ZERO SETS OF ENTIRE FUNCTIONS OF EXPONENTIAL TYPE WITH ADDITIONAL CONDITIONS ON THE REAL AXIS

S. YU. FAVOROV

ABSTRACT. A complete description of the zero sets is obtained for some well-known classes of entire functions of exponential growth.

We denote by B the class of entire functions of exponential type bounded on the real axis; by C , the class of Cartwright's entire functions, i.e., entire functions f of exponential type with the property

$$(1) \quad \int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1+x^2} dx < \infty;$$

and by D , the class of entire functions f of exponential type with the following property: for each sequence $h_m \in \mathbb{R}$ there exists a subsequence h'_m and a function $\tilde{f} \not\equiv 0$ such that the sequence $f(z + h'_m)$ converges uniformly to a function $\tilde{f}$ on every compact subset of $\mathbb{C}$.

It is easily seen that $D \subset B \subset C$. The classes B and C are well known and play an important role in various topics of complex and functional analysis (see [1, 2, 3]); the class D is an extension of the well-known class of sine-type functions (see [2]), and it inherits some properties of sine-type functions.

Here we consider sequences $\{a_k\}$ of complex numbers without finite limit points; note that each $\{a_k\}$ may appear in the sequence with a finite multiplicity. We say that a sequence $\{a_k\}$ is the *zero set for an entire function f* if $f(a) \neq 0$ for $a \notin \{a_k\}$ and $f(a) = f'(a) = \dots = f^{(p-1)}(a) = 0$, $f^{(p)}(a) \neq 0$, where p is the multiplicity of a in $\{a_k\}$.

The zero set of any function $f \in C$ satisfies the following conditions:

$$(2) \quad \sum_{a_k \neq 0} |\operatorname{Im} a_k^{-1}| < \infty;$$

$$(3) \quad \text{the limit } \lim_{R \rightarrow \infty} \sum_{0 < |a_k| < R} a_k^{-1} \text{ exists;}$$

and

$$(4) \quad \lim_{R \rightarrow \infty} R^{-1} \operatorname{card}\{a_k : 0 < |a_k| < R, |\arg a_k| \leq \alpha\} = d/2\pi,$$

$$(5) \quad \lim_{R \rightarrow \infty} R^{-1} \operatorname{card}\{a_k : 0 < |a_k| < R, |\arg a_k - \pi| \leq \alpha\} = d/2\pi,$$

for every $0 < \alpha \leq \pi/2$, where d is the width of the indicator diagram of f (see [2, p. 127]). Thus, every function $f \in C$ has the form

$$(6) \quad f(z) = Az^s e^{i\lambda z} \lim_{R \rightarrow \infty} \prod_{0 < |a_k| < R} (1 - z/a_k)$$

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with $A \in \mathbb{C}$, $\lambda \in \mathbb{R}$ (see [2, p. 130]). However, there are zero sets satisfying (2)–(5) and such that no entire function with this zero set belongs to C .¹

In [1, Appendix VI], B. Levin investigated zero sets $\{a_k\}$, $k \in \mathbb{Z}$, for functions of class B with the additional property $\sup_k |a_k - hk| < \infty$ for some $h \in \mathbb{R}$. In the present paper we obtain a complete description of the zero sets for the classes B , C , and D .

In what follows, we put $n(c, t) = \text{card}\{a_k : |a_k - c| \leq t\}$ for $c \in \mathbb{C}$. Assume also that the sequence $\{a_k\}$ satisfies the conditions

$$(7) \quad n(0, t) = O(t), \quad t \rightarrow \infty,$$

and

$$(8) \quad n(0, t+1) - n(0, t) = o(t), \quad t \rightarrow \infty.$$

It is clear that (7) and (8) are preserved if we replace $n(0, t)$ with $n(c, t)$.

Theorem 1. *A sequence $\{a_k\} \subset \mathbb{C}$ is the zero set of some function $f \in C$ if and only if it satisfies conditions (3), (7), and (8) and*

$$(9) \quad \int_{-\infty}^{\infty} \left[\int_0^{\infty} [n(b, t) - n(x, t)] t^{-1} dt \right]^+ \frac{dx}{1+x^2} < \infty$$

for some point $b \in \mathbb{R} \setminus \{a_k\}$.

Theorem 2. *A sequence $\{a_k\} \subset \mathbb{C}$ is the zero set of some function $f \in B$ if and only if it satisfies conditions (3), (7), and (8), and*

$$(10) \quad \sup_{x \in \mathbb{R}} \int_0^{\infty} [n(b, t) - n(x, t)] t^{-1} dt < \infty$$

for some point $b \in \mathbb{R} \setminus \{a_k\}$.²

Theorem 3. *A sequence $\{a_k\} \subset \mathbb{C}$ is the zero set of some function $f \in D$ if and only if it satisfies conditions (3), (7), and (8), and*

$$(11) \quad \sup_{x \in \mathbb{R}} \left| \int_1^{\infty} [n(0, t) - n(x, t)] t^{-1} dt \right| < \infty.$$

The proofs of Theorems 1–3 are based on the following lemma.

Lemma 1. *Suppose a sequence $\{a_k\} \subset \mathbb{C} \setminus \{0\}$ satisfies (3), (7), and (8). Then*

$$(12) \quad g(z) = \lim_{R \rightarrow \infty} \prod_{|a_k| < R} (1 - z/a_k)$$

is a well-defined entire function of a finite exponential type, and

$$(13) \quad \log |g(z)| = \int_0^{\infty} [n(0, t) - n(z, t)] t^{-1} dt$$

for all $z \in \mathbb{C}$. Moreover, if z_0 is a point of the sequence $\{a_k\}$ with multiplicity $l = n(z_0, 0) > 0$, then

$$(14) \quad \log \left| \frac{g^{(l)}(z_0)}{l!} \right| = \int_1^{\infty} \frac{n(0, t) - n(z_0, t)}{t} dt + \int_0^1 \frac{n(0, t) - n(z_0, t) + l}{t} dt.$$

¹For example, take $a_1 = -e^2$ and consider reals $a_k < a_1$, $k = 2, \dots$, such that the number $n(r) = \text{card}\{a_k : a_k \geq -r\}$ is the integral part of $r/\log^2 r$ for each $r \geq e^2$. Then (2)–(5) are fulfilled and the function (6) with $s = 0$ does not satisfy (1) because for $x > e^2$ we have

$$\log |f(x)| = \int_1^{\infty} \log(1 + x/t) dn(t) = \int_1^{\infty} \frac{xn(t)dt}{t(x+t)} > 1 + \int_x^{\infty} \frac{xdt}{2t \log^2 t} = 1 + \frac{x}{2 \log x}$$

(this idea belongs to Professor A. Grishin).

²Theorems 1 and 2 were announced (with my permission) in the survey [4, p. 45].

Proof of Lemma 1. Relation (7) implies that the integral $\int_0^\infty n(0, t)t^{-3}dt$ is finite; hence,

$$(15) \quad \sum |a_k|^{-2} = \int_0^\infty t^{-2} dn(0, t) < \infty.$$

Let K be a fixed disk in the complex plane. Then

$$\sum_{R < |a_k| < R'} \log \left(1 - \frac{z}{a_k}\right) - \sum_{R < |a_k| < R'} \frac{z}{a_k} \rightarrow 0$$

as $R, R' \rightarrow \infty$ uniformly in $z \in K$. Hence, uniformly in $z \in K$ we have

$$(16) \quad \lim_{R \rightarrow \infty} \sum_{|a_k| > R} \log \left(1 - \frac{z}{a_k}\right) = 0,$$

and the function g is well defined. Now, conditions (3) and (7) show that the function (12) is of a finite exponential type.

Note that (13) is trivial for $z = a_k$. Fix $z \notin \{a_k\}$. Suppose $\varepsilon > 0$ is such that $n(0, \varepsilon) = n(z, \varepsilon) = 0$. Then for any $R < \infty$ we have

$$(17) \quad \begin{aligned} & \sum_{|a_k| \leq R} (\log |z - a_k| - \log |a_k|) \\ &= \sum_{\varepsilon \leq |a_k - z| \leq R} \log |z - a_k| - \sum_{\varepsilon \leq |a_k| \leq R} \log |a_k| \\ &+ \sum_{|a_k| \leq R, |z - a_k| > R} \log |z - a_k| - \sum_{|z - a_k| \leq R, |a_k| > R} \log |z - a_k|. \end{aligned}$$

Furthermore,

$$\sum_{\varepsilon \leq |a_k - z| \leq R} \log |z - a_k| = n(z, R) \log R - \int_\varepsilon^R n(z, t)t^{-1}dt$$

and

$$(18) \quad \begin{aligned} & (\log R)[n(z, R) - n(0, R)] + \sum_{|a_k| \leq R, |z - a_k| > R} \log |z - a_k| \\ & - \sum_{|z - a_k| \leq R, |a_k| > R} \log |z - a_k| \\ &= \sum_{|a_k| \leq R, |z - a_k| > R} \log \frac{|z - a_k|}{R} + \sum_{|z - a_k| \leq R, |a_k| > R} \log \frac{R}{|z - a_k|}. \end{aligned}$$

Observe that if $|z - a_k| > R$ and $|a_k| \leq R$, then $|a_k| > R - |z|$ and $|z - a_k| \leq |z| + R$. Also, if $|a_k| > R$ and $|z - a_k| \leq R$, then $|a_k| \leq R + |z|$ and $|z - a_k| > R - |z|$. Hence, as $R \rightarrow \infty$, the right-hand side of (18) does not exceed the quantity

$$(19) \quad O(|z|/R)[n(0, R) - n(0, R - |z|)] + O(|z|/R)[n(0, R + |z|) - n(0, R)].$$

Recalling (8), we see that (19) tends to zero for fixed z as $R \rightarrow \infty$. Therefore, (13) follows from (17) and (16). In order to get (14), it suffices to apply (13) to the function $g(z)(z - z_0)^{-l}z_0^l$. Lemma 1 is proved. $\square$

Proof of Theorems 1-3. Relation (15) shows that the limits as $R \rightarrow \infty$ of the sums $\sum_{|a_k - c| < R} (a_k - c)^{-1}$ and $\sum_{|a_k - c| < R} (a_k)^{-1}$ exist simultaneously; the latter sum differs from $\sum_{|a_k| < R} (a_k)^{-1}$ in no more than $n(0, R + |c|) - n(0, R - |c|) = o(R)$ terms, and the modulus of each of these terms does not exceed $(R - |c|)^{-1}$. Therefore, (3) is invariant with respect to the shift of the origin. Applying Lemma 1 to the function

$\lim_{R \rightarrow \infty} \prod_{|a_k - c| < R} (1 - z/(a_k - c))$, we see that $[\int_0^\infty [n(c, t) - n(x, t)] t^{-1} dt]^+ = O(|x|)$ as $|x| \rightarrow \infty$. Therefore, (9) is also invariant with respect to the shift of the real parameter b . Hence, there is no loss of generality in assuming that $0 \notin \{a_k\}$ and $b = 0$ in (9) and (10).

Lemma 1 immediately shows that, under the hypotheses of Theorems 1 or 2, the function (12) belongs to C or to B , respectively. Let $\{a_k\}$ satisfy the hypothesis of Theorem 3. Using (13), (14), and the Jensen formula

$$\frac{1}{2\pi} \int_0^{2\pi} \log |g(z + e^{i\theta})| d\theta = \log \left| \frac{g^{(l)}(z)}{l!} \right| + \int_0^1 \frac{n(z, t) - n(z, 0)}{t} dt$$

with $l = n(z, 0)$, we get

$$(20) \quad \frac{1}{2\pi} \int_0^{2\pi} \log |g(z + e^{i\theta})| d\theta = \int_1^\infty \frac{n(0, t) - n(z, t)}{t} dt + \int_0^1 \frac{n(0, t)}{t} dt.$$

Hence, (11) and the inequality

$$\log |g(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |g(z + e^{i\theta})| d\theta$$

imply that g is bounded on $\mathbb{R}$. The Phragmén–Lindelöf theorem yields the boundedness of $g(z)$ on every horizontal strip of finite width. Therefore, for each sequence $t_m \in \mathbb{R}$ there exists a subsequence t'_m such that the $f(z + t'_m)$ converge uniformly to a function $h(z)$ on every compact subset of $\mathbb{C}$. If $h(z) \equiv 0$ on $\mathbb{C}$, then $\int_0^{2\pi} \log |g(t'_m + e^{i\theta})| d\theta \rightarrow -\infty$. By (20), this contradicts (11); hence $g \in D$.

To prove the “only if” part, note that, by (6), every $f \in C$ with $f(0) \neq 0$ has the form

$$f(z) = A e^{i\lambda z} g(z)$$

with g as in (12). Conditions (7) and (8) follow immediately from (4) and (5). Using Lemma 1, we see that the zero set of f satisfies condition (9) and, when $f \in B$, condition (10). If $f \in D$, then f and g are bounded on $\mathbb{R}$ and, by the Phragmén–Lindelöf theorem, on every horizontal strip of finite width. Hence, if (11) is false, then, by (20),

$$\frac{1}{2\pi} \int_0^{2\pi} \log |g(t_l + e^{i\theta})| d\theta \rightarrow -\infty$$

for some sequence t_j of reals. By the properties of averages of subharmonic functions, we have

$$\int_{|u+iv-t_j|<1} \log |g(u+iv)| du dv \rightarrow -\infty.$$

Since $\log |g|$ is bounded from above on the strip $|\operatorname{Im} z| < 2$, we see that

$$\int_{|u+iv-t_j-s|<2} \log |g(u+iv)| du dv \rightarrow -\infty$$

uniformly in $s \in [-1, 1]$. Using the properties of averages of subharmonic functions once again, we obtain

$$(21) \quad \sup_{-1 \leq s \leq 1} \log |g(t_j + s)| \rightarrow -\infty.$$

On the other hand, $f(z + t'_j) \rightarrow h(z) \not\equiv 0$ for some subsequence t'_j uniformly on the compact subsets of $\mathbb{C}$, and so (21) is impossible. Theorems 1–3 are proved completely. $\square$

Remark 1. It can be shown that all these theorems are valid with the function $\tilde{n}(c, t) = \operatorname{card}\{a_k : |\operatorname{Re}(a_k - c)| \leq t, |\operatorname{Im}(a_k - c)| \leq t\}$ instead of $n(c, t)$ as well.

Remark 2. The additional condition

$$\limsup_{y \rightarrow \pm\infty} \int_0^\infty [n(b, t) - n(iy, t)](|y|t)^{-1} dt \leq \sigma$$

in Theorems 1–3 gives a complete description of the zero sets for the corresponding subclasses of functions of exponential type at most σ .

Example. Let $\alpha(t)$ be a strictly monotone increasing concave function on $\mathbb{R}^+$ with the properties $\alpha(0) = 0$, $1 \leq \alpha'(t) \leq 1 + O(t^{-1})$ as $t \rightarrow \infty$, and $\alpha(t) \geq t + 1 + \varepsilon$ for large t with some $\varepsilon > 0$. Consider a sequence of reals $\{a_k\}$, $k \in \mathbb{Z} \setminus \{0\}$, such that $\alpha(a_k) = k$ and $a_{-k} = -a_k$, $k \in \mathbb{N}$. Note that

$$n(0, t) - n(x, t) = 2E[\alpha(t)] + E[\alpha(x - t)] - E[\alpha(x + t)]$$

for $x > t$, and

$$n(0, t) - n(x, t) = 2E[\alpha(t)] - E[\alpha(x - t)] - E[\alpha(x + t)]$$

for $0 < x < t$; here $E[x]$ is the integral part of x . Since $a_{n+1} - a_n = (\alpha'(\tilde{t}))^{-1}$ with some $\tilde{t} \in (a_n, a_{n+1})$, we see that

$$0 \leq 1 - (a_{n+1} - a_n) \leq (a_{n+1} - a_n)^{-1} - 1 \leq \alpha'(a_n) - 1 < C/a_n < 1/2$$

for $n \geq n_0$, and

$$\left| \int_{a_n}^{a_{n+1}} (\alpha(t) - E[\alpha(t)] - 1/2) dt \right| \leq \left| \int_{a_n}^{a_{n+1}} \left(\int_{a_n}^t \alpha'(u) du - 1/2 \right) dt \right| \leq C'/a_n.$$

Therefore for any $N < \infty$ we have

$$\left| \int_{a_{n_0}}^{a_N} (\alpha(t) - E[\alpha(t)] - 1/2) dt \right| \leq C' \log a_N.$$

Now it easily follows that all the integrals

$$\begin{aligned} \int_1^x \frac{\alpha(t) - E[\alpha(t)] - 1/2}{t} dt, \quad \int_1^x \frac{\alpha(x \pm t) - E[\alpha(x \pm t)] - 1/2}{t} dt, \\ \int_x^\infty \frac{\alpha(t \pm x) - E[\alpha(t \pm x)] - 1/2}{t} dt, \end{aligned}$$

are bounded uniformly in $x \rightarrow \infty$. Hence, up to a uniformly bounded term, the integral $\int_0^\infty [n(0, t) - n(x, t)]t^{-1} dt$ equals

$$(22) \quad \begin{aligned} \int_1^x \frac{2E[\alpha(t)] - 2t}{t} dt + \int_1^{x-1} \frac{\alpha(x-t) - \alpha(x+t) + 2t}{t} dt \\ + \int_x^\infty \frac{2\alpha(t) - \alpha(t-x) - \alpha(x+t)}{t} dt. \end{aligned}$$

Since $\alpha(t)$ is concave, the third integral in (22) is positive. Furthermore, for $1 < t < x-1$ we have

$$\alpha(x-t) - \alpha(x+t) + 2t = \int_{-t}^t 1 - \alpha'(x+u) du \geq -C \log \frac{x+t}{x-t}.$$

Now it is easy to check that the second integral in (22) is uniformly bounded from below. Obviously, the first integral in (22) is unbounded from above as $x \rightarrow \infty$. Thus, the sequence $\{a_k\}$ is not the zero set of any function $f \in B$.

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DEPARTMENT OF MATHEMATICS, KHARKOV NATIONAL UNIVERSITY, SVOBODY SQUARE 4, KHARKOV 61077, UKRAINE

E-mail address: favorov_s@mail.ru, fav@univer.kharkov.ua

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