ON $(2,3)$ -GENERATION OF MATRIX GROUPS OVER THE RING OF INTEGERS

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To the centenary of D. K. Faddeev's birth

ABSTRACT. The groups $GL(5, \mathbb{Z})$, $SL(5, \mathbb{Z})$, $SL(6, \mathbb{Z})$, $GL(7, \mathbb{Z})$ and $SL(7, \mathbb{Z})$ are $(2,3)$ -generated.

§1. INTRODUCTION

A group is said to be $(2,3)$ -generated if it can be generated by an involution and an element of order 3. It is well known that the group $PSL(2, \mathbb{Z})$ is isomorphic to the free product of the cyclic group of order 2 and the cyclic group of order 3: $PSL(2, \mathbb{Z}) \simeq C_2 * C_3$. Therefore, the problem of classification of $(2,3)$ -generated groups is, in essence, the problem of finding the quotients of the modular group different from $\{1\}$, C_2 , and C_3 . In this generality, the problem looks absolutely hopeless. Thus, what is usually studied is $(2,3)$ -generation in some wide class of groups.

Many authors investigated the problem of $(2,3)$ -generation for classical matrix groups over finite fields [1, 2, 4, 9, 10] and over the ring $\mathbb{Z}$ of integers [5, 7, 8, 11, 12, 13]. In the case of groups over finite fields the problem is solved (see, e.g., [4]). However, for matrix groups over $\mathbb{Z}$, the answer is not known for all dimensions.

As was shown in [8], the groups $SL(n, \mathbb{Z})$ and $GL(n, \mathbb{Z})$ are $(2,3)$ -generated provided $n \geq 13$ and $n \geq 19$, respectively. On the other hand, $SL(4, \mathbb{Z})$ and $GL(4, \mathbb{Z})$ are not $(2,3)$ -generated, since even the group $SL(4, 2) \simeq GL(4, 2) \simeq \text{Alt}(8)$ is not $(2,3)$ -generated [6]. It is also known that $SL(3, \mathbb{Z})$ and $GL(3, \mathbb{Z})$ are not $(2,3)$ -generated [7, 11].

For the groups $GL(5, \mathbb{Z})$ and $SL(6, \mathbb{Z})$, some preliminary results were given in [5] and [12], respectively. Those papers did not give a complete answer, but described a reduction of the original problem to the analysis of finitely many cases listed explicitly.

For a long time it has not even been clear where the border between the positive and negative results lies. In a private correspondence, M. C. Tamburini conjectured that the groups $SL(n, \mathbb{Z})$ for $n \leq 12$ might be non- $(2,3)$ -generated. In this respect, the author's result in [13], showing that the group $GL(6, \mathbb{Z})$ can be generated by an involution and an element of order 3, was all the more astonishing.

In this paper we present further positive results on $(2,3)$ -generation of the groups $SL(n, \mathbb{Z})$ and $GL(n, \mathbb{Z})$. In particular we prove the following theorem.

Theorem 1.1. *The groups $GL(5, \mathbb{Z})$, $SL(5, \mathbb{Z})$, $SL(6, \mathbb{Z})$, $GL(7, \mathbb{Z})$, and $SL(7, \mathbb{Z})$ are $(2,3)$ -generated.*

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This result is proved separately for each value of n ; see Theorem 1.2 for $n = 5$, Theorem 7.1 for $n = 6$ and Theorem 9.1 for $n = 7$ below. In addition, in Theorem 8.1 we construct yet another set of $(2,3)$ -generators for the group $\mathrm{PSL}(6, \mathbb{Z})$, which is not inherited from $\mathrm{SL}(6, \mathbb{Z})$.

For $n = 5$ we can achieve more. Namely, for $\mathrm{GL}(5, \mathbb{Z})$ and $\mathrm{SL}(5, \mathbb{Z})$ we list all pairs of $(2,3)$ -generators up to $\mathrm{GL}(5, \mathbb{Z})$ -conjugation.

Theorem 1.2. *The groups $\mathrm{GL}(5, \mathbb{Z})$ and $\mathrm{SL}(5, \mathbb{Z})$ are $(2,3)$ -generated. Moreover, any pair of $(2,3)$ -generators of $\mathrm{GL}(5, \mathbb{Z})$ is conjugate in $\mathrm{GL}(5, \mathbb{Z})$ to one of the six pairs of matrices X, Y , while any pair of $(2,3)$ -generators of $\mathrm{SL}(5, \mathbb{Z})$ is conjugate in $\mathrm{GL}(5, \mathbb{Z})$ to one of the six pairs of matrices $-X, Y$, where*

$$X = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 & 0 & 0 & a_1 \\ -1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 1 & a_3 \\ 0 & 0 & -1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and (a_1, a_2, a_3, a_4) is one of the following sets:

$$\begin{aligned} (1, -1, -2, -2), & \quad (0, -1, -2, -2), \\ (-1, 1, -2, -2), & \quad (0, 1, -2, -2), \\ (1, -1, 1, -3), & \quad (0, -1, 0, -1). \end{aligned}$$

The proof of Theorem 1.2 is given in §§3–6; in particular, see Theorems 3.1, 3.2, 3.3, 4.1, 5.1, and 6.1.

The results of the present paper, together with the results of [8], give hope for the complete solution of the $(2,3)$ -generation problem for the groups $\mathrm{SL}(n, \mathbb{Z})$ and $\mathrm{GL}(n, \mathbb{Z})$. For the group $\mathrm{SL}(6, \mathbb{Z})$, a classification similar to Theorem 1.2 may be expected. For partial results in this direction, see [12]. However, at the moment I am unable to give a complete answer.

Some matrix manipulations used in the proofs of the main results of the present paper were made by means of the MAGMA package.

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§2. NOTATION AND AUXILIARY STATEMENTS

In the present paper we use the following notation. The identity matrix of size n is denoted by I_n or simply by I if the size is clear from the context. As usual, the e_{ij} , $1 \leq i, j \leq n$, are the elements of the standard basis of the matrix algebra (the so-called “matrix units”) and the $t_{ij}(\xi)$ are the elementary transvections

$$t_{ij}(\xi) = I + \xi e_{ij}, \quad i \neq j, \quad \xi \in \mathbb{Z}.$$

The proof of the main result is constructive; in each case the corresponding pairs of $(2,3)$ -generators, say x and y , are given explicitly. It is well known that, for $n \geq 3$, the group $\mathrm{SL}(n, \mathbb{Z})$ is generated by the transvections

$$t_{ij}(1), \quad 1 \leq i \neq j \leq n;$$

see [3]. The main idea of the proof of Theorem 1.1 is to find enough transvections within the group $\langle x, y \rangle$. The required calculations can be reduced by using the following well-known commutator identity:

$$(1) \quad t_{ij}(\xi)t_{jk}(\zeta)t_{ij}(\xi)^{-1}t_{jk}(\zeta)^{-1} = t_{ij}(\xi)t_{jk}(\zeta)t_{ij}(-\xi)t_{jk}(-\zeta) = t_{ik}(\xi\zeta),$$

which is valid for any pairwise distinct i, j , and k .

As the following easy lemma shows, for odd n the groups $\mathrm{SL}(n, \mathbb{Z})$ and $\mathrm{GL}(n, \mathbb{Z})$ can be considered simultaneously.

Lemma 2.1. *Let n be odd. Let $x, y \in \mathrm{GL}(n, \mathbb{Z})$ and $x^2 = y^3 = I$. We have $\langle x, y \rangle = \mathrm{GL}(n, \mathbb{Z})$ if and only if $\langle -x, y \rangle = \mathrm{SL}(n, \mathbb{Z})$.*

Proof. Clearly $\det y = 1$ and $\det x = -\det(-x)$.

Assume that $\langle x, y \rangle = \mathrm{GL}(n, \mathbb{Z})$. Then $\det x = -1$. Consequently, $\langle -x, y \rangle \subseteq \mathrm{SL}(n, \mathbb{Z})$. It is well known that the derived subgroup of $\mathrm{GL}(n, \mathbb{Z})$ is $\mathrm{SL}(n, \mathbb{Z})$, i.e., any matrix $g \in \mathrm{SL}(n, \mathbb{Z})$ can be represented in the form

$$g = \prod_{i=1}^m [g_{2i-1}(x, y), g_{2i}(x, y)],$$

where $g_1, \dots, g_{2m}$ are some words in the alphabet x, y, y^{-1} . For any j , we have $g_j(x, y) = \pm g_j(-x, y)$. Therefore,

$$[g_{2i-1}(x, y), g_{2i}(x, y)] = [g_{2i-1}(-x, y), g_{2i}(-x, y)],$$

and the inclusion $\mathrm{SL}(n, \mathbb{Z}) \subseteq \langle -x, y \rangle$ follows.

Now assume that $\langle -x, y \rangle = \mathrm{SL}(n, \mathbb{Z})$. The group $\mathrm{SL}(n, \mathbb{Z})$ coincides with its derived subgroup. Hence, any matrix $g \in \mathrm{SL}(n, \mathbb{Z})$ can be represented in the form

$$g = \prod_{i=1}^m [g_{2i-1}(-x, y), g_{2i}(-x, y)] = \prod_{i=1}^m [g_{2i-1}(x, y), g_{2i}(x, y)],$$

where $g_1, \dots, g_{2m}$ are some words in x, y, y^{-1} . Therefore, $\langle x, y \rangle$ contains $\mathrm{SL}(n, \mathbb{Z})$. Since $-x \in \mathrm{SL}(n, \mathbb{Z})$ and n is odd, we have $\det x = -1$. Hence, $\langle x, y \rangle = \mathrm{GL}(n, \mathbb{Z})$. $\square$

To prove Theorem 1.2, we also need the following condition sufficient for the group $\langle x, y \rangle$ to be a proper subgroup of $\mathrm{SL}(n, \mathbb{Z})$ or $\mathrm{GL}(n, \mathbb{Z})$.

Lemma 2.2. *Let $x, y \in \mathrm{SL}(n, \mathbb{Z})$, $x^2 = y^3 = I$. Assume that there exists a set $\mathcal{W} \subseteq \mathbb{R}^n$ and a vector $u \in \mathbb{R}^n \setminus \mathcal{W}$ such that*

- (i) $xy\mathcal{W} \subseteq \mathcal{W}$, $xy^2\mathcal{W} \subseteq \mathcal{W}$;
- (ii) $xyu \in \mathcal{W}$, $xy^2u \in \mathcal{W}$.

Then $\langle x, y \rangle \simeq \mathrm{PSL}(2, \mathbb{Z})$. In particular, $\langle x, y \rangle \neq \mathrm{GL}(n, \mathbb{Z})$, $\langle x, y \rangle \neq \mathrm{SL}(n, \mathbb{Z})$.

Proof. Since $\mathrm{PSL}(2, \mathbb{Z}) \simeq C_2 * C_3$, it suffices to show that the group $\langle x, y \rangle$ is the free product of two cyclic groups of orders 2 and 3. In other words, we must show that there are no (nontrivial) relations between x and y except $x^2 = y^3 = I$.

Let g be a reduced word in the alphabet x, y such that $g = I$. Reducibility means that g does not contain substrings xx and yyy .

The assumptions imply that $x \neq I$ and $y \neq I$. Therefore, if g is nonempty, then it contains both x and y . Replacing g by a conjugate word (if necessary), we may assume without loss of generality that g is cyclically reduced, i.e., g starts with x and ends with y or y^2 or g starts with y or y^2 and ends with x . Passing to a conjugate word (if necessary), again we may assume that the first alternative holds; namely,

$$g = \prod_{i=1}^m xy^{\varepsilon_i},$$

where $\varepsilon_i \in \{1, 2\}$. If $m > 0$, then the assumptions of the lemma show that $gu \in \mathcal{W}$, whence $gu \neq u$ and $g \neq I$. This proves that there are no nontrivial relations between x and y . $\square$

§3. $n = 5$: PRELIMINARY REMARKS

The problem on $(2, 3)$ -generation of the group $\mathrm{GL}(5, \mathbb{Z})$ was considered for the first time by Luzgarëv and Pevzner [5]. They proved the following reduction theorem.

Theorem 3.1 (see [5]). *Assume that the group $\mathrm{GL}(5, \mathbb{Z})$ is $(2, 3)$ -generated. Then the corresponding generating involution and element of order 3 are simultaneously conjugate to*

$$(2) \quad X = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 & 0 & 0 & a_1 \\ -1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 1 & a_3 \\ 0 & 0 & -1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where (a_1, a_2, a_3, a_4) is one of the following sets :

- (3) $(1, -1, -2, -2), \quad (0, -1, -2, -2),$
- (4) $(1, -1, -2, 4), \quad (0, -1, 4, -8),$
- (5) $(-1, 1, -2, -2), \quad (0, 1, -2, -2),$
- (6) $(-1, 1, -2, 4), \quad (0, 1, 4, -8),$
- (7) $(1, -1, 1, -3), \quad (0, -1, 0, -1).$

However, the paper [5] gave no final answer as to whether the group $\mathrm{GL}(5, \mathbb{Z})$ is $(2, 3)$ -generated.

Remark 3.1. In fact, as was already observed in [5], it suffices to consider only 5 cases out of 10. Namely, two pairs X, Y and X, Y^2 generate or do not generate $\mathrm{GL}(5, \mathbb{Z})$ simultaneously. The sets in the statement of Theorem 3.1 are grouped into pairs as follows. If a pair X, Y corresponds to the first set in some line (3)–(7), then the second set in the same line corresponds to a pair of matrices that are simultaneously conjugate to X, Y^2 . Thus, it suffices to consider only one quadruple in each line.

It turns out that, in the cases of (3), (5), or (7), the corresponding matrices generate $\mathrm{GL}(5, \mathbb{Z})$; see Theorems 4.1, 5.1, and 6.1 below. In this section we concentrate on negative results and show that in the cases of (4) and (6), the matrices X and Y generate proper subgroups of $\mathrm{GL}(5, \mathbb{Z})$.

Theorem 3.2. *Let X, Y be defined by (2) with $a_1 = 1, a_2 = -1, a_3 = -2$, and $a_4 = 4$. Then $\langle X, Y \rangle \simeq \mathrm{PSL}(2, \mathbb{Z})$, $\langle -X, Y \rangle \simeq \mathrm{PSL}(2, \mathbb{Z})$.*

Proof. We shall apply Lemma 2.2. We start with the following auxiliary construction. Consider the vectors

$$\begin{aligned} v_1 &= (0, 0, 2, -1, -2)^t, \\ v_2 &= -XYv_1 = (-2, 2, 3, 11, -1)^t, \\ v_3 &= (-2, 2, 1, 7, 1)^t, \\ v_4 &= -XYv_3 = (3, -1, 5, 1, -6)^t, \\ v_5 &= XY^2v_3 = (0, 3, 6, 3, -5)^t, \\ v_6 &= -XYv_4 = (-7, 4, 13, 37, -7)^t, \\ v_7 &= XY^2v_4 = (2, -9, 18, -9, -24)^t, \\ v_8 &= -XYv_5 = (-2, 2, 13, 31, -8)^t, \\ v_9 &= XY^2v_7 = (-7, -26, 57, -23, -81)^t, \end{aligned}$$

$$\begin{aligned}
v_{10} &= (-1, -1, 3, -3, -5)^t, \\
v_{11} &= XY^2 v_{10} = (-2, -4, 10, -5, -15)^t, \\
v_{12} &= (0, 0, 1, 2, -2)^t,
\end{aligned}$$

where t means transposition. Let

$$\mathcal{W}_0 = \text{conv}\{\lambda v_i : \lambda > 0, i = 1, \dots, 12\}$$

be the convex hull of the vectors λv_i , $\lambda > 0$, $i = 1, \dots, 12$.

Lemma 3.1. *We have $(XY^2)\mathcal{W}_0 \subseteq \mathcal{W}_0$ and $(-XY)\mathcal{W}_0 \subseteq \mathcal{W}_0$.*

Proof. The proof is based on a direct calculation that employs the convexity of the set $\mathcal{W}_0$. For the reader's convenience, the relevant numerical data is presented. It suffices to check that the images of v_i under the actions of XY^2 and $-XY$ lie in $\mathcal{W}_0$. From the definition of the vectors $v_1, \dots, v_{12}$, it readily follows that

$$\begin{aligned}
-XY v_i &\in \mathcal{W}_0 \quad \text{for } i = 1, 3, 4, 5, \\
XY^2 v_i &\in \mathcal{W}_0 \quad \text{for } i = 3, 4, 7, 10.
\end{aligned}$$

The remaining inclusions follow from the relations

$$\begin{aligned}
XY^2 v_1 &= \frac{25}{123} v_1 + \frac{20}{123} v_7 + \frac{14}{123} v_{10} + \frac{13}{123} v_{11} + \frac{11}{41} v_{12}, \\
-XY v_2 &= v_1 + \frac{2}{3} v_3 + v_4 + \frac{1}{3} v_8 + v_{12}, \\
XY^2 v_2 &= \frac{241}{67} v_1 + \frac{11}{67} v_2 + \frac{48}{67} v_5 + \frac{11}{67} v_7 + \frac{71}{67} v_{12}, \\
XY^2 v_5 &= \frac{103}{44} v_1 + \frac{49}{44} v_4 + \frac{7}{22} v_7 + \frac{41}{44} v_{10} + \frac{1}{44} v_{11}, \\
-XY v_6 &= \frac{403}{46} v_3 + \frac{127}{23} v_4 + \frac{8}{23} v_8 + \frac{31}{46} v_{11} + \frac{155}{46} v_{12}, \\
XY^2 v_6 &= \frac{93}{4} v_1 + \frac{3}{10} v_3 + \frac{9}{20} v_8 + \frac{3}{2} v_{10} + \frac{137}{20} v_{12}, \\
-XY v_7 &= \frac{11}{5} v_1 + \frac{24}{5} v_2 + \frac{56}{5} v_3 + v_{10} + 6v_{12}, \\
-XY v_8 &= 3v_1 + 3v_2 + \frac{14}{5} v_4 + \frac{6}{5} v_6 + \frac{12}{5} v_{12}, \\
XY^2 v_8 &= \frac{147}{13} v_1 + \frac{18}{13} v_3 + \frac{16}{13} v_5 + \frac{18}{13} v_7 + \frac{48}{13} v_{12}, \\
-XY v_9 &= \frac{29}{2} v_2 + \frac{373}{10} v_3 + \frac{23}{5} v_5 + \frac{17}{5} v_{10} + \frac{102}{5} v_{12}, \\
XY^2 v_9 &= \frac{17}{3} v_1 + \frac{1}{3} v_2 + \frac{1}{3} v_6 + \frac{4}{3} v_9 + \frac{31}{3} v_{11}, \\
-XY v_{10} &= \frac{813}{256} v_3 + \frac{1}{8} v_4 + \frac{35}{128} v_5 + \frac{3}{256} v_{11} + \frac{369}{256} v_{12}, \\
-XY v_{11} &= \frac{3}{2} v_1 + \frac{23}{12} v_2 + \frac{91}{12} v_3 + \frac{2}{3} v_5 + \frac{14}{3} v_{12}, \\
XY^2 v_{11} &= \frac{13}{25} v_1 + \frac{1}{25} v_2 + \frac{2}{5} v_3 + \frac{7}{25} v_7 + \frac{71}{25} v_{11}, \\
-XY v_{12} &= v_3 + \frac{11}{25} v_4 + \frac{4}{25} v_6 + \frac{1}{5} v_{10} + \frac{3}{25} v_{12}, \\
XY^2 v_{12} &= \frac{148}{123} v_1 + \frac{20}{123} v_7 + \frac{14}{123} v_{10} + \frac{13}{123} v_{11} + \frac{11}{41} v_{12}.
\end{aligned}$$

□

Consider the set $\mathcal{W} = \mathcal{W}_0 \cup (-\mathcal{W}_0)$. By Lemma 3.1, we have

$$(8) \quad XY\mathcal{W} \subseteq \mathcal{W}, \quad XY^2\mathcal{W} \subseteq \mathcal{W},$$

$$(9) \quad -XY\mathcal{W} \subseteq \mathcal{W}, \quad -XY^2\mathcal{W} \subseteq \mathcal{W}.$$

Take $v = (0, 1, 2, 0, -2)^t$ and $u = (-1, 2, -2, 17, -1)^t$. A direct calculation shows that $v^t v_i > 0$, $i = 1, \dots, 12$. Therefore, $v^t w > 0$ for any $w \in \mathcal{W}_0$, and $v^t w < 0$ for any $w \in -\mathcal{W}_0$. But $v^t u = 0$. Hence, $u \notin \mathcal{W}$. On the other hand,

$$(10) \quad XY u = -\frac{1}{3}v_1 - \frac{5}{6}v_2 - \frac{3}{2}v_3 - \frac{5}{3}v_4 - \frac{1}{3}v_7 \in -\mathcal{W}_0 \subseteq \mathcal{W},$$

$$(11) \quad XY^2 u = \frac{153}{28}v_1 + \frac{13}{28}v_4 + \frac{5}{14}v_5 + \frac{5}{28}v_{10} + \frac{3}{28}v_{11} \in \mathcal{W}_0 \subseteq \mathcal{W},$$

and in a similar way we obtain

$$(12) \quad -XY u \in \mathcal{W}, \quad -XY^2 u \in \mathcal{W}.$$

Relations (8), (10), (11), and Lemma 2.2 applied to X, Y show that $\langle X, Y \rangle \simeq \text{PSL}(2, \mathbb{Z})$. Similarly, (9), (12), and Lemma 2.2 applied to $-X, Y$ imply $\langle -X, Y \rangle \simeq \text{PSL}(2, \mathbb{Z})$. $\square$

Theorem 3.3. *Let X, Y be defined by (2) with $a_1 = -1$, $a_2 = 1$, $a_3 = -2$, $a_4 = 4$. Then $\langle X, Y \rangle \simeq \text{PSL}(2, \mathbb{Z})$, $\langle -X, Y \rangle \simeq \text{PSL}(2, \mathbb{Z})$.*

Proof. The proof is similar to the proof of Theorem 3.2. Consider

$$\begin{aligned} v_1 &= (-1, -1, -1, 3, -2)^t, \\ v_2 &= -XY v_1 = (1, 0, 7, 9, -5)^t, \\ v_3 &= XY^2 v_1 = (-2, 3, 6, -3, -8)^t, \\ v_4 &= XY^2 v_3 = (1, 10, 19, -11, -27)^t, \\ v_5 &= (0, 0, 1, -1, -1)^t, \\ v_6 &= (2, -2, 1, 4, 1)^t, \\ v_7 &= -XY v_6 = (-3, 1, 2, 4, -3)^t, \\ v_8 &= XY^2 v_6 = (0, -3, 3, 3, -2)^t, \\ v_9 &= -XY v_7 = (4, -1, 10, 14, -7)^t, \\ v_{10} &= XY^2 v_7 = (-2, 6, 12, -2, -15)^t, \\ v_{11} &= -XY v_8 = (-1, 1, 7, 15, -5)^t, \\ v_{12} &= (1, 2, 4, -4, -6)^t, \\ v_{13} &= XY^2 v_{12} = (3, 5, 12, -11, -18)^t. \end{aligned}$$

Let

$$\mathcal{W}_0 = \text{conv}\{\lambda v_i : \lambda > 0, i = 1, \dots, 13\}$$

be the convex hull of the vectors λv_i , $\lambda > 0$, $i = 1, \dots, 13$.

Lemma 3.2. *We have $(XY^2)\mathcal{W}_0 \subseteq \mathcal{W}_0$ and $(-XY)\mathcal{W}_0 \subseteq \mathcal{W}_0$.*

Proof. The definitions of the vectors $v_1, \dots, v_{13}$ imply immediately that

$$\begin{aligned} -XY v_i &\in \mathcal{W}_0 \quad \text{for } i = 1, 6, 7, 8, \\ XY^2 v_i &\in \mathcal{W}_0 \quad \text{for } i = 1, 3, 6, 7, 12. \end{aligned}$$

The remaining inclusions follow from the relations

$$\begin{aligned}
-XYv_2 &= \frac{5}{4}v_1 + \frac{9}{4}v_2 + \frac{7}{4}v_5 + 2v_6 + \frac{1}{4}v_8, \\
XY^2v_2 &= \frac{8}{47}v_3 + \frac{86}{47}v_8 + \frac{6}{47}v_9 + \frac{35}{47}v_{10} + \frac{109}{47}v_{12}, \\
-XYv_3 &= \frac{26}{61}v_1 + \frac{20}{61}v_4 + \frac{377}{61}v_6 + \frac{25}{61}v_7 + \frac{1}{61}v_{10}, \\
-XYv_4 &= \frac{215}{63}v_1 + \frac{167}{126}v_2 + \frac{1555}{126}v_5 + \frac{1184}{63}v_6 + \frac{3}{2}v_{12}, \\
XY^2v_4 &= \frac{1}{2}v_2 + \frac{5}{2}v_5 + v_7 + \frac{15}{2}v_{12} + 2v_{13}, \\
-XYv_5 &= \frac{27}{641}v_1 + \frac{10}{641}v_5 + \frac{310}{641}v_6 + \frac{18}{641}v_9 + \frac{24}{641}v_{11}, \\
XY^2v_5 &= \frac{1}{24}v_1 + \frac{1}{12}v_4 + \frac{1}{24}v_5 + \frac{1}{24}v_7 + \frac{1}{12}v_{12}, \\
XY^2v_8 &= \frac{1}{8}v_1 + \frac{11}{8}v_5 + \frac{5}{8}v_7 + \frac{1}{2}v_8 + \frac{1}{2}v_{10}, \\
-XYv_9 &= \frac{92}{41}v_1 + \frac{426}{41}v_5 + \frac{413}{82}v_6 + \frac{165}{82}v_{11} + \frac{5}{82}v_{13}, \\
XY^2v_9 &= v_4 + \frac{5}{2}v_5 + \frac{19}{6}v_8 + \frac{1}{6}v_9 + \frac{4}{3}v_{12}, \\
-XYv_{10} &= \frac{47}{27}v_1 + \frac{40}{27}v_2 + \frac{110}{27}v_5 + \frac{269}{27}v_6 + \frac{4}{3}v_{12}, \\
XY^2v_{10} &= \frac{1}{2}v_2 + \frac{2}{3}v_3 + \frac{5}{6}v_4 + \frac{2}{3}v_5 + \frac{4}{3}v_{13}, \\
-XYv_{11} &= \frac{61}{42}v_1 + \frac{101}{42}v_2 + \frac{65}{21}v_5 + \frac{32}{21}v_6 + \frac{1}{2}v_9, \\
XY^2v_{11} &= \frac{15}{43}v_1 + \frac{523}{43}v_5 + \frac{20}{43}v_7 + \frac{48}{43}v_{11} + \frac{41}{43}v_{13}, \\
-XYv_{12} &= \frac{1}{2}v_1 + v_5 + 4v_6 + \frac{1}{2}v_8 + \frac{1}{2}v_{12}, \\
-XYv_{13} &= 2v_1 + \frac{620}{71}v_5 + \frac{941}{71}v_6 + \frac{56}{71}v_7 + \frac{61}{71}v_{12}, \\
XY^2v_{13} &= \frac{1}{6}v_1 + \frac{1}{4}v_3 + \frac{1}{3}v_8 + \frac{35}{12}v_{12} + \frac{23}{12}v_{13}.
\end{aligned}$$

□

Set $\mathcal{W} = \mathcal{W}_0 \cup (-\mathcal{W}_0)$. By Lemma 3.2, we have

$$\begin{aligned}
XY\mathcal{W} &\subseteq \mathcal{W}, & XY^2\mathcal{W} &\subseteq \mathcal{W}, \\
-XY\mathcal{W} &\subseteq \mathcal{W}, & -XY^2\mathcal{W} &\subseteq \mathcal{W}.
\end{aligned}$$

Take $v = (0, 0, 2, 1, 0)^t$ and $u = (0, 0, -1, 2, 0)^t$. A direct calculation shows that $v^t v_i > 0$ for all $i = 1, \dots, 13$. Therefore, $v^t w \neq 0$ for any $w \in \mathcal{W}$. On the other hand, $v^t u = 0$. Hence, $u \notin \mathcal{W}$. Moreover,

$$\begin{aligned}
XYu &= -\frac{1}{11}v_1 - \frac{5}{44}v_2 - \frac{39}{44}v_5 - \frac{1}{22}v_{11} - \frac{1}{44}v_{12} \in -\mathcal{W}_0 \subseteq \mathcal{W}, \\
XY^2u &= v_5 \in \mathcal{W}_0 \subseteq \mathcal{W}, \\
-XYu &\in \mathcal{W}, & -XY^2u &\in \mathcal{W}.
\end{aligned}$$

To complete the proof, we apply Lemma 2.2 just as we did in Theorem 3.2.

□

§4. $n = 5$: THE FIRST GENERATING SET

In this section we consider the case where X and Y are given by (2) with $a_1 = 1$, $a_2 = -1$, $a_3 = -2$, and $a_4 = -2$. For technical reasons, it is more convenient to work not with the matrices X and Y but rather with their conjugates. Let

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 3 & 0 & 0 & 1 \end{pmatrix}.$$

We set

$$(13) \quad x = QXQ^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ -1 & -6 & 19 & 6 & -11 \\ -1 & -4 & 13 & 4 & -8 \\ 2 & 5 & -13 & -3 & 7 \\ 0 & -1 & 5 & 2 & -4 \end{pmatrix},$$

$$(14) \quad y = QYQ^{-1} = \begin{pmatrix} 0 & -2 & 6 & 2 & -3 \\ -1 & 8 & -24 & -7 & 13 \\ -2 & 5 & -14 & -4 & 7 \\ 0 & -3 & 6 & 1 & -3 \\ -3 & 3 & -9 & -3 & 4 \end{pmatrix}.$$

Theorem 4.1. *Let x, y be given by (13), (14). Then $\langle x, y \rangle = \mathrm{GL}(5, \mathbb{Z})$ and $\langle -x, y \rangle = \mathrm{SL}(5, \mathbb{Z})$.*

The key step of the proof is the following lemma.

Lemma 4.1. *Let G be a subgroup of $\mathrm{GL}(5, \mathbb{Z})$ containing x, y , and $t_{51}(a)$ for some $a \in \mathbb{Z}$. Then G also contains all the transvections $t_{ij}(a)$ with $1 \leq i \neq j \leq 5$.*

Proof. In what follows, the elements of G that do not depend on a will be denoted by h with subscripts, while the elements that do depend on a will be denoted by z with subscripts. Let

$$z_1 := t_{51}(a).$$

We set

$$(15) \quad h_1 := ((xyxy^2)^2(xy)^3x)^6,$$

$$h_2 := xyxy^2(xy)^5xy^2,$$

$$(16) \quad h_3 := xy^2xyx,$$

$$h_4 := h_2x,$$

$$h_5 := xyxy^2xy(xy^2)^2(xy)^4x,$$

$$h_6 := (xy^2)^3(xy)^3(xy^2)^2xy.$$

Next, we define

$$z_2 := h_2z_1h_2^{-1} = I + a(10e_{13} - 16e_{23} - 3e_{43} - 2e_{53}),$$

$$z_3 := h_4z_1h_4^{-1} = I + a(e_{13} - 3e_{23} + 4e_{43} + 3e_{53}),$$

$$z_4 := h_3h_4z_1h_4^{-1}h_3^{-1} = I + a(e_{13} - 3e_{23} - 7e_{43} - 2e_{53}),$$

$$z_5 := h_2h_5h_1h_5^{-1}z_1h_5h_1^{-1}h_5^{-1}h_2^{-1} = I + a(-8e_{13} + 11e_{23} + 6e_{43} + 4e_{53}),$$

$$z_6 := h_6h_3h_6^{-1}h_4z_1h_4^{-1}h_6h_3^{-1}h_6^{-1} = I + a(13e_{13} - 17e_{23} - 32e_{43} + 3e_{53}).$$

It is clear that the elements $z_2, \dots, z_6$ commute pairwise; we can use them for finding an elementary transvection of level a . Namely,

$$(17) \quad z_7 := z_2^{-565} z_3^{617} z_4^{151} z_5^{-651} z_6^{-25} = t_{13}(a),$$

$$(18) \quad z_8 := z_2^{-46} z_3^{50} z_4^{12} z_5^{-53} z_6^{-2} = t_{23}(a),$$

$$(19) \quad z_9 := z_2^{-1059} z_3^{1157} z_4^{284} z_5^{-1220} z_6^{-47} = t_{43}(a),$$

$$(20) \quad z_{10} := z_2^{585} z_3^{-639} z_4^{-157} z_5^{674} z_6^{26} = t_{53}(a).$$

Next, we introduce

$$(21) \quad z_{11} := h_2^{-1} z_7^5 z_8^{-7} z_9^{-3} z_{10}^{-2} h_2 = t_{21}(a),$$

$$(22) \quad z_{12} := h_2^{-1} z_7^{-17} z_8^{26} z_9^7 z_{10}^5 h_2 = t_{31}(a),$$

$$(23) \quad z_{13} := h_2^{-1} z_7^{-6} z_8^{10} z_9 z_{10} h_2 = t_{41}(a).$$

Now, we set

$$h_7 := xy(xy^2)^2 xyxy^2(xy)^2 xy^2 x,$$

$$h_8 := xy^2(xy)^2 xy^2 xyxy^2(xy)^3 x,$$

$$h_9 := xy^2(xy)^2(xy^2)^2 xyx,$$

$$h_{10} := (xy)^3(xy^2)^2$$

and consider

$$z_{14} := h_7 z_7 h_7^{-1} = I + a(-2e_{12} - 2e_{14} + e_{15}),$$

$$z_{15} := h_8 z_7 h_8^{-1} = I + a(-3e_{12} + 9e_{13} + 3e_{14} - 4e_{15}),$$

$$z_{16} := h_9 z_7 h_9^{-1} = I + a(5e_{12} - 17e_{13} - 5e_{14} + 8e_{15}),$$

$$z_{17} := h_{10} z_7 h_{10}^{-1} = I + a(-9e_{12} + 31e_{13} + 8e_{14} - 16e_{15}).$$

The matrices $z_7, z_{14}, z_{15}, z_{16}$, and z_{17} commute pairwise, and

$$(24) \quad z_{18} := z_7^{-1} z_{15}^{-4} z_{16}^{-4} z_{17}^{-1} = t_{12}(a),$$

$$(25) \quad z_{19} := z_7^{-2} z_{15}^{-2} z_{16}^{-3} z_{17}^{-1} = t_{14}(a),$$

$$(26) \quad z_{20} := z_7^{-6} z_{14} z_{15}^{-12} z_{16}^{-14} z_{17}^{-4} = t_{15}(a).$$

In a similar way, let

$$h_{11} := xyxy^2 xyx,$$

$$h_{12} := (xy^2)^3 xyx,$$

$$h_{13} := (xy)^2 xy^2(xy)^3 x,$$

$$z_{21} := h_{11} z_7 h_{11}^{-1} = I + a(2e_{41} + 3e_{42} - 3e_{43} + e_{45}),$$

$$z_{22} := h_{12} z_7 h_{12}^{-1} = I + a(-2e_{42} + e_{45}),$$

$$z_{23} := h_{13} z_7 h_{13}^{-1} = I + a(-2e_{41} - e_{42} + e_{43} - e_{45}).$$

Using these matrices, we find

$$(27) \quad z_{24} := z_9^2 z_{21} z_{22} z_{23} = t_{45}(a),$$

$$(28) \quad z_{25} := z_9^{-1} z_{13}^{-2} z_{21}^{-1} z_{22}^{-1} z_{23}^{-2} = t_{42}(a).$$

Finally, consider

$$\begin{aligned}
 (29) \quad & h_{14} := xyxy^2, \\
 & z_{26} := h_{14}z_{19}h_{14}^{-1}z_{18}z_{25}^{-2} = t_{52}(a), \\
 & h_{15} := (xyxy^2)^3x, \\
 (30) \quad & z_{27} := h_{15}z_{25}^{-1}h_{15}^{-1}z_1^{-6}z_{12}^{-3}z_{26}^{-2} = t_{32}(a), \\
 (31) \quad & z_{28} := h_{15}z_{10}^{-1}h_{15}^{-1}z_{26}^{-1} = t_{54}(a), \\
 (32) \quad & z_{29} := h_{15}z_9h_{15}^{-1}z_{26}^{-2}z_{27}^{-1}z_{28}^{-2} = t_{34}(a), \\
 (33) \quad & z_{30} := h_{14}^{-1}z_{18}^{-1}z_{25}^2z_{26}^4z_{27}^2h_{14} = t_{24}(a), \\
 & h_{16} := xyxy^2(xy)^2xy^2x, \\
 (34) \quad & z_{31} := h_{16}z_8^{-1}h_{16}^{-1}z_{11}^{-1} = t_{25}(a), \\
 & h_{17} := xyxy^2(xy)^3(xy^2)^2(xy)^2(xy^2)^2(xy)^2(xy^2)^2, \\
 & h_{18} := h_{17}xh_{17}^{-1}, \\
 (35) \quad & z_{32} := h_{18}z_{12}h_{18}z_{12}z_{27}z_{29} = t_{35}(a).
 \end{aligned}$$

The assumption that $t_{51}(a), x, y \in G$, and relations (17)–(35) imply that G contains all elementary transvections $t_{ij}(a)$ of level a , $1 \leq i \neq j \leq 5$. $\square$

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. We keep the notation used in the proof of Lemma 4.1. First, observe that the matrix h_1 defined by (15) is an elementary transvection of level 3:

$$h_1 = t_{51}(3).$$

By Lemma 4.1, $\langle x, y \rangle$ contains $t_{ij}(3)$ for all $1 \leq i \neq j \leq 5$.

Also, we note that the matrix h_3 defined by (16) has a rather simple shape:

$$h_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 3 & 0 & 1 & 1 & 0 \\ 2 & 1 & -3 & 0 & 1 \end{pmatrix}.$$

Our next aim is to find other matrices of the same block-triangular shape. Set

$$h_{19} := (xyx^2)^2(y^2xyx)(y^2x)^3xyx^2yx(yx)^4(y^2x)^2xyx^2xy.$$

The matrix h_{19} modulo 3 is a block-triangular matrix. This suggests that we can find block-triangular matrices by using h_{19} and transvections of level 3. Namely, let

$$\begin{aligned}
 h_{20} &:= h_{19} \cdot (t_{14}(3))^{-239}(t_{24}(3))^{704}(t_{34}(3))^{189}(t_{15}(3))^{-229}(t_{25}(3))^{673}(t_{35}(3))^{181}, \\
 h_{21} &:= (t_{14}(3))^{2026}(t_{15}(3))^{3016}(t_{24}(3))^{-1115}(t_{25}(3))^{-1661}(t_{34}(3))^{26}(t_{35}(3))^{38}h_{20}.
 \end{aligned}$$

The matrix h_{21} is block-triangular. Consider

$$\begin{aligned}
 h_{22} &:= h_3h_{21}h_3h_{21}^{-1}, \\
 h_{23} &:= h_3h_{21}^2h_3h_{21}^{-2}, \\
 h_{24} &:= h_3h_{21}^3h_3h_{21}^{-3}, \\
 h_{25} &:= h_3h_{21}^{-2}h_3h_{21}^2.
 \end{aligned}$$

It is easy to check that

$$\begin{aligned} h_{26} &:= h_{22}h_{23}h_{24}h_{25} \\ &= I - 39868892529699498e_{41} - 66045130609368789e_{42} \\ &\quad + 276830690835290373e_{43} + 24288462735300964e_{51} \\ &\quad + 40192071970237800e_{52} - 168478035844626270e_{53}. \end{aligned}$$

In particular, h_{26} , $t_{41}(3)$, $t_{42}(3)$, $t_{43}(3)$, $t_{51}(3)$, $t_{52}(3)$, and $t_{53}(3)$ commute pairwise, and

$$\begin{aligned} h_{27} &:= h_{26} \cdot (t_{41}(3))^{13289630843233166} \cdot (t_{42}(3))^{22015043536456263} \\ &\quad \times (t_{43}(3))^{-92276896945096791} \cdot (t_{51}(3))^{-8096154245100321} \\ &\quad \times (t_{52}(3))^{-13397357323412600} \cdot (t_{53}(3))^{56159345281542090} \\ &= t_{51}(1). \end{aligned}$$

By Lemma 4.1, all the transvections $t_{ij}(1)$, $1 \leq i \neq j \leq 5$, lie in $\langle x, y \rangle$. Therefore, $\langle x, y \rangle$ contains $\text{SL}(5, \mathbb{Z})$. Since $\langle x, y \rangle$ also contains matrices of determinant -1 (e.g., x), the group $\langle x, y \rangle$ coincides with $\text{GL}(5, \mathbb{Z})$. Lemma 2.1 implies that $\langle -x, y \rangle = \text{SL}(5, \mathbb{Z})$. $\square$

§5. $n = 5$: THE SECOND GENERATING SET

In this section we consider the matrices X and Y given by (2) with $a_1 = -1$, $a_2 = 1$, $a_3 = -2$, and $a_4 = -2$. As in the preceding section, it is more convenient to work with conjugates of X and Y instead of X and Y themselves. Put

$$Q = \begin{pmatrix} 1 & 13 & 2 & 0 & -8 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$(36) \quad x = QXQ^{-1} = \begin{pmatrix} -1 & 0 & -8 & 0 & -16 \\ 0 & -1 & -1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & -13 & 0 & 1 & -5 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$$(37) \quad y = QYQ^{-1} = \begin{pmatrix} -13 & 157 & 28 & 2 & 53 \\ -1 & 12 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & -3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 5.1. *Let x and y be defined by (36) and (37). Then $\langle x, y \rangle = \text{GL}(5, \mathbb{Z})$ and $\langle -x, y \rangle = \text{SL}(5, \mathbb{Z})$.*

We start with the following auxiliary lemma.

Lemma 5.1. *Let G be a subgroup of $\text{GL}(5, \mathbb{Z})$ containing x , y , and $t_{32}(a)$ for some $a \in \mathbb{Z}$. Then G also contains $t_{12}(a)$, $t_{42}(a)$, and $t_{52}(a)$.*

Proof. As before, by h with subscripts we denote elements of the group G that do not depend on a , and by z with subscripts we denote elements that depend on a . Set

$$\begin{aligned} h_1 &:= y^2xy, \\ h_2 &:= yxy^2, \\ h_3 &:= (xy^2)^3(xy)^3x, \\ h_4 &:= xy^2xy(xy^2)^3(xy)^2xy^2(xy)^2(xy^2)^2x, \\ z_1 &:= t_{32}(a) \end{aligned}$$

and consider

$$\begin{aligned} z_2 &:= h_1z_1h_1 = I + a(-5e_{12} + e_{32} - 2e_{42} - e_{52}), \\ z_3 &:= h_2z_1h_2 = I + a(-8e_{12} - 7e_{32} + 8e_{42} + 2e_{52}), \\ z_4 &:= h_3h_1z_1h_1h_3 = I + a(-34e_{12} + 65e_{32} - 58e_{42} - 35e_{52}), \\ z_5 &:= h_4h_1z_1h_1h_4^{-1} = I + a(-138e_{12} + 313e_{32} - 325e_{42} - 161e_{52}). \end{aligned}$$

In particular, the elements z_1, z_2, z_3, z_4 , and z_5 commute pairwise. Using easy linear algebra, we see that

$$\begin{aligned} z_6 &:= z_1^{808} z_2^{-6983} z_3^{3414} z_4^{-1059} z_5^{316} = t_{12}(a), \\ z_7 &:= z_1^{146} z_2^{-1260} z_3^{616} z_4^{-191} z_5^{57} = t_{42}(a), \\ z_8 &:= z_1^{-341} z_2^{2962} z_3^{-1448} z_4^{449} z_5^{-134} = t_{52}(a). \end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 5.1. Our first aim is to find sufficiently many matrices of the form

$$(38) \quad \begin{pmatrix} I_2 & 0 \\ * & I_3 \end{pmatrix}.$$

However, we start with slightly more general matrices

$$\begin{pmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{pmatrix},$$

where Q_1 and Q_3 are square matrices of size 2 and 3, respectively. Consider

$$\begin{aligned} h_5 &:= xy(xy^2)^7(xy)^6 = \begin{pmatrix} 11 & -120 & 0 & 0 & 0 \\ 1 & -11 & 0 & 0 & 0 \\ -39 & 532 & -14 & -28 & 45 \\ 57 & -773 & 15 & 34 & -60 \\ 17 & -232 & 5 & 11 & -19 \end{pmatrix}, \\ h_6 &:= xyxy^2x = \begin{pmatrix} -11 & 120 & 0 & 0 & 0 \\ -1 & 11 & 0 & 0 & 0 \\ -3 & 38 & -1 & -2 & 6 \\ 5 & -65 & 0 & 4 & -15 \\ 2 & -25 & 0 & 1 & -4 \end{pmatrix}, \\ h_7 &:= (y^2x)^2xy^2xyxy = \begin{pmatrix} 1 & -30 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 11 & -129 & -49 & -24 & -48 \\ -14 & 165 & 60 & 29 & 60 \\ -4 & 45 & 20 & 10 & 19 \end{pmatrix}. \end{aligned}$$

TABLE 1

	k_{31}	k_{32}	k_{41}	k_{42}	k_{51}	k_{52}
h_8	8878	-88380	-11492	115170	-3814	38190
h_9	5628	-47130	-6992	58920	-2314	19440
h_{10}	15504	-144690	-17218	160680	-5756	53760
h_{11}	9378	-94380	-11492	115170	-3814	38190
h_{12}	23858	-449160	-29428	554310	-9826	185070
h_{13}	-187422	2085600	232352	-2585550	77334	-860550
h_{14}	1253860	-13921500	-1401298	15558450	-467216	5187450
h_{15}	-586936	7760460	657316	-8691870	219582	-2905870

Now, it is easy to verify that the following matrices have the desired shape (38):

$$\begin{aligned}
h_8 &:= (h_5^{-6} h_6 h_7 h_5^{-2} h_7)^2, \\
h_9 &:= (h_7 h_5^{-2} h_7 h_6)^2, \\
h_{10} &:= h_5 h_8 h_5^{-1}, \\
h_{11} &:= h_6 h_8 h_6^{-1}, \\
h_{12} &:= h_7 h_8 h_7^{-1}, \\
h_{13} &:= h_6 h_7 h_8 h_7^{-1} h_6^{-1}, \\
h_{14} &:= h_5 h_7 h_5 h_8 h_5^{-1} h_7^{-1} h_5^{-1}, \\
h_{15} &:= (h_5 h_6 h_7 (h_6 h_7 h_6 h_7 h_5^5)^2 h_6 h_7 h_6 h_5^3)^2.
\end{aligned}$$

Namely, each of the eight matrices $h_8, \dots, h_{15}$ can be written as

$$h_i = I + k_{31}e_{31} + k_{32}e_{32} + k_{41}e_{41} + k_{42}e_{42} + k_{51}e_{51} + k_{52}e_{52},$$

where the coefficients k_{ij} are listed in Table 1. In particular, the matrices $h_8, \dots, h_{15}$ commute pairwise, and we can use their combination in order to find an elementary transvection of level 2:

$$(39) \quad h_{16} := h_8^{-177312135} h_9^{12142834959} h_{10}^{-3022311905} h_{11}^{-18496419129}$$

$$\begin{aligned}
(40) \quad &\times h_{12}^{-438573050} h_{13}^{-495339791} h_{14}^{56766741} \\
&= t_{31}(2).
\end{aligned}$$

We also need another transvection of level 10. Namely, set

$$\begin{aligned}
h_{17} &:= h_8^{250005876} h_9^{-92554481} h_{10}^{5425894} h_{11}^{-47781509} h_{12}^{2141478} h_{13}^{12452569} h_{14}^{577930} h_{15}^{-380735} \\
&= I + 30e_{32} - 30e_{42} - 10e_{52}
\end{aligned}$$

and consider the following product of conjugates of h_{17} :

$$h_{18} := h_{17}^{-510} h_1 h_{17}^{-2520} h_1^{-1} h_3 h_{17}^{472} h_3^{-1} h_4 h_{17}^{50} h_4^{-1} h_1 h_4 h_{17}^{-71} h_4^{-1} h_1^{-1} = t_{32}(10).$$

(It is clear that h_1 is an involution and we can write h_1 instead of h_1^{-1} . However, here we prefer to keep the notation h_1^{-1} , in order to emphasize that the above expression for h_{18} is a product of conjugates of h_{17} .)

Now we use suitable conjugates of h_{16} for finding further transvections, Take

$$\begin{aligned}
h_{19} &:= y^2(xy^2)^5 xy(xy^2)^2, \\
h_{20} &:= yxy^2x(yx)^3(y^2x)^4y^2, \\
h_{21} &:= yxy^2x(yx)^2y^2xyx(y^2x)^2(yx)^3(y^2x)^5(yx)^3(y^2x)^3yxy^2,
\end{aligned}$$

and set

$$\begin{aligned}
h_{22} &:= h_{19}h_{16}h_{19}^{-1}, \\
h_{23} &:= h_{20}h_{16}h_{20}^{-1}, \\
h_{24} &:= h_{21}h_{16}h_{21}^{-1}, \\
h_{25} &:= h_{22}h_{16}h_{22}^{-1} = I + 3194e_{31} - 40964e_{32} - 3724e_{34} + 9044e_{35}, \\
h_{26} &:= h_{23}h_{16}h_{23}^{-1} = I + 11930e_{31} - 115020e_{32} + 21300e_{34} - 32376e_{35}, \\
h_{27} &:= h_{24}h_{16}h_{24}^{-1} = I + 120283562e_{31} - 1382839856e_{32} \\
&\quad + 152542256e_{34} - 229142928e_{35}.
\end{aligned}$$

In particular, h_{16} , h_{18} , h_{25} , h_{26} , and h_{27} commute pairwise, and

$$\begin{aligned}
h_{28} &:= h_{16}^{-124140789441885} h_{18}^{413807222290788} h_{25}^{-6525940705} h_{26}^{-59250292539} h_{27}^{8114005} \\
&= t_{35}(4), \\
h_{29} &:= h_{16}^{-1105501046986} h_{18}^{3685044372190} h_{25}^{-58114938} h_{26}^{-527636893} h_{27}^{72257} = t_{34}(4), \\
(41) \quad h_{30} &:= h_{16}^{68914022536559} h_{18}^{-229715956946515} h_{25}^{3622732116} h_{26}^{32891493712} h_{27}^{-4504311} = t_{32}(2).
\end{aligned}$$

By (41), we have $t_{32}(2) \in \langle x, y \rangle$. Lemma 5.1 implies that

$$\begin{aligned}
h_{31} &:= t_{12}(2), \\
h_{32} &:= t_{42}(2), \\
h_{33} &:= t_{52}(2)
\end{aligned}$$

are also in $\langle x, y \rangle$. Next, we have

$$\begin{aligned}
h_{34} &:= h_8^{4840} h_9^{-2962} h_{10}^{420} h_{11}^{929} h_{12}^{86} h_{13}^{291} h_{14}^5 h_{15}^{-8} h_{30}^{9150} h_{32}^{8445} h_{33}^{1895} = I + 2e_{41} + 4e_{51}, \\
h_{35} &:= h_8^{-3480} h_9^{1882} h_{10}^{25} h_{11}^{330} h_{12}^{-12} h_{13}^{-168} h_{14}^{-11} h_{15} h_{30}^{-3585} h_{32}^{-2400} h_{33}^{-7660} = t_{51}(10).
\end{aligned}$$

Setting

$$\begin{aligned}
h_{36} &:= h_{19}^{-1} h_{16} h_{19}, \\
h_{37} &:= h_{20}^{-1} h_{16} h_{20},
\end{aligned}$$

we see that

$$\begin{aligned}
h_{38} &:= h_{36}h_{16}h_{36}^{-1} = I + 532e_{21} - 11702e_{31} + 11704e_{41} + 4256e_{51}, \\
h_{39} &:= h_{37}h_{16}h_{37}^{-1} = I - 1704e_{21} - 30670e_{31} + 11076e_{41} + 17892e_{51}.
\end{aligned}$$

Clearly, h_{16} , h_{34} , h_{35} , h_{38} , and h_{39} commute pairwise. By an easy calculation,

$$\begin{aligned}
h_{40} &:= h_{16}^{9064162} h_{34}^{-6459011} h_{35}^{1745066} h_{38}^{852} h_{39}^{266} = t_{41}(2), \\
h_{41} &:= h_{16}^{-4532081} h_{34}^{3229506} h_{35}^{-872533} h_{38}^{-426} h_{39}^{-133} = t_{51}(2), \\
h_{42} &:= h_{16}^{2180895} h_{34}^{-1554092} h_{35}^{419880} h_{38}^{205} h_{39}^{64} = t_{21}(4).
\end{aligned}$$

Set

$$h_{43} := y^2 x (yx)^2 y^2 h_{31} y (xy^2)^2 xy;$$

then

$$h_{44} := h_{43} h_{31} h_{43}^{-1} h_{31}^3 = I + 4e_{13} + 8e_{15}.$$

We take

$$h_{45} := (h_5 h_6 h_7)^2 (h_6 h_7 h_5^{-1})^2 h_5^{36} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -31227 & 274086 & 1 & 36 & -108 \\ 32975 & -286842 & 0 & 1 & 0 \\ 11019 & -95904 & 0 & 0 & 1 \end{pmatrix}.$$

It is easily seen that

$$h_{46} := h_{45} h_{28}^{27} h_{29}^{-9} h_{16}^{15613} h_{30}^{-137043} h_{32}^{143421} h_{33}^{47952} h_{40}^{-16485} h_{41}^{-5509} = I - e_{31} + 5e_{41} + e_{51}.$$

(For further purposes, the matrix h_{46} is slightly more convenient than, say, the matrix $I + e_{31} + e_{41} + e_{51}$, which can be obtained in a similar way. For instance, the shape of $xh_{46}x$ is simpler; see also the definition of h_{47} below.) Now we take

$$\begin{aligned} h_{47} &:= h_{44}^2 x h_{46} x h_{44}^{-2} = I + e_{21} - e_{31}, \\ h_{48} &:= h_{16}^{-1} h_{42} h_{47}^{-2} = t_{21}(2), \\ h_{49} &:= h_{31}^{-6} h_6 h_{47} h_6^{-1} h_{31}^5 (h_{48} h_{31}^{-1})^6 h_{31}^{-5} h_{48} \\ &= I + e_{21} - 897e_{31} + 1495e_{41} + 575e_{51} - 468e_{32} + 780e_{42} + 300e_{52}, \\ (42) \quad h_{50} &:= h_{46} h_{47}^{-1} h_{49} h_{16}^{449} h_{30}^{234} h_{32}^{-390} h_{33}^{-150} h_{40}^{-750} h_{41}^{-288} = t_{31}(1), \\ h_{51} &:= h_{47} h_{50} = t_{21}(1). \end{aligned}$$

The next step is to find other transvections of level 1. Consider

$$\begin{aligned} h_{52} &:= (yx)^3 y^2 x (yx)^4 y^2 x (yx)^2 (y^2 x)^2 (yx)^4 y^2, \\ h_{53} &:= h_{52} h_{50} h_{52}^{-1}, \\ h_{54} &:= h_{19} h_{50} h_{19}^{-1}, \\ h_{55} &:= h_{20} h_{50} h_{20}^{-1}. \end{aligned}$$

We arrive at

$$\begin{aligned} h_{56} &:= h_{53} h_{50} h_{53}^{-1} = I - 56979e_{31} + 197395e_{32} + 85470e_{34} + 93610e_{35}, \\ h_{57} &:= h_{54} h_{50} h_{54}^{-1} = I + 799e_{31} - 10241e_{32} - 931e_{34} + 2261e_{35}, \\ h_{58} &:= h_{55} h_{50} h_{55}^{-1} = I + 2983e_{31} - 28755e_{32} + 5325e_{34} - 8094e_{35}. \end{aligned}$$

In particular, h_{28} , h_{29} , h_{30} , h_{50} , h_{56} , h_{57} , and h_{58} commute pairwise. An easy calculation shows that

$$(43) \quad h_{59} := h_{56} h_{57}^2 h_{28}^{-24533} h_{29}^{-20902} h_{30}^{-88456} h_{50}^{55381} = t_{32}(1),$$

$$(44) \quad h_{60} := h_{58}^{-1} h_{57}^{-2} h_{28}^{-893} h_{29}^{866} h_{50}^{4581} h_{59}^{-49237} = t_{34}(1),$$

$$(45) \quad h_{61} := h_{57} h_{28}^{-565} h_{50}^{-799} h_{59}^{10241} h_{60}^{931} = t_{35}(1).$$

By (43) and Lemma 5.1, we have

$$(46) \quad t_{12}(1), t_{42}(1), t_{52}(1) \in \langle x, y \rangle.$$

Finally, set

$$\begin{aligned}
h_{62} &:= yxy^2x(yx)^2y^2xyh_{60}y^2xy(xy^2)^2xyxy^2, \\
h_{63} &:= (xy)^2xy^2(xy)^5xy^2h_{60}yx(y^2x)^5yx(y^2x)^2, \\
h_{64} &:= xh_{61}x, \\
h_{65} &:= yxyh_{61}y^2xy^2, \\
h_{66} &:= y^2xyh_{61}y^2xy, \\
h_{67} &:= h_{62}h_{51}h_{62}^{-1} = I - 48e_{21} - 1617e_{23} - 980e_{24} - 1813e_{25}, \\
h_{68} &:= h_{63}h_{51}h_{63}^{-1} = I + 155e_{21} - 2772e_{23} - 1771e_{24} + 77e_{25}, \\
h_{69} &:= h_{64}h_{51}h_{64}^{-1} = I + e_{21} + 8e_{23} + 8e_{25}, \\
h_{70} &:= h_{65}h_{51}h_{65}^{-1} = I + e_{21} + 40e_{23} + 20e_{24} + 60e_{25}, \\
h_{71} &:= h_{66}h_{51}h_{66}^{-1} = I + e_{21} - 5e_{23} - 5e_{24} + 5e_{25}.
\end{aligned}$$

In particular, the elements $h_{51}, h_{67}, \dots, h_{71}$ commute pairwise, and

$$(47) \quad h_{72} := h_{51}^{2164} h_{67}^{26} h_{68}^5 h_{69}^{216} h_{70}^{992} h_{71}^{-2899} = t_{23}(1).$$

Relations (47), (42), (44), and (45) imply that $t_{23}(1)$, $t_{31}(1)$, $t_{34}(1)$, and $t_{35}(1)$ belong to $\langle x, y \rangle$. Using the commutator identity (1), we obtain

$$t_{21}(1), t_{24}(1), t_{25}(1) \in \langle x, y \rangle.$$

Using this and also (43), (46), (47), and again the commutator identity (1), we conclude that $\langle x, y \rangle$ contains all the transvections $t_{ij}(1)$, $1 \leq i \neq j \leq 5$. Therefore, $\langle x, y \rangle$ contains $\text{SL}(5, \mathbb{Z})$. Since $\langle x, y \rangle$ also contains matrices of determinant -1 (e.g., x), the group $\langle x, y \rangle$ coincides with $\text{GL}(5, \mathbb{Z})$. Lemma 2.1 implies that $\langle -x, y \rangle = \text{SL}(5, \mathbb{Z})$. $\square$

§6. $n = 5$: THE THIRD GENERATING SET

In this section we consider the case where X and Y are given by (2) with $a_1 = 1$, $a_2 = -1$, $a_3 = 1$, and $a_4 = -3$. Again, it is more convenient to work with a pair of conjugates of X and Y . Let

$$Q = \begin{pmatrix} 1 & 0 & 0 & 2 & 4 \\ 1 & 0 & 1 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and let

$$(48) \quad x = QXQ^{-1} = \begin{pmatrix} -3 & 0 & 4 & 0 & 4 \\ -1 & -1 & 4 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 \\ 2 & 0 & -2 & -1 & -6 \\ -1 & 0 & 1 & 0 & 2 \end{pmatrix},$$

$$(49) \quad y = QYQ^{-1} = \begin{pmatrix} 2 & -2 & 0 & 1 & 1 \\ 3 & -2 & -1 & 1 & -2 \\ 2 & -1 & -1 & 1 & 0 \\ -1 & 2 & -2 & -1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 6.1. *Let x and y be defined by (48) and (49). Then $\langle x, y \rangle = \text{GL}(5, \mathbb{Z})$ and $\langle -x, y \rangle = \text{SL}(5, \mathbb{Z})$.*

Proof. First, we find suitable transvections of level 2. We start with the following matrices:

$$\begin{aligned}
h_1 &:= (yx)^3(y^2x)^3xyx^2y, \\
h_2 &:= h_1^{-4} = t_{52}(2), \\
h_3 &:= yxy^2xyx^2y^2, \\
h_4 &:= yxyxyx^2y^2xyx^2y^2, \\
h_5 &:= (xy^2)^2xyx^2y^2(xy)^2(xy^2)^2xyx^2y^2xyx^2y^2, \\
h_6 &:= h_3h_2h_3^{-1} = I + 8e_{12} + 2e_{32} - 12e_{42} + 4e_{52}, \\
h_7 &:= h_4^2h_2h_4^{-2} = I - 4e_{12} - 2e_{32} + 8e_{42}, \\
h_8 &:= h_4^{-2}h_2h_4^2 = I - 4e_{12} - 2e_{32} + 6e_{42}, \\
h_9 &:= h_5h_2h_5^{-1} = I + 10e_{12} + 2e_{32} - 14e_{42} + 6e_{52}.
\end{aligned}$$

It is easy to check that $h_2, h_6, \dots, h_9$ commute pairwise and generate the same subgroup as $t_{i2}(2)$ with $i = 1, 3, 4, 5$. More precisely,

$$\begin{aligned}
h_{10} &:= h_2^{-1}h_6^{-1}h_7h_8^{-1}h_9 = t_{12}(2), \\
h_{11} &:= h_2^2h_6^{-1}h_8^{-2} = t_{32}(2), \\
h_{12} &:= h_7h_8^{-1} = t_{42}(2).
\end{aligned}$$

Next, consider

$$\begin{aligned}
h_{13} &:= (yx)^3y^2xy, \\
h_{14} &:= (yx)^6(y^2x)^2y(yx)^{-6}, \\
h_{15} &:= (yx)^2y^2xyx^2y, \\
h_{16} &:= h_{13}h_2h_{13}^{-1} = I + 6e_{51} - 4e_{52} + 4e_{53} + 4e_{54}, \\
h_{17} &:= h_{13}^2h_2h_{13}^{-2} = I + 4e_{51} - 2e_{52} + 8e_{53} + 4e_{54}, \\
h_{18} &:= h_{14}h_2h_{14}^{-1} = I + 16e_{51} - 12e_{52} + 20e_{53} + 14e_{54}, \\
h_{19} &:= h_{15}^2h_2h_{15}^{-2} = I - 12e_{51} + 8e_{52} - 6e_{53} - 8e_{54}.
\end{aligned}$$

The matrices $h_2, h_{16}, \dots, h_{19}$ allow us to find transvections $t_{5i}(2)$, $i = 1, 3$. Namely,

$$\begin{aligned}
h_{20} &:= h_2^{-2}h_{16}^{-1}h_{17}^4h_{18}^{-2}h_{19}^{-2} = t_{51}(2), \\
h_{21} &:= h_{16}^2h_{19} = t_{53}(2).
\end{aligned}$$

There is a similar expression for $t_{54}(2)$, but we omit it because it is not used in what follows. Now, we set

$$\begin{aligned}
h_{22} &:= yxy^2xyx^2y^2(xy)^3x, \\
h_{23} &:= (y^2x)^2(yx)^4y, \\
h_{24} &:= h_{22}^{-2}h_{20}h_{22}^2 = I - 6e_{21} - 2e_{31} - 4e_{41} + 4e_{51}, \\
h_{25} &:= h_{23}^4h_{20}h_{23}^{-4} = I + 4e_{21} + 2e_{31} + 2e_{41} + 2e_{51}, \\
h_{26} &:= h_{23}^{-4}h_{20}h_{23}^4 = I + 2e_{21} + 2e_{51}.
\end{aligned}$$

The matrices h_{20} , h_{24} , h_{25} , and h_{26} commute pairwise and generate the same subgroup as $t_{i1}(2)$ with $i = 2, 3, 4, 5$. Namely,

$$\begin{aligned} h_{27} &:= h_{20}^{-1} h_{26} = t_{21}(2), \\ h_{28} &:= h_{20}^{-3} h_{24} h_{25}^2 h_{26}^{-1} = t_{31}(2), \\ h_{29} &:= h_{20}^4 h_{24}^{-1} h_{25}^{-1} h_{26}^{-1} = t_{41}(2). \end{aligned}$$

We put

$$\begin{aligned} h_{30} &:= yxy, \\ h_{31} &:= h_{30} h_{21} h_{30}^{-1}, \\ h_{32} &:= h_{30}^{-1} h_{21} h_{30}, \\ h_{33} &:= h_{30}^2 h_{21} h_{30}^{-2}, \\ h_{34} &:= h_{21}^{-6} h_{31}^5 h_{32}^3 h_{33}^{-4} = t_{13}(2), \\ h_{35} &:= h_{21}^2 h_{31}^{-2} h_{32}^{-1} h_{33} = t_{23}(2), \\ h_{36} &:= h_{21}^{-3} h_{31}^3 h_{32}^2 h_{33}^{-2} = t_{43}(2) \end{aligned}$$

and consider

$$\begin{aligned} h_{37} &:= yx(y^2x)^2(yx)^2y^2, \\ h_{38} &:= h_{29} h_{37} h_{36}^{-1} h_{37} h_{36}^{-2} = t_{45}(2), \\ h_{39} &:= (xy)^3(xy^2)^3x, \\ h_{40} &:= (yx)^3(y^2x)^2y^2, \\ h_{41} &:= h_{27} h_{35} h_{11} h_{35}^{-1} h_{10}^{-1} h_{39} h_{10}, \\ h_{42} &:= h_{41} h_{40} h_1^2 h_{34}^5 h_{40}^{-1} h_{41}^{-1}, \\ h_{43} &:= (h_{41} h_{40})^2 h_1^2 h_{34}^5 (h_{41} h_{40})^{-2}, \\ h_{44} &:= h_{42} h_{43}, \\ h_{45} &:= h_{28}^{-1} h_{44} h_{38} h_2^{-6} h_{20}^3 h_{21}^5 h_{29}^{-20} h_{12}^{37} h_{36}^{-26} = I + e_{43} + e_{51} + e_{52}. \end{aligned}$$

Let

$$\begin{aligned} h_{46} &:= (yxy^2x)^2y^2xyx(y^2x)^2y^2, \\ h_{47} &:= h_{46} h_{45} h_{46}^{-1}, \\ h_{48} &:= h_{41} h_{46} h_{45} h_{46}^{-1} h_{45}^{-1} h_{41}^{-1}, \\ h_{49} &:= h_{46} h_{47} h_{48} h_{46}^{-1}, \\ h_{50} &:= h_{41} h_{48} h_{49} h_{41}^{-1}. \end{aligned}$$

Now we are ready to find transvections of level 1:

$$\begin{aligned} (50) \quad h_{51} &:= h_{47} h_{48} h_{49} h_{50} h_2^6 h_{20}^{-1} h_{21}^{-13} h_{29}^{-15} h_{12}^{15} h_{36}^{-8} = t_{53}(1), \\ h_{52} &:= h_{45} h_{48} h_{49} h_{50} h_2^6 h_{12}^{15} h_{20}^{-2} h_{21}^{-13} h_{29}^{-15} h_{36}^{-7} = t_{41}(1). \end{aligned}$$

Next, we set

$$\begin{aligned} h_{53} &:= h_{13} h_{51} h_{13}^{-1} = I + e_{51} - e_{52} + 2e_{53} + e_{54}, \\ h_{54} &:= h_{13}^2 h_{51} h_{13}^{-2} = I + e_{51} + 3e_{53} + e_{54}, \\ h_{55} &:= h_{13}^{-1} h_{51} h_{13} = I - 2e_{51} + e_{52} + e_{53} - e_{54} \end{aligned}$$

and find more transvections:

$$\begin{aligned}
(51) \quad & h_{56} := h_{51}^3 h_{53}^{-1} h_{55}^{-1} = t_{51}(1), \\
(52) \quad & h_{57} := h_{51}^{-1} h_{53}^{-1} h_{54} = t_{52}(1), \\
(53) \quad & h_{58} := h_{51}^{-6} h_{53} h_{54} h_{55} = t_{54}(1), \\
& h_{59} := h_{37} h_{52} h_{37} = I + 2e_{41} - 3e_{43} - 3e_{45}, \\
(54) \quad & h_{60} := h_{38}^2 h_{36}^{-18} h_{12}^{16} h_{29}^{-13} h_{56}^{-7} h_{21}^{-7} h_2^5 h_{47} h_{50} h_{59} = t_{45}(1), \\
& h_{61} := yx(y^2x)^3yx, \\
& h_{62} := xy^2(xy)^3xy^2xy(xy^2)^2(xy)^2, \\
& h_{63} := y^2h_{60}y = I - e_{15} - e_{25} - e_{35}, \\
& h_{64} := yh_{61}^2h_{60}h_{61}^{-2}y^2 = I + 4e_{25} + 4e_{35} + 3e_{45}, \\
& h_{65} := h_{62}^2h_{60}h_{62}^{-2} = I + 6e_{15} + 11e_{25} + 4e_{35} - 3e_{45}, \\
& h_{66} := h_{62}^{-2}yh_{60}y^2h_{62}^2 = I - 7e_{15} - 12e_{25} - 4e_{35} + 5e_{45}, \\
(55) \quad & h_{67} := h_{60}^8h_{64}h_{65}^{-8}h_{66}^{-7} = t_{15}(1), \\
(56) \quad & h_{68} := h_{60}^{-6}h_{64}^{-1}h_{65}^7h_{66}^6 = t_{25}(1), \\
(57) \quad & h_{69} := h_{60}^{-2}h_{63}^{-1}h_{65}h_{66} = t_{35}(1).
\end{aligned}$$

Relations (50)–(57) imply that the transvections $t_{5i}(1)$, $t_{i5}(1)$, $i = 1, 2, 3, 4$, are in $\langle x, y \rangle$. Using the commutator identity (1), we conclude that $\langle x, y \rangle$ contains all the transvections $t_{ij}(1)$, $1 \leq i \neq j \leq 5$. Therefore, $\langle x, y \rangle$ contains $\text{SL}(5, \mathbb{Z})$. Since $\langle x, y \rangle$ also contains matrices of determinant -1 (e.g., x), the group $\langle x, y \rangle$ coincides with $\text{GL}(5, \mathbb{Z})$. Lemma 2.1 implies that $\langle -x, y \rangle = \text{SL}(5, \mathbb{Z})$. $\square$

Proof of Theorem 1.2. This follows immediately from Theorem 3.1, Remark 3.1, and Theorems 3.2, 3.3, 4.1, 5.1, 6.1. $\square$

Remark 6.1. It is easy to check that all six pairs of matrices listed in Theorem 1.2 are pairwise nonconjugate.

§7. THE GROUP $\text{SL}(6, \mathbb{Z})$

As was shown in [12], the problem on (2,3)-generation of $\text{SL}(6, \mathbb{Z})$ and the classification of the corresponding generators can be reduced to the analysis of eight cases. Using methods similar to those employed in the proof of Theorems 3.2, 3.3, one can exclude four cases. The answer in the fifth case is positive (see Theorem 7.1 below). However, the author was unable to settle the remaining three cases. For this reason, in the present paper we restrict ourselves to the positive result only.

Consider the matrices

$$(58) \quad x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 5 & 1 & -5 \\ 1 & 2 & 0 & 1 & 0 & -1 \\ 5 & 10 & 0 & 0 & 1 & -5 \\ 2 & 4 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$(59) \quad y = \begin{pmatrix} -1 & 1 & 2 & 2 & 0 & -2 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 3 & 1 & -1 & 0 & 0 \\ 1 & 2 & 2 & -5 & -1 & 4 \\ 5 & 7 & 1 & -10 & 0 & 5 \\ 2 & 3 & 2 & -7 & -1 & 5 \end{pmatrix}.$$

It can be shown that the pair x, y is conjugate to the pair X, Y in [12, Theorem 1] with $a_1 = 3, a_2 = 1, a_3 = -1, a_4 = 1, b_1 = 0, b_2 = 2, b_3 = -2$, and $b_4 = -3$. The choice of x, y instead of X, Y is for technical convenience and simplification of the proof only.

Theorem 7.1. *Let x and y be defined by (58) and (59). Then $\langle x, y \rangle = \text{SL}(6, \mathbb{Z})$.*

Lemma 7.1. *Let G be a subgroup of $\text{SL}(6, \mathbb{Z})$ containing x, y , and $t_{36}(a)$ for some integer a . Then G also contains*

- (i) $t_{3j}(a), j \in \{1, 2, 4, 5\}$;
- (ii) $t_{i6}(a), i \in \{1, 2, 4, 5\}$;
- (iii) $t_{4i}(a), i \in \{1, 2, 3\}$;
- (iv) $t_{63}(a)$.

Proof. As before, by h with subscripts we denote elements of the group G that do not depend on a , while by z_i ($i = 1, 2, \dots$) we denote elements that do depend on a . We start with the matrix

$$z_1 = t_{36}(a)$$

and set

$$\begin{aligned} h_1 &:= yxy^2x(yx)^2(y^2x)^2(yx)^4y^2, \\ h_2 &:= (yx)^5y^2xy^2, \\ z_2 &:= h_1z_1h_1^{-1} = I + a(e_{31} + 3e_{32} + 3e_{34} - 2e_{36}), \\ z_3 &:= h_1^{-1}z_1h_1 = I + a(-8e_{31} - 11e_{32} - 4e_{34} - 2e_{35} + 7e_{36}), \\ z_4 &:= h_1^2z_1h_1^{-2} = I + a(17e_{31} + 28e_{32} + 8e_{34} + 3e_{35} - 10e_{36}), \\ z_5 &:= h_1^{-2}z_1h_1^2 = I + a(9e_{31} + 16e_{32} - 8e_{34} - 3e_{35} + 12e_{36}), \\ z_6 &:= h_2^4z_1h_2^{-4} = I + a(e_{31} + e_{32} - 2e_{34}). \end{aligned}$$

In particular, the matrices $z_1, \dots, z_6$ commute pairwise. It is easy to check that

$$\begin{aligned} z_7 &:= z_1^{101}z_2^{46}z_3^{-9}z_4^{-9}z_5^{-3}z_6^{63} = t_{31}(a), \\ z_8 &:= z_1^{123}z_2^{56}z_3^{-9}z_4^{-10}z_5^{-4}z_6^{78} = t_{32}(a), \\ z_9 &:= z_1^{-156}z_2^{-71}z_3^{12}z_4^{13}z_5^5z_6^{-99} = t_{34}(a), \\ z_{10} &:= z_1^{-229}z_2^{-106}z_3^{19}z_4^{20}z_5^7z_6^{-145} = t_{35}(a), \end{aligned}$$

which proves (i). Now we set

$$\begin{aligned} h_3 &:= (yxy^2x)^2x, \\ h_4 &:= (y^2xyx)^2x, \\ h_5 &:= y^2xyx(y^2x)^3(yx)^3y^2xy, \\ h_6 &:= (yx)^2y^2xyx(y^2x)^2(yx)^2y^2x(yx)^3y^2, \\ z_{11} &:= h_3z_1h_3^{-1} = I + a(2e_{36} + 3e_{56}), \\ z_{12} &:= h_4z_1h_4^{-1} = I + a(-e_{16} + e_{26} - e_{56}), \\ z_{13} &:= h_5z_1h_5^{-1} = I + a(-2e_{16} + e_{26} + e_{36} + e_{46} - 5e_{56}), \end{aligned}$$

$$\begin{aligned} z_{14} &:= h_6 z_1 h_6^{-1} = I + a(7e_{16} - 5e_{26} + 9e_{36} - e_{46} + 16e_{56}), \\ z_{15} &:= h_6^{-1} z_{12} h_6 = I + a(4e_{16} - 2e_{26} + e_{36} - 3e_{56}). \end{aligned}$$

The matrices $z_1, z_{11}, z_{12}, z_{13}, z_{14}$, and z_{15} commute pairwise. A direct calculation shows that

$$\begin{aligned} z_{16} &:= z_1 z_{11}^4 z_{12}^{-2} z_{13}^{-1} z_{14}^{-1} z_{15} = t_{16}(a), \\ z_{17} &:= z_1^{-6} z_{11}^{-2} z_{12}^5 z_{13} z_{14} = t_{26}(a), \\ z_{18} &:= z_1^{-9} z_{11}^{-1} z_{12}^3 z_{13}^2 z_{14} = t_{46}(a), \\ z_{19} &:= z_1^{-7} z_{11}^{-6} z_{12}^6 z_{13}^2 z_{14}^2 z_{15}^{-1} = t_{56}(a). \end{aligned}$$

This proves (ii). Next, we set

$$\begin{aligned} h_7 &:= yx(y^2x)^5(yx)^2y^2, \\ (60) \quad z_{20} &:= z_1^5 z_7^{-10} z_8^{-20} z_{18} x z_{18} h_7^4 z_{18} h_7^{-4} = t_{42}(a), \\ h_8 &:= (xyxy^2)^2(xy)^2(xy^2)^2(xy)^4xy^2, \\ h_9 &:= h_8^2 x h_8^{-2} x, \\ (61) \quad z_{21} &:= z_1^{-125} z_7^{81} z_8^{500} h_9 z_7 h_9^{-1} x z_{18}^{-25} x z_{18}^{-25} z_{20}^{100} = t_{41}(a), \\ h_{10} &:= (xy(xy^2)^2(xy)^2(xy^2)^2xy)^2, \\ h_{11} &:= (xy^2xy^2xyx)^4, \\ z_{22} &:= h_{11} z_{19} h_{11}^{-1} z_{19}^{-1} = t_{54}(3a), \\ z_{23} &:= z_1^{12} z_7^{-18} z_8^{-36} z_9^{-18} h_{10}^{-1} z_{19}^{-1} h_{10} z_{19}^{10} z_{22}^{-5} = I + a(15e_{51} + 30e_{52}), \\ z_{24} &:= h_{10} z_{20}^2 z_{21} h_{10}^{-1} \\ &= I + a(30e_{31} + 60e_{32} + 4e_{41} + 8e_{42} + 30e_{51} + 60e_{52} + 3e_{61} + 6e_{62}), \\ z_{25} &:= z_7^{-30} z_8^{-60} z_{20}^{-10} z_{21}^{-5} z_{23}^{-1} z_{24} \\ &= I + a(-e_{41} - 2e_{42} + 15e_{51} + 30e_{52} + 3e_{61} + 6e_{62}), \\ h_{12} &:= xy^2xyx, \\ (62) \quad z_{26} &:= h_{12} z_{25} h_{12}^{-1} = t_{43}(a). \end{aligned}$$

Relations (60), (61), (62) prove (iii). Finally, let

$$\begin{aligned} z_{27} &:= h_{11} z_{20}^{-6} z_{21}^{-3} h_{11}^{-1} z_{23}^{-2} z_{20}^{16} z_{21}^8 \\ &= I + a(5e_{41} + 10e_{42} + 15e_{51} + 30e_{52} + 9e_{61} + 18e_{62}), \\ h_{13} &:= (y^2xyx)^3, \\ z_{28} &:= h_{13}^{-2} z_{27} h_{13}^2 = I + a(e_{61} + 2e_{62}), \\ z_{29} &:= h_{12} z_{28}^{-4} z_{23}^{-1} h_{12} = t_{63}(a). \end{aligned}$$

The last-written relation completes the proof of the lemma. $\square$

Proof of Theorem 7.1. We keep the notation introduced in the proof of Lemma 7.1. Set

$$\begin{aligned} h_{14} &:= (y^2xyxy^2xy^2xy^2)^4, \\ h_{15} &:= (xyy^2x)^4, \\ h_{16} &:= h_{14}h_{15}h_{14}^{-1}h_{15}^{-1}, \\ h_{17} &:= xyxy^2x, \\ h_{18} &:= h_{17}h_{16}^{-1}h_{17}^{-1}xh_{16}x = t_{36}(3). \end{aligned}$$

Lemma 7.1 applied to $a = 3$ shows that the matrices

$$\begin{aligned} h_{19} &:= t_{31}(3), & h_{20} &:= t_{32}(3), & h_{21} &:= t_{34}(3), & h_{22} &:= t_{35}(3), \\ h_{23} &:= t_{41}(3), & h_{24} &:= t_{42}(3), & h_{25} &:= t_{43}(3), & h_{26} &:= t_{46}(3) \end{aligned}$$

are in $\langle x, y \rangle$. Now, we consider

$$\begin{aligned} h_{27} &:= h_{21}^{-2}xh_{25}^{-1}xh_{21}^2, \\ h_{28} &:= h_7^4xh_9xh_7^{-4}, \\ h_{29} &:= h_{27}h_{21}^{-1}h_{28}h_{21}h_{25}h_{21}^2, \\ h_{30} &:= h_{18}^{-15}h_{19}^{14}h_{20}^{22}h_{22}^4h_{26}^{-1}h_{24}^2h_{23}h_{29} = t_{32}(1), \\ h_{31} &:= xy^2(xy)^2xy^2(xy)^3x, \\ h_{32} &:= h_1^{-1}h_{30}h_1 = I + 9e_{31} + 15e_{32} - 2e_{34} - e_{35} + 3e_{36}, \\ h_{33} &:= h_2^{-4}h_{30}h_2^4 = I + e_{31} + 2e_{32} - 2e_{34} + 2e_{36}, \\ h_{34} &:= h_{31}^2h_{30}h_{31}^{-2} = I + 2e_{31} + 4e_{32} - 3e_{34} + e_{36}, \\ h_{35} &:= h_1^2h_{30}h_1^{-2} = I - 24e_{31} - 37e_{32} - 9e_{34} - 5e_{35} + 14e_{36}. \end{aligned}$$

In particular, h_{18} , h_{30} , h_{32} , h_{33} , h_{34} , and h_{35} commute pairwise. An easy calculation shows that

$$h_{36} := h_{18}^{-91}h_{30}^{26}h_{32}^5h_{33}^{205}h_{34}^{-137}h_{35}^{-1} = t_{36}(1).$$

In particular,

$$(63) \quad t_{36}(1) \in \langle x, y \rangle.$$

Applying Lemma 7.1 with $a = 1$, we conclude that

$$(64) \quad \{t_{31}(1), t_{32}(1), t_{34}(1), t_{35}(1)\} \subset \langle x, y \rangle,$$

$$(65) \quad \{t_{16}(1), t_{26}(1), t_{46}(1), t_{56}(1)\} \subset \langle x, y \rangle,$$

$$(66) \quad t_{63}(1) \in \langle x, y \rangle.$$

Relations (64), (66) and the commutator identity (1) imply that

$$(67) \quad \{t_{61}(1), t_{62}(1), t_{64}(1), t_{65}(1)\} \subset \langle x, y \rangle.$$

Applying (63), (65), (66), (67), and the commutator identity (1), we see that all the transvections $t_{ij}(1)$ with $1 \leq i \neq j \leq 6$ are in $\langle x, y \rangle$. Hence, $\langle x, y \rangle = \text{SL}(6, \mathbb{Z})$. $\square$

§8. YET ANOTHER (2,3)-GENERATING SET FOR $\text{PSL}(6, \mathbb{Z})$

Since the group $\text{SL}(6, \mathbb{Z})$ is (2,3)-generated, so is the group $\text{PSL}(6, \mathbb{Z})$. On the other hand, it is natural to ask whether $\text{PSL}(6, \mathbb{Z})$ has a pair of (2,3)-generators different from those “inherited” from $\text{SL}(6, \mathbb{Z})$. For instance, we may look for generators that are the projective images of matrices satisfying the relations

$$(68) \quad x^2 = -I, \quad y^3 = I.$$

The following theorem gives an affirmative answer.

Theorem 8.1. *Let*

$$x = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $x^2 = -I$, $y^3 = I$, and $\langle x, y \rangle = \mathrm{SL}(6, \mathbb{Z})$. In particular, the projective images of x and y are of order 2 and 3, respectively, and generate the group $\mathrm{PSL}(6, \mathbb{Z})$.

Remark 8.1. In contrast to the preceding sections, it is not known whether (up to $\mathrm{GL}(6, \mathbb{Z})$ -conjugation) there are finitely many generating pairs for $\mathrm{SL}(6, \mathbb{Z})$ that satisfy (68). Moreover, it seems plausible that there are infinitely many pairwise nonconjugate generating sets. For this reason, the following method of finding specific generators as in Theorem 8.1 was used. Matrices x_0 and y_0 of the form

$$x_0 = \begin{pmatrix} 0 & 0 & -1 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & -1 & a_3 & a_4 \\ 1 & 0 & 0 & 0 & a_2 & -a_1 \\ 0 & 1 & 0 & 0 & a_4 & -a_3 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad y_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & a_5 & a_6 \\ 0 & 1 & 0 & 0 & a_7 & a_8 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

were sought so as to satisfy the following natural necessary condition: for any prime p not exceeding some given bound, the matrices $x_0 \bmod p$ and $y_0 \bmod p$ generate $\mathrm{SL}(6, p)$. The following set, which passed all tests, was found:

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = (1, -1, 0, 1, -1, -1, 1, 0).$$

It turns out that, for this set, the matrices x_0 and y_0 generate $\mathrm{SL}(6, \mathbb{Z})$. However, for technical reasons it is more convenient to work not with x_0, y_0 , but rather with a conjugate pair x, y ($x = Qx_0Q^{-1}$, $y = Qy_0Q^{-1}$ for a suitable integral matrix Q). Note that the corresponding generators were found by means of a computer search; however, the rigorous proof below is computer-independent.

Proof of Theorem 8.1. Relations (68) are trivial.

Consider the following matrices:

$$\begin{aligned} h_1 &:= xyx^{-1}y^{-1}, \\ h_2 &:= (y^2x^{-1}yx)^2, \\ h_3 &:= (h_2h_1)^2h_2^{-1}(h_2h_1)^{-2}h_2, \\ h_4 &:= (h_2h_1)^4h_2^{-1}(h_2h_1)^{-4}h_2, \\ h_5 &:= h_1h_2h_1h_2^{-1}h_1^{-1}h_2^{-1}h_1^{-1}h_2, \\ h_6 &:= h_3^{-9}h_4h_5^3. \end{aligned}$$

In particular, they are all of block shape

$$\begin{pmatrix} * & 0 \\ * & * \end{pmatrix},$$

where the upper left-hand block is of size 2×2 , and the lower right-hand block is of size 4×4 . Moreover, h_6 has the form

$$(69) \quad I + k_{31}e_{31} + k_{32}e_{32} + k_{41}e_{41} + k_{42}e_{42} + k_{51}e_{51} + k_{52}e_{52} + k_{61}e_{61} + k_{62}e_{62}.$$

TABLE 2

	k_{31}	k_{32}	k_{41}	k_{42}	k_{51}	k_{52}	k_{61}	k_{62}
h_6	3	5	1	-3	2	3	-2	5
h_7	3	5	9	10	2	3	3	13
h_8	3	5	10	12	2	3	4	14
h_9	3	5	7	7	2	3	2	11
h_{10}	2	-5	4	-2	1	-3	-4	-2
h_{11}	0	-8	2	3	-2	-8	-3	-5
h_{12}	1	2	15	-9	1	1	-24	17
h_{13}	-3	-13	-1	-10	9	10	-12	-15
h_{14}	-5	-16	2	3	6	5	-3	-5
h_{15}	-6	-17	2	3	7	7	-3	-5

Now we consider matrices of the same form (69). (For each matrix, the values of the coefficients k_{ij} are listed in Table 2.)

$$\begin{aligned}
h_7 &:= h_3^{-8} h_4 h_3^{-1} h_5^3, \\
h_8 &:= h_3^{-9} h_5^3 h_4, \\
h_9 &:= h_3^{-9} h_5^2 h_4 h_5, \\
h_{10} &:= h_1 h_6 h_1^{-1}, \\
h_{11} &:= h_1^{-1} h_6 h_1, \\
h_{12} &:= h_2^{-1} h_6 h_2, \\
h_{13} &:= h_1 h_7 h_1^{-1}, \\
h_{14} &:= h_1^{-1} h_7 h_1, \\
h_{15} &:= h_1^{-1} h_8 h_1.
\end{aligned}$$

Clearly, the matrices $h_7, \dots, h_{15}$ commute pairwise. We show that, in fact, the subgroup generated by them coincides with

$$\langle t_{31}(1), t_{41}(1), t_{51}(1), t_{61}(1), t_{32}(1), t_{42}(1), t_{52}(1), t_{62}(1) \rangle.$$

Indeed, we have

$$\begin{aligned}
h_{16} &:= h_7^{1034} h_8^{-808} h_9^{-149} h_{10}^7 h_{11}^{121} h_{12}^{-21} h_{13}^{-8} h_{14}^{-685} h_{15}^{612} = t_{31}(1), \\
h_{17} &:= h_7^{-2087} h_8^{1686} h_9^{221} h_{10}^{-17} h_{11}^{-286} h_{12}^{46} h_{13}^{19} h_{14}^{1623} h_{15}^{-1450} = t_{41}(1), \\
h_{18} &:= h_7^{-4604} h_8^{3681} h_9^{543} h_{10}^{-36} h_{11}^{-603} h_{12}^{99} h_{13}^{40} h_{14}^{3423} h_{15}^{-3058} = t_{51}(1), \\
h_{19} &:= h_7^{-4701} h_8^{3803} h_9^{490} h_{10}^{-39} h_{11}^{-649} h_{12}^{104} h_{13}^{43} h_{14}^{3685} h_{15}^{-3292} = t_{61}(1), \\
h_{20} &:= h_7^{128} h_8^{-121} h_9^{12} h_{10}^2 h_{11}^{31} h_{12}^{-4} h_{13}^{-2} h_{14}^{-177} h_{15}^{158} = t_{32}(1), \\
h_{21} &:= h_7^{4698} h_8^{-3801} h_9^{-489} h_{10}^{39} h_{11}^{649} h_{12}^{-104} h_{13}^{-43} h_{14}^{-3685} h_{15}^{3292} = t_{42}(1), \\
h_{22} &:= h_7^{-2664} h_8^{2186} h_9^{233} h_{10}^{-24} h_{11}^{-391} h_{12}^{61} h_{13}^{26} h_{14}^{2222} h_{15}^{-1985} = t_{52}(1), \\
h_{23} &:= h_7^{-2609} h_8^{2114} h_9^{267} h_{10}^{-22} h_{11}^{-363} h_{12}^{58} h_{13}^{24} h_{14}^{2062} h_{15}^{-1842} = t_{62}(1).
\end{aligned}$$

Now we use these matrices for finding more transvections in $\langle x, y \rangle$. Consider

$$\begin{aligned}
 h_{24} &:= y h_{17} y^{-1} = t_{46}(1), \\
 h_{25} &:= y h_{18}^{-1} h_{17} y^{-1} = t_{26}(1), \\
 h_{26} &:= y h_{19} y^{-1} = t_{36}(1), \\
 h_{27} &:= y h_{16} h_{17} h_{19} y^{-1} = t_{16}(1), \\
 h_{28} &:= y^{-1} h_{16} y h_{19}^{-1} = t_{63}(1), \\
 h_{29} &:= y^{-1} h_{20}^{-1} y h_{23}^{-1} = t_{65}(1), \\
 h_{30} &:= x h_{24}^{-1} x^{-1} h_{28} = t_{64}(1), \\
 h_{31} &:= h_{22} h_{25} h_{22}^{-1} h_{25}^{-1} = t_{56}(1).
 \end{aligned}$$

In particular, we already know that $t_{6k}(1)$ and $t_{i6}(1)$ for all $i, k = 1, \dots, 5$ are in $\langle x, y \rangle$. Using identity (1) we conclude that $t_{ik}(1) \in \langle x, y \rangle$ for all $i, k, i \neq k$. Therefore, $\langle x, y \rangle = \text{SL}(6, \mathbb{Z})$, which proves the theorem. $\square$

§9. THE GROUP $\text{SL}(7, \mathbb{Z})$

In this section we prove the following theorem.

Theorem 9.1. *The groups $\text{SL}(7, \mathbb{Z})$ and $\text{GL}(7, \mathbb{Z})$ are (2, 3)-generated. More precisely, let*

$$(70) \quad x = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16 & 1 & -18 & 3 & 0 & -3 & 3 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -25 & 0 & 25 & 0 & 0 & 1 & -5 \\ -11 & 0 & 12 & 0 & 1 & 0 & -3 \\ -25 & 0 & 25 & 1 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$(71) \quad y = \begin{pmatrix} 19 & -4 & -6 & -12 & 18 & 0 & -3 \\ 18 & -1 & -12 & -6 & 10 & -1 & 3 \\ 20 & -5 & -4 & -15 & 23 & 0 & -6 \\ -2 & 8 & -22 & 25 & -40 & -1 & 21 \\ -12 & 8 & -12 & 24 & -37 & -1 & 16 \\ -12 & 15 & -33 & 45 & -73 & -1 & 40 \\ -7 & 1 & 4 & 3 & -4 & 0 & 0 \end{pmatrix}.$$

Then $x^2 = y^3 = 1$ and $\langle x, y \rangle = \text{SL}(7, \mathbb{Z})$, $\langle -x, y \rangle = \text{GL}(7, \mathbb{Z})$.

Remark 9.1. The method of finding the matrices x, y is similar to that used in [13] for $\text{GL}(6, \mathbb{Z})$. We looked for matrices x_0, y_0 of the form

$$x_0 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & a_3 \\ 1 & 0 & 0 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & a_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$y_0 = \begin{pmatrix} 1 & 0 & 0 & a_4 & a_7 & 0 & a_{10} \\ 0 & 1 & 0 & a_5 & a_8 & 0 & a_{11} \\ 0 & 0 & 1 & a_6 & a_9 & 0 & a_{12} \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Obviously, $x_0^2 = y_0^3 = I$. Moreover, we imposed an extra condition on the characteristic polynomial of the commutator of x_0 and y_0 :

$$(72) \quad \chi_{x_0 y_0 x_0^{-1} y_0^{-1}}(\lambda) = (\lambda - 1)(\lambda + 1)^2(\lambda^2 - \lambda + 1)(\lambda^2 + \lambda + 1).$$

If, in addition, the eigenspace of $x_0 y_0 x_0^{-1} y_0^{-1}$ relative to -1 is one-dimensional, then

$$(x_0 y_0 x_0^{-1} y_0^{-1})^6$$

is a transvection (not necessarily an elementary transvection). Condition (72) can be written as a system of polynomial equations. Next, we looked for “small” solutions of this system that satisfy the following natural necessary condition: for primes p not exceeding some given bound, the matrices $x_0 \bmod p$ and $y_0 \bmod p$ generate $\mathrm{SL}(7, p)$. In this way we found the following set of parameters that passed all the tests: $a_1 = 1$, $a_2 = 1$, $a_3 = -2$, $a_4 = 1$, $a_5 = 0$, $a_6 = 1$, $a_7 = -1$, $a_8 = -1$, $a_9 = 0$, $a_{10} = 1$, $a_{11} = 0$, $a_{12} = -3$. It turns out that, for this set, the matrices x_0 and y_0 actually generate $\mathrm{SL}(7, \mathbb{Z})$. It is more convenient to work not with x_0, y_0 , but rather with a conjugate pair x, y given by (69), (70); $x = Qx_0Q^{-1}$, $y = Qy_0Q^{-1}$ for a suitable integral matrix Q . The choice of conjugates was made, in particular, so as to guarantee that $(xyx^{-1}y^{-1})^6$ is an elementary transvection.

Proof of Theorem 9.1. First, we note that

$$(73) \quad h_1 := (xyxy^2)^6 = t_{21}(1).$$

Take

$$h_2 := xyxy^2xyxy^2xy^2xy.$$

Then

$$(74) \quad h_3 := h_2h_1^{-1}h_2^{-1} = t_{12}(1).$$

We set

$$\begin{aligned} h_4 &:= xyxy^2, \\ h_5 &:= h_2h_4h_2^{-1}, \end{aligned}$$

and

$$\begin{aligned} Q_{m,j,k} &:= h_1^{-k-1}h_4^mh_3^{-1}h_4^{-m}h_1^kh_3h_1h_3^{-1}h_1^{-k}h_4^mh_3h_4^{-m}h_1^{k+1}h_3^j, \\ R_{m,j,k} &:= h_1^{-1}h_3^{-k}h_5^mh_1^{-1}h_5^{-m}h_3^kh_1h_3h_1^{-1}h_3^{-k}h_5^mh_1h_5^{-m}h_3^kh_1h_3^j. \end{aligned}$$

A direct calculation shows that

$$\begin{aligned} h_6 &:= Q_{1,12,-9} = I + 121e_{13} - 22e_{14} + 77e_{15} - 11e_{16} - 88e_{17}, \\ h_7 &:= Q_{-1,12,-4} = I + 44e_{13} - 44e_{14} + 44e_{15} + 11e_{16} - 11e_{17}, \\ h_8 &:= Q_{2,-14,-10} = I - 255e_{13} + 75e_{14} - 60e_{15} - 45e_{16} - 30e_{17}, \\ h_9 &:= Q_{-2,-14,23} = I + 285e_{13} + 45e_{14} - 105e_{15} + 30e_{16} - 30e_{17}, \\ h_{10} &:= Q_{-3,-6,2} = I + 14e_{13} + 14e_{14} - 7e_{16} + 14e_{17}, \\ h_{11} &:= R_{1,-12,-4} = I + 253e_{13} - 110e_{14} + 198e_{15} + 11e_{16} - 77e_{17}, \end{aligned}$$

$$h_{12} := R_{2,14,23} = I + 1155e_{13} - 735e_{14} + 1305e_{15} + 30e_{16} - 480e_{17},$$

$$h_{13} := R_{-3,6,3} = I + 140e_{13} - 63e_{14} + 140e_{15} - 7e_{16} - 105e_{17}.$$

In particular, $h_6, h_7, \dots, h_{13}$ generate an Abelian subgroup. By standard linear algebra, we can find another basis for the same subgroup:

$$(75) \quad h_{14} := h_6^{219} h_7^2 h_8^{-149} h_9^{-100} h_{10}^{-20} h_{11}^{-52} h_{12}^{-23} h_{13}^{28} = t_{13}(1),$$

$$(76) \quad h_{15} := h_6^{476} h_7^{460} h_8^{-246} h_9^{-308} h_{10}^{109} h_{11}^{456} h_{12}^{-164} h_{13}^{141} = I + e_{14} + e_{17},$$

$$(77) \quad h_{16} := h_6^{200} h_7^{244} h_8^{-105} h_9^{-143} h_{10}^{71} h_{11}^{240} h_{12}^{-82} h_{13}^{86} = t_{15}(1),$$

$$(78) \quad h_{17} := h_6^{-370} h_7^{-516} h_8^{398} h_9^{350} h_{10}^{-325} h_{11}^{-212} h_{12}^{176} h_{13}^{-542} = t_{16}(1),$$

$$(79) \quad h_{18} := h_6^{138} h_7^{-87} h_8^{161} h_9^{46} h_{10}^{-258} h_{11}^{249} h_{12}^{18} h_{13}^{-465} = t_{17}(3).$$

Remark 9.2. It is not too difficult to show that $h_{14}, \dots, h_{18}$ is, in fact, a basis of the group generated by $h_6, \dots, h_{13}$. Thus, it is impossible to define $t_{17}(1)$ and $t_{14}(1)$ by using only $h_6, \dots, h_{13}$.

Observe that

$$h_5 h_{14}^{-1} h_5^{-1} = I + 7e_{12} + e_{13} + 7e_{14} - 13e_{15} + e_{16} + 2e_{17}.$$

Therefore,

$$(80) \quad h_{19} := h_5 h_{14}^{-1} h_5^{-1} h_3^{-7} h_{14}^{-1} h_{15}^{-7} h_{16}^{13} h_{17}^{-1} h_{18}^2 = t_{17}(1).$$

Combining this with (76), we conclude that

$$(81) \quad h_{20} := h_{15} h_{19}^{-1} = t_{14}(1).$$

In a similar way, we define

$$S_{m,j,k} := h_1^j h_3^{-1} h_1^{-k} h_4^m h_3^{-1} h_4^{-m} h_1^k h_3 h_1 h_3^{-1} h_1^{-k} h_4^m h_3 h_4^{-m} h_1^k h_3,$$

$$T_{m,j,k} := h_1^j h_3^{-k-1} h_5^m h_1^{-1} h_5^{-m} h_3^k h_1 h_3 h_1^{-1} h_3^{-k} h_5^m h_1 h_5^{-m} h_3^{k+1}$$

and consider

$$h_{21} := S_{1,10,2} = I - 121e_{31} + 913e_{41} + 550e_{51} + 1518e_{61} - 88e_{71},$$

$$h_{22} := S_{-1,10,7} = I + 33e_{31} - 165e_{41} - 99e_{51} - 451e_{61} + 66e_{71},$$

$$h_{23} := S_{2,-16,-25} = I + 120e_{31} - 585e_{41} - 435e_{51} - 1230e_{61} + 120e_{71},$$

$$h_{24} := S_{-2,-16,8} = I - 45e_{31} + 525e_{41} + 345e_{51} + 915e_{61} - 45e_{71},$$

$$h_{25} := S_{-3,-8,-5} = I + 112e_{41} + 56e_{51} + 133e_{61} - 21e_{71},$$

$$h_{26} := T_{-1,-10,2} = I - 77e_{31} + 671e_{41} + 396e_{51} + 1221e_{61} - 66e_{71},$$

$$h_{27} := T_{2,16,8} = I - 675e_{31} + 3990e_{41} + 2550e_{51} + 6945e_{61} - 360e_{71},$$

$$h_{28} := T_{-3,8,-4} = I - 126e_{31} + 805e_{41} + 497e_{51} + 1484e_{61} - 84e_{71}.$$

The matrices $h_{21}, \dots, h_{28}$ generate an Abelian subgroup, and we can find its basis. Set

$$(82) \quad h_{29} := h_{21}^{778} h_{22}^{-448} h_{23}^{92} h_{24}^{143} h_{25}^{-1089} h_{26}^{-106} h_{27}^{-6} h_{28}^{-731} = I + e_{31} + 2e_{71},$$

$$(83) \quad h_{30} := h_{21}^{1618} h_{22}^{185} h_{23}^{-424} h_{24}^{-412} h_{25}^{-1854} h_{26}^{475} h_{27}^{-194} h_{28}^{-1013} = I + e_{41} + 2e_{71},$$

$$(84) \quad h_{31} := h_{21}^{-1151} h_{22}^{-1167} h_{23}^{988} h_{24}^{1197} h_{25}^{1131} h_{26}^{-1694} h_{27}^{225} h_{28}^{1143} = I + e_{51} + 2e_{71},$$

$$(85) \quad h_{32} := h_{21}^{-213} h_{22}^{-155} h_{23}^{99} h_{24}^{76} h_{25}^{148} h_{26}^{33} h_{27}^{67} h_{28}^{-148} = t_{61}(1),$$

$$(86) \quad h_{33} := h_{21}^{624} h_{22}^{-1038} h_{23}^{326} h_{24}^{287} h_{25}^{-1326} h_{26}^{240} h_{27}^{191} h_{28}^{-1833} = t_{71}(3).$$

Note that

$$(87) \quad h_{34} := x h_{32}^{-1} x h_1^3 = t_{41}(1).$$

Using (82), (83), (84), (86) and (87), we conclude that

$$(88) \quad h_{35} := h_{34}h_{30}^{-1}h_{33} = t_{71}(1),$$

$$(89) \quad h_{36} := h_{29}h_{35}^{-2} = t_{31}(1),$$

$$(90) \quad h_{37} := h_{31}h_{35}^{-2} = t_{51}(1).$$

Relations (73)–(75), (77), (78), (80), (81), (85), (87), and (88)–(90) show that the group $\langle x, y \rangle$ contains the transvections $t_{i1}(1)$, $t_{1k}(1)$ for all $i, k \in \{2, 3, 4, 5, 6, 7\}$. The commutator identity (1) implies that $t_{ik}(1) \in \langle x, y \rangle$ for all i, k , $i \neq k$. Hence, $\langle x, y \rangle = \mathrm{SL}(7, \mathbb{Z})$. Now, the fact that $\langle -x, y \rangle = \mathrm{GL}(7, \mathbb{Z})$ follows from Lemma 2.1. $\square$

REFERENCES

- [1] L. Di Martino and N. Vavilov, *(2, 3)-generation of $\mathrm{SL}(n, q)$. I. Cases $n = 5, 6, 7$* , Comm. Algebra **22** (1994), no. 4, 1321–1347. MR1261262 (95f:20076)
- [2] ———, *(2, 3)-generation of $\mathrm{SL}(n, q)$. II. Cases $n \geq 8$* , Comm. Algebra **24** (1996), 487–515. MR1373489 (97e:20067)
- [3] A. J. Hahn and O. T. O’Meara, *The classical groups and K-theory*, Grundlehren Math. Wiss., Bd. 291, Springer-Verlag, Berlin, 1989. MR1007302 (90i:20002)
- [4] M. W. Liebeck and A. Shalev, *Classical groups, probabilistic methods, and the (2, 3)-generation problem*, Ann. of Math. (2) **144** (1996), 77–125. MR1405944 (97e:20106a)
- [5] A. Yu. Luzgarëv and I. M. Pevzner, *Some facts about the life of $\mathrm{GL}(5, \mathbb{Z})$* , Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **305** (2003), 153–162; English transl., J. Math. Sci. (N. Y.) **130** (2005), no. 3, 4729–4733. MR2033620
- [6] G. A. Miller, *On the groups generated by two operators*, Bull. Amer. Math. Soc. **7** (1901), no. 10, 424–426. MR1557828
- [7] Ya. N. Nuzhin, *On a question of M. Conder*, Mat. Zametki **70** (2001), no. 1, 79–87; English transl., Math. Notes **70** (2001), no. 1–2, 71–78. MR1883773 (2002m:20076)
- [8] P. Sanchini and M. C. Tamburini, *Constructive (2, 3)-generation: A permutational approach*, Rend. Sem. Mat. Fis. Milano **64** (1994), 141–158 (1996). MR1397469 (97i:20040)
- [9] M. C. Tamburini and J. S. Wilson, *On the (2, 3)-generation of some classical groups. II*, J. Algebra **176** (1995), 667–680. MR1351631 (96h:20085)
- [10] M. C. Tamburini, J. S. Wilson, and N. Gavioli, *On the (2, 3)-generation of some classical groups. I*, J. Algebra **168** (1994), 353–370. MR1289105 (95f:20075)
- [11] M. C. Tamburini and P. Zucca, *On a question of M. Conder*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **11** (2000), no. 1, 5–7. MR1797048 (2001i:20097)
- [12] M. A. Vsemirnov, *Is the group $\mathrm{SL}(6, \mathbb{Z})$ (2, 3)-generated?* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **330** (2006), 101–130; English transl., J. Math. Sci. (N. Y.) **140** (2007), no. 5, 660–675. MR2253569 (2008e:20074)
- [13] ———, *The group $\mathrm{GL}(6, \mathbb{Z})$ is (2, 3)-generated*, J. Group Theory **10** (2007), no. 4, 425–430. MR2334749 (2008f:20118)

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