'_3 = p_3 u'_1 v, \quad t'_4 = p_3 u'_2 v. \end{aligned}$$

Proof. These relations are verified directly, as in the proof of Proposition 4.5. $\square$

We also need the following auxiliary statement.

Lemma 4.8. *Set $\bar{u} := u_1 + u'_1 \in \mathrm{HH}^2(R)$. For any $r \geq 3$, the translates $T^r(\bar{u})$ can be taken so that their matrices have the following block form:*

$$T^r(\bar{u}) = \begin{pmatrix} \star & \mathbf{O} \\ \star & \mathrm{id}_{\Lambda^4} \end{pmatrix},$$

where id_{Λ^4} is the matrix of the identity map of the module Λ^4 .

Proof. By Proposition 4.5, we have

$$T^2(\bar{u}) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ \star & 0 & 0 & 1 & 0 \\ \star & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence, $T^2(\bar{u})d_4^Q = \begin{pmatrix} \star & d_2^Q \end{pmatrix}$. Since we need the identity $T^2(\bar{u})d_4^Q = d_2^Q T^3(\bar{u})$, we can choose the translate $T^3(\bar{u})$ with a matrix of the desired form. With the help of similar arguments, we complete the proof by induction on r . $\square$

We continue the proof of Proposition 4.6. Using induction on n , we check that $\mathrm{HH}^n(R) \subset \mathcal{H}$. Assume that $n \geq 6$, and put

$$(4.19) \quad f = (f', f'') : Q_n = Q_{n-4} \oplus X_n \rightarrow R$$

with $f' \in \mathrm{Hom}_\Lambda(Q_{n-4}, R)$ and $f'' \in \mathrm{Hom}_\Lambda(X_n, R)$. Observe that $f'' \in \mathrm{Ker} \delta_\Lambda^n$.

a) Assume additionally that, in (4.19), we have $f'' = 0$. Clearly, also $f' \in \mathrm{Ker} \delta^{n-4}$. Furthermore, by Remark 4.1 we have $f = f' \circ \pi_n = f' \circ T^{n-4}(w)$. Consequently, the relation $f = f'w$ is fulfilled in $\mathrm{HH}^*(R)$, and it remains to apply the induction hypothesis to f' .

b) Now we consider the case where n is even: $n = 2l$. Using the form of the matrix of the differential d_{2l}^Q (see (3.81)) and Lemmas 3.25 and 3.10, we see that $g := (O_{n-5}, f'')$ lies in $\text{Ker } \delta^{n-2}$. By Lemma 4.8, it follows that $g \circ T^{n-2}(\bar{u}) = (\star, f'')$. Therefore, $f_1 := f - g\bar{u} = (\star, O_4)$, and it remains to apply part a) of the proof to f_1 and the induction hypothesis to g .

c) Now we assume that n is odd: $n = 2l+1$. Set $f'' = (r_1, r_2, r_3, r_4)$ with $r_i \in R$. Since $B^+ * r_3 = 0 = C^+ * r_4$, by Lemma 3.15 we have $r_3 = B^- * u$, $r_4 = C^- * u'$ for some $u, u' \in R$. Replacing f by the element $f - \delta^{2l}(O_{2l-1}, u, u')$, we may assume that $r_3 = 0 = r_4$ for the initial f . From the relation $\delta_{\mathcal{X}}^{2l+1}(f'') = 0$ it follows that $B^- * r_1 = 0$ and $C^- * r_2 = 0$. It is easily seen that $g := (O_{2l-6}, -r_1(bc)^m, r_2(bc)^m, r_1, r_2, 0, 0) \in \text{Hom}_\Lambda(Q_{2l-1}, R)$ is a cocycle. Hence, $f_1 := f - g\bar{u} = (\star, O_4)$, and the desired statement is established by an argument similar to part b) of the proof. $\square$

Corollary 4.9. *If $R = \mathbb{Z}[D_{4m}]$, then the algebra $\text{HH}^*(R)$ is commutative.*

Proof. This follows immediately from Proposition 4.6 and from the relations satisfied by the elements of the set $\mathcal{Y}$ (see Proposition 4.5), because there is a unique generator with odd degree, namely v , and for this generator we have $2v = 0$. $\square$

Let $\mathcal{A} = \mathcal{A}_m = \mathbb{Z}[\mathcal{X}]/I_m$ be a graded $\mathbb{Z}$ -algebra defined in §1, where $\mathcal{X}$ is as in (1.5), and I_m is the corresponding ideal of relations (see (1.6)–(1.26)). For an element $x \in \mathbb{Z}[\mathcal{X}]$, we keep the notation x also for its image in $\mathcal{A}$ under the canonical map $\mathbb{Z}[\mathcal{X}] \rightarrow \mathcal{A}$. The nonzero images of monomials in $\mathbb{Z}[\mathcal{X}]$ under this map are also called monomials. Any element $x \in \mathcal{A}$ is represented as a linear combination of monomials (with coefficients in $\mathbb{Z}$). Propositions 4.5 and 4.6 imply that there exists a surjective homomorphism $\varphi: \mathcal{A} \rightarrow \text{HH}^*(R)$ of graded $\mathbb{Z}$ -algebras that takes the generators belonging to $\mathcal{X}$ to the corresponding generators in $\mathcal{Y}$ (see (4.1)). Note that no ambiguity is caused by employing the same letter to denote elements of both sets that correspond to each other. Let $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}^n$ be the direct decomposition of the algebra $\mathcal{A}$ into homogeneous direct summands, and let $\varphi_n: \mathcal{A}^n \rightarrow \text{HH}^n(R)$ be the restriction of φ to the corresponding homogeneous component.

Now, Theorem 1.1 is a consequence of the following statement.

Proposition 4.10. *For any $n \geq 0$, $\varphi_n: \mathcal{A}^n \rightarrow \text{HH}^n(R)$ is an isomorphism.*

Proof. By definition, *reduction* of a monomial a in $\mathcal{A}$ is the process of replacement of some submonomials of a by other elements of $\mathcal{A}$ in accordance with the following rules ($x \mapsto y$ means the replacement of every occurrence of the monomial x by the element y):

$$(4.20) \quad p_1^2 \mapsto \begin{cases} m \left(1 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \right) & \text{if } m \text{ is odd,} \\ m \left(1 + p_3 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \right) & \text{if } m \text{ is even;} \end{cases}$$

$$(4.21) \quad (p'_1)^2 \mapsto \begin{cases} m \left(1 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \right) & \text{if } m \text{ is odd,} \\ m \left(1 + p_3 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \right) & \text{if } m \text{ is even;} \end{cases}$$

$$(4.22) \quad p_1 p'_1 \mapsto \begin{cases} m \left(p_3 + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} q_{2i-1} \right) & \text{if } m \text{ is odd,} \\ m \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} q_{2i-1} & \text{if } m \text{ is even;} \end{cases}$$

$$(4.23) \quad p_1 p_3 \mapsto \begin{cases} p'_1 & \text{if } m \text{ is odd,} \\ p_1 & \text{if } m \text{ is even;} \end{cases} \quad p'_1 p_3 \mapsto \begin{cases} p_1 & \text{if } m \text{ is odd,} \\ p'_1 & \text{if } m \text{ is even;} \end{cases}$$

$$(4.24) \quad p_1 p_2 \mapsto 2p'_1, \quad p'_1 p_2 \mapsto 2p_1, \quad p_3^2 \mapsto 1;$$

$$(4.25) \quad p_2 p_3 \mapsto q_{m-1}, \quad p_2^m \mapsto 2p_3 - g_m(p_2);$$

$$p_2^{m-1} u_0 \mapsto \begin{cases} - \left(1 + g_{m-1}(p_2) + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} q_{m-1-2i} \right) u_0 & \text{if } m \text{ is odd,} \\ - \left(g_{m-1}(p_2) + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} q_{m-1-2i} \right) u_0 & \text{if } m \text{ is even;} \end{cases}$$

$$p_3 u_0 \mapsto \begin{cases} - \sum_{i=0}^{\lfloor \frac{m-3}{2} \rfloor} q_{m-2-2i} \cdot u_0 & \text{if } m \text{ is odd,} \\ - \left(1 + \sum_{i=0}^{\lfloor \frac{m-3}{2} \rfloor} q_{m-2-2i} \right) u_0 & \text{if } m \text{ is even;} \end{cases}$$

$$p_1 u_1 \mapsto \begin{cases} u_2 & \text{if } m \text{ is odd,} \\ u_2 + p_3 u_2 & \text{if } m \text{ is even;} \end{cases}$$

$$p'_1 u'_1 \mapsto \begin{cases} u'_2 & \text{if } m \text{ is odd,} \\ u'_2 + p_3 u'_2 & \text{if } m \text{ is even;} \end{cases}$$

$$p_1 u'_1 \mapsto p_3 u'_2 \text{ if } m \text{ is odd; } p'_1 u_1 \mapsto p_3 u_2 \text{ if } m \text{ is odd;}$$

$$p_2 u_1 \mapsto m p_3 u_0, \quad p_2 u'_1 \mapsto m p_3 u_0;$$

$$p_1 u_2 \mapsto \begin{cases} u_1 + m \left(\lfloor \frac{m}{2} \rfloor \cdot 1 + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\lfloor \frac{m}{2} \rfloor - i \right) q_{2i} \right) u_0 & \text{if } m \text{ is odd,} \\ u_1 + p_3 u_1 + m \left(\sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\lfloor \frac{m}{2} \rfloor - i \right) q_{2i-1} \right) u_0 & \text{if } m \text{ is even;} \end{cases}$$

$$p'_1 u'_2 \mapsto \begin{cases} u'_1 + m \left(\lfloor \frac{m}{2} \rfloor \cdot 1 + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\lfloor \frac{m}{2} \rfloor - i \right) q_{2i} \right) u_0 & \text{if } m \text{ is odd,} \\ u'_1 + p_3 u'_1 + m \left(\sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\lfloor \frac{m}{2} \rfloor - i \right) q_{2i-1} \right) u_0 & \text{if } m \text{ is even;} \end{cases}$$

$$p'_1 u_2 \mapsto \begin{cases} p_3 u_1 + m \left(\sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} \left(\lfloor \frac{m}{2} \rfloor - i \right) q_{2i+1} \right) u_0 & \text{if } m \text{ is odd,} \\ m \left(\lfloor \frac{m}{2} \rfloor \cdot 1 + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\lfloor \frac{m}{2} \rfloor - i \right) q_{2i} \right) u_0 & \text{if } m \text{ is even;} \end{cases}$$

$$p_1 u'_2 \mapsto \begin{cases} p_3 u'_1 + m \left(\sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} \left(\lfloor \frac{m}{2} \rfloor - i \right) q_{2i+1} \right) u_0 & \text{if } m \text{ is odd,} \\ m \left(\lfloor \frac{m}{2} \rfloor \cdot 1 + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\lfloor \frac{m}{2} \rfloor - i \right) q_{2i} \right) u_0 & \text{if } m \text{ is even;} \end{cases}$$

$$\begin{aligned}
u_0^2 &\mapsto (2 - q_2)w, & u_0u_1 &\mapsto mq_{m-1}w, & u_0u'_1 &\mapsto mq_{m-1}w; \\
u_1u'_1 &\mapsto mw & \text{if } m \text{ is odd;} \\
u_1u'_2 &\mapsto p'_1w & \text{if } m \text{ is odd;} \\
u'_1u_2 &\mapsto p_1w & \text{if } m \text{ is odd;} \\
u_1^2 &\mapsto \begin{cases} u_2^2 + m \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \cdot w & \text{if } m \text{ is odd,} \\ u_2^2 + m \left(1 + p_3 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \right) w & \text{if } m \text{ is even;} \end{cases} \\
(u'_1)^2 &\mapsto \begin{cases} (u'_2)^2 + m \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \cdot w & \text{if } m \text{ is odd,} \\ (u'_2)^2 + m \left(1 + p_3 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \right) w & \text{if } m \text{ is even;} \end{cases} \\
u_2u'_2 &\mapsto \begin{cases} m \left(p_3 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i-1} \right) w & \text{if } m \text{ is odd,} \\ m \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i+1} \cdot w & \text{if } m \text{ is even;} \end{cases} \\
v^2 &\mapsto (u_1 + u'_1)w.
\end{aligned}$$

Any replacement on the above list is called an *elementary step of reduction*.

On the ring $\mathbb{Z}[\mathcal{X}]$, we introduce a new grading with the help of the function $\chi: \mathbb{Z}[\mathcal{X}] \rightarrow \mathbb{N}_0$ such that, for any elementary step of reduction $x \mapsto y$, we have $\chi(x) > \chi(y)$ (more precisely, here x and y are the corresponding elements of $\mathbb{Z}[\mathcal{X}]$). For example, it is easily seen that the degree function has such properties if we introduce a new grading χ on $\mathbb{Z}[\mathcal{X}]$ as follows:

$$\begin{aligned}
\chi(p_1) &= \chi(p'_1) = 2m - 1, & \chi(p_2) &= 1, & \chi(p_3) &= m - 1, \\
\chi(u_0) &= 2, & \chi(u_1) &= \chi(u'_1) = \chi(v) = m + 1, \\
\chi(u_2) &= \chi(u'_2) = m, & \chi(w) &= 1.
\end{aligned}$$

Since the values of χ strictly decrease under reduction, after finitely many steps we obtain monomials or linear combinations of monomials that do not admit any elementary step of reduction. We say that a presentation of an element $f \in \mathcal{A}$ as a linear combination of monomials has *normal form* if reduction cannot be applied to any of these monomials. Since these elementary steps of reduction correspond to some relations in the algebra $\mathcal{A}$ (see (1.6)–(1.26)), any element $f \in \mathcal{A}$ admits at least one presentation in the normal form.

1) First, assume that $n = 0$. Let

$$(4.26) \quad f = p_1^{\alpha_1} (p'_1)^{\alpha'_1} p_2^{\alpha_2} p_3^{\alpha_3} \in \mathcal{A}^0$$

be a monomial in the normal form. By the steps of reduction in (4.20)–(4.25), we have $\alpha_1 + \alpha'_1 + \alpha_3 \in \{0, 1\}$ and $0 \leq \alpha_2 \leq m - 1$. Hence, f coincides with one of the monomials

$$p_1, p'_1, p_3, p_2^i \quad (0 \leq i \leq m - 1).$$

Consequently, the Abelian group $\mathcal{A}^0$ is generated by the set consisting of $m + 3$ elements and simultaneously admits an epimorphism onto the free Abelian group $\mathrm{HH}^0(R)$ of rank $m + 3$. This immediately implies that $\varphi: \mathcal{A}^0 \rightarrow \mathrm{HH}^0(R)$ is an isomorphism.

2) Assume that $n \geq 1$. We shall prove that the cardinality $|\mathcal{A}^n|$ does not exceed the cardinality $|\mathrm{HH}^n(R)|$, which implies that $\varphi_n: \mathcal{A}^n \rightarrow \mathrm{HH}^n(R)$ is an isomorphism.

Let $f \in \mathcal{A}^n$ be a monomial in the normal form. Put $f = f_0 f'$, where $f_0 \in \mathcal{A}^0$ and

$$f' = u_0^{\beta_0} u_1^{\beta_1} u_2^{\beta_2} (u'_1)^{\gamma_1} (u'_2)^{\gamma_2} v^\varepsilon w^l.$$

Using the list of elementary steps of reduction, we see that $\beta_0, \varepsilon \in \{0, 1\}$, $\beta_1 + \gamma_1 \leq 1$ and $\beta_2 \gamma_2 = 0$. Next, we consider several cases successively.

a) Assume additionally that $\beta_0 = 1$. Then f' does not involve any of the following elements: u_1 and u'_1 (because the products $u_0 u_1$ and $u_0 u'_1$ can be reduced), u_2 , u'_2 , and v (because $u_0 u_2 = u_0 u'_2 = u_0 v = 0$), p_1, p'_1 , and p_3 (because $p_3 u_0$ can be reduced and $p_1 u_0 = p'_1 u_0 = 0$). Moreover, in (4.26) we have $\alpha_2 \leq m - 2$ (because $p_2^{m-1} u_0$ can be reduced). Consequently, f coincides with one of the monomials

$$(4.27) \quad p_2^i u_0 w^l \quad (0 \leq i \leq m - 2, l \in \mathbb{N}_0).$$

b) Assume that f' involves u_1 . Then f' cannot involve u'_1 and u'_2 (because $u_1 u'_1$ and $u_1 u'_2$ are zero if m is even, or can be reduced if m is odd). Moreover, f_0 cannot involve p_1, p'_1, p_2 (because $p_1 u_1, p_2 u_1$ can be reduced, and $p'_1 u_1$ either can be reduced or is equal to zero). Consequently, f coincides with one of the monomials

$$(4.28) \quad u_1 u_2^t w^l, u_1 u_2^t v w^l, p_3 u_1 u_2^t w^l, p_3 u_1 u_2^t v w^l \quad (t, l \in \mathbb{N}_0).$$

c) If f' involves u'_1 , then, by symmetry with case b), we see that f coincides with one of the following monomials:

$$(4.29) \quad u'_1 (u'_2)^t w^l, u'_1 (u'_2)^t v w^l, p_3 u'_1 (u'_2)^t w^l, p_3 u'_1 (u'_2)^t v w^l \quad (t, l \in \mathbb{N}_0).$$

d) Assume that u_0, u_1, u'_1 do not occur in f' and u_2 occurs in f' . As in case b), we prove that f has the form $f = p_3^{\alpha_3} u_2^{\beta_2} v^\varepsilon w^l$. Consequently, f coincides with one of the following monomials:

$$(4.30) \quad u_2^t w^l, u_2^t v w^l, p_3 u_2^t w^l, p_3 u_2^t v w^l \quad (t \in \mathbb{N}, l \in \mathbb{N}_0).$$

e) If u_0, u_1, u'_1, u_2 do not occur in f' and u'_2 occurs in f' , then, by symmetry with case d), we see that f coincides with one of the following monomials:

$$(4.31) \quad (u'_2)^t w^l, (u'_2)^t v w^l, p_3 (u'_2)^t w^l, p_3 (u'_2)^t v w^l \quad (t \in \mathbb{N}, l \in \mathbb{N}_0).$$

f) Assume that $u_0, u_1, u'_1, u_2, u'_2$ do not occur in f' and v occurs in f' . Then f_0 does not involve p_2 (because $p_2 v = 0$). Consequently, f coincides with one of the following monomials:

$$(4.32) \quad v w^l, p_1 v w^l, p'_1 v w^l, p_3 v w^l.$$

g) Finally, if $u_0, u_1, u'_1, u_2, u'_2, v$ do not occur in f' , then, clearly, f coincides with one of the monomials

$$(4.33) \quad w^l, p_1 w^l, p'_1 w^l, p_3 w^l, p_2^i w^l \quad (l \in \mathbb{N}, 0 \leq i \leq m - 1).$$

Now, considering the degrees of the elements in (4.27)–(4.33) (with respect to the initial grading on the algebra $\mathcal{A}$), we form Table 1.

The defining relations of the algebra $\mathcal{A}$ show that the monomials $p_2^i w^k$ ($0 \leq i \leq m - 1$), $p_2^i u_0 w^k$ ($0 \leq i \leq m - 2$), and $p_3 w^k$ admit scalar coefficients modulo $2m$, and the remaining monomials on the above list admit coefficients modulo 2. The listed monomials of degree $4k + r$ with $r \in \{0, 1, 2, 3\}$ generate the group $\mathcal{A}^{4k+r}$. From this, it is easily seen that for $k \geq 1$ we have $|\mathcal{A}^{4k}| \leq (2m)^{m+1} 2^{8k+2}$, and for $k \geq 0$ we have $|\mathcal{A}^{4k+1}| \leq 2^{8k}$, $|\mathcal{A}^{4k+2}| \leq (2m)^{m-1} 2^{8(k+1)}$, $|\mathcal{A}^{4k+3}| \leq 2^{8k+4}$. This completes the proof of the inequality $|\mathcal{A}^n| \leq |\mathrm{HH}^n(R)|$ for all $n \geq 1$. $\square$

TABLE 1

Degree	Normal forms
$4k, k > 0$	$p_3^\alpha u_1^\beta u_2^{2(k-l)-1} w^l$ ($0 \leq l \leq k-1; \alpha, \beta \in \{0, 1\}$), $p_3^\alpha (u_1')^\beta (u_2')^{2(k-l)-1} w^l$ ($0 \leq l \leq k-1; \alpha, \beta \in \{0, 1\}$), $p_2^i w^k$ ($0 \leq i \leq m-1$), $p_1 w^k$, $p_1' w^k$, $p_3 w^k$
$4k+1$	$p_3^\alpha u_1 u_2^{2(k-l)-1} v w^l$ ($0 \leq l \leq k-1; \alpha \in \{0, 1\}$), $p_3^\alpha u_2^{2(k-l)-1} v w^l$ ($0 \leq l \leq k-1; \alpha \in \{0, 1\}$), $p_3^\alpha u_1' (u_2')^{2(k-l)-1} v w^l$ ($0 \leq l \leq k-1; \alpha \in \{0, 1\}$), $p_3^\alpha (u_2')^{2(k-l)-1} v w^l$ ($0 \leq l \leq k-1; \alpha \in \{0, 1\}$),
$4k+2$	$p_3^\alpha u_1^\beta u_2^{2(k-l)} w^l$ ($0 \leq l \leq k, \alpha, \beta \in \{0, 1\}$), $p_3^\alpha (u_1')^\beta (u_2')^{2(k-l)-1} w^l$ ($0 \leq l \leq k, \alpha, \beta \in \{0, 1\}$), $p_2^i u_0 w^k$ ($0 \leq i \leq m-2$)
$4k+3$	$p_3^\alpha u_1 u_2^{2(k-l)-1} w^l$ ($0 \leq l \leq k-1; \alpha \in \{0, 1\}$), $p_3^\alpha u_1' (u_2')^{2(k-l)-1} w^l$ ($0 \leq l \leq k-1; \alpha \in \{0, 1\}$), $p_3^\alpha u_2^{2(k-l)} w^l$ ($0 \leq l \leq k; \alpha \in \{0, 1\}$), $p_3^\alpha (u_2')^{2(k-l)} w^l$ ($0 \leq l \leq k-1; \alpha \in \{0, 1\}$), $p_1 v w^k$, $p_1' v w^k$

Corollary 4.11. *If m is odd, then the cohomology algebra $\mathrm{HH}^*(\mathbb{Z}[D_{4m}])$ is generated by the homogeneous elements*

$$p_1, p_2, p_3, u_0, u_1, u_1', v, w$$

of degrees

$$\deg p_i = 0 \ (i = 1, 2, 3), \deg u_0 = \deg u_1 = \deg u_1' = 2, \deg v = 3, \deg w = 4,$$

and these elements satisfy the following defining relations:

— *of degree 0:*

$$(4.34) \quad p_1^2 = m \left(1 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \right),$$

$$(4.35) \quad p_1 p_2 = 2p_1 p_3, \quad q_m = 2p_3, \quad p_2 p_3 = q_{m-1}, \quad p_3^2 = 1;$$

— *of degree 2:*

$$(4.36) \quad p_1 u_0 = 2m u_0 = 2u_1 = 2u_1' = 0,$$

$$(4.37) \quad \left(1 + \sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} q_{m-1-2i} \right) u_0 = 0,$$

$$(4.38) \quad p_3 u_0 = - \left(\sum_{i=0}^{\lfloor \frac{m-3}{2} \rfloor} q_{m-2-2i} \right) u_0,$$

$$(4.39) \quad p_2 u_1 = p_2 u_1' = m p_3 u_0;$$

— *of degree 3:*

$$(4.40) \quad 2v = p_2 v = 0;$$

— of degree 4:

$$(4.41) \quad 2mw = 2p_1w = 0, \quad u_0^2 - (2 - q_2)w,$$

$$(4.42) \quad u_0u_1 = u_0u_1 = mq_{m-1}w, \quad u_1u'_1 = mw;$$

— of degree 5:

$$(4.43) \quad u_0v = 0;$$

— of degree 6:

$$(4.44) \quad v^2 = (u_1 + u'_1)w.$$

Proof. We can exclude the elements p'_1, u_2, u'_2 from the set $\mathcal{X}$ (see (1.5)), by using the generators (1.8) and (1.14) of the ideal I_m of relations. It can be checked that the relations that involve excluded elements follow from relations (4.34)–(4.44). $\square$

Corollary 4.12. *If m is even, then the cohomology algebra $\mathrm{HH}^*(\mathbb{Z}[D_{4m}])$ is generated by the homogeneous elements*

$$p_1, p'_1, p_2, p_3, u_0, u_1, u'_1, u_2, u'_2, v, w$$

of degrees

$$\begin{aligned} \deg p'_1 &= \deg p_i = 0 \quad (i = 1, 2, 3); \\ \deg u_0 &= \deg u_i = \deg u'_i = 2 \quad \text{for } i = 1, 2; \quad \deg v = 3, \deg w = 4, \end{aligned}$$

and these elements satisfy the following defining relations:

— of degree 0:

$$\begin{aligned} p_1^2 &= (p'_1)^2 = m \left(1 + p_3 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \right), \\ p_1p'_1 &= m \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} q_{2i-1}, \\ p_1p_3 &= p_1, \quad p'_1p_3 = p'_1, \\ p_1p_2 &= 2p'_1, \quad p'_1p_2 = 2p_1, \quad q_m = 2p_3, \quad p_2p_3 = q_{m-1}, \quad p_3^2 = 1; \end{aligned}$$

— of degree 2:

$$\begin{aligned} p_1u_0 &= p'_1u_0 = 2mu_0 = 2u_i = 2u'_i = 0 \quad \text{for } i \in \{1, 2\}; \\ \sum_{i=0}^{\lfloor \frac{m-2}{2} \rfloor} q_{m-1-2i} \cdot u_0 &= 0, \\ p_3u_0 &= - \left(1 + \sum_{i=0}^{\lfloor \frac{m-3}{2} \rfloor} q_{m-2-2i} \right) u_0, \\ p_1u_1 &= u_2 + p_3u_2, \quad p'_1u'_1 = u'_2 + p_3u'_2, \\ p'_1u_1 &= p_1u'_1 = p_2u_2 = p_2u'_2 = 0, \\ p_2u_1 &= p_2u'_1 = mp_3u_0, \\ p_1u_2 &= u_1 + p_3u_1 + m \left(\sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i-1} \right) u_0, \end{aligned}$$

$$p'_1 u'_2 = u'_1 + p_3 u'_1 + m \left(\sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i-1} \right) u_0,$$

$$p'_1 u_2 = m \left(\left\lfloor \frac{m}{2} \right\rfloor \cdot 1 + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i} \right) u_0,$$

$$p_1 u'_2 = m \left(\left\lfloor \frac{m}{2} \right\rfloor \cdot 1 + \sum_{i=1}^{\lfloor \frac{m-2}{2} \rfloor} \left(\left\lfloor \frac{m}{2} \right\rfloor - i \right) q_{2i} \right) u_0;$$

— of degree 3:

$$2v = p_2 v = 0;$$

— of degree 4:

$$2mw = 2p_1 w = 2p'_1 w = u_0 u_2 = u_0 u'_2 = 0,$$

$$u_1 u'_1 = u_1 u'_2 = u'_1 u_2 = 0,$$

$$u_0^2 - (2 - q_2)w, u_0 u_1 = u_0 u'_1 = m q_{m-1} w,$$

$$u_2^2 - u_1^2 = (u'_2)^2 - (u'_1)^2 = m \left(1 + p_3 + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i} \right) w,$$

$$u_2 u'_2 = m \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} q_{2i+1} \cdot w;$$

— of degree 5:

$$u_0 v = 0;$$

— of degree 6:

$$v^2 = (u_1 + u'_1)w.$$

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