TAME AND PURELY WILD EXTENSIONS OF VALUED FIELDS

YU. L. ERSHOV

Dedicated to the centenary of the birth of the outstanding Russian mathematician D. K. Faddeev

ABSTRACT. A systematic and concise exposition of the basic results concerning two complementary classes (tame and purely wild) of extensions of (Henselian) valued fields is given. These notions proved to be quite useful both for the general theory and for the model theory of such fields. Along with new results, new proofs of old results are presented. Thus, in the proof of the well-known Pank theorem on the existence of a complement to the ramification group in the absolute Galois group of a Henselian valued field, the properties of maximal immediate extensions are employed instead of cohomological methods.

Let F be a field. By definition, a *valuation ring* of F is an arbitrary subring R of F such that, for each $a \in F^\times = F \setminus \{0\}$, we have either $a \in R$ or $a^{-1} \in R$. Every valuation ring R is a local ring; i.e., it has a unique maximal ideal $\mathfrak{m}(R)$. The quotient ring $F_R = R/\mathfrak{m}(R)$ of R by the maximal ideal $\mathfrak{m}(R)$ is called the *residue field*. The quotient group $\Gamma_R = F^\times/U(R)$ of the multiplicative group $F^\times$ of the field F by the multiplicative group $U(R)$ of units (invertible elements) of R is called the *valuation group*. The group Γ_R is equipped with the structure of an ordered group in which the cone of nonnegative elements is the image of the set $R \setminus \{0\}$ in Γ_R . We use the additive notation for the operation in Γ_R .

The homomorphism $v_R : F^\times \rightarrow \Gamma_R = F^\times/U(R)$ is called the *valuation* (determined by the valuation ring R).

A pair $\mathbb{F} = \langle F, R \rangle$, where F is a field and R is a valuation ring of F , is called a *valued field*. If $\mathbb{F} = \langle F, R \rangle$, $\mathbb{F}_0 = \langle F_0, R_0 \rangle$ are valued fields, $F \leq F_0$, and $R_0 \cap F = R$, then $\mathbb{F}_0$ is called an *extension* of $\mathbb{F}$ ($\mathbb{F} \leq \mathbb{F}_0$). If $\mathbb{F} \leq \mathbb{F}_0$, then the residue field F_R is naturally identified with a subfield of the residue field F_{R_0} ($F_R \leq F_{R_0}$), and the valuation group Γ_R is naturally identified with a subgroup of Γ_{R_0} ($\Gamma_R \leq \Gamma_{R_0}$).

A valued field $\mathbb{F}$ is said to be *Henselian* if, for every algebraic extension $F_0 \geq F$, there exists a unique valuation ring R_0 in F_0 such that $\langle F_0, R_0 \rangle \geq \mathbb{F}$.

A valued field $\mathbb{F} = \langle F, R \rangle$ is Henselian if and only if the following statement is valid.

Hensel lemma. *Let $f \in R[x]$ be a monic polynomial such that the image $\bar{f}$ of f in $F_R[x]$ has a simple root $\bar{\alpha}$ in F_R . Then the polynomial f has a root $\alpha \in R$ such that $\bar{\alpha} = \alpha + \mathfrak{m}(R)$.*

If $\mathbb{F}_0 \geq \mathbb{F}$ is a finite extension (i.e., $[F_0 : F] < \omega$) of valued fields, then the degree $f = [F_{R_0} : F_R]$ of the extension $F_{R_0} \geq F_R$ and the index $e = [\Gamma_{R_0} : \Gamma_R]$ of the subgroup Γ_R in Γ_{R_0} are finite and we have the following *basic inequality*: $e \cdot f \leq n = [F_0 : F]$.

2000 *Mathematics Subject Classification*. Primary 12F15.

Key words and phrases. Henselian valued fields, valuation ring, valuation group, ramified extension, totally unramified extension.

Partially supported by the Council on Grants of President of the Russian Federation for the state support of leading scientific schools (project no. NSh-4787.2006.1).

A valued field $\mathbb{F}$ is said to be *algebraically complete* if, for every finite extension $\mathbb{F}_0 \geq \mathbb{F}$, equality occurs in the basic inequality: $n = e \cdot f$.

An extension $\mathbb{F}_0 \geq \mathbb{F}$ of valued fields is *immediate* if $F_{R_0} = F_R$ and $\Gamma_{R_0} = \Gamma_R$.

A finite extension $\mathbb{F}_0 \geq \mathbb{F}$ is *unramified* (respectively, *totally ramified*) if $[F_0 : F] = [F_{R_0} : F_R] (= f)$ (respectively, $[F_0 : F] = [\Gamma_{R_0} : \Gamma_R] (= e)$).

We say that an arbitrary algebraic extension $\mathbb{F}_0 \geq \mathbb{F}$ is *unramified* (respectively, *totally ramified*) if, for every intermediate extension $(\mathbb{F}_0 \geq) \mathbb{F}_1 \geq \mathbb{F}$ finite over $\mathbb{F}$, the extension $\mathbb{F}_1 \geq \mathbb{F}$ is unramified (respectively, totally ramified).

In what follows, we fix a Henselian valued field $\mathbb{F} = \langle F, R \rangle$ such that the residue field F_R has characteristic $p > 0$.

A finite extension $\mathbb{F}_0 \geq \mathbb{F}$ is said to be *tame* if the following conditions are fulfilled:

- 1) F_{R_0} is a separable extension of F_R ;
- 2) p does not divide the index $[\Gamma_{R_0} : \Gamma_R]$;
- 3) the extension $\mathbb{F}_0 \geq \mathbb{F}$ is defectless, i.e., $[F_0 : F_1] = [F_{R_0} : F_R] \cdot [\Gamma_{R_0} : \Gamma_R]$.

An algebraic extension $\mathbb{F}_0 \geq \mathbb{F}$ is *tame* (see [3, 4]) if, for every intermediate subextension $(\mathbb{F}_0 \geq) \mathbb{F}_1 \geq \mathbb{F}$ finite over $\mathbb{F}$ ($[F_1 : F] < \omega$), the extension $\mathbb{F}_1 \geq \mathbb{F}$ is tame.

Corollary 1 (to the definition). *Let $\mathbb{F}_0 \geq \mathbb{F}_1 \geq \mathbb{F}$ be a tower of algebraic extensions. The extension $\mathbb{F}_0 \geq \mathbb{F}$ is tame if and only if the extensions $\mathbb{F}_0 \geq \mathbb{F}_1$ and $\mathbb{F}_1 \geq \mathbb{F}$ are tame.*

For finite extensions, this follows directly from the relations $F_{R_0} \geq F_{R_1} \geq F_R$, $[F_{R_0} : F_R] = [F_{R_0} : F_{R_1}][F_{R_1} : F_R]$, $\Gamma_{R_0} \geq \Gamma_{R_1} \geq \Gamma_R$, and $[\Gamma_{R_0} : \Gamma_R] = [\Gamma_{R_0} : \Gamma_{R_1}][\Gamma_{R_1} : \Gamma_R]$.

Corollary 2 (to the definition). *If $\mathbb{F}$ is algebraically complete, the residue field F_R is perfect, and the group Γ_R is p -divisible ($p\Gamma_R = \Gamma_R$), then every algebraic extension $\mathbb{F}_0 \geq \mathbb{F}$ is tame.*

It is convenient to define the complementary notion of a purely wild extension not only for algebraic extensions. An extension $\mathbb{F}_0 \geq \mathbb{F}$ is said to be *purely wild* (see [3, 4]) if F_{R_0} is a purely inseparable extension of F_R and the quotient group Γ_{R_0}/Γ_R is a p -group.

Remark 1. Every immediate extension is purely wild.

Remark 2. If $\mathbb{F}_0 \geq \mathbb{F}$ is a purely inseparable extension, then $\mathbb{F}_0 \geq \mathbb{F}$ is a purely wild extension.

Corollary 3 (to the definition). *Let $\mathbb{F}_0 \geq \mathbb{F}_1 \geq \mathbb{F}$ be a tower of extensions. The extension $\mathbb{F}_0 \geq \mathbb{F}$ is purely wild if and only if the extensions $\mathbb{F}_0 \geq \mathbb{F}_1$ and $\mathbb{F}_1 \geq \mathbb{F}$ are purely wild.*

This fact can be proved in the same way as Corollary 1.

Corollary 4 (to the definition). *An extension $\mathbb{F}_0 \geq \mathbb{F}$ is tame and purely wild if and only if $\mathbb{F}_0 = \mathbb{F}$.*

Indeed, such an extension $\mathbb{F}_0 \geq \mathbb{F}$ must be immediate. Since it is defectless, we see that $\mathbb{F}_0 = \mathbb{F}$.

Corollary 5. *If $\mathbb{F}_0 \geq \mathbb{F}$ is tame, then F_0 is a separable extension of F .*

Proof. Indeed, let $F_{0,s}$ be the separable closure of F in F_0 , and let $\mathbb{F}_{0,s} \Leftarrow \langle F_{0,s}, R_0 \cap F_{0,s} \rangle$. Then the extension $\mathbb{F}_0 \geq \mathbb{F}_{0,s}$ is tame and purely wild simultaneously (Corollary 1 and Remark 2). By Corollary 3, we have $\mathbb{F}_0 = \mathbb{F}_{0,s}$. $\square$

Proposition 1. *Let $\mathbb{F}' \geq \mathbb{F}_i \geq \mathbb{F}$ be extensions such that $\mathbb{F}_0 \geq \mathbb{F}$ is tame and $\mathbb{F}_0 \geq \mathbb{F}$ is purely wild. Then F_0 and F_1 are linearly disjoint over F ($F_0 \parallel_F F_1$), and for $F'' \Leftarrow F_0 F_1 = F_0 \otimes_F F_1$ and $\mathbb{F}'' \Leftarrow \langle F'', R' \cap F'' \rangle$ the extension $\mathbb{F}'' \geq \mathbb{F}_0$ is purely wild and $\mathbb{F}'' \geq \mathbb{F}_1$ is tame.*

Proof. Without loss of generality, we may assume that the extension $\mathbb{F}_0 \geq \mathbb{F}$ is finite and $F'' = F'$. We have $F_R \leq F_{R_i} \leq F_{R'}$, $i = 0, 1$, F_{R_0} is a separable extension of F_R , and F_{R_1} is a purely inseparable extension of F_R . Consequently, $F_R \leq F_{R_0}F_{R_1} = F_{R_0} \otimes_{F_R} F_{R_1} \leq F_{R'}$ and $[F_{R'} : F_{R_1}] \geq [F_{R_0} : F_R]$. We have $\Gamma_R \leq \Gamma_{R_i} \leq \Gamma_{R'}$ ($i = 0, 1$), p does not divide the index $[\Gamma_{R_0} : \Gamma_R]$, and Γ_{R_1}/Γ_R is a p -group, whence $\Gamma_{R_0} \cap \Gamma_{R_1} = \Gamma_R$ and $[\Gamma_{R'} : \Gamma_{R_1}] \geq [\Gamma_{R_0} : \Gamma_R]$ ($\Gamma_{R'}/\Gamma_{R_1} \geq \Gamma_{R_0}\Gamma_{R_1}/\Gamma_{R_1} \simeq \Gamma_{R_0}/\Gamma_{R_0} \cap \Gamma_{R_1} = \Gamma_{R_0}/\Gamma_R$). Next, $[F_{R_0} : F_R] \cdot [\Gamma_{R_0} : \Gamma_R] = [F_0 : F] \geq [F' : F_1] \geq [F_{R'} : F_{R_1}] \cdot [\Gamma_{R'} : \Gamma_{R_1}] \geq [F_{R_0} : F_R] \cdot [\Gamma_{R_0} : \Gamma_R]$. Consequently, $[F_0 : F] = [F' : F_1]$, $F_{R'} = F_{R_0} \otimes F_{R_1}$, $\Gamma'_{R'} = \Gamma_{R_0}\Gamma_{R_1}$, $F_{R'}$ is a separable extension of F_{R_1} and a purely inseparable extension of F_{R_0} , $[\Gamma_{R'} : \Gamma_{R_1}] = [\Gamma_{R_0} : \Gamma_R]$, and $\Gamma_{R'}/\Gamma_{R_0} = \Gamma_{R_0}\Gamma_{R_1}/\Gamma_{R_0} = \Gamma_{R_1}/\Gamma_{R_1} \cap \Gamma_{R_0} = \Gamma_{R_1}/\Gamma_R$ is a p -group. Now, all conclusions of the proposition follow from the relations obtained. $\square$

Proposition 2. *Let $\mathbb{F}_0 \geq \mathbb{F}$ be a finite extension, and let $(\mathbb{F}_0 \geq) \mathbb{F}_1 \geq \mathbb{F}$ be the maximal (with respect to inclusion) tame subextension of $\mathbb{F}_0$. Then $\mathbb{F}_0 \geq \mathbb{F}_1$ is a purely wild extension.*

Proof. We verify that F_{R_1} coincides with the separable closure F_s of F_R in F_{R_0} . Indeed, since $\mathbb{F}_1 \geq \mathbb{F}$ is a finite extension, we see that F_{R_1} is separable over F_R and $F_{R_1} \leq F_s$. We assume that $F_{R_1} \neq F_s$. Let $\bar{\alpha} \in F_s$ be such that $F_s = F_{R_1}(\bar{\alpha})$, and let $f \in R_1[x]$ be a unitary polynomial such that $\bar{f} \in F_{R_1}[x]$ is the minimal polynomial of $\bar{\alpha}$ over F_{R_1} . The element $\bar{\alpha}$ is separable over F_{R_1} . Therefore, by the Hensel lemma, there exists a root α of f in R_0 such that $\bar{\alpha} = \alpha + \mathbf{m}(R_0)$. Let $F'_1 \Leftarrow F_1(\alpha)$ and $\mathbb{F}'_1 = \langle F'_1, R_0 \cap F'_1 \rangle$. Then $[F'_1 : F_1] = [F_{R_1}(\bar{\alpha}) : F_{R_1}] \leq [F_{R'_1} : F_{R_1}]$. Consequently, $F_{R_1}(\bar{\alpha}) = F_{R'_1} = F_s$, and the multiplicativity of relative degrees of extensions and ramification indices shows that $[\Gamma_{R'_1} : \Gamma_R] = [\Gamma_{R_1} : \Gamma_R]$ and $[F'_1 : F] = [F'_1 : F_1][F_1 : F] = [F_s : F_{R_1}][F_{R_1} : F_R][\Gamma_{R'_1} : \Gamma_R] = [F_{R'_1} : F_R][F_{R'_1} : \Gamma_R]$. Therefore, $\mathbb{F}'_1 \geq \mathbb{F}_1$ is a tame extension of $\mathbb{F}$ and $\mathbb{F}'_1 \neq \mathbb{F}_1$, which contradicts maximality. It follows that F_{R_0} is a purely inseparable extension of $F_{R_1} (= F_s)$.

Now, we prove that the quotient group $\Gamma_{R_0}/\Gamma_{R_1}$ is a p -group. Suppose the contrary. Then there exists a prime number $q \neq p$ and an element $\gamma_0 \in \Gamma_{R_0} \setminus \Gamma_{R_1}$ such that $q\gamma_0 \in R_1$. Let a be an element of $F^\times$ such that $v_{R_0}(a) = \gamma_0$. Then $v_{R_0}(a^q) = qv_{R_0}(a) = q\gamma_0 \in \Gamma_{R_1}$. We choose $b \in F_1^\times$ so that $v_{R_1}(b) = q\gamma_0$. Then $v_{R_0}(a^qb^{-1}) = 0$ and $0 \neq a^qb^{-1} \in F_{R_0}$. Since F_{R_0} is a purely inseparable extension of F_{R_1} , we see that, for an appropriate $n \in \omega$, we have $(a^qb^{-1})^{p^n} \in F_{R_1}$. Then there exists $c \in R_1 \setminus \mathbf{m}(R_1)$ such that $(a^qb^{-1})^{p^n}c^{-1} = 1$. By the Hensel lemma, the polynomial $x^q - (a^qb^{-1})^{p^n}c^{-1} \in R_0[x]$ has a root α such that $\bar{\alpha} = 1$. We have $\alpha^q = (a^qb^{-1})^{p^n}c^{-1}$ and $b^{p^n}c = (a^{p^n}\alpha^{-1})^q$; i.e., the polynomial $x^q - b^{p^n}c$, $b^{p^n}c \in F_1$, has a root $a^{p^n}\alpha^{-1}$ in F_0 . We obtain $v_{R_0}(a^{p^n}\alpha^{-1}) = q^{-1}v_{R_0}(b^{p^n}c) = p^nq^{-1}v_{R_0}(b) = p^nq^{-1}q\gamma_0 = p^n\gamma_0$. However, since $q \neq p$, $\gamma_0 \notin \Gamma_{R_1}$, and $q\gamma_0 \in \Gamma_{R_1}$, we have $p^n\gamma_0 \notin \Gamma_{R_1}$. Consequently, for $F_2 \Leftarrow F_1(a^{p^n}\alpha^{-1})$ we obtain $[F_2 : F_1] = q$ and $[\Gamma_{R_2} : \Gamma_{R_1}] = q$, where $R_2 \Leftarrow R_0 \cap F_2$, which contradicts the maximality of $\mathbb{F}_1$.

Therefore, $\mathbb{F}_0 \geq \mathbb{F}_1$ is a purely wild extension.

The proposition is proved. $\square$

We shall prove that the field $\mathbb{F}_1$ in Proposition 2 is unique. For this, we need more definitions.

We say that a finite extension $\mathbb{F}_0 \geq \mathbb{F}$ is *inertial* if F_{R_0} is a separable extension of F_R and $[F_0 : F] = [F_{R_0} : F_R]$.

An algebraic extension $\mathbb{F}_0 \geq \mathbb{F}$ is *inertial* if, for every intermediate subextension $(\mathbb{F}_0 \geq) \mathbb{F}_1 \geq \mathbb{F}$ finite over $\mathbb{F}$ ($[F_1 : F] < \omega$), the extension $\mathbb{F}_1 \geq \mathbb{F}$ is inertial.

From this definition it is clear that every inertial extension is tame.

Corollary 6 (to the definition). *Let $\mathbb{F}_0 \geq \mathbb{F}_1 \geq \mathbb{F}$ be a tower of extensions. The extension $\mathbb{F}_0 \geq \mathbb{F}$ is inertial if and only if the extensions $\mathbb{F}_0 \geq \mathbb{F}_1$ and $\mathbb{F}_1 \geq \mathbb{F}$ are inertial.*

Proposition 3. *Let $\mathbb{F}_0 \geq \mathbb{F}$ be an algebraic extension. Then there exists a maximal subextension $(\mathbb{F}_0 \geq) \mathbb{F}_{0,i} \geq \mathbb{F}$ that is an inertial extension of $\mathbb{F}$; the residue field $F_{R_{0,i}}$ of $\mathbb{F}_{0,i}$ coincides with the separable closure of F_R in F_{R_0} , and $\Gamma_{R_{0,i}} = \Gamma_R$.*

Proof. We start with a lemma interesting in its own right. □

Lemma 1. *Let $\mathbb{F}' \geq \mathbb{F}_i \geq \mathbb{F}$, $i = 0, 1$, be extensions such that the extension $\mathbb{F}_0 \geq \mathbb{F}$ is inertial. Then, for $F'_1 \Leftarrow F_1 F_0$ and $\mathbb{F}'_1 \Leftarrow \langle F'_1, R' \cap F'_1 \rangle$, the extension $F'_1 \geq \mathbb{F}_1$ is inertial, and if $F_{R_0} \leq F_{R_1}$, then $F'_1 = F_1$ (i.e., $\mathbb{F}_0 \leq \mathbb{F}_1$).*

Essentially, this lemma was proved in the course of the proof of Proposition 2.

Corollary 7 (to Lemma 1). *Let $(\mathbb{F}' \geq) \mathbb{F}_i \geq \mathbb{F}$, $i = 0, 1$, be inertial extensions of $\mathbb{F}$. Then $\mathbb{F}'_1 \geq \mathbb{F}$ is an inertial extension ($F'_1 \Leftarrow F_1 F_0$).*

This implies the existence of a maximal inertial extension $F_{0,i} \geq \mathbb{F}$ that is a subextension of $\mathbb{F}_0$. The fact that $\Gamma_{R_{0,i}} = \Gamma_R$ follows from the definition of an inertial extension. Now, we prove that $F_{R_{0,i}}$ is the separable closure of F_R in F_{R_0} . Assuming the contrary, we can find an element $\bar{\alpha} \in F_{R_0} \setminus F_{R_{0,i}}$ separable over F_R . Let $f \in R[x]$ be a unitary polynomial such that $\bar{f} \in F_R[x]$ is the minimal polynomial of $\bar{\alpha}$ over F_R . By the Hensel lemma, the polynomial f has a root α in R_0 such that $\bar{\alpha} = \alpha + \mathbf{m}(R_0)$. Putting $F_1 \Leftarrow F(\alpha)$ and $\mathbb{F}_1 = \langle F_1, R_0 \cap F_1 \rangle$, we obtain an inertial extension $\mathbb{F}_1 \geq \mathbb{F}$. However, in this case we have $\mathbb{F}_1 \leq \mathbb{F}_{0,i}$, $\alpha \in R_{0,i}$, and $\bar{\alpha} \in F_{R_{0,i}}$, a contradiction.

The proposition is proved. □

We call the field $F_{0,i}(\mathbb{F}_{0,i})$ in Proposition 3 the *inertial closure* of $\mathbb{F}$ in $\mathbb{F}_0$.

We recall the basics of the decomposition theory of normal valued field extensions (see [5, 6]).

Let $\mathbb{F}_0 \geq \mathbb{F}$ be a normal extension of valued fields, and let $G = G(F_0/F)$ be the group of all F -automorphisms of F_0 . If $F_{0,s}$ is the separable closure of F in F_0 , then $F_{0,s}$ is a Galois extension of F , $G(F_0/F) = G(F_{0,s}/F)$, and $F_0 = F_{0,s} \otimes_F F_{0,\text{ins}}$, where $F_{0,\text{ins}}$ is a purely inseparable closure of F in F_0 .

By definition, the *inertia group* G_i of a (normal) extension $\mathbb{F}_0 \geq \mathbb{F}$ is the (closed) subgroup $\{\varphi \mid \varphi \in G, \forall a \in R_0(a - \varphi(a) \in \mathbf{m}(R_0))\}$ of the group G . The group G_i is the kernel of the natural epimorphism $\varepsilon : G = G(F_0/F) \twoheadrightarrow G(F_{R_0}/F_R)$ (see Proposition 1.3.3 in [1]). Therefore, G_i is a normal subgroup of G .

Remark 3. The group $G(F_{R_0}/F_R)$ is naturally isomorphic to the group $G(F_{0,i}/F)$. Therefore, $F_{0,i} = F_{0,s}^{G_i}$ by Galois theory.

The field $F_{0,s}^{G_i}$ is called the *inertia field* of the extension $\mathbb{F}_0 \geq \mathbb{F}$.

By definition, the *ramification group* G_r of a (normal) extension $\mathbb{F}_0 \geq \mathbb{F}$ is the subgroup $\{\varphi \mid \varphi \in G_i, \forall a \in F_0^\times(\varphi(a)a^{-1} - 1 \in \mathbf{m}(R_0))\}$ of the inertia group G_i .

The field $F_{0,s}^{G_r}$ is called the *ramification field* of the extension $\mathbb{F}_0 \geq \mathbb{F}$.

Now, we prove two technical statements needed in the sequel.

Lemma 2. *Let $\mathbb{F}_0 \geq \mathbb{F}$ be a finite normal extension. Then the ramification group G_r is a p -group.*

Proof. Passing from F to $F_0^{G_r}$, we may assume that $G_r = G = G(F_0/F)$ and $F_0 \geq F$ is a Galois extension. Assuming that the ramification group G_r is not a p -group, we can find a prime $q \neq p$ and an element $\varphi \in G_r \setminus \{e\}$ such that $\varphi^q = \text{id}_{F_0}$. We consider the field $F_1 = F_0^\varphi = \{a \mid a \in F_0, \varphi(a) = a\}$. Then F_0 is a Galois extension of F_0^φ whose

Galois group $\langle \varphi \rangle$ is cyclic of order q . We note that $\langle \varphi \rangle$ is also the ramification group of the extension $\mathbb{F} \geq \mathbb{F}_1$. Let $\alpha \in R_0 \setminus R_1$. Then $F_0 = F_1(\alpha)$ ($F_0 \geq F_1(\alpha) > F_1$) and $q = [F_0 : F_1]$. Let $f = x^q + a_1 x^{q-1} + \dots + a_q \in R_1[x]$ be the minimal polynomial of α over F_1 . Without loss of generality, we may assume that $a_1 = 0$ (otherwise, we can replace α by the element $\alpha' = \alpha + q^{-1}a_1$, where $q^{-1}a_1 \in R_1$). Then the trace $\text{Tr}(\alpha)$ of α is equal to 0. Let $\alpha_0 = \alpha$, $\alpha_1 = \varphi(\alpha)$, $\dots$, $\alpha_{q-1} = \varphi^{q-1}(\alpha)$ be all roots of f . Then $\text{Tr}(\alpha) = \sum \alpha_i$. Since $\varphi^i \in G_r$, the definition of G_r implies that $\varphi^i(\alpha)\alpha^{-1} - 1 \in \mathfrak{m}(R_0)$, $i < q$, whence $v_{R_0}(\alpha_i\alpha^{-1} - 1) > 0$ and $\overline{\alpha_i\alpha^{-1}} = 1$, $i < q$. Therefore, $0 = \alpha^{-1} \text{Tr} \alpha = \sum_{i < q} \alpha_i\alpha^{-1}$, and passing to the residue field F_{R_0} , we obtain

$$\overline{\alpha^{-1} \text{Tr}(\alpha)} = \sum_{i < q} \overline{\alpha_i\alpha^{-1}} = q \neq 0.$$

This contradiction proves the lemma. $\square$

Lemma 3. *Let $\mathbb{F}_0 \geq \mathbb{F}$ be a finite, normal, purely wild extension. Then $[F_0 : F]$ is a power of p , and the group $G = G(F_0/F)$ coincides with the ramification group G_r of the normal extension $\mathbb{F}_0 \geq \mathbb{F}$.*

Proof. Passing from $\mathbb{F}_0$ to the separable closure $\mathbb{F}_{0,s}$ of $\mathbb{F}$ in $\mathbb{F}_0$ ($[F_1 : F_{0,s}]$ is a power of p !), we may assume that $\mathbb{F}_0 \geq \mathbb{F}$ is a finite Galois extension. We note that the condition $\varphi(a)a^{-1} - 1 \in \mathfrak{m}(R_0)$ for $a \in F_0^\times$ is equivalent to the condition $v_R(\varphi(a) - a) > v_R(a)$.

Let $\varphi \in G$. If $a \in R_0 \setminus \mathfrak{m}(R_0)$, then $\varphi(a) - a \in \mathfrak{m}(R_0)$, because F_{R_0} is a purely inseparable extension of F_R and φ induces the trivial F_R -automorphism of F_{R_0} . Consequently, for $a \in R_0 \setminus \mathfrak{m}(R_0)$ we have $\varphi(a)a^{-1} - 1 \in \mathfrak{m}(R_0)$. Let $a \in F_0^\times$. Then $v_{R_0}(a^{p^k}) \in \Gamma_R$ for an appropriate $k \in \omega$ (the quotient group Γ_{R_0}/Γ_R is a p -group!). Let $b \in F$ be such that $v_{R_0}(a^{p^k}) = v_{R_0}(b) = v_R(b)$. Then $a^{p^k}b^{-1} \in R_0 \setminus \mathfrak{m}(R_0)$, $v_{R_0}(\varphi(a^{p^k}b^{-1}) - a^{p^k}b^{-1}) > 0$, and $v_{R_0}(\varphi(a)^{p^k} - a^{p^k}) > v_R(b) = v_{R_0}(a^{p^k})$. To obtain the required relation $v_{R_0}(\varphi(a) - a) > v_{R_0}(a)$, it suffices to verify that, for all $c, d \in F_0^\times$, the relation $v_{R_0}(c) = v_{R_0}(d) = v_{R_0}(c - d)$ implies $v_{R_0}(c^p) (= v_{R_0}(d^p)) = v_{R_0}(c^p - d^p)$.

If p is the characteristic of F_0 , this is obvious. Suppose the characteristic of F_0 is zero and $p > 2$. Then

$$(c - d)^p = c^p + \sum_{i=1}^{p-1} (-1)^i C_p^i c^{p-i} d^i - d^p.$$

Observe that p divides C_p^i , $0 < i < p$, and therefore, $v_{R_0}(C_p^i c^{p-i} d^i) > v_{R_0}(c^{p-i} d^i) = v_{R_0}(c^p) (= v_{R_0}(d^p))$, $0, 1 < i < p$. Hence,

$$v_{R_0} \left(\sum_{i=1}^{p-1} (-1)^i C_p^i c^{p-i} d^i \right) > v_{R_0}(c^p) = v_{R_0}(d^p),$$

and if $v_{R_0}(c^p - d^p) > v_{R_0}(c^p)$, then also $v_{R_0}((c - d)^p) > v_{R_0}(c^p)$. This contradicts the assumption that $v_{R_0}(c - d) = v_{R_0}(c)$.

If $p = 2$, then the same argument shows that $v_{R_0}(c^2 + d^2) = v_{R_0}(c^2)$. However, $c^2 - d^2 = c^2 + d^2 - 2d^2$ and $v_{R_0}(2d^2) > v_{R_0}(d^2) = v_{R_0}(c^2)$. Thus, $v_{R_0}(c^2 + d^2) = v_{R_0}(c^2 + d^2) = v_{R_0}(c^2)$.

This completes the proof of the lemma. $\square$

Now we prove a proposition that will imply the uniqueness of $\mathbb{F}_1$ from Proposition 2 in the case where $\mathbb{F}_0 \geq \mathbb{F}$ is a (finite) normal extension.

Proposition 4. *Let $\mathbb{F}_0 \geq \mathbb{F}$ be a finite normal extension, and let $\mathbb{F}_1$ be the maximal tame extension of $\mathbb{F}$ in $\mathbb{F}_0$. Then F_1 is the ramification field of the extension $\mathbb{F}_0 \geq \mathbb{F}$.*

By Proposition 2, the extension $\mathbb{F}_0 \geq \mathbb{F}_1$ is normal and purely wild. By Lemma 3, we have $G(F_0/F_1) \leq G_r$, and the degree $[F_0 : F_1]$ is a power of p . Since the extension $\mathbb{F}_1 \geq \mathbb{F}$ is tame, the field F_1 is a separable extension of F (see Corollary 4). Then $F_1 \leq F_{0,s}$, and we may assume that $F_{0,s} = F_0$, i.e., that F_0 is a Galois extension of F . Let $F_{0,i}$ be the inertial closure of F in F_0 , $F_{0,i} \leq F_1$. We have $[F_1 : F_{0,i}] = [\Gamma_{R_1} : \Gamma_{R_{0,i}}] = [\Gamma_{R_1} : \Gamma_R]$ and p does not divide the index $[\Gamma_{R_1} : \Gamma_R]$. By Galois theory, $[F_1 : F_{0,i}] = [G(F_0/F_{0,i}) : G(F_0/F_1)]$, but $G(F_0/F_{0,i}) = G_i$ by the remark preceding the definition of the inertia field. Moreover, by Galois theory, $G(F_0/F_1)$ is a p -group. We note that, by Lemma 2, the ramification group G_r is a p -group. Since $G(F_0/F_1) \leq G_r$, p does not divide the index $[G_i : G(F_0/F_1)]$, and $G_r \leq G_i$ is a p -group, it follows that $G(F_0/F_1) = G_r$, and $F_1 = F_0^{G_r} (= F_{0,s}^{G_r}$ in the general case) is the ramification field of the extension $\mathbb{F}_0 \geq \mathbb{F}_1$.

The proposition is proved. $\square$

Corollary 8 (to the proof). $[F_0 : F_1]$ is a power of p .

Corollary 9. Let $(\bar{\mathbb{F}} \geq) \mathbb{F}_j \geq \mathbb{F}$, $j = 0, 1$, be tame finite extensions. Then $\mathbb{F}' \geq \mathbb{F}$ is a tame extension, where $F' \rightleftharpoons F_0 F_1$ and $\mathbb{F}' \rightleftharpoons \langle F', \bar{R} \cap F' \rangle$.

Proof. Indeed, let F_3 be the smallest Galois extension of F containing F' (F' is a finite separable extension of $F!$), and let $\mathbb{F}_3 \rightleftharpoons \langle F_3, \bar{R} \cap F_3 \rangle$. Then $\mathbb{F} \leq \mathbb{F}_0$ and $\mathbb{F}_1 \leq \mathbb{F}_3$ are separable subextensions. Consequently, if $\mathbb{F}'_0 \geq \mathbb{F}_0$ and $\mathbb{F}'_1 \geq \mathbb{F}_1 \geq \mathbb{F}$ are maximal tame subextensions, then, by Proposition 4, we have $F'_0 = F'_1 = F_0^{G_r}$, and the extension $(\mathbb{F}'_0 \geq) \mathbb{F}' \geq \mathbb{F}$ is tame. $\square$

Corollary 10. Let $\mathbb{F}_0 \geq \mathbb{F}$ be a finite extension, and let $\mathbb{F}_0 \geq \mathbb{F}_1 \geq \mathbb{F}$ be a tower of extensions such that $\mathbb{F}_1 \geq \mathbb{F}$ is tame and $\mathbb{F}_0 \geq \mathbb{F}_1$ is purely wild. Then $\mathbb{F}_1$ is the largest tame subextension of $\mathbb{F}$ in $\mathbb{F}_0$.

Corollary 11. a) If $\mathbb{F}_0 \geq \mathbb{F}$ is a finite purely wild extension, then $[F_0 : F]$ is a power of p .

b) If $\mathbb{F}_0 \geq \mathbb{F}$ is a finite extension and p does not divide $[F_0 : F]$, then $\mathbb{F}_0 \geq \mathbb{F}$ is a tame extension.

Proof. a) Let $\mathbb{F}_0 \geq \mathbb{F}$ be a purely wild extension, $\mathbb{F}'_0 \geq \mathbb{F}_0$ the smallest normal extension, and $\mathbb{F}'_{0,t}$ the largest tame subextension of $\mathbb{F}$ in $\mathbb{F}'_0$. Then $\mathbb{F}'_{0,t} \cap F_0$ is a tame extension of $\mathbb{F}$. Therefore, $F'_{0,t} \cap F_0 = F$, and $[F_0 : F] = [F'_{0,t} F_0 : F]$ (see Proposition 1) divides $[F'_0 : F'_{0,t}]$, but $[F'_0 : F'_{0,t}]$ is a power of p (see Corollary 8).

b) Let $\mathbb{F}_0 \geq \mathbb{F}$ be a finite extension such that p does not divide $[F_0 : F]$. Let $\mathbb{F}_1 \geq \mathbb{F}$ be the largest tame subextension of $\mathbb{F}$ in $\mathbb{F}_0$. Then p does not divide $[F_0 : F_1]$ because $[F_0 : F] = [F_0 : F_1][F_1 : F]$ and $[F_0 : F_1]$ are powers of p by part a). Hence, $F_0 = F_1$. $\square$

Corollary 12 (Ostrowski lemma). If $\mathbb{F}_0 \geq \mathbb{F}$ is a finite extension, $f \rightleftharpoons [F_{R_0} : F_R]$, and $e \rightleftharpoons [\Gamma_{R_0} : \Gamma_R]$, then $[F_0 : F] = f \cdot e \cdot p^k$ for some $k \in \omega$.

Proof. Indeed, if $\mathbb{F}_1 \geq \mathbb{F}$ is the largest tame extension of $\mathbb{F}$ in $\mathbb{F}_0$, then $[F_1 : F] = f_1 \cdot e_1$, where $f_1 \rightleftharpoons [F_{R_1} : F_R]$, $e_1 \rightleftharpoons [\Gamma_{R_1} : \Gamma_R]$, $[F_0 : F_1] = p^K$ for some $K \in \omega$ (by Corollary 11); we have $[F_{R_0} : F_{R_1}] = p^{k_0}$, $[\Gamma_{R_0} : \Gamma_{R_1}] = p^{k_1}$ for some k_0 and $k_1 \in \omega$ (by the definition of a wild extension); also, $k_0 + k_1 \leq K$, $f = f_1 \cdot p^{k_0}$, and $e = e_1 \cdot p^{k_0}$. This implies the corollary. $\square$

From Proposition 2 and Corollary 9 we deduce the following statement.

Theorem 1. For every algebraic extension $\mathbb{F}_0 \geq \mathbb{F}$, there exists a maximal tame subextension $\mathbb{F}_0 \geq \mathbb{F}_{0,t} \geq \mathbb{F}$ such that $\mathbb{F}_0 \geq \mathbb{F}_{0,t}$ is purely wild.

Proof. Let $F_{0,t}$ be the union of all finite extensions $F_1 \geq F$ such that $F_0 \geq F_1$ and $\mathbb{F}_1 = \langle F_1, R_0 \cap F_1 \rangle \geq \mathbb{F}$ is tame. Then $\mathbb{F}_{0,t} = \langle F_{0,t}, R_0 \cap F_{0,t} \rangle$ is the largest tame subextension of $\mathbb{F}$ in $\mathbb{F}_0$. We prove that the extension $\mathbb{F}_0 \geq \mathbb{F}_{0,t}$ is purely wild. Let $\alpha \in F_0$, and let $\mathbb{F}' = \langle F(\alpha), R_0 \cap F(\alpha) \rangle$. If $\mathbb{F}'_t$ is the largest tame subextension of $\mathbb{F}$ in $\mathbb{F}'$, then $\mathbb{F}'_t = F_{0,t} \cap F'$ and $\mathbb{F}'_t \leq \mathbb{F}_{0,t}$. By Proposition 2, the extension $\mathbb{F}' \geq \mathbb{F}'_t$ is purely wild. In particular, if $\alpha \in R_0 \cap F' = R'$, then $\bar{\alpha} = \alpha + \mathbf{m}(R') \in F_{R'}$ and $\bar{\alpha}^{p^k} \in F_{R'_t} \leq F_{R_{0,t}}$ for some $k \in \omega$, and $v_{R'}(\alpha)^{p^{k'}} \in \Gamma_{R'_t} \leq \Gamma_{R_{0,t}}$ for $\alpha \in R' \setminus \{0\}$ and some $k' \in \omega$. Thus, F_{R_0} is purely inseparable over $F_{R_{0,n}}$, and the quotient group $\Gamma_{R_0}/\Gamma_{R_{0,t}}$ is a p -group; i.e., the extension $\mathbb{F}_0 \geq \mathbb{F}_{0,t}$ is purely wild.

The theorem is proved. $\square$

The maximal tame subextension $\mathbb{F}_{0,t} \geq \mathbb{F}$ of an algebraic extension $\mathbb{F}_0 \geq \mathbb{F}$ is called the *tame closure* of $\mathbb{F}$ in $\mathbb{F}_0$. For the tame closure we use the notation $\mathbb{F}_{0,t}$ ($(\mathbb{F}_0)_t$ would be more proper); for $\mathbb{F}' \geq \mathbb{F}$ we use $\mathbb{F}'_t$, etc.

Corollary 13. *Let $\mathbb{F}_0 \geq \mathbb{F}$ be an algebraic extension and $\mathbb{F}_{0,t} \geq \mathbb{F}$ the tame closure of $\mathbb{F}$ in $\mathbb{F}_0$. Then, for every intermediate valued field $\mathbb{F}_0 \geq \mathbb{F}_1 \geq \mathbb{F}$, the tame closure of $\mathbb{F}$ in $\mathbb{F}_1$ coincides with $\mathbb{F}_1 \cap \mathbb{F}_{0,t}$; in particular, if $\mathbb{F}_1 \cap \mathbb{F}_{0,t} = \mathbb{F}$, then the extension $\mathbb{F}_1 \geq \mathbb{F}$ is purely wild.*

We say that an algebraic extension $\mathbb{F}_0 \geq \mathbb{F}$ is *split* if there is a subextension $\mathbb{F}_0 \geq \mathbb{F}_w \geq \mathbb{F}$ such that the extension $\mathbb{F}_w \geq \mathbb{F}$ is purely wild and $\mathbb{F}_0 \geq \mathbb{F}_w$ is tame.

The following theorem of Pank (see [3]) on splitting of the algebraic closure is of great importance.

Theorem 2. *Let $\bar{\mathbb{F}} \geq \mathbb{F}$, and let $\bar{F}$ be the algebraic closure of F . Then, for every maximal purely wild extension $\mathbb{F}_w \geq \mathbb{F}$ in $\bar{\mathbb{F}}$, the extension $\bar{\mathbb{F}} \geq \mathbb{F}_w$ is tame.*

Proof. It can easily be checked that, for a valued field $\mathbb{F}_w$ (as in the theorem), the field F_{R_w} is perfect (actually, $F_{R_w} = F_R^{p^{-\infty}}$ is the smallest perfect field containing F_R) and the group Γ_{R_w} is p -divisible ($p\Gamma_{R_w} = \Gamma_{R_w}$) (actually, Γ_{R_w} is the smallest p -divisible group containing Γ_R). Indeed, if $\beta \in F_{\bar{R}} \setminus F_{R_w}$ and $\beta^p \in F_{R_w}$, then there is $b \in R_w$ such that $\beta^p = b + \mathbf{m}(\bar{R})$, and so $\mathbb{F}_w(b^{p^{-1}})$ is a purely wild extension of $\mathbb{F}$. We act in the same way in the case where the group Γ_{R_w} is not p -divisible (but $\Gamma_{\bar{R}}$ is p -divisible!). Similarly, we check that the field F_w is perfect.

Let $\mathbb{F}' \geq \mathbb{F}_w$ be a maximal intermediate extension (which exists by Proposition 1.5.7 in [1]). By Proposition 1.5.10 in [1], the field $\mathbb{F}'$ is algebraically complete, $F_{R'} = F_{R_w}$ is perfect, and $\Gamma_{R'} = \Gamma_{R_w}$ is p -divisible. By Corollary 2, every algebraic extension $\mathbb{F}'_0 \geq \mathbb{F}'$ is tame. Let $(\bar{\mathbb{F}} \geq) \mathbb{F}_0 \geq \mathbb{F}_w$ be a finite extension. If $F'_0 = F'F_w$, then $\mathbb{F}'_0 \geq \mathbb{F}'$ is a finite tame extension. Since $\mathbb{F}_w$ is algebraically closed in $\mathbb{F}'$ (this follows from the fact that $\mathbb{F}_w$ is maximal) and F_w is perfect, we see that F' is a regular extension of F_w and therefore, $F_0 \parallel_{F_w} F'$, $F'_0 = F_0 \otimes_{F_w} F'$, and F_0 is algebraically closed in $\mathbb{F}'_0$. The extension $F_{R'_0} \geq F_{R'} = F_{R_w}$ is finite separable, and the field F_{R_w} is a purely inseparable extension of F_R . Therefore, there is an element $\bar{\alpha} \in F_{R'_0}$ separable over F_R and such that $F_{R'_0} = F_{R_w}(\bar{\alpha})$. Let $f \in R[x]$ be a monic polynomial such that $\bar{f} \in F_R[x]$ is the minimal polynomial of $\bar{\alpha}$ over F_R (and over F_{R_w}). By the Hensel lemma, there exists a root α of f in R_0 such that $\bar{\alpha} = \alpha + \mathbf{m}(R'_0)$. Then the extension $(\mathbb{F}_0 \geq) \mathbb{F}_w(\alpha) \geq \mathbb{F}_w$ is tame (even inertial). Next, p does not divide $[F'_0 : F'] \cdot [F_{R'_0} : F_{R'}]^{-1} = [\Gamma_{R'_0} : \Gamma_{R'}]$, but $[F'_0 : F'] = [F_0 : F_w] = [F_0 : F_w(\alpha)] \cdot [F_w(\alpha) : F_w]$ and $[F_w(\alpha) : F_w] = [F_{R'_0} : F_{R'}]$. Consequently, $[F_0 : F_w(\alpha)] = [\Gamma_{R'_0} : \Gamma_{R'}]$. However, in this case the extension $\mathbb{F}_0 \geq \mathbb{F}_w(\alpha)$ is tame (Corollary 11 b), whence $\mathbb{F}_0(\geq F_w(\alpha)) \geq \mathbb{F}_w$ is tame.

The theorem is proved. $\square$

Now, we prove a splitting criterion for a normal extension $\mathbb{F}_0 \geq \mathbb{F}$ in terms of Galois groups.

Proposition 5. *A normal extension $\mathbb{F}_0 \geq \mathbb{F}$ is split if and only if the epimorphism*

$$\pi : G(F_0/F) \rightarrow G(F_{0,t}/F)$$

is split, i.e., there exists a (closed) subgroup $G' \leq G(F_0/F)$ such that $G' \cap G_r = \{e\}$ and $G'G_r = G(F_0/F_1)$, where $G_r = \text{Ker } \pi$ is the ramification group of the extension $\mathbb{F}_0 \geq \mathbb{F}$ (such a subgroup G' is called the complement to G_r).

Proof. We assume that the extension $\mathbb{F}_0 \geq \mathbb{F}$ is split and $\mathbb{F}_0 \geq \mathbb{F}' \geq \mathbb{F}$ is a tower of extensions such that the extension $\mathbb{F}' \geq \mathbb{F}$ is purely wild and $\mathbb{F}_0 \geq \mathbb{F}'$ is tame. Let $\mathbb{F}_{0,t}$ be the tame closure of $\mathbb{F}$ in $\mathbb{F}_0$. We prove that $F_0 = F' \otimes_F F_{0,t}$. Let $F'_0 = F'F_{0,t}$. By Proposition 1, we have $F'_0 = F' \otimes_F F_{0,t}$. Consequently, we must prove that $F'_0 = F_0$. Since $\mathbb{F}_0 \geq \mathbb{F}'_0 \geq \mathbb{F}' \geq \mathbb{F}$, we see that the extension $\mathbb{F}_0 \geq \mathbb{F}'_0$ is tame. Since $\mathbb{F}_0 \geq \mathbb{F}'_0 \geq \mathbb{F}_{0,t} \geq \mathbb{F}$, the extension $\mathbb{F}_0 \geq \mathbb{F}'_0$ is purely wild. By Proposition 3, we have $\mathbb{F}_0 = \mathbb{F}'_0$.

We put $G' = G(F_0/F') (\geq G(F_0/F))$ and prove that G' is a complement to G_r . Since $F' \cap F_{0,t} = F$ ($\mathbb{F}' \cap \mathbb{F}_{0,t} \geq \mathbb{F}$ is tame and purely wild), $F_{0,t} = (F_{0,s})^{G_r}$, and $F' = F_0^{G'} (F' \geq F_{0,\text{ins}}$ and $F' = (F' \cap F_{0,s}) \otimes_F F_{0,\text{ins}}$), we obtain $G'G_r = G(F_0/F_1)$. The property $G' \cap G_r = \{e\}$ follows from the fact that $F_0 = F' \otimes_F F_{0,t}$.

Conversely, suppose that the epimorphism $\pi : G(F_0/F') \rightarrow G(F_{0,t}/F)$ is split and $G' \geq G(F_0/F_1)$ is a complement to G_r . Consider the field $F' = F_0^{G'}$ and put $\mathbb{F}' = \langle F', R_0 \cap F' \rangle$; we prove that $\mathbb{F}' \geq \mathbb{F}$ is a wild extension. Since $F_{0,t} = F_0^{G_r}$ and $G_r G' = G(F_0/F')$, we obtain $F' \cap F_{0,t} = F$, and, by Corollary 13, the extension $\mathbb{F}' \geq \mathbb{F}$ is purely wild. By Proposition 1, the extension $\mathbb{F}'_0 (= \langle F_0 F', R_0 \cap F_0 F' \rangle) \geq \mathbb{F}'$ is tame. It remains to note that $F_0 F' = F_0$, which follows directly from the condition $G_r \cap G' = \{e\}$.

The proposition is proved. $\square$

The statement below follows from Theorem 2 and Proposition 5.

Theorem 3 (Theorem 2.2 in [3]). *The absolute ramification group G_r (= the ramification group of the extension $\bar{\mathbb{F}} \geq \mathbb{F}$) has a complement in $G(F) = G(\bar{F}/F)$ (the epimorphism $\pi : G(F) \rightarrow G(\bar{F}_t/F) \simeq G(F)/G_r$ is split).*

The next statement says that, for every finite normal extension $\mathbb{F}_0 \geq \mathbb{F}$, there exists a “not very large” tame extension such that the extension $\mathbb{F}'_0 \geq \mathbb{F}$ is split ($F'_0 = F'F_0$).

Proposition 6. *Let $\mathbb{F}_0 \geq \mathbb{F}$ be a finite normal extension. Then there is a finite tame Galois extension $\mathbb{F}' \geq \mathbb{F}$ such that, for $F'_0 = F_0 F'$ and $\mathbb{F}'_0 = \langle F'_0, \bar{R} \cap F'_0 \rangle$, the following conditions are fulfilled:*

- 1) $F'_{0,t} = F'$ (then $F_{0,t} \leq F'$);
- 2) if $G' = G(F'/F)$, $G'_0 = G(F'_0/F)$, and G'_r is the ramification group of the normal extension $\mathbb{F}'_0 \geq \mathbb{F}$, then G' is a complement to G'_r in G'_0 ;
- 3) there exists an embedding $\eta : G' \rightarrow G(F_0/F)$ such that the composition $\pi\eta : G' \rightarrow G(F_{0,t}/F)$ coincides with the restriction to G' of the composition of the epimorphism $\pi' : G'_0 \rightarrow G(F'_{0,t}/F)$ (with kernel G'_r) and the restriction epimorphism $G(F'_{0,t}/F) \rightarrow G(F_{0,t}/F)$.

Proof. Let $(\bar{\mathbb{F}} \geq) \mathbb{F}_w \geq \mathbb{F}$ be as in Theorem 2. Let $F' = F_0 F_w$, let $\mathbb{F}'_w = \langle F'_w, \bar{R} \cap F'_w \rangle$, and let $\mathbb{F}'$ be the tame closure of $\mathbb{F}$ in $\mathbb{F}'_w$. The proof of Theorem 2 shows that $F'_w = F' \otimes_F F_w$, and $G' = G(F'/F) \simeq G(F'_w/F_w)$ is identified (via the embedding η !) with a subgroup of $G(F_0/F)$. It is easily seen that $F'_0 = F \otimes_{F_{0,t}} F_0$, and therefore, $\mathbb{F}'_0 \geq \mathbb{F}'$ is a wild extension and $\mathbb{F}'_{0,t} = \mathbb{F}'$.

We omit the routine proof of statements 2) and 3). $\square$

In conclusion, we present yet another statement on splitting.

Proposition 7. *Let $\mathbb{F}_0 \geq \mathbb{F}$ be a finite extension. Then there exists a finite purely wild extension $\mathbb{F}' \geq \mathbb{F}$ such that, for $F'_0 = F'\mathbb{F}_0$ and $\mathbb{F}'_0 = \langle F'_0, \bar{R} \cap F'_0 \rangle$, the extension $\mathbb{F}'_0 \geq \mathbb{F}'$ is tame (and therefore, the extension $\mathbb{F}'_0 \geq \mathbb{F}$ is split).*

Proof. Let $(\bar{\mathbb{F}} \geq) \mathbb{F}_w \geq \mathbb{F}$ be as in Theorem 2, let $F'_w = F_0 F_w$, and let $\mathbb{F}'_w = \langle F_w, \bar{R} \cap F_w \rangle$. Then $\mathbb{F}'_w \geq \mathbb{F}_w$ is a split tame extension. If $\mathbb{F}'' = \langle F'', \bar{R} \cap F'' \rangle$ is the tame closure of $\mathbb{F}$ in $\mathbb{F}'_w$, then $F'_w = F'' \otimes_F F_w$. However, $F'_w = F_0 F_w$. Therefore, there exists a finite extension $F' \geq F$ in F_w such that $F_0 \leq F'' \otimes_F F'$. It is easily seen that $\mathbb{F}' = \langle F', \bar{R} \cap F' \rangle \geq \mathbb{F}$ is the required extension. $\square$

The results presented in this paper show that the classes of tame and purely wild extensions of (Henselian) valued fields play the same role as the classes of separable and purely inseparable extensions for arbitrary fields. Furthermore, for improper valued fields $\langle F, F \rangle$ these classes coincide.

For (Henselian) valued fields, there is yet another pair of classes of extensions with the same behavior, namely, the inertial extensions and the extensions $\mathbb{F}_0 \geq \mathbb{F}$ such that F_{R_0} is inseparable over $\mathbb{F}$. For this pair of classes of extensions, analogs of all results of the present paper are valid. In particular, we have the following version of Theorem 3.

Theorem 3'. *The inertia group G_i of the absolute Galois group $G(F)$ of the Henselian valued field $\mathbb{F}$ has a complement in $G(F)$ (or equivalently, the epimorphism $\varepsilon : G(F) \rightarrow G(F_R)$ is split).*

For the first time, this theorem appeared in the paper [7]. The book [1] contains a proof of this result (see Proposition 1.5.11) known to the author since 1965. Unfortunately, an uncritical view of that proof (maximal intermediate extensions were involved in the arguments) has led to a more general formulation of Proposition 1.5.11, which (in this generality) turned out to be wrong (at the conference in Oberwolfach (Germany) in 2003, I. B. Fesenko and M. V. Bondarko called the author's attention to this fact).

REFERENCES

- [1] Yu. L. Ershov, *Multiply normed fields*, "Nauchn. Kniga", Novosibirsk, 2000. (Russian)
- [2] S. Lang, *Algebra*, 3rd ed., Grad. Texts in Math., vol. 211, Springer-Verlag, New York, 2002. MR1878556 (2003e:00003)
- [3] F.-V. Kuhlmann, M. Pank, and P. Roquette, *Immediate and purely wild extensions of valued fields*, Manuscripta Math. **55** (1986), no. 1, 39–67. MR0828410 (87d:12012)
- [4] F.-V. Kuhlmann, *Henselian function fields and tame fields*, Heidelberg, 1994 (manuscript).
- [5] O. Zariski and P. Samuel, *Commutative algebra*. Vol. II, Grad. Texts in Math., vol. 29, Springer-Verlag, New York – Heidelberg, 1975. MR0389876 (52:10706)
- [6] A. J. Engler and A. Prestel, *Valued fields*, Springer-Verlag, Berlin, 2005. MR2183496 (2007a:12005)
- [7] O. V. Mel'nikov and O. I. Tavgen', *The absolute Galois group of a Henselian field*, Dokl. Akad. Nauk BSSR **29** (1985), no. 7, 581–583, 667. (Russian) MR0801082 (87a:12007)

SOBOLEV INSTITUTE OF MATHEMATICS, SIBERIAN BRANCH OF THE RUSSIAN ACADEMY OF SCIENCES,
4 ACAD. KOPTYUG AVENUE, 630090 NOVOSIBIRSK, RUSSIA
E-mail address: `ershov@math.nsc.ru`

Received 20/APR/2007

Translated by B. M. BEKKER