SUBSEQUENCES OF ZEROS FOR CLASSES OF HOLOMORPHIC FUNCTIONS, THEIR STABILITY, AND THE ENTROPY OF ARCWISE CONNECTEDNESS. II

B. N. KHABIBULLIN, F. B. KHABIBULLIN, AND L. YU. CHEREDNIKOVA

ABSTRACT. Let Ω be a domain in the complex plane $\mathbb{C}$, $H(\Omega)$ the space of functions holomorphic in Ω , and $\mathcal{P}$ a family of functions subharmonic in Ω . Denote by $H_{\mathcal{P}}(\Omega)$ the class of functions $f \in H(\Omega)$ satisfying $|f(z)| \leq C_f \exp p_f(z)$ for all $z \in \Omega$, where $p_f \in \mathcal{P}$ and C_f is a constant. Conditions are found ensuring that a sequence $\Lambda = \{\lambda_k\} \subset \Omega$ be a subsequence of zeros for various classes $H_{\mathcal{P}}(\Omega)$. As a rule, the results and the method are new already when $\Omega = \mathbb{D}$ is the unit circle and $\mathcal{P}$ is a system of radial majorants $p(z) = p(|z|)$.

We continue the enumeration of Part I.

CHAPTER III. ON THE ENTROPY OF ARCWISE CONNECTEDNESS AGAIN

We recall some definitions and notation of Part I (see [1]), retaining the enumeration adopted there.

Let D be a domain in $\mathbb{C}$. For a point $z \in D$ (a subset $S \subset D$), we denote by $d_D(z) := \text{dist}(z, \partial D)$ (respectively, $d_D(S) := \text{dist}(S, \partial D)$) the distance from z (or S) to the boundary ∂D of D .

Definition III.1. *The entropy of arcwise connectedness of a nonempty subset of a bounded domain $D \subset \mathbb{C}$ is defined to be the quantity*

$$(III.1) \quad \ell(S; D) \stackrel{\text{def}}{=} \sup_{z, w \in S} \inf_{l(z, w) \subset D} \frac{|l(z, w)|}{\text{dist}(l(z, w), \partial D)}$$

$$(III.1d) \quad = \sup_{z, w \in S} \inf_{l(z, w) \subset D} \frac{|l(z, w)|}{d_D(l(z, w))},$$

where the inner infimum is taken over all rectifiable arcs $l(z, w) \subset D$ that connect z with w , and $l(z, w) \subset D$ is the Euclidean length of such an arc.

§9. ENTROPY OF ARCWISE CONNECTEDNESS AND FAMILIES OF SETS

The main result of this section is Proposition 9.2. It says that a family of subsets of a bounded domain Ω with uniformly bounded entropy of arcwise connectedness can be inscribed in an admissible (see Definition 5.1) family of domains with special properties. A key role in the proof will be played by the following statement.

2000 *Mathematics Subject Classification.* Primary 30C15.

Key words and phrases. Holomorphic function, algebra of functions, weighted space, nonuniqueness sequence.

Supported by RFBR, grant no. 06-01-00067, and by the Program of state subventions for leading scientific schools, grant NSh-10052.2006.1.

Proposition 9.1. *Suppose S is a subset of a bounded domain $\Omega \subset \mathbb{C}$ and*

$$(9.1) \quad \ell(S; \Omega) < l < +\infty.$$

Then

$$(S1) \quad b_\Omega(S) \stackrel{(2.1b)}{:=} \sup\{d_\Omega(z) : z \in S\} \leq (\ell(S; \Omega) + 1)d_\Omega(S);$$

(S2) *for every $z, w \in S$ there exists a rectifiable arc $\tilde{l}(z, w) \subset \Omega$ joining z and w and such that*

$$(9.2d) \quad \frac{1}{2(l+1)} d_\Omega(S) \leq d_\Omega(\tilde{l}(z, w)) \leq (l+1)d_\Omega(S),$$

$$(9.2l) \quad |\tilde{l}(z, w)| \leq (l+1)^2 d_\Omega(S);$$

(S3) *there exists a squarable domain $D \Subset \Omega$ that includes S and satisfies*

$$(9.3S) \quad \ell(S, D) \leq 6(l+1)^3,$$

$$(9.3D) \quad \ell(D, \Omega) \leq 10(l+1)^3.$$

Proof. Under condition (9.1), Proposition 2.3(14) implies $S \Subset \Omega$. Thus, by Proposition 2.3(11), there is no loss of generality in assuming that S is a compact subset of Ω .

Proof of property (S1). Since S is compact, there exist $z_b, z_d \in S$ such that $b_\Omega(S) = d_\Omega(z_b)$ and $d_\Omega(S) = d_\Omega(z_d)$. By (9.1), there is a rectifiable arc $l(z_b, z_d)$ joining z_b and z_d and such that

$$ld_\Omega(z_d) \stackrel{(d2)}{\geq} ld_\Omega(l(z_b, z_d)) \stackrel{(9.1)}{\geq} |l(z_b, z_d)| \geq |z_b - z_d| \stackrel{(d1)}{\geq} d_\Omega(z_b) - d_\Omega(z_d),$$

which implies (S1). ■

Proof of property (S2). By Proposition 2.1(d2) and property (S1), for every arc $\tilde{l}(z, w)$ joining two points $z, w \in S$ we have

$$(9.4) \quad d_\Omega(\tilde{l}(z, w)) \stackrel{(d2)}{\leq} d_\Omega(z) \stackrel{(2.1b)}{\leq} b_\Omega(S) \stackrel{(S1)}{\leq} (l+1)d_\Omega(S).$$

If $|z - w| \leq d_\Omega(S)/2$, we take the segment $[z, w]$ for the role of $\tilde{l}(z, w)$. Then Proposition 2.1(d3) with $d = d_\Omega(S)$ shows that

$$(9.5) \quad d_\Omega(\tilde{l}(z, w)) = d_\Omega([z, w]) \geq \frac{d_\Omega(S)}{2}$$

and $\tilde{l}(z, w) \subset \Omega$. Thus, the choice $\tilde{l}(z, w) = [z, w]$ is indeed possible in this case, and, along with (9.5) and (9.4), we obtain

$$(9.6) \quad |\tilde{l}(z, w)| = |z - w| \leq \frac{d_\Omega(S)}{2}.$$

Suppose now that $|z - w| > d_\Omega(S)/2$ (and $z, w \in S$). By (9.1) and the definition of the entropy of arcwise connectedness, there is an arc $\tilde{l}(z, w) \subset \Omega$ such that, along with (9.4), it satisfies the following inequalities:

$$\frac{d_\Omega(S)}{2} < |z - w| \leq |\tilde{l}(z, w)| \stackrel{(III.1), (9.1)}{\leq} ld_\Omega(\tilde{l}(z, w)) \stackrel{(9.4)}{\leq} l(l+1)d_\Omega(S).$$

Combining this with (9.4), (9.5), and (9.6), we obtain

$$(9.7d) \quad \min\{1/2, 1/(2l)\} d_\Omega(S) \leq d_\Omega(\tilde{l}(z, w)) \leq (l+1)d_\Omega(S),$$

$$(9.7l) \quad |\tilde{l}(z, w)| \leq \max\{1/2, l(l+1)\} d_\Omega(S).$$

Clearly, these inequalities imply (9.2d) and (9.2l) (respectively). ■

Proof of property (S3). By (S2), every two points $z, w \in S$ can be joined by a rectifiable arc $\tilde{l}(z, w) \subset \Omega$ that satisfies (9.2). Consider the connected set

$$(9.8) \quad \tilde{S} \stackrel{\text{def}}{=} \bigcup \{ \tilde{l}(z, w) : z, w \in S \} \supset S$$

and put

$$(9.9) \quad \varepsilon = \frac{1}{3} \frac{1}{2(l+1)} d_\Omega(S) > 0.$$

By (9.2d) and the definition (9.8) of $\tilde{S}$, we obtain

$$(9.10) \quad d_\Omega(\tilde{S}) = \inf \{ d_\Omega(\tilde{l}(z, w)) : z, w \in S \} \geq \frac{1}{2(l+1)} d_\Omega(S) = 3\varepsilon.$$

Consider two domains:

$$\tilde{S}^\varepsilon = \bigcup \{ D(z, \varepsilon) : z \in \tilde{S} \}, \quad \tilde{S}^{2\varepsilon} = \bigcup \{ D(z, 2\varepsilon) : z \in \tilde{S} \}$$

(these are the ε - and the 2ε -blowup of $\tilde{S}$). It is easy to see that there exists an “intermediate” squarable domain D (it can even be chosen so that its boundary be a polygonal line) with

$$(9.11) \quad S \subset \tilde{S} \subset \tilde{S}^\varepsilon \subset D \subset \tilde{S}^{2\varepsilon}.$$

By (d4) in Proposition 2.1, using also (9.10), we obtain

$$(9.12) \quad d_\Omega(\tilde{S}^{2\varepsilon}) \stackrel{(d4)}{\geq} d_\Omega(\tilde{S}) - 2\varepsilon \stackrel{(9.10)}{\geq} \varepsilon,$$

so that the domains $\tilde{S}^\varepsilon \subset \tilde{S}^{2\varepsilon}$ are included in Ω . By (d2) and (d5) in Proposition 2.1, we have

$$(9.13) \quad d_D(\tilde{S}) \stackrel{(d2)}{\geq} d_{\tilde{S}^\varepsilon}(\tilde{S}) \stackrel{(d5)}{\geq} \varepsilon.$$

Together with the definition (9.8) of $\tilde{S}$, (9.13) implies

$$\begin{aligned} \ell(S; D) &\leq \sup_{z, w \in S} \frac{|\tilde{l}(z, w)|}{d_D(\tilde{l}(z, w))} \stackrel{(9.8)}{\leq} \sup_{\tilde{l}(z, w) \subset \tilde{S}} \frac{|\tilde{l}(z, w)|}{d_D(\tilde{S})} \\ &\stackrel{(9.21)}{\leq} \frac{(l+1)^2 d_\Omega(S)}{d_D(\tilde{S})} \stackrel{(9.13)}{\leq} \frac{(l+1)^2 d_\Omega(S)}{\varepsilon} \stackrel{(9.9)}{=} 6(l+1)^3, \end{aligned}$$

i.e., (9.3S) is fulfilled.

Concerning the domain D , we observe that arbitrary two points $z, w \in D$ can be joined by a rectifiable arc $l(z, w)$ which is the union of two segments $[z, z'], [w', w] \subset \tilde{S}^{2\varepsilon}$ (here $z', w' \in \tilde{S}$) and an arc $\tilde{l}(z', w') \subset \tilde{S} \subset \tilde{S}^{2\varepsilon}$ that satisfies (9.21). This yields the following estimate:

$$\begin{aligned} \ell(D; \Omega) &\stackrel{(III.1)}{\leq} \sup_{z, w \in S} \frac{|[z, z']| + |\tilde{l}(z, w)| + |[w', w]|}{\min \{ d_\Omega([z, z']), d_\Omega(\tilde{l}(z, w)), d_\Omega([w', w]) \}} \\ &\stackrel{(9.21)}{\leq} \frac{2\varepsilon + (l+1)^2 d_\Omega(S) + 2\varepsilon}{d_\Omega(\tilde{S}^{2\varepsilon})} \stackrel{(9.12)}{\leq} \frac{(l+1)^2 d_\Omega(S) + 4\varepsilon}{\varepsilon} \\ &\stackrel{(9.9)}{=} 6(l+1)^3 + 4 \leq 10(l+1)^3, \end{aligned}$$

i.e., (9.3D) is fulfilled and, in particular, $D \Subset \Omega$ by (I3). $\square$

Proposition 9.2. *Let $\Sigma = \{S_k\}$, $k = 1, 2, \dots$, be a locally finite family of precompact sets in a bounded domain Ω , and let*

$$(9.14) \quad \limsup_{k \rightarrow \infty} \ell(S_k; \Omega) < +\infty.$$

Then Σ can be combinatorially inscribed in a family of domains $\mathcal{D} = \{D_k\}$ admissible for Ω (see Definition 5.1) and such that

$$(9.15S) \quad \limsup_{k \rightarrow \infty} \ell(S_k; D_k) < +\infty, \quad S_k \Subset D_k, \quad k = 1, 2, \dots,$$

$$(9.15D) \quad \limsup_{k \rightarrow \infty} \ell(D_k; \Omega) < +\infty, \quad D_k \Subset \Omega, \quad k = 1, 2, \dots$$

Proof. By (9.14), there exists l such that

$$(9.16) \quad \ell(S_k; \Omega) < l$$

for all sufficiently large k . But since all S_k are precompact in Ω , statement (13) in Proposition 2.3 allows us, increasing l if necessary, to ensure (9.16) for all $k = 1, 2, \dots$. By (9.14) and (S3) in Proposition 9.1, for every $k \in \mathbb{N}$ there exists a *squarable* subdomain $D_k \Subset \Omega$ that includes S_k and satisfies an estimate of the form (9.3) uniformly in k , specifically,

$$(9.17) \quad \ell(S_k, D_k) \leq 6(l+1)^3, \quad \ell(D_k, \Omega) \leq 10(l+1)^3, \quad k = 1, 2, \dots$$

This implies (9.15).

It remains to show that the family $\mathcal{D} = \{D_k\}$ constructed above is locally finite in Ω . In the notation (2.1b), the property of the family Σ to be locally finite in Ω is equivalent to the condition $\lim_{k \rightarrow \infty} b_\Omega(S_k) = 0$. *A fortiori*, we have $\lim_{k \rightarrow \infty} d_\Omega(D_k) = 0$, because by the definitions (2.1z) and (2.1b) and property (d2) in Proposition 2.1, the inclusions $S_k \subset D_k$ obviously imply the inequalities $d_\Omega(D_k) \leq d_\Omega(S_k) \leq b_\Omega(S_k)$. Now, by (S1) in Proposition 9.1, the second relation in (9.17) yields

$$\limsup_{k \rightarrow \infty} b_\Omega(D_k) \leq (10(l+1)^3 + 1) \limsup_{k \rightarrow \infty} d_\Omega(D_k) = 0,$$

as required. $\square$

§10. ENTROPY OF ARCWISE CONNECTEDNESS, BLOWUPS, AND STARS OF SUBSETS

For $\delta > 0$, on a domain $\Omega \Subset \mathbb{C}$, we introduce the following functions:

$$(10.1) \quad r_\delta(z) \stackrel{\text{def}}{=} \delta d_\Omega(z) \stackrel{(2.1z)}{=} \delta \text{dist}(z, \partial\Omega), \quad 0 < \delta \leq 1, \quad z \in \Omega.$$

Clearly, if $\delta < 1$, they satisfy (7.3). We also introduce special notation for the r_δ -blowup (defined by (5.6)) of a subset D of Ω :

$$(10.2r) \quad D^{(\delta)} \stackrel{\text{def}}{=} D^{r_\delta} \stackrel{(2.1z)}{=} \bigcup \{D(z, r_\delta(z)) : z \in D\}$$

$$(10.2d) \quad \stackrel{(10.1)}{=} \{w \in \Omega : \exists z \in D, |w - z| < \delta \text{dist}(z, \partial\Omega)\} \subset \Omega, \quad 0 < \delta \leq 1.$$

To distinguish this from the notions of blowup used before, the set $D^{(\delta)}$ in (10.2) will be called the *relative* (δ) -blow up of D in Ω .

Proposition 10.1. *The domain $D \Subset \Omega \Subset \mathbb{C}$ have the following properties:*

($\delta 1$) *if $z, w \in \Omega$ satisfy $|z - w| < r_\delta(z)$, then*

$$(10.3) \quad d_\Omega(w) < (1 + \delta)d_\Omega(z), \quad d_\Omega(z) < \frac{1}{1 - \delta} d_\Omega(w), \quad |z - w| < \frac{1}{1 - \delta} r_\delta(w);$$

($\delta 2$) *if $0 < \delta < 1$, then $D^{(\delta)} \Subset \Omega$ and $d_\Omega(D^{(\delta)}) \geq (1 - \delta)d_\Omega(D)$;*

- (δ3) if $0 < \delta < 1/2$, $z \in \Omega$, and $D(z, r_\delta(z)) \cap D \neq \emptyset$, then $z \in D^{(2\delta)} \Subset \Omega$;
 (δ4) if $q > 0$ and $\delta > 0$ satisfy $q + 2\delta < 1$, then $(S^{(q)})^{(\delta)} \subset S^{(q+2\delta)} \Subset \Omega$ for every $S \Subset \Omega$.

Proof. If $|z - w| < r_\delta(z) \stackrel{(10.1)}{=} \delta d_\Omega(z)$, then, by Proposition 2.1(d1), we obtain $d_\Omega(w) \leq d_\Omega(z) + |z - w| < d_\Omega(z) + \delta d_\Omega(z)$, which proves the first inequality in (10.3). Next, $d_\Omega(z) \leq d_\Omega(w) + |z - w| < d_\Omega(w) + \delta d_\Omega(z)$, whence $d_\Omega(z) < \frac{1}{1-\delta} d_\Omega(w)$, and the second inequality in (10.3) is proved. Multiplying this by δ and combining it with the initial assumption $|z - w| < \delta d_\Omega(z)$, we obtain the last inequality in (δ1).

Let $\delta < 1$, and let w be an arbitrary point in $D^{(\delta)}$. By (10.2r), there exists $z \in D$ with $|z - w| < r_\delta(z)$, and then (δ1) shows that $d_\Omega(w) \geq (1 - \delta)d_\Omega(z) \stackrel{(2.1z)}{\geq} (1 - \delta)d_\Omega(D)$. This implies the lower estimate $d_\Omega(D^{(\delta)}) \stackrel{(2.1z)}{\geq} (1 - \delta)d_\Omega(D) > 0$ (as required in (δ2)) and the inclusion $D^{(\delta)} \Subset \Omega$.

Suppose $w \in D(z, r_\delta(z)) \cap D$. The condition $\delta < 1/2$ implies $\delta/(1 - \delta) < 2\delta < 1$. Then $|z - w| < r_\delta(z)$ and $|z - w| \stackrel{(\delta 1)}{<} \frac{1}{1-\delta} r_\delta(w) = \frac{\delta}{1-\delta} d_\Omega(w) \leq 2\delta d_\Omega(w)$. By (10.2r) and (δ2), this shows that $z \in D^{(2\delta)} \Subset \Omega$, and (δ3) is proved.

If w is an arbitrary point of $(S^{(q)})^{(\delta)}$, then by (10.2d) it is possible to find $w' \in S^{(q)}$ and then $z \in S$ with $|w - w'| < \delta d_\Omega(w')$ and $|z - w'| < q d_\Omega(z)$. The latter inequality, combined with the first inequality in (10.3), implies $d_\Omega(w') < (1 + q)d_\Omega(z)$, whence we deduce that $|z - w| \leq |z - w'| + |w - w'| < q d_\Omega(z) + \delta(1 + q)d_\Omega(z) \leq (q + 2\delta)d_\Omega(z)$, because $1 + q < 2$. Since $w \in (S^{(q)})^{(\delta)}$ is arbitrary, this means that $(S^{(q)})^{(\delta)} \subset S^{(q+2\delta)}$, and (δ2) shows that $S^{(q+2\delta)} \Subset \Omega$ because $q + 2\delta < 1$. $\square$

Proposition 10.2. Suppose Ω is a bounded domain in $\mathbb{C}$ and $0 < \delta < 1/2$. Then

- (ℓ1) for every $D \subset \Omega$ with $\ell(D; \Omega) < l < +\infty$ we have $\ell(D^{(\delta)}; \Omega) \leq 4(l + 1)^3$.

Furthermore, let $\mathcal{D} = \{D\}$ be a family of subsets of Ω with

$$(10.4) \quad \sup_{D \in \mathcal{D}} \ell(D; \Omega) < l < +\infty.$$

Then:

- (ℓ2) for every star $\mathcal{D}_*[z]$ with kernel (see (5.5)) we have $\ell(\mathcal{D}_*[z]; \Omega) \leq 4(l + 1)^4$, $z \in \Omega$;
 (ℓ2) for $\delta < 1/6$ we have $\ell((\mathcal{D}_*[z, r_\delta(z)])^{(\delta)}; \Omega) \leq 10^{11}(l + 1)^{36}$, $z \in \Omega$.

Proof. Under the assumptions of (ℓ1), let $z, w \in D^{(\delta)}$. By (10.2d), there exist $z', w' \in D$ with $|z - z'| < r_\delta(z')$ and $|w - w'| < r_\delta(w')$. Now, Proposition 9.1(S1) (with D substituted for S) shows that

$$(10.5) \quad |z - z'| + |w' - w| < \delta(d_\Omega(z') + d_\Omega(w')) \stackrel{(2.1b)}{\leq} 2\delta b_\Omega(D) \stackrel{(S1)}{\leq} 2\delta(l + 1)d_\Omega(D).$$

By Proposition 10.1(δ2), for the pair of segments $[z, z'], [w, w'] \subset D^{(\delta)}$ we obtain

$$(10.6) \quad \min\{d_\Omega([z, z']), d_\Omega([w, w'])\} \geq d_\Omega(D^{(\delta)}) \stackrel{(\delta 2)}{\geq} (1 - \delta)d_\Omega(D).$$

Now, Proposition 9.1(S2) (with $D \Subset \Omega$ substituted for S) implies the existence of a rectifiable arc $\tilde{l}(z', w') \subset \Omega$ that joins z' with w' and satisfies

$$(10.7) \quad d_\Omega(\tilde{l}(z', w')) \stackrel{(9.2d)}{\geq} \frac{1}{2(l + 1)} d_\Omega(D), \quad |\tilde{l}(z', w')| \stackrel{(9.2l)}{\leq} (l + 1)^2 d_\Omega(D).$$

For every $z, w \in D^{(\delta)}$, we fix the arc $l(z, w) = [z, z'] \cup \tilde{l}(z', w') \cup [w', w]$, where $\tilde{l}(z', w')$ is as described above. The arc $l(z, w)$ lies in Ω and joins z and w . Using (10.5)–(10.7) and

the restriction $\delta < 1/2$, we arrive at the estimate claimed in $(\ell 1)$:

$$\begin{aligned}
\ell(D^{(\delta)}; \Omega) &\stackrel{(\text{III.1})}{\leq} \sup_{z, w \in D^{(\delta)}} \frac{|l(z, w)|}{d_\Omega(\tilde{l}(z, w))} \\
&\leq \sup_{z, w \in D} \frac{|z - z'| + |\tilde{l}(z, w)| + |w' - w|}{\min\{d_\Omega([z, z']), d_\Omega(\tilde{l}(z, w)), d_\Omega([w', w])\}} \\
&\stackrel{(10.5)-(10.7)}{\leq} \frac{2\delta(l+1)d_\Omega(D) + (l+1)^2d_\Omega(D)}{\min\{(1-\delta)d_\Omega(D), \frac{1}{2(l+1)}d_\Omega(D)\}} \\
&\leq \frac{(l+1)^2(2\delta+1)}{\min\{1/2, 1/(2(l+1))\}} \\
&\leq 4(l+1)^3.
\end{aligned}$$

We pass to property $(\ell 2)$. If z belongs to at most one set $D \in \mathcal{D}$, the definitions of $\mathcal{D}_*[z]$ (see (5.5)) and of the entropy of arcwise connectedness show that $\ell(\mathcal{D}_*[z]; \Omega) < l$. Otherwise, for every $w, w' \in \mathcal{D}_*[z]$ there exist two members D, D' of $\mathcal{D}$ (possibly coinciding as sets) with $w \in D, w' \in D'$, and $z \in D \cap D'$. Proposition 9.1(S2) (with $D, D' \Subset \Omega$ in place of S) implies the existence of rectifiable curves $\tilde{l}(z, w), \tilde{l}(z, w') \subset \Omega$ that join z , respectively, with w and w' , for which estimates (9.2) take the form

$$(10.8w) \quad \frac{1}{2(l+1)}d_\Omega(D) \stackrel{(9.2d)}{\leq} d_\Omega(\tilde{l}(z, w)), \quad |\tilde{l}(z, w)| \stackrel{(9.2l)}{\leq} (l+1)^2d_\Omega(D),$$

$$(10.8w') \quad \frac{1}{2(l+1)}d_\Omega(D') \stackrel{(9.2d)}{\leq} d_\Omega(\tilde{l}(z, w')), \quad |\tilde{l}(z, w')| \stackrel{(9.2l)}{\leq} (l+1)^2d_\Omega(D').$$

Now Proposition 9.1(S1) and the condition $z \in D \cap D'$ show that

$$\begin{aligned}
(10.9) \quad \min\{d_\Omega(D), d_\Omega(D')\} &\stackrel{(S1)}{\geq} \frac{1}{l+1} \min\{b_\Omega(D), b_\Omega(D')\} \\
&\stackrel{(2.1b)}{\geq} \frac{1}{l+1} d_\Omega(z) \stackrel{(2.1S)}{\geq} \frac{1}{l+1} \max\{d_\Omega(D), d_\Omega(D')\}.
\end{aligned}$$

By (10.8), it follows that the arc $l(w, w') = \tilde{l}(z, w) \cup \tilde{l}(z, w')$ (which joins w and w') satisfies

$$\begin{aligned}
\frac{|l(w, w')|}{d_\Omega(l(w, w'))} &\leq \frac{|\tilde{l}(z, w)| + |\tilde{l}(z, w')|}{\min\{d_\Omega(\tilde{l}(z, w)), d_\Omega(\tilde{l}(z, w'))\}} \\
&\stackrel{(10.8)}{\leq} \frac{(l+1)^2(d_\Omega(D) + d_\Omega(D'))}{\frac{1}{2(l+1)} \min\{d_\Omega(D), d_\Omega(D')\}} \leq 4(l+1)^3 \frac{\max\{d_\Omega(D), d_\Omega(D')\}}{\min\{d_\Omega(D), d_\Omega(D')\}} \\
&\stackrel{(10.9)}{\leq} 4(l+1)^4.
\end{aligned}$$

Since $w, w' \in \mathcal{D}_*[z]$ are arbitrary, this gives the estimate claimed in $(\ell 2)$, by the definition of the entropy of arcwise connectedness.

For $0 < \delta \leq 1$, the family $\mathcal{D} = \{D\}$ generates the family $\mathcal{D}^{(\delta)} = \{D^{(\delta)}\}$ of relative (δ) -blowups in Ω of the sets $D \in \mathcal{D}$.

To prove $(\ell 3)$, we show that

$$(10.10) \quad (\mathcal{D}_*[z, r_\delta(z)])^{(\delta)} \subset \left(\left(\mathcal{D}^{(2\delta)} \right)_* [z] \right)^{(3\delta)} \Subset \Omega.$$

For this, we first verify the inclusions

$$(10.11) \quad \mathcal{D}_*[z, r_\delta(z)] \subset \left(\mathcal{D}^{(2\delta)} \right)_* [z] \cup D(z, r_\delta(z)) \subset \left(\left(\mathcal{D}^{(2\delta)} \right)_* [z] \right)^{(\delta)}, \quad z \in \Omega.$$

If $D \in \mathcal{D}$ and $D \cap D(z, r_\delta(z)) \neq \emptyset$, then $z \in D^{(2\delta)}$ by Proposition 10.1 ($\delta 3$), whence $D \subset (\mathcal{D}^{(2\delta)})_*[z]$. This proves the first inclusion in (10.11). Since the rightmost term in (10.11) contains the disk $D(z, r_\delta(z))$, the second inclusion in (10.11) also follows. Applying the operation of relative (δ) -blowup to the leftmost and the rightmost term in (10.11), we arrive at (10.10).

Furthermore, by (10.4), applying ($\ell 1$) to every $D \in \mathcal{D}$, we obtain the estimate $\ell(D^{(2\delta)}; \Omega) < 4(l+1)^3$ (which is uniform in $D \in \mathcal{D}$). Property ($\ell 2$) (already proved) now shows that

$$\ell\left(\left(\mathcal{D}^{(2\delta)}\right)_*[z]; \Omega\right) \leq 4(4(l+1)^3 + 1)^4 \leq 2500(l+1)^{12}.$$

Using ($\ell 1$) again, this time with $3\delta < 1/2$ in place of δ and for subsets of the form $(\mathcal{D}^{(2\delta)})_*[z]$, we see that

$$\ell\left(\left(\left(\mathcal{D}^{(2\delta)}\right)_*[z]\right)^{(3\delta)}; \Omega\right) \leq 4(2500(l+1)^{12} + 1)^3 \leq 10^{11}(l+1)^{36},$$

which implies ($\ell 3$) by (10.10). $\square$

§11. RELATIONSHIP WITH THE DIAMETER AND SEGMENTAL HULL OF A SUBSET

For a set $S \subset \mathbb{C}$, we denote¹⁴ by $\boxtimes S \stackrel{\text{def}}{=} \bigcup\{[z, w] : z, w \in S\}$ its *segmental hull*. Each of the obvious inclusions $S \subset \boxtimes S \subset \text{conv } S$ may happen to be strict. However, the following elementary statement is true.

Proposition 11.1. $\text{diam } S = \text{diam } \overline{S} = \text{diam } \boxtimes S = \text{diam } \text{conv } S$ for $S \subset \mathbb{C}$.

Proof. The first two identities are easy consequences of the definitions. By the Carathéodory theorem [3, Theorem 2.4] the convex hull of S is the union of all (including degenerate) triangles with vertices in S . Consequently, $\text{diam } \text{conv } S$ is the least upper bound of the diameters of all triangles with vertices in S or, what is the same, the least upper bound of the maximum side lengths for these triangles. The last quantity coincides with the diameter of S . $\square$

By construction, the set $\boxtimes S$ is connected. If it is precompact in a bounded domain Ω , then for every $\delta \in (0, 1)$ its relative (δ) -blowup $(\boxtimes S)^{(\delta)}$ in Ω is a precompact subdomain of Ω . Any of the obvious inequalities $d_\Omega(S) \geq d_\Omega(\boxtimes S) \geq d_\Omega(\text{conv } S)$ (and even the two at once) may happen to be strict (depending on the geometry of Ω and S), but the following holds true.

Proposition 11.2. *If the domain Ω is simply connected, then $d_\Omega(\boxtimes S) = d_\Omega(\text{conv } S)$ provided $\boxtimes S \subset \Omega$. If Ω is convex, then $d_\Omega(S) = d_\Omega(\boxtimes S) = d_\Omega(\text{conv } S)$ for $S \subset \Omega$.*

Proof. By the Carathéodory theorem, the convex hull $\text{conv } S$ is obtained from $\boxtimes S$ by adjoining the interiors of all triangles formed by the segments in $\boxtimes S$ whose ends belong to S . If $\boxtimes S \subset \Omega$, then the interiors of such triangles must lie in Ω if Ω is simply connected. Therefore, $\text{conv } S \subset \Omega$ and $d_\Omega(\boxtimes S) = d_\Omega(\text{conv } S)$.

If Ω is convex, then it is simply connected and $\boxtimes S \subset \Omega$ for every $S \subset \Omega$, whence $d_\Omega(\boxtimes S) = d_\Omega(\text{conv } S)$. Moreover, if Ω is convex, then the function $d_\Omega : \Omega \rightarrow \mathbb{R}$ is continuous and concave (see Proposition 2.1(d6)). Therefore, by the minimum principle for concave functions, the greatest lower bound of d_Ω on a segment is attained at one of its endpoints. Thus, $d_\Omega(\boxtimes S) = \inf_{z \in \boxtimes S} d_\Omega(z) = \inf_{z \in S} d_\Omega(z) = d_\Omega(S)$, and the proposition is proved. $\square$

¹⁴This notation is borrowed from Azarin's paper [2, 3.1], where a similar notion arose for different reasons and was called the "frame" of a convex hull.

Proposition 11.3. *Let $S \neq \emptyset$ be a subset of a domain $\Omega \Subset \mathbb{C}$. Then:*

(ld1) *if $\boxtimes S \Subset \Omega$, then $\ell(S; \Omega) \leq \frac{\text{diam } S}{d_\Omega(\boxtimes S)}$;*

(ld2) *if Ω is convex, then $\ell(S; \Omega) \leq \frac{\text{diam } S}{d_\Omega(S)}$;*

(ld3) *if*

$$(11.1) \quad \frac{\text{diam } S}{d_\Omega(S)} \leq q < 2,$$

then $\boxtimes S \Subset \Omega$ and we have

$$(11.2) \quad d_\Omega(\boxtimes S) \geq \frac{\sqrt{4-q^2}}{2} d_\Omega(S), \quad \ell(S; \Omega) \leq \ell(\boxtimes S; \Omega) \leq \frac{2q}{\sqrt{4-q^2}};$$

(ld4) *if the restriction*

$$(11.3) \quad \frac{\text{diam } S}{d_\Omega(S)} \leq q < 1,$$

(which is stronger than (11.1)) holds, then, along with (11.2), we have the following: for every $\delta > 0$ with

$$(11.4) \quad q + 30\delta < 1,$$

there exists a squarable subdomain $D \Subset \Omega$ that satisfies the following three conditions:

(q1) *$\text{conv } S \Subset \Omega$ and $\boxtimes S \Subset D$;*

(q2) *for every $z \in \Omega$ with $D(z, \delta d_\Omega(z)) \cap D \neq \emptyset$, the union $(D(z, r_\delta(z)))^{(\delta)} \cup D^{(\delta)}$ is included in the disk $D(z, (q + 30\delta)d_\Omega(z)) \Subset \Omega$;*

(q3) *we have*

$$(11.5) \quad \ell(S; D) \leq \frac{2q}{\sqrt{3}\delta}, \quad \frac{\text{diam } S}{d_D(S)} \leq \frac{2q}{\sqrt{3}\delta}, \quad \ell(D; \Omega) \leq \frac{1}{\delta}.$$

Proof. To prove (ld1), we take $z, w \in S$ and choose the segment $[z, w] \subset \boxtimes S \subset \Omega$ for the role of an arc joining them. Then Definition III.1 shows that

$$\ell(S; D) \leq \sup_{z, w \in S} \frac{|z - w|}{d_\Omega([z, w])} \leq \frac{1}{d_\Omega(\boxtimes S)} \sup_{z, w \in S} |z - w| = \frac{\text{diam } S}{d_\Omega(\boxtimes S)}.$$

If Ω is convex, Proposition 11.2 implies $\ell(S; \Omega) \leq \frac{\text{diam } S}{d_\Omega(\boxtimes S)} = \frac{\text{diam } S}{d_\Omega(S)}$.

Now, we prove (11.2) in (ld3) for $q < 2$. Let $z, w \in S$. Condition (11.1) yields

$$(11.6) \quad |z - w| \leq \text{diam } S \leq q d_\Omega(S) < 2 d_\Omega(S) = 2 \text{dist}(S, \partial\Omega).$$

Since $d_\Omega(z) \geq d_\Omega(S)$ and $d_\Omega(w) \geq d_\Omega(S)$, the disks $D(z, d_\Omega(S))$ and $D(w, d_\Omega(S))$ are included in Ω . By (11.6), their intersection is nonempty and their union contains the entire segment $[z, w]$ (Figure 1). Consequently, outside the union of these disks, the points nearest to $[z, w]$ are the intersection points of the circles $\partial D(z, d_\Omega(S))$ and $\partial D(w, d_\Omega(S))$. Let ζ be one of these points, and let h be the foot of the perpendicular dropped from ζ to the segment $[z, w]$. Considering the isosceles triangle with vertices at z, w , and ζ , and using the first inequality in (11.6) and the relation $|z - \zeta| = |w - \zeta| = d_\Omega(S)$, we obtain the estimate $\text{dist}([z, w], \zeta) = |\zeta, h| \geq \sqrt{d_\Omega^2(S) - (qd_\Omega(S)/2)^2} = \sqrt{1 - q^2/4} d_\Omega(S)$, whence $d_\Omega([z, w]) \geq \sqrt{1 - q^2/4} d_\Omega(S)$. Since $z, w \in S$ are arbitrary and $q < 2$, this implies the first estimate in (11.2) and the relation $\boxtimes S \Subset \Omega$. Now, we can use (ld1), the relation $\text{diam } \boxtimes S = \text{diam } S$, (11.1), and the first estimate in (11.2) to complete the proof of (11.2) and statement (ld3).

We pass to the proof of statement (ld4). For this, we need a lemma.

FIGURE 1. To the proof of (1d3).

FIGURE 2. To the proof of Lemma 11.1.

Lemma 11.1. *Under the assumption (11.3), for every $w \in \boxtimes S$ the closed disk*

$$\overline{D(w, qd_\Omega(w))} \subset \Omega$$

includes the convex hull $\text{conv } S$.

Proof of the lemma. By the definition of the segmental hull, given $w \in \boxtimes S$, there exist $z_1, z_2 \in S$ (see Figure 2), possibly coinciding, such that w lies in the segment $[z_1, z_2] \subset \boxtimes S$. We consider the pair of concentric circles centered at z_1 and of radii $\text{diam } S$ and $d_\Omega(S)$, and a similar pair of circles centered at z_2 . (Observe that, by (11.3), $\text{diam } S < d_\Omega(S)$.) Since $\text{diam } S = \text{diam conv } S$, the closure of each of the disks $\overline{D(z_1, \text{diam } S)}$ and $\overline{D(z_2, \text{diam } S)}$ includes $\text{conv } S$:

$$(11.7) \quad \text{conv } S \subset \overline{D(z_1, \text{diam } S)} \cap \overline{D(z_2, \text{diam } S)}.$$

Now, let z be a point of intersection of the circles $\partial D(z_1, \text{diam } S)$ and $\partial D(z_2, \text{diam } S)$, and let ζ be that of $\partial D(z_1, d_\Omega(S))$ and $\partial D(z_2, d_\Omega(S))$. It is easily seen that the *closed disk* $\overline{D(w, |z - w|)}$ (bounded by the smaller circle shown in bold in Figure 2) *includes* the intersection on the right in (11.7) and, consequently, also $\text{conv } S$. Moreover, the disk $D(w, |\zeta - w|)$ (bounded by the larger circle shown in bold in Figure 2) is included in $D(z_1, d_\Omega(S)) \cup D(z_2, d_\Omega(S))$, which lies in Ω by construction. This means that the distance $d_\Omega(w)$ from w to $\partial\Omega$ is at least $|\zeta - w|$. Thus, if we prove the inequality

$$(11.8) \quad \frac{|z - w|}{|\zeta - w|} \leq q,$$

then $|z - w| \leq qd_\Omega(w)$ and $D(w, |z - w|) \subset D(w, qd_\Omega(w))$. We have seen that $\text{conv } S$ lies in the closure of the disk on the left, and the required inclusion $\text{conv } S \subset \overline{D(w, qd_\Omega(w))}$ follows.

In the proof of (11.8), there is no loss of generality in assuming that the middle point o of the segment $[z_1, z_2]$ coincides with the origin. Then $|w| \leq |z_2|$ and $\frac{|z - w|^2}{|\zeta - w|^2} = \frac{|z|^2 + |w|^2}{|\zeta|^2 + |w|^2} = 1 - \frac{|\zeta|^2 - |z|^2}{|\zeta|^2 + |w|^2}$. Since the last fraction is positive, it follows that $\frac{|z - w|^2}{|\zeta - w|^2} \leq 1 - \frac{|\zeta|^2 - |z|^2}{|\zeta|^2 + |z_2|^2}$. Together with the identities $|\zeta|^2 = (d_\Omega(S))^2 - |z_2|^2$ and $|z|^2 = (\text{diam } S)^2 - |z_2|^2$, this implies

$$\frac{|z - w|^2}{|\zeta - w|^2} \leq 1 - \frac{(d_\Omega(S))^2 - (\text{diam } S)^2}{(d_\Omega(S))^2} = \frac{(\text{diam } S)^2}{(d_\Omega(S))^2} \stackrel{(11.3)}{\leq} q^2,$$

yielding (11.8). This completes the proof of Lemma 11.1. ■

We return to the proof of (ld4). We repeatedly use condition (11.4) without stipulation (in particular, this condition includes the restrictions $\delta < 1/30$ and $q < 1$).

By Lemma 11.1, we have $\text{conv } S \subseteq \Omega$ and, *a fortiori*, $\boxtimes S \subseteq \Omega$. Consequently, the domain $((\boxtimes S)^{(\delta)})^{(\delta/2)}$ is precompact in Ω . We use statements ($\delta 4$) and ($\delta 2$) of Proposition 10.1 (in ($\delta 4$) we replace q and δ by δ and $\delta/2$, respectively) to conclude that $(\boxtimes S)^{(\delta)} \subseteq ((\boxtimes S)^{(\delta)})^{(\delta/2)} \subset (\boxtimes S)^{(2\delta)} \subseteq \Omega$. Therefore, there exists an “intermediate” squarable domain D with

$$(11.9) \quad S \subset \boxtimes S \subseteq (\boxtimes S)^{(\delta)} \subset D \subset ((\boxtimes S)^{(\delta)})^{(\delta/2)} \subset (\boxtimes S)^{(2\delta)} \subseteq \Omega,$$

and, finally, we obtain (q1) by construction. Now, we pass to (q2).

Let $D(z, \delta d_\Omega(z)) \cap D \neq \emptyset$. Then by Proposition 10.1($\delta 3$), the point z belongs to $D^{(2\delta)} \stackrel{(11.9)}{\subset} ((\boxtimes S)^{(2\delta)})^{(2\delta)} \subset (\boxtimes S)^{(6\delta)}$, where the last inclusion follows from Proposition 10.1($\delta 4$) (in which we take 2δ for the roles of q and δ). Therefore, there is $w \in \boxtimes S$ with $|z - w| < 6\delta d_\Omega(w)$, and, changing the roles of z and w in (10.3) (in Proposition 10.1($\delta 1$)), we obtain

$$(11.10) \quad d_\Omega(w) < \frac{1}{1-6\delta} d_\Omega(z), \quad |z - w| < \frac{6\delta}{1-6\delta} d_\Omega(z) < 12\delta d_\Omega(z).$$

By Lemma 11.1, the set $\boxtimes S$ is included in $\overline{D(w, qd_\Omega(w))}$. Then the first inequality in (11.10) yields

$$(11.11) \quad \boxtimes S \subset D(w, (q/(1-6\delta))d_\Omega(z)) \subset D(w, q(1+12\delta)d_\Omega(z)).$$

The last inequality in (11.10) means that $w \in D(z, 12\delta d_\Omega(z))$. By (11.11), this implies

$$(11.12) \quad \boxtimes S \subset D(z, 12\delta d_\Omega(z)) \cup D(w, (q+12\delta)d_\Omega(z)) \subset D(z, (q+24\delta)d_\Omega(z)).$$

But $D(z, (q+24\delta)d_\Omega(z)) = \{z\}^{(q+24\delta)}$ is the relative $(q+24\delta)$ -blowup of the singleton $\{z\}$, so that, taking relative (2δ) -blowup of both sides of (11.12) yields (by (11.9))

$$(11.13) \quad D \subset (\boxtimes S)^{(2\delta)} \subset (D(z, (q+24\delta)d_\Omega(z)))^{(2\delta)} = (\{z\}^{(q+24\delta)})^{(2\delta)}.$$

We apply Proposition 10.1($\delta 4$) (with $q+24\delta$ and 2δ in place of q and δ , respectively) to the set $\{z\}$ to obtain (taking (14.9) and (11.13) into account)

$$(11.14) \quad D \subset (\{z\}^{(q+24\delta)})^{(2\delta)} \subset \{z\}^{(q+28\delta)}.$$

Relative (δ) -blowup applied to all terms in (11.14), followed by application of Proposition 10.1($\delta 4$) to the set $\{z\}$ (with $q+28\delta$ and δ in the roles of q and δ) gives (by (11.4))

$$D^{(\delta)} \subset (\{z\}^{(q+28\delta)})^{(\delta)} \subset \{z\}^{(q+30\delta)} = D(z, (q+30\delta)d_\Omega(z)).$$

Clearly, the disk in the rightmost term includes $(D(z, r_\delta(z)))^{(\delta)}$ (because, by Proposition 10.1($\delta 4$) this set is included in $D(z, 3\delta d_\Omega(z))$), and property (q2) is proved.

To establish (q3), we observe that, by (ld1) (applied to $D \supset \boxtimes S$ in the role of Ω), the inclusions (11.9) imply

$$(11.15) \quad \ell(S; D) \stackrel{(\text{ld1})}{\leq} \frac{\text{diam } S}{d_D(\boxtimes S)} \stackrel{(11.3)}{\leq} \frac{qd_\Omega(S)}{d_D(\boxtimes S)} \stackrel{(11.9)}{\leq} \frac{qd_\Omega(S)}{d_{(\boxtimes S)^{(\delta)}}(\boxtimes S)}.$$

Obviously, the first inequality in (11.15) remains true if we replace $\ell(S; D)$ on the left with $\frac{\text{diam } S}{d_D(S)}$. By the definition (10.2d), we obtain $d_{(\boxtimes S)^{(\delta)}}(\boxtimes S) \geq \delta d_\Omega(\boxtimes S)$. Hence, by

the first inequality in (11.2), estimates (11.15) yield

$$(11.16) \quad \max \left\{ \ell(S; D), \frac{\text{diam } S}{d_D(S)} \right\} \stackrel{(11.15)}{\leq} \frac{qd_\Omega(S)}{\delta d_\Omega(\boxtimes S)} \stackrel{(11.2)}{\leq} \frac{qd_\Omega(S)}{\delta \frac{\sqrt{4-q^2}}{2} d_\Omega(S)} \stackrel{(11.3)}{\leq} \frac{2q}{\sqrt{3}\delta},$$

which proves the first two inequalities in (11.5) in (q3).

Let $z \in D$. Then (11.14) shows that $D \subset \{z\}^{(q+28\delta)}$. By Proposition 2.3(12), this implies $\ell(D; \Omega) \stackrel{(12)}{\leq} \ell(\{z\}^{(q+28\delta)}; \Omega)$. Since $\{z\}^{(q+28\delta)}$ is a disk (in particular, a convex set), from (ld2) we deduce that

$$(11.17) \quad \ell(D; \Omega) \stackrel{(ld2)}{\leq} \frac{\text{diam}\{z\}^{(q+28\delta)}}{d_\Omega(\{z\}^{(q+28\delta)})} = \frac{2(q+28\delta)d_\Omega(z)}{d_\Omega(\{z\}^{(q+28\delta)})}.$$

By the obvious estimate $d_\Omega(\{z\}^{(q+28\delta)}) \geq (1 - (q+28\delta))d_\Omega(z)$, the restriction (11.4), when applied to (11.17), implies the inequality

$$(11.18) \quad \ell(D; \Omega) \stackrel{(11.17)}{\leq} \frac{2(q+28\delta)d_\Omega(z)}{(1 - (q+28\delta))d_\Omega(z)} \stackrel{(11.4)}{\leq} \frac{2(1-2\delta)}{1 - (1-2\delta)} \leq \frac{1}{\delta};$$

this gives the last inequality in (11.5) and completes the proof of (ld4). $\square$

Proposition 11.4. *Let $\Sigma = \{S_k\}$, $k = 1, 2, \dots$, be a locally finite family of precompact subsets of a bounded domain Ω . Suppose one of the following four conditions is fulfilled:*

(Sd1) $\boxtimes S_k \Subset \Omega$, $k = 1, 2, \dots$, and $\limsup_{k \rightarrow \infty} \frac{\text{diam } S_k}{d_\Omega(\boxtimes S_k)} < +\infty$;

(Sd2) the domain Ω is convex and $\limsup_{k \rightarrow \infty} \frac{\text{diam } S_k}{d_\Omega(S_k)} < +\infty$;

(Sd3) $\limsup_{k \rightarrow \infty} \frac{\text{diam } S_k}{d_\Omega(S_k)} < 2$;

(Sd4) $\text{conv } S_k \Subset \Omega$, $k = 1, 2, \dots$, and $\limsup_{k \rightarrow \infty} \frac{\text{diam } S_k}{d_\Omega(\text{conv } S_k)} < +\infty$.

Then the family Σ can be combinatorially inscribed in a family $\mathcal{D} = \{D_k\}$ of admissible domains for Ω satisfying (9.15).

Moreover, if $\limsup_{k \rightarrow \infty} \frac{\text{diam } S_k}{d_\Omega(S_k)} < q < 1$ (this condition is stronger than (Sd3)), then for every $\delta > 0$ with $q + 30\delta < 1$ an admissible family $\mathcal{D} = \{D_k\}$ satisfying (9.15) can be chosen in such a way that there exists $k_0 \in \mathbb{N}$ with

$$(11.19) \quad \frac{\text{diam } S_k}{d_{D_k}(S_k)} \leq \frac{2}{\sqrt{3}\delta} \frac{\text{diam } S_k}{d_\Omega(S_k)} \leq \frac{2q}{\sqrt{3}\delta} \text{ for all } k \geq k_0,$$

and $(\mathcal{D}_*[z, r_\delta(z)))^{(\delta)} \subset D(z, (q+30\delta)d_\Omega(z))$ whenever z is sufficiently close to $\partial\Omega$ in the sense that $d_\Omega(z) < \varepsilon$ for a suitable $\varepsilon > 0$.

Proof. Each of the conditions (Sd1)–(Sd3) implies condition (9.14) of Proposition 9.2 (it suffices to apply (ld1)–(ld3) in the previous Proposition 11.3 to each case, respectively). Moreover, (Sd4) implies (Sd1). Thus, the first statement of Proposition 11.4 is a consequence of Propositions 11.3 and 9.2. The assumption of the last statement says that, uniformly in $k \geq k_0$, we have

$$(11.20) \quad q_k \stackrel{\text{def}}{=} \frac{\text{diam } S_k}{d_\Omega(S_k)} \leq q < 1$$

(this is an estimate of type (11.3)) and also that $q_k + 30\delta < 1$ (as in (11.4)). For every $S = S_k$, we take a subdomain $D = D_k \Subset \Omega$ satisfying $S \Subset D$ and otherwise arbitrary for $k < k_0$, but enjoying the conclusion of item (ld4) for $k \geq k_0$ with the constant q_k from (11.20) in place of q . By (11.5) in (q3), relation (11.19) is true, and we have (9.15). By the definitions (5.4)–(5.5) and (10.2), we have

$$(\mathcal{D}_*[z, r_\delta(z)))^{(\delta)} \subset (D[z, r_\delta(z)))^{(\delta)} \cup \{D_k^{(\delta)} : D_k \cap D(z, r_\delta(z)) \neq \emptyset\};$$

so, the final inclusion of Proposition 11.4 follows from (q2) in (ld4) for D_k with $k \geq k_0$, and from the fact that there are only finitely many k 's with $k < k_0$ and $D_k \Subset \Omega$ in any case. $\square$

Remark. It can easily be shown that the restriction $q < 2$ in (ld3) and (Sd3) (see (11.1)) cannot be weakened, and inequalities (11.2) are sharp. The same can be said about the restriction $q < 1$ in (11.3) (Lemma 11.1) and in (ld4) if we interpret (q2) as a qualitative condition concerning the existence, for every $z \in S$, of a disk in Ω centered at z and containing S . Still the same is true for the final statement of Proposition 11.4. At the same time, using, for instance, the Young theorem (an estimate of the diameter of a convex set in terms of the radius of a circumscribed circle; see [3, Theorem 7.5]), makes it possible to show that the condition $q < \sqrt{3}$ (intermediate between (11.3) and (11.1)) implies the inclusion $\text{conv } S \Subset \Omega$, i.e., (q1). The restriction $q < \sqrt{3}$ is sharp in this situation.

The notion of the segmental hull on the plane extends naturally to subsets $S \subset \mathbb{R}^n$. For instance, it is possible to define the k -simplicial hull $\boxtimes^k S$, $S \subset \mathbb{R}^n$, for $n \geq 1$, $k = 0, 1, \dots, n$, as the union of the convex hulls of all $(k+1)$ -tuples of points in S (i.e., the union of all k -simplexes, including degenerate, with vertices in S). Clearly, $\boxtimes^0 S = S$, $\boxtimes^1 S = \boxtimes S$ is the segmental hull, and $\boxtimes^n S = \text{conv } S$ by the Carathéodory theorem [3, Theorem 2.4], mentioned at the beginning of §11. Recalling the remarks about the entropy of arcwise connectedness for a subset of a domain in $\mathbb{R}^n$ (see §2), and extending the notions of blowup and a star to the multidimensional case by analogy, it is possible to generalize the preceding (and some of the forthcoming) material (for instance, Proposition 12.1) related to the notions under study to $\mathbb{R}^n$. Most of the statements can also easily be extended to metric spaces.

§12. THE ENTROPY OF ARCWISE CONNECTEDNESS, AND SYSTEMS OF POSITIVE WEIGHTS

Proposition 12.1. *Suppose a system $\mathcal{P}$ of positive subharmonic functions in a domain $\Omega \subset \mathbb{C}$ (i.e., a system of positive weights on Ω) has property $(A^\dagger)$ (see the Introduction), and the following holds true:*

(D₀) *there exists ε , $0 < \varepsilon < 1$, such that for every $p \in \mathcal{P}$ there is $p_1 \in \mathcal{P}$ and a constant C_1 with*

$$(12.1) \quad \frac{1}{2\pi} \int_0^{2\pi} p(z + \varepsilon d_\Omega(z) e^{i\theta}) d\theta \leq p_1(z) + C_1, \quad z \in \Omega.$$

Then for every $p \in \mathcal{P}$ and every positive number l there exists $\tilde{p} \in \mathcal{P}$ and a constant $\tilde{C}$ such that for every $S \subset \Omega$ satisfying $\ell(S; \Omega) < l$ we have

$$(12.2) \quad \sup_{z \in S} p(z) \leq \inf_{z \in S} \tilde{p}(z) + \tilde{C}.$$

Proof. We need a more detailed quantitative version of Proposition 2.2.

Lemma 12.1. *Suppose $\varepsilon > 0$ and $\ell(S; \Omega) < l < +\infty$. Then for every $z, w \in S$ there is a finite sequence of circles $D(\zeta_k, d)$, $k = 0, \dots, n+1$, of radius*

$$(12.3) \quad d = \min \left\{ \varepsilon, 1/(4(l+1)) \right\} d_\Omega(S)$$

that are compactly embedded in Ω and satisfy the condition

$$(D1) \quad z = \zeta_0, w = \zeta_{n+1} \text{ and } |\zeta_{k-1} - \zeta_k| \leq d/2 \text{ for all } k = 1, \dots, n+1$$

of Proposition 2.2; moreover,

$$(12.4) \quad n \leq 2(l+1)^2 \max \{1/\varepsilon, 4(l+1)\}.$$

Proof of Lemma 12.1. By Proposition 9.1(S2), there is a rectifiable arc $\tilde{l}(z, w) \subset \Omega$ such that, in the notation (12.3), we have

$$(12.5d) \quad d_\Omega(\tilde{l}(z, w)) \geq \frac{1}{2(l+1)} d_\Omega(S) \geq 2d,$$

$$(12.5l) \quad |\tilde{l}(z, w)| \leq (l+1)^2 d_\Omega(S).$$

As in the proof of Proposition 2.2, moving along $\tilde{l}(z, w)$ from z to w , we split this arc by consecutive points

$$(12.6) \quad \zeta_0 = z, \zeta_1, \dots, \zeta_{n+1} = w, \quad \zeta_k \in \tilde{l}(z, w) \subset \Omega,$$

into subarcs $\tilde{l}(\zeta_{k-1}, \zeta_k)$ whose lengths satisfy

$$(12.7) \quad |\tilde{l}(\zeta_{k-1}, \zeta_k)| = \frac{d}{2}, \quad k = 1, 2, \dots, n; \quad |\tilde{l}(\zeta_n, \zeta_{n+1})| \leq \frac{d}{2},$$

that is, we have (D1). From (12.5d) we deduce that $d \leq \frac{1}{2} \text{dist}(\tilde{l}(z, w), \partial\Omega)$, and the construction (12.6) of the $\{\zeta_k\}$ implies that every disk $D(\zeta_k, d)$, $k = 0, 1, \dots, n+1$, is precompact in Ω . Moreover, (12.7) shows that $nd/2 = |\bigcup_{k=1}^n \tilde{l}(\zeta_{k-1}, \zeta_k)| \leq |\tilde{l}(z, w)|$, whence

$$n \leq 2 \frac{|\tilde{l}(z, w)|}{d} \stackrel{(12.5l)}{\leq} 2 \frac{(l+1)^2 d_\Omega(S)}{d} \stackrel{(12.3)}{=} \frac{2(l+1)^2}{\min\{\varepsilon, 1/(4(l+1))\}}. \quad \blacksquare$$

Now we turn to the proof of Proposition 12.1.

Let $p = p_0 \in \mathcal{P}$. In view of (D₀), we can choose by recursion a sequence of *nonnegative* subharmonic functions $\{p_k\} \subset \mathcal{P}$ and a sequence of nonnegative numbers C_k such that

$$(12.8) \quad \frac{1}{2\pi} \int_0^{2\pi} p_k(z + \varepsilon d_\Omega(z) e^{i\theta}) d\theta \leq p_{k+1}(z) + C_{k+1}, \quad z \in \Omega, \quad k = 0, 1, \dots$$

By the mean-value inequality for subharmonic functions, (12.8) implies

$$(12.9) \quad p_k(z) \leq p_{k+1}(z) + C_{k+1}, \quad z \in \Omega, \quad k = 0, 1, \dots$$

Suppose $z, w \in S$, $\ell(S; \Omega) < l$, and, in accordance with Lemma 12.1, a sequence of circles $D(\zeta_k, d) \Subset \Omega$ has been chosen such that the centers ζ_k satisfy (D1), the radius d is defined by (12.3), and their total number is restricted by (12.4). We put $H_k = \mathcal{H}_{D(\zeta_{k+1}, d)}^{p_k}$, where on the right we have the harmonic extension¹⁵ of the subharmonic function p_k inside $D(\zeta_{k+1}, d)$, $k = 0, 1, \dots, n$. Since p_k is positive, H_k is also positive in $D(\zeta_{k+1}, d)$; by the condition $|\zeta_k - \zeta_{k+1}| \leq d/2$ in (D1) and the Harnack inequality, we obtain

$$p_k(\zeta_k) \leq H_k(\zeta_k) \leq \frac{d + d/2}{d - d/2} H_k(\zeta_{k+1}) = 3 \cdot \frac{1}{2\pi} \int_0^{2\pi} p_k(\zeta_{k+1} + d e^{i\theta}) d\theta$$

for $k = 0, 1, \dots, n$. By the choice of d (see (12.3)), it follows that

$$p_k(\zeta_k) \stackrel{(12.3)}{\leq} 3 \cdot \frac{1}{2\pi} \int_0^{2\pi} p_k(\zeta_{k+1} + \varepsilon d_\Omega(\zeta_{k+1}) e^{i\theta}) d\theta \stackrel{(12.8)}{\leq} 3(p_{k+1}(\zeta_{k+1}) + C_{k+1})$$

for $k = 0, \dots, n$. We apply this consecutively $k+1$ times starting with $k = 0$ to obtain

$$(12.10) \quad p(z) = p_0(\zeta_0) \leq 3^{n+1} p_{n+1}(\zeta_{n+1}) + \sum_{k=1}^{n+1} 3^k C_k = 3^{n+1} p_{n+1}(w) + \sum_{k=1}^{n+1} 3^k C_k.$$

¹⁵The notation is coordinated with that for *balayage*; see (4.3).

Let $N = \lceil 2(l+1)^2 \max\{1/\varepsilon, 4(l+1)\} \rceil$ be the integral part of the right-hand side of (12.4). We have $n \leq N$. Using (2.1) and applying (12.9) $N - n$ times starting with $k = n + 1$ in the case where $n < N$, we obtain

$$(12.11) \quad p(z) \leq 3^{n+1} \left(p_{N+1}(w) + \sum_{k=n+2}^{N+1} C_k \right) + \sum_{k=1}^{n+1} 3^k C_k \leq 3^{N+1} p_{N+1}(w) + \sum_{k=1}^{N+1} 3^k C_k.$$

By (12.8) and the choice of N , we see that, for fixed p , we can choose $p_{N+1} \in \mathcal{P}$ and the constant $C' = \sum_{k=1}^{N+1} 3^k C_k$ in (12.11) entirely on the basis of ε and l , independently of $S \subset \Omega$ (provided $\ell(S; \Omega) < l$) and of $z, w \in S$. Hence, (12.11) implies

$$(12.12) \quad \sup_{z \in S} p(z) \leq \inf_{w \in S} 3^{N+1} p_{N+1}(w) + C'$$

for every $S \subset \Omega$ with $\ell(S; \Omega) < l$. Finally, applying $(A^\dagger)$ at most $3^{N+1} - 1$ times, we find, starting with p_{N+1} , a function $\tilde{p} \in \mathcal{P}$ and a constant C'' such that

$$(12.13) \quad 3^{N+1} p_{N+1}(w) = \underbrace{p_{N+1}(w) + \cdots + p_{N+1}(w)}_{3^{N+1} \text{ times}} \stackrel{(A^\dagger)}{\leq} \tilde{p}(w) + C'', \quad w \in \Omega.$$

The choice of $\tilde{p}$ and C'' is based entirely on p_{N+1} and N , i.e., in the long run, entirely on p, ε , and l . Inequalities (12.12) and (12.13) yield (12.2) with $\tilde{C} = C' + C''$. $\square$

CHAPTER IV. NONUNIQUENESS AND STABILITY THEOREMS

§13. NONUNIQUENESS THEOREM FOR ALGEBRAS DETERMINED BY POSITIVE WEIGHTS

Theorem 13.1 (Nonuniqueness). *Suppose $\mathcal{P} \subset SH^+(\Omega)$ is a system of weights satisfying $(A^\dagger)$ and also the condition*

(LD₀) *there exists $\varepsilon, 0 < \varepsilon < 1$, such that for every $p \in \mathcal{P}$ there is $p_1 \in \mathcal{P}$ and a constant C_1 with*

$$(13.1) \quad \frac{1}{2\pi} \int_0^{2\pi} p(z + \varepsilon d_\Omega(z) e^{i\theta}) d\theta + \log \left(1 + \frac{1}{d_\Omega(z)} \right) \leq p_1(z) + C_1, \quad z \in \Omega.$$

Let $\Sigma = \{S_k\}$, $k = 1, 2, \dots$, be a locally finite family of precompact Borel subsets of Ω satisfying¹⁶

$$(13.2) \quad \limsup_{k \rightarrow \infty} \ell(S_k; \Omega) < +\infty$$

or one of the conditions (Sd1)–(Sd4) in Proposition 11.4. If for a sequence Λ in Ω there exists a function $p \in \mathcal{P}$ such that

(RS) *for the measure n_Λ defined by (0.1) and the Riesz measure ν_p of p there exist representations (all summands are positive; compare with (7.1))*

$$(13.3) \quad n_\Lambda = \lambda^{(0)} + \sum_{k=1}^{\infty} \lambda^{(k)}, \quad \nu_p = \nu_p^{(0)} + \sum_{k=1}^{\infty} \nu_p^{(k)},$$

where $\lambda^{(0)}$ is a measure with compact support in Ω , the measures $\lambda^{(k)}, \nu_p^{(k)}$ are supported on S_k , $k = 1, 2, \dots$, and we have

$$(13.4) \quad \limsup_{k \rightarrow \infty} \frac{\lambda^{(k)}(S_k)}{\nu_p^{(k)}(S_k)} < +\infty,$$

then Λ is a subsequence of zeros (i.e., a nonuniqueness sequence) for $A_p^\dagger(\Omega)$.

¹⁶This condition coincides with (9.14). See Definition III.1 concerning the entropy of acwise connectedness.

Proof. Under condition (13.2) (which coincides with (9.14)), we apply Proposition 9.2, and under one of conditions (Sd1)–(Sd4) we apply Proposition 11.4, to conclude that Σ can be combinatorially inscribed in some family of domains $\mathcal{D} = \{D_k\}$ admissible for Ω and satisfying relations (9.15). The first of them, (9.15S), coincides with (7.2d) in the preparatory Theorem 7.1 for algebras. The assumptions (13.4) and (13.3) imply condition (7.2s) and agreements (i)–(iii) in the same preparatory theorem if we take $\log |f_\Lambda|$ for the role of u ($f_\Lambda \not\equiv 0$ being a function holomorphic in Ω with $\text{Zero}_{f_\Lambda} = \Lambda$), and put $\lambda_u = n_\Lambda$, $\nu^{(k)} = \nu_p^{(k)}$. Thus, all assumptions of the preparatory Theorem 7.1 hold true.

For the role of r in (7.3), we take the function $r_\delta(z) = \delta d_\Omega(z)$, $z \in \Omega$, from (10.1) with $0 < \delta < 1/6$. For every $z \in \Omega$, put $t = t(z) = r_\delta(z)$. By the preparatory Theorem 7.1 and the remark at the end of §8, this choice of t implies the existence of a function $h \not\equiv 0$ holomorphic in Ω and such that the following is true: $(\mathcal{D}_*[z, t])^r = (\mathcal{D}_*[z, r_\delta(z)])^{(\delta)}$ (the relative (δ) -blowup (see (10.2)), of the star with kernel for the disk $D(z, r_\delta(z))$ (see (5.4)–(5.5))), and (7.4) is fulfilled without the summand $B \log(1 + |z|)$, i.e.,

$$(13.5) \quad \log |f_\Lambda(z)| + \log |h(z)| \leq A \mathcal{H}_{(\mathcal{D}_*[z, r_\delta(z)])^{(\delta)}}^p(z) + \log(1 + 1/r_\delta(z)), \quad z \in \Omega,$$

where A is a constant. For short, we put $S_\delta(z) = (\mathcal{D}_*[z, r_\delta(z)])^{(\delta)}$. By the maximum principle for subharmonic functions and by the definition (4.3) of *balayage*, with the help of the elementary inequality

$$(13.6) \quad \log\left(1 + \frac{1}{r_\delta(z)}\right) = \log\left(1 + \frac{1}{\delta d_\Omega(z)}\right) \leq \log\left(1 + \frac{1}{d_\Omega(z)}\right) + \log\left(1 + \frac{1}{\delta}\right), \quad z \in \Omega,$$

we can rewrite (13.5) in the following weak form:

$$(13.7) \quad \log |(f_\Lambda h)(z)| \leq A \sup_{w \in S_\delta(z)} p(w) + \log(1 + 1/d_\Omega(z)) + \log(1 + 1/\delta), \quad z \in \Omega.$$

By (9.15D), since $\mathcal{D}$ is locally finite and all D_k are precompact in Ω , we obtain a uniform estimate $\sup_{D_k \in \mathcal{D}} \ell(D_k, \Omega) < l < +\infty$ with some constant l , and this coincides with (10.4) in Proposition 10.2. Since $\delta < 1/6$, statement ($\ell 3$) in Proposition 10.2 implies that $\ell(S_\delta(z); \Omega) = \ell((\mathcal{D}_*[z, r_\delta(z)])^{(\delta)}; \Omega) \leq 10^{11}(l+1)^{36} = l'$ for all $z \in \Omega$, with a constant $l' < +\infty$. Condition (LD₀) is equivalent to the simultaneous validity of (D₀) in Proposition 12.1 and (L) in the Introduction. In Proposition 12.1, we take $S_\delta(z)$, $z \in \Omega$, for the role of S , and l' for the role of l ; then it implies the existence of a function $\tilde{p} \in \mathcal{P}$ and a constant $\tilde{C}$ such that relations of the form (12.2) are satisfied, i.e., $\sup_{w \in S_\delta(z)} p(w) \leq \inf_{w \in S_\delta(z)} \tilde{p}(w) + \tilde{C} \leq \tilde{p}(z) + \tilde{C}$, $z \in \Omega$. In the context of (13.7), these inequalities yield

$$(13.8) \quad \log |(f_\Lambda h)(z)| \leq A \tilde{p}(z) + \log(1 + 1/d_\Omega(z)) + \tilde{C} + \log(1 + 1/\delta), \quad z \in \Omega.$$

Applying (L) (once) and (A[†]) (several times; at most $[A] + 1$ steps suffice) to the right-hand side of (13.8), we find $\hat{p} \in \mathcal{P}$ and a constant $\hat{C}$ such that $\log |(f_\Lambda h)(z)| \leq \hat{p}(z) + \hat{C}$ for all $z \in \Omega$. This means that the function $f_\Lambda h \not\equiv 0$ belongs to $\mathcal{A}_\mathcal{P}$. Since the sequence of zeros of $f_\Lambda h$ includes the sequence $\text{Zero}_{f_\Lambda} = \Lambda$, the theorem is proved. $\square$

Remark. It is possible to consider a version of Theorem 13.1 in which conditions (13.2), (Sd1), (Sd2), and (Sd4) “interlace”: the family $\Sigma = \{S_k\}$ splits in finitely many subfamilies, each satisfying some of these conditions.

The conditions on p in statement (U₁) of Theorem 0.1 make the system (0.2) of weights obey a restriction equivalent to (LD₀) in Theorem 13.1. The assumption of statement (U₁) of Theorem 0.1 implies the assumptions of (Sd3) and (RS) in Theorem 13.1 if for the roles of $\lambda^{(k)}$ and $\nu_p^{(k)}$ we take the restrictions of n_Λ as in (0.1) and of the Riesz

measure ν_p to S_k and the sets S_k are mutually disjoint (that is, $\lambda^{(k)}(S_k) = n_\Lambda(S_k)$, $\nu_p^{(k)}(S_k) = \nu_p(S_k)$). Thus, statement (U₁) of Theorem 0.1 is indeed a particular case of Theorem 13.1; moreover, if Ω is convex, it suffices to require (Sd2).

§14. ZERO SUBSEQUENCES FOR CLASSES $A_{\mathcal{P}}^\uparrow(\Omega)$
WITH WEIGHTS OF VARIABLE SIGN

Theorem 14.1. *Suppose that a system $\mathcal{P}$ of functions subharmonic on a bounded domain Ω and not identically equal to zero satisfies (A[†]) and the following condition:*

(LD₁) *for each number γ , $0 < \gamma < 1$, and each function $p \in \mathcal{P}$ there is a function $p_\gamma \in \mathcal{P}$ and a constant C_γ such that*

$$(14.1) \quad \frac{1}{2\pi} \int_0^{2\pi} p(z + \gamma d_\Omega(z) e^{i\theta}) d\theta + \log\left(1 + \frac{1}{d_\Omega(z)}\right) \leq p_\gamma(z) + C_\gamma, \quad z \in \Omega.$$

Let $\Sigma = \{S_k\}$, $k = 1, 2, \dots$, be a locally finite family of precompact Borel subsets of Ω with

$$(14.2) \quad \limsup_{k \rightarrow \infty} \frac{\text{diam } S_k}{d_\Omega(S_k)} < 1.$$

If for a sequence Λ in Ω there is $p \in \mathcal{P}$ for which condition (RS) in Theorem 13.1 is fulfilled (that is, (13.3)–(13.4) hold true), then Λ is a subsequence of zeros for $A_{\mathcal{P}}^\uparrow(\Omega)$.

Proof. By (14.2), there exist $q > 0$ and $\delta > 0$ such that

$$(14.3) \quad \limsup_{k \rightarrow \infty} \frac{\text{diam } S_k}{d_\Omega(S_k)} < q < 1, \quad q + 30\delta = \gamma < 1.$$

This means that the assumptions of the final statement of Proposition 11.4 are satisfied. Applying that statement, we combinatorially inscribe Σ in a system of domains $\mathcal{D} = \{D_k\}$, admissible for Ω , in such a way that (9.15S) holds true (that condition coincides with assumption (7.2d) of the preparatory theorem for algebras), and at the same time, for some $\varepsilon > 0$ we have

$$(14.4) \quad (\mathcal{D}_*[z, r_\delta(z)))^{(\delta)} \subset D(z, \gamma d_\Omega(z)) \quad \text{provided } d_\Omega(z) < \varepsilon, \quad z \in \Omega.$$

Together, conditions (7.2d) and (RS) infer the validity of all assumptions of the preparatory Theorem 7.1 in which we take $\log |f_\Lambda|$ for the role of u , where $f_\Lambda \not\equiv 0$ is a holomorphic function in Ω with $\text{Zero}_{f_\Lambda} = \Lambda$, and put $\lambda_u = n_\Lambda$, $\nu^{(k)} = \nu_p^{(k)}$. Starting with δ in (14.3), we take the function $r_\delta(z) = \delta d_\Omega(z)$, $z \in \Omega$, from (10.1) for the role of r in (7.3). Now, the preparatory Theorem 7.1 and the remark at the end of §8 show that there exists a function $h \not\equiv 0$ holomorphic in Ω and such that (7.4) is fulfilled with $t = t(z) = r_\delta(z)$ and $(\mathcal{D}_*[z, t))^r = (\mathcal{D}_*[z, r_\delta(z)))^{(\delta)}$, but without the summand $B \log(1 + |z|)$, and this takes the form (13.5) in our situation. By the remark at the end of §7, we can adjust A to be a natural number. By the subordination principle (4.4) for *balayage* and by (14.4), inequality (13.5) implies

$$(14.5) \quad \log |(f_\Lambda h)(z)| \leq A \mathcal{H}_{D(z, \gamma d_\Omega(z))}^p(z) + \log\left(1 + \frac{1}{\delta d_\Omega(z)}\right), \quad d_\Omega(z) < \varepsilon, \quad z \in \Omega.$$

But the value taken at the center of a disk by the *balayage* of p from this disk is equal to the mean value of p on the boundary circle. Therefore, since $A \geq 1$, (14.5) yields

$$(14.6) \quad \begin{aligned} \log|(f_\Lambda h)(z)| &\leq A \frac{1}{2\pi} \int_0^{2\pi} p(z + \gamma d_\Omega(z) e^{i\theta}) d\theta + \log\left(1 + \frac{1}{\delta d_\Omega(z)}\right) \\ &\stackrel{(13.6)}{\leq} A \left(\frac{1}{2\pi} \int_0^{2\pi} p(z + \gamma d_\Omega(z) e^{i\theta}) d\theta + \log\left(1 + \frac{1}{d_\Omega(z)}\right) \right) + C_\delta, \\ &\quad d_\Omega(z) < \varepsilon, \end{aligned}$$

where $C_\delta = \log(1 + 1/\delta)$ is a constant and $A \in \mathbb{N}$. Since $\gamma < 1$ in (14.3), condition (LD₁) shows that for $p \in \mathcal{P}$ there is $p_\gamma \in \mathcal{P}$ and $C_\gamma \geq 0$ for which (14.1) is fulfilled. Then (14.6) implies $\log|(f_\Lambda h)(z)| \leq A p_\gamma(z) + A C_\gamma + C_\delta$ if $d_\Omega(z) < \varepsilon$, where A is a natural number. Moreover, (14.1) also ensures that p_γ is *locally bounded below* in Ω (i.e., bounded below on every compact subset of Ω). Representing $A p_\gamma$ as the sum of A copies of p and applying (A[†]) to this sum $A - 1$ times, we find $p_0 \in \mathcal{P}$ and a constant C such that $\log|(f_\Lambda h)(z)| \leq p_0(z) + C$ provided $d_\Omega(z) < \varepsilon$. The function p_0 is bounded below on the compact set $\{z \in \Omega : d_\Omega(z) \geq \varepsilon\}$ because p_γ is locally bounded below on Ω . Then, if $c > 0$ is sufficiently small, we have $\log|c(f_\Lambda h)(z)| \leq p_0(z)$ for all $z \in \Omega$. Furthermore, Λ is a subsequence of zeros for the function $c f_\Lambda h \not\equiv 0$; consequently, this is a subsequence of zeros for $A_p^\dagger(\Omega)$. $\square$

Remark. In order to obtain statement (U₂) of Theorem 0.1, it suffices to repeat the last paragraph in the remark at the end of the preceding section nearly word-for-word, but with replacement of (U₁) by (U₂), conditions (LD₀) and (Sd3) by (LD₁) and (14.2), and Theorem 13.1 by Theorem 14.1; also, the final remark about a convex domain Ω should be omitted.

§15. NONUNIQUENESS THEOREMS FOR $H_p^1(\Omega)$

We remind the reader (see the Introduction) that for $p \in SH^+(\Omega)$ the space $H_p^1(\Omega)$ was defined to be the space of all $f \in H(\Omega)$ satisfying

$$(15.1) \quad |f(z)| \leq C_f \exp(c_f p(z)), \quad z \in \Omega,$$

with some positive constants $c_f < 1$ and C_f .

Theorem 15.1. *Let p be a positive subharmonic function in a bounded domain $\Omega \Subset \mathbb{C}$ such that*

(LD₀¹) *for every $b > 1$ there exist numbers ε , $0 < \varepsilon < 1$, and C_b with*

$$(15.2) \quad \frac{1}{2\pi} \int_0^{2\pi} p(z + \varepsilon d_\Omega(z) e^{i\theta}) d\theta + \log\left(1 + \frac{1}{d_\Omega(z)}\right) \leq b p(z) + C_b, \quad z \in \Omega.$$

Let $\Sigma = \{S_k\}$, $k = 1, 2, \dots$, be a locally finite family of precompact Borel subsets of Ω satisfying

$$(15.3) \quad \lim_{k \rightarrow \infty} \frac{\text{diam } S_k}{d_\Omega(S_k)} = 0.$$

If a sequence Λ in Ω satisfies a stronger form of condition (RS) in Theorem 3.1, namely, if (13.4) is replaced with

$$(15.4) \quad \limsup_{k \rightarrow \infty} \frac{\lambda^{(k)}(S_k)}{\nu_p^{(k)}(S_k)} < 1,$$

then Λ is a nonuniqueness sequence for $H_p^1(\Omega)$.

Proof. First, we specify the choice of some constants to be used in the proof. By (15.4), there exists β , $0 < \beta < 1$, such that condition (8.1s) of the preparatory Theorem 8.1 is fulfilled with $\nu^{(k)} = \nu_p^{(k)}$. Starting with this β , we can find $\alpha > 0$ and then $b > 1$ such that

$$(15.5) \quad 0 < \frac{1+\alpha}{1-\alpha} \beta b < 1.$$

For $b > 1$ chosen above, we fix ε , $0 < \varepsilon < 1$, in such a way that (15.2) in condition (LD₀¹) will be true. Next, we choose $\delta > 0$ with $30\delta < \varepsilon < 1$. After that we choose $q > 0$ with

$$(15.6) \quad q + 30\delta < \varepsilon < 1, \quad \frac{2q}{\sqrt{3}\delta} < \alpha.$$

Formula (15.3) ensures the condition $\limsup_{k \rightarrow \infty} \frac{\text{diam } S_k}{d_\Omega(S_k)} < q < 1$ in the final statement of Proposition 11.4. By (15.6), that proposition shows that Σ can be combinatorially inscribed in a family of admissible domains $\mathcal{D} = \{D_k\}$ for Ω , so that (11.19) holds true (which, by (15.6), implies condition (8.1d) of Theorem 8.1), and *for all $z \in \Omega$ sufficiently close to $\partial\Omega$* we have

$$(15.7) \quad (\mathcal{D}_*[z, r_\delta(z)))^{(\delta)} \subset D(z, (q + 30\delta)d_\Omega(z)) \stackrel{(15.6)}{\subset} D(z, \varepsilon d_\Omega(z)).$$

We apply the preparatory Theorem 8.1 with $r = r_\delta$ and $\lambda = n_\Lambda$ to the function $u = \log|f_\Lambda| \in SH(\Omega)$, where $f_\Lambda \not\equiv 0$ is a holomorphic function in Ω such that $\text{Zero}_{f_\Lambda} = \Lambda$. This yields a holomorphic function $h \not\equiv 0$ in Ω that satisfies (8.2) without the summand $B \log(1 + |z|)$ on the right (see the remark at the end of §8). In our situation, when $t = t(z) = r_\delta(z)$ and $(\mathcal{D}_*[z, t))^r = (\mathcal{D}_*[z, r_\delta(z)))^{(\delta)}$, inequality (8.2) takes the form

$$(15.8) \quad \log|(f_\Lambda h)(z)| \leq \frac{1+\alpha}{1-\alpha} \beta \mathcal{H}_{(\mathcal{D}_*[z, r_\delta(z)))^{(\delta)}}^p(z) + \log(1 + 1/r_\delta(z)), \quad z \in \Omega.$$

By (15.7) and the subordination principle (4.4) for *balayage*, there is $\varepsilon' > 0$ such that, whenever $d_\Omega(z) < \varepsilon'$, we have

$$(15.9) \quad \mathcal{H}_{(\mathcal{D}_*[z, r_\delta(z)))^{(\delta)}}^p(z) \leq \mathcal{H}_{D(z, \varepsilon d_\Omega(z))}^p(z) = \frac{1}{2\pi} \int_0^{2\pi} p(z + \varepsilon d_\Omega(z) e^{i\theta}) d\theta.$$

Together with (LD₀¹) and (13.6), this implies that (15.8) takes the following form:

$$\begin{aligned} \log|(f_\Lambda h)(z)| &\stackrel{(15.2), (13.6)}{\leq} \frac{1+\alpha}{1-\alpha} \beta b p(z) - \frac{1+\alpha}{1-\alpha} \beta \log(1 + 1/d_\Omega(z)) + \frac{1+\alpha}{1-\alpha} \beta C_b \\ &\quad + \log(1 + 1/d_\Omega(z)) + \log(1 + 1/\delta), \quad d_\Omega(z) < \varepsilon', \quad z \in \Omega. \end{aligned}$$

By (15.5), this shows that, taking

$$c' \stackrel{\text{def}}{=} \max \left\{ \frac{1+\alpha}{1-\alpha} \beta b, 1 - \frac{1+\alpha}{1-\alpha} \beta \right\} < 1, \quad C' = \frac{1+\alpha}{1-\alpha} \beta C_b + \log(1 + 1/\delta),$$

we obtain

$$(15.10) \quad \log|(f_\Lambda h)(z)| \leq c' (p(z) + \log(1 + 1/d_\Omega(z))) + C'$$

whenever $d_\Omega(z) < \varepsilon'$. Increasing C' if necessary, we can ensure (15.10) *for all $z \in \Omega$* . Since $c' < 1$, there exists $b' > 1$ with $c'b' < 1$. We weaken (LD₀¹), substituting $p(z)$ for its average over a circle on the right-hand side of (15.2) and replacing b by b' , to deduce from (15.10) that $\log|(f_\Lambda h)(z)| \leq c'b'p(z) + c'C_{b'} + C'$ for all $z \in \Omega$. This means that the function $f = f_\Lambda h \not\equiv 0$ (holomorphic in Ω and having Λ as a subsequence of zeros) satisfies (15.1) with constants $c_f = c'b' < 1$ and $C_f = \exp(c'C_{b'} + C')$, i.e., $f \in H_p^1(\Omega)$ and Λ is a subsequence of zeros for $H_p^1(\Omega)$. $\square$

Remark. The assumptions for $p \in SH^+(\Omega)$ in statement (U₃) of Theorem 0.1 impose restrictions (equivalent to (LD₀¹) in Theorem 15.1) on the system of weights $\mathcal{P} = \{cp: c \in \mathbb{R}, 0 \leq c < 1\}$. In the same way as in the final remark in §13, it can be shown that (U₃) follows from Theorem 15.1.

§16. UNIQUENESS THEOREM FOR $H_{p+\log}(\Omega)$

We remind the reader (see the Introduction) that, for $p \in SH(\Omega)$, the space $H_{p+\log}(\Omega)$ was defined as the set of functions $f \in H(\Omega)$ satisfying

$$(16.1) \quad |f(z)| \leq C_f \left(\frac{1}{d_\Omega(z)} \right)^{c_f} \exp p(z), \quad z \in \Omega,$$

with some constants $C_f, c_f \geq 0$ (Ω is a bounded domain).

Theorem 16.1 (Nonuniqueness). *Let $p \not\equiv -\infty$ be a subharmonic function in a bounded domain $\Omega \Subset \mathbb{C}$ that satisfies the following condition:*

(LD₀⁰) *there exist numbers $\varepsilon, 0 < \varepsilon < 1$, and $C \geq 0$ such that*

$$(16.2) \quad \frac{1}{2\pi} \int_0^{2\pi} p(z + \varepsilon d_\Omega(z) e^{i\theta}) d\theta \leq p(z) + C \log \left(1 + \frac{1}{d_\Omega(z)} \right) + C, \quad z \in \Omega.$$

Suppose $\Sigma = \{S_k\}$, $k = 1, 2, \dots$, is a locally finite family of precompact Borel subsets of Ω that satisfies (15.3). Suppose also that a sequence Λ in Ω satisfies a stronger form of (RS), namely, with (13.4) replaced by the requirement that $\lambda^{(k)}(S_k) \leq \nu_p^{(k)}(S_k)$ for all sufficiently large k . If the measure (compare with (8.8))

$$(16.3) \quad \sigma \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{\text{diam } S_k}{d_\Omega(S_k)} \nu_p^{(k)}$$

is finite on Ω , or merely satisfies the Blaschke condition (8.17), but Ω is simply connected, then Λ is a sequence of nonuniqueness for $H_{p+\log}(\Omega)$.

Proof. We show that the functions p and $u = \log |f_\Lambda|$ (where $f_\Lambda \not\equiv 0$ is a holomorphic function in Ω with $\text{Zero}_{f_\Lambda} = \Lambda$) and the measure $\lambda_u = n_\Lambda$ satisfy the assumptions of Theorem 8.3. The inequalities $\lambda^{(k)}(S_k) \leq \nu_p^{(k)}(S_k)$ coincide with condition (8.7s) of Theorems 8.2 and 8.3 for $\nu^{(k)} = \nu_p^{(k)}$.

We choose $q > 0$ and $\delta > 0$ so small that $q + 30\delta < \varepsilon$, where ε ensures (16.2) in condition (LD₀⁰). Then by (15.3) and the final part of Proposition 11.4, the family Σ can be combinatorially inscribed in a family of domains $\mathcal{D} = \{D_k\}$, admissible for Ω , in such a way that the following is true. First, inequalities (11.19) are fulfilled for all sufficiently large k . In fact, we shall content ourselves with the existence of k_0 and a constant a , $a = 2/(\sqrt{3}\delta) > 0$, such that

$$(16.4) \quad \frac{\text{diam } S_k}{d_{D_k}(S_k)} \leq a \frac{\text{diam } S_k}{d_\Omega(S_k)} \quad \text{for all } k \geq k_0.$$

Second, the inclusions (15.7) are true for all $z \in \Omega$ close to $\partial\Omega$.

Once $\mathcal{D}$ is constructed, conditions (i)–(iii) from §7 occur in Theorem 16.1 as part of condition (RS). By (16.4), condition (15.3) ensures (8.7d) in Theorem 8.3. Finally, the same conditions (16.4) show that the measure σ in (8.8) has the same properties as σ in (16.3). Thus, by Theorem 8.3 and the remark at the end of §8, putting $r = r_\delta$ in (10.1), we see that there exists a holomorphic function $h \not\equiv 0$ in Ω for which inequality (8.16) with $t = r_\delta(z)$ takes the form

$$(16.5) \quad \log |(f_\Lambda h)(z)| \leq \mathcal{H}_{(\mathcal{D}_\star[z, r_\delta(z)])^{(\delta)}}^p(z) + \log(1 + 1/r_\delta(z)), \quad z \in \Omega.$$

By (16.5) and the subordination principle (4.4) for *balayage*, there exists $\varepsilon' > 0$ such that (15.9) is fulfilled whenever $d_\Omega(z) < \varepsilon'$. Then from (16.5) and (13.6) it follows that

$$\log |(f_\Lambda h)(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} p(z + \varepsilon d_\Omega(z) e^{i\theta}) d\theta + \log \left(1 + \frac{1}{d_\Omega(z)}\right) + \log \left(1 + \frac{1}{\delta}\right)$$

whenever $d_\Omega(z) < \varepsilon'$, $z \in \Omega$. Hence, using (16.2) and condition (LD_0^0) , we obtain

$$(16.6) \quad \log |(f_\Lambda h)(z)| \leq p(z) + C_1 \log \frac{1}{d_\Omega(z)} + C_2$$

with some constants $C_1, C_2 \geq 0$, whenever $d_\Omega(z) < \varepsilon'$, $z \in \Omega$. At the same time, (LD_0^0) shows that p is locally bounded below in Ω . Increasing C_1 and then C_2 if necessary, we can therefore extend (16.6) to all $z \in \Omega$. Thus, the function $f_\Lambda h \not\equiv 0$ belongs to $H_{p+\log}(\Omega)$ and has Λ as a subsequence of zeros. $\square$

Remark. If $\Omega = \mathbb{D}$ is the unit disk or, more generally, a simply connected domain with nonpolar boundary, it is possible to relax the restrictions on the measure σ defined by (16.3) (the nonuniqueness Theorem 16.1), and also on the measure σ in (8.8) (the preparatory Theorem 8.3). Specifically, the Blaschke condition (8.17) can be replaced by a strictly weaker condition of finiteness for a certain analog of Korenblum's density (related to the Carleson characteristic; see [5]–[11]) of a measure.

Furthermore, if in Theorem 16.1 the family Σ consists of *mutually disjoint sets*, condition (15.3) in this theorem can be lifted provided that $\lambda^{(k)}(S_k) = n_\Lambda(S_k)$, $\nu_p^{(k)}(S_k) = \nu_p(S_k)$ (as in the remark in §13) and the measure σ is finite. Indeed, the S_k with $n_\Lambda(S_k) = 0$ can be eliminated from Σ (because they give no information). Then $\nu_p(S_k) \geq n_\Lambda(S_k) \geq 1$ for the remaining S_k with sufficiently large k . Now, the convergence of the series (16.3) on Ω implies that the coefficients of $\nu_p^{(k)}$ tend to zero, which is (15.3). That is why no condition of type (15.3) occurs in the simpler version of Theorem 16.1 stated in the Introduction as (U_4) in Theorem 0.1.

§17. NONUNIQUENESS THEOREM FOR THE CLASSES $H_{p+\mathcal{S}}(\Omega)$

We recall (see the definition before Theorem 8.2) that, given $p \in SH(\Omega)$, $p \not\equiv -\infty$, and a system $\mathcal{S}$ of weights on Ω , the class $H_{p+\mathcal{S}}(\Omega)$ is defined to be the set of functions $f \in H(\Omega)$ satisfying

$$(17.1) \quad |f(z)| \leq C_f \exp(p(z) + s_f(z)), \quad z \in \Omega,$$

with some weight $s_f \in \mathcal{S}$ and some constant $C_f \geq 0$.

Theorem 17.1 (Nonuniqueness). *Suppose a system $\mathcal{S} \subset SH^+(\Omega)$ of weights on a domain $\Omega \Subset \mathbb{C}$ has properties $(A^\dagger)$ and (LD_0) of Theorem 13.1 (where $\mathcal{P}$ should be replaced with $\mathcal{S}$). Suppose also that a subharmonic function $p \not\equiv -\infty$ in Ω satisfies the condition (LDS) there exist numbers ε , $0 < \varepsilon < 1$, and $C \geq 0$ and a weight $s \in \mathcal{S}$ such that*

$$(17.2) \quad \frac{1}{2\pi} \int_0^{2\pi} p(z + \varepsilon d_\Omega(z) e^{i\theta}) d\theta + \log \left(1 + \frac{1}{d_\Omega(z)}\right) \leq p(z) + s(z) + C, \quad z \in \Omega.$$

Let $\Sigma = \{S_k\}$, $k = 1, 2, \dots$, be a locally finite family of precompact Borel subsets of Ω satisfying (15.3), and let a sequence Λ in Ω satisfy condition (RS) in Theorem 13.1, in which (13.4) is replaced with the stronger requirement $\lambda^{(k)}(S_k) \leq \nu_p^{(k)}(S_k)$ for sufficiently large k . If there exists a weight $s \in \mathcal{S}$ whose Riesz measure σ_s admits a representation (compare with (13.3))

$$(17.3) \quad \sigma_s \stackrel{\text{def}}{=} \sigma^{(0)} + \sum_{k=1}^{\infty} \sigma^{(k)},$$

where $\sigma^{(0)}$ is a finite measure or satisfies the Blaschke condition (8.17) in case Ω is simply connected, and $\sigma^{(k)}$ is supported on S_k for $k = 1, 2, \dots$, and

$$(17.4) \quad \limsup_{k \rightarrow \infty} \frac{\text{diam } S_k}{d_\Omega(S_k)} \cdot \frac{\nu^{(k)}(S_k)}{\sigma^{(k)}(S_k)} < +\infty,$$

then Λ is a nonuniqueness sequence for $H_{p+\mathcal{S}}(\Omega)$.

Unlike the preceding statement, here the “gap” between weights in the system $\mathcal{P} = p + \mathcal{S}$ is, generally speaking, greater than logarithmic. So, this theorem covers intermediate situations between those of Theorems 16.1 and 15.1. It is proved by a combination of ideas used in the proofs of Theorems 16.1 (for spaces) and 13.1 (for algebras). So, we only outline the arguments.

Outline of the proof. Let f_Λ be a holomorphic function in Ω with Λ as the sequence of zeros. Take $\varepsilon > 0$ so small that (17.2) and condition (LDS) in Theorem 13.1 are fulfilled (for S in place of $\mathcal{P}$). As in the proof of Theorem 16.1, Proposition 11.4 makes it possible to inscribe Ω combinatorially in a system of domains $\mathcal{D} = \{D_k\}$, admissible for Ω and satisfying (9.15), such that for all $z \in \Omega$ close to $\partial\Omega$ we have (15.7) and (16.4).

By Theorem 8.2, for the function $s_1 \in SH(\Omega)$ whose Riesz measure σ_1 coincides with the right-hand side of (8.8), there exists a holomorphic function $h_1 \not\equiv 0$ in Ω satisfying

$$\log |f_\Lambda(z)| + \log |h_1(z)| \leq \mathcal{H}_{(\mathcal{D}_*[z,t])^r}^p(z) + \mathcal{H}_{(\mathcal{D}_*[z,t])^r}^{s_1}(z) + \log(1 + 1/t)$$

whenever $0 < t < d_\Omega(z)$ and $z \in \Omega$. By (15.7) and (LDS), it follows that there exists $s \in \mathcal{S}$ with

$$(17.5) \quad \log |(f_\Lambda h_1)(z)| \leq p(z) + s(z) + \frac{1}{2\pi} \int_0^{2\pi} s_1(z + \varepsilon d_\Omega(z) e^{i\theta}) d\theta + C.$$

On the other hand, as in the proof of the nonuniqueness Theorem 13.1 for algebras, the representation (17.3) together with condition (17.4) (these are analogs of (13.3) and (13.4) with $\sigma^{(k)}$ in place of $\nu_p^{(k)}$ and $c_k \nu_p^{(k)}$ in place of $\lambda^{(k)}$, where c_k is taken from (8.12)) make it possible to find a holomorphic function $h_2 \not\equiv 0$ in Ω such that $s_1 + \log |h_2| \leq s$ for some $s \in \mathcal{S}$. From this, using (17.5) and (LD₀) in Theorem 13.1, we deduce the existence of $s \in \mathcal{S}$ with $\log |f_\Lambda h_1 h_2| = \log |f_\Lambda h_1| + \log |h_2| \leq p + s$. $\square$

Remark. The condition of positivity for $s \in \mathcal{S}$ can be dropped in Theorem 17.1 if, in place of (LD₀) and (LDS), we impose stronger restrictions on $\mathcal{S}$, namely, (LD₁) in Theorem 14.1 (with $\mathcal{S}$ in place of $\mathcal{P}$) and

(LDS₁) for every ε , $0 < \varepsilon < 1$, there is $s \in \mathcal{S}$ and a constant $C \geq 0$ such that (17.2) is fulfilled.

This version can be proved along the lines of the proof of Theorem 14.1.

§18. STABILITY THEOREMS FOR SEQUENCES OF ZEROS AND NONUNIQUENESS SEQUENCES

Given two sequences $\Lambda = (\lambda_k)$ and $\Gamma = (\gamma_k)$, $k = 1, 2, \dots$, in a domain $\Omega \subset \mathbb{C}$, without limit points in Ω , we consider two characteristics that show how close Λ and Γ are to each other relative to Ω , and that are defined in terms of the distance d_Ω to $\partial\Omega$ (see (2.1)):

$$(18.1) \quad \begin{aligned} \Delta_\Omega(\Lambda, \Gamma) &= \limsup_{k \rightarrow \infty} \frac{|\lambda_k - \gamma_k|}{\min\{d_\Omega(\lambda_k), d_\Omega(\gamma_k)\}}, \\ \Delta_\Omega^{\text{seg}}(\Lambda, \Gamma) &= \limsup_{k \rightarrow \infty} \frac{|\lambda_k - \gamma_k|}{d_\Omega([\lambda_k, \gamma_k])}. \end{aligned}$$

If Ω is *convex*, equality occurs in the inequality $\Delta_\Omega(\Lambda, \Gamma) \leq \Delta_\Omega^{\text{seg}}(\Lambda, \Gamma)$.

In all stability theorems that follow, Ω is a *bounded* domain and Λ and Γ are as above.

Theorem 18.1 (Stability). *Suppose one of the conditions*

$$(18.2) \quad \Delta_\Omega^{\text{seg}}(\Lambda, \Gamma) < +\infty \quad \text{or} \quad \Delta_\Omega(\Lambda, \Gamma) < 2$$

is fulfilled, and the algebra $A_{\mathcal{P}}^\uparrow(\Omega)$ is determined by a system $\mathcal{P}$ of positive weights satisfying $(A^\uparrow)$ and (LD_0) . Then Λ and Γ can be uniqueness sequences for $A_{\mathcal{P}}^\uparrow(\Omega)$ only simultaneously.

Since conditions (18.2) (and the corresponding conditions (18.8), (18.9), and (18.12) in the forthcoming uniqueness theorems 18.2, 18.3, and 18.4) are symmetric with respect to Λ and Γ , it is enough to prove this statement only “one way”; that is, it suffices to show that if Γ is a nonuniqueness sequence for the space in question, then so is Λ . Furthermore, in the proof of all stability theorems, the family $\Sigma = \{S_k\}$ in Ω will consist of two-point sets:

$$(18.3) \quad S_k = \{\lambda_k, \gamma_k\}, \quad k = 1, 2, \dots; \quad \text{diam } S_k = |\lambda_k - \gamma_k|, \quad \boxtimes S_k = \text{conv } S_k = [\lambda_k, \gamma_k].$$

This family is locally finite because Λ and Γ have no limit points in Ω . Next, in these proofs, $f_\Lambda \not\equiv 0$ is a holomorphic function with zero sequence Λ .

Proof of Theorem 18.1. Conditions (18.2) are special forms of conditions (Sd1) or (Sd3) in Proposition 11.4. So, if one of them is fulfilled, then Σ can be combinatorially inscribed in some family of domains $\mathcal{D} = \{D_k\}$ admissible for Ω and such that relations (9.15) hold true. The first of them, namely, (9.15S) coincides with condition (7.2D) of Theorem 7.1. In particular, the agreement (ii) at the beginning of §7 applies.

Suppose that Γ is a nonuniqueness sequence for $A_{\mathcal{P}}^\uparrow(\Omega)$. Then there is a nonzero function $g \in A_{\mathcal{P}}^\uparrow(\Omega)$ with $\text{Zero}_g \supset \Gamma$; this function g obeys the restriction

$$(18.4) \quad \log |g(z)| \leq p_g(z), \quad z \in \Omega,$$

for some $p_g \in A_{\mathcal{P}}^\uparrow(\Omega)$. In the notation (0.1), we can write n_Λ and n_Γ as sums of Dirac measures δ_z (i.e., point masses at points $z \in \Omega$):

$$(18.5) \quad n_\Lambda = \sum_{k=1}^{\infty} \delta_{\lambda_k}, \quad n_\Gamma = \sum_{k=1}^{\infty} \delta_{\gamma_k}.$$

Now, in the preparatory Theorem 7.1 for algebras, we take $u = \log |f_\Lambda|$ and $p = \log |g|$. In the representations (7.1), we take S_k from (18.3), and, in accordance with (18.5), choose the following measures (we write ν_p in place of ν):

$$(18.6l) \quad \lambda_u = n_\Lambda, \quad \lambda^{(0)} = 0, \quad \lambda^{(k)} = \delta_{\lambda_k}, \quad k = 1, 2, \dots,$$

$$(18.6n) \quad \nu = \nu_{\log |g|} \geq n_\Gamma, \quad \nu^{(0)} = \nu_{\log |g|} - n_\Gamma, \quad \nu^{(k)} = \delta_{\gamma_k}, \quad k = 1, 2, \dots$$

Then (with (18.5) taken into account) the agreements (i) and (iii) at the beginning of §7 apply. The obvious relations $\delta_{\lambda_k}(S_k) = \delta_{\gamma_k}(S_k) = 1$ imply condition (7.2s) of the preparatory Theorem 7.1. Thus, all the assumptions of Theorem 7.1 are fulfilled. For the role of r in (7.3), we take the function $r_\delta(z)$, $z \in \Omega$, from (10.1) with $0 < \delta < 1/6$. By Theorem 7.1 and the remark at the end of §8, there exists a function $h \not\equiv 0$ holomorphic in Ω and such that, with $t = t(z) = r_\delta(z)$ and $(\mathcal{D}_*[z, t])^r = (\mathcal{D}_*[z, r_\delta(z)))^{(\delta)}$, inequality (7.4) is fulfilled without the summand $B \log(1 + |z|)$; i.e., we have

$$(18.7) \quad \log |f_\Lambda(z)| + \log |h(z)| \leq A \mathcal{H}_{(\mathcal{D}_*[z, r_\delta(z)))^{(\delta)}}^{\log |g|}(z) + \log(1 + 1/r_\delta(z)), \quad z \in \Omega,$$

with some constant A . By (18.4), this estimate transforms to (13.5) if we replace the *balayage* of $\log |g|$ by that of p_g in (18.7).

Now, a word-for-word repetition of the arguments and calculations of the proof of Theorem 13.1 after (13.5) (with p_g in place of p) shows that the function $f_\Lambda h \not\equiv 0$, having Λ as a subsequence of zeros, belongs to $A_p^\dagger(\Omega)$. In other words, Λ is a nonuniqueness sequence for $A_p^\dagger(\Omega)$. $\square$

The other three stability theorems are established by combining the ideas of the preceding proof and the corresponding theorems in §§14–16. So, we only outline the arguments.

Theorem 18.2 (Stability). *Suppose that*

$$(18.8) \quad \Delta_\Omega(\Lambda, \Gamma) < 1,$$

and the class $A_p^\dagger(\Omega)$ is determined by a system $\mathcal{P}$ of weights satisfying $(A^\dagger)$ and (LD_1) . Then Λ and Γ can be subsequences of zeros for this class only simultaneously.

Outline of the proof. For the family (18.3) of sets, condition (18.8) means that the assumption of the final statement in Proposition 11.4 is satisfied. Consequently, there exists a family $\mathcal{D}$ as in the proof of Theorem 14.1. Suppose that Γ is a subsequence of zeros for $A_p^\dagger(\Omega)$ and consider a function g that has Λ as a subsequence of zeros and satisfies (18.4) with some $p_g \in A_p^\dagger(\Omega)$. The preparatory Theorem 7.1 (with $\log |g|$ in the role of p , and with (18.5)–(18.6)) implies (18.7), which yields (14.5) with p_g in place of p . Combining the arguments after (14.5) and (18.7) in the proofs of Theorems 14.1 and 18.1, we deduce that Λ is also a subsequence of zeros for $A_p^\dagger(\Omega)$. $\square$

Theorem 18.3 (Stability). *Suppose that*

$$(18.9) \quad \Delta_\Omega(\Lambda, \Gamma) = 0,$$

and $p \in SH^+(\Omega)$ satisfies (LD_0^1) . Then Λ and Γ can be uniqueness sequences for $H_p^1(\Omega)$ only simultaneously.

Outline of the proof. Let Γ be a nonuniqueness sequence for $H_p^1(\Omega)$. Consider a nonzero function $g \in H_p^1(\Omega)$ for which Γ is a subsequence of zeros. By the definition of $H_p^1(\Omega)$, there exists c strictly greater than 1 (and close to 1) such that

$$(18.10) \quad c \log |g(z)| \leq p(z), \quad z \in \Omega.$$

Now, we take β with $1/c < \beta < 1$. After that, we choose $\alpha, b, \varepsilon, q$, and δ so as to ensure (15.5)–(15.6) (as at the beginning of the proof of Theorem 15.1).

For the family (18.3) of sets, relation (18.9) means that (15.3) is fulfilled. Consequently, there exists an admissible family $\mathcal{D}$ with the same properties as in the proof of Theorem 15.1 and satisfying (15.7). For the subharmonic function $c \log |g|$, with Riesz measure

$$(18.11) \quad \nu_{c \log |g|} = c \Gamma = \sum_{k=1}^{\infty} c \delta_{\gamma_k},$$

we represent Λ as in (18.5) and in the notation (18.6l) but, instead of (18.6n), we put $\nu = c \nu_{\log |g|}$, $\nu^{(0)} = c(\nu_{\log |g|} - n_\Gamma)$, and $\nu^{(k)} = c \delta_{\gamma_k}$ for $k = 1, 2, \dots$. Now, in the preparatory Theorem 8.1, we put $u = \log |f_\Lambda|$ and $p = c \log |g|$. Then, accounting for (18.11) and the amendments made in (18.6), we ensure the agreements (i)–(iii) (see the beginning of §7), and also condition (8.1s), because $\frac{\lambda^{(k)}(S_k)}{\nu^{(k)}(S_k)} = \frac{\delta_{\gamma_k}(S_k)}{c \delta_{\gamma_k}(S_k)} = \frac{1}{c} < \beta$ for $k = 1, 2, \dots$. Therefore, Theorem 8.1 yields a holomorphic function $h \not\equiv 0$ satisfying (18.8) with $c \log |g|$ in the role of p . But by (8.10), this function can be replaced with p in the inequality in question. After that we argue exactly as in the proof of Theorem 15.1 after (15.8) to conclude that Λ is also a nonuniqueness sequence for $H_p^1(\Omega)$. $\square$

Theorem 18.4 (Stability). *Suppose that*

$$(18.12) \quad \sum_{k=1}^{\infty} \frac{|\lambda_k - \gamma_k|}{\min\{d_{\Omega}(\lambda_k), d_{\Omega}(\gamma_k)\}} < +\infty$$

and $p \in SH(\Omega)$ satisfies (LD_0^0) . Then Λ and Γ can be uniqueness sequences for $H_{p+\log}(\Omega)$ only simultaneously.

Outline of the proof. If Γ is a nonuniqueness sequence for $H_p(\Omega)$, then there exists a function $g \not\equiv 0$ in $H_p(\Omega)$ that has Γ as a subsequence of zeros and satisfies $\log |g| \leq p$ on G . Again, we consider the measures n_{Λ} and n_{Γ} (see (18.5)) corresponding to Λ and Γ . Then, in the notation (18.3), condition (18.12) can be treated as the finiteness of the measure

$$(18.13) \quad \sigma = \sum_{k=1}^{\infty} \frac{|\lambda_k - \gamma_k|}{\min\{d_{\Omega}(\lambda_k), d_{\Omega}(\gamma_k)\}} \delta_{\gamma_k} = \sum_{k=1}^{\infty} \frac{\text{diam } S_k}{d_{\Omega}(S_k)} \delta_{\gamma_k}.$$

It follows that, in the notation (18.6), for $u = \log |f_{\Lambda}|$ and for $\log |g|$ in the role of p , the agreements (i) and (iii) from the beginning of §7 apply. Since the sum in (18.13) is finite, we have $\lim_{k \rightarrow \infty} \frac{\text{diam } S_k}{d_{\Omega}(S_k)} = 0$; hence condition (15.3) is fulfilled. In our setting, the finiteness of σ in (18.13) is the same as the finiteness of the measure (8.8) in the preparatory Theorem 8.3, and the same as the finiteness of the measure (16.3) in the nonuniqueness Theorem 16.1. For $\lambda^{(k)} = \delta_{\lambda_k}$ and $\nu_p^{(k)} = \delta_{\gamma_k}$, the condition $\lambda^{(k)}(S_k) \leq \nu_p^{(k)}(S_k)$ of Theorem 16.1 is fulfilled for $k = 1, 2, \dots$ (we even have equality now).

We choose an admissible system $\mathcal{D}$ of domains as in the proof of Theorem 16.1. In particular, this means that the agreement (ii) at the beginning of §7 applies, and (16.4) (with the same a) and (15.7) hold true. By Theorem 8.3 (with $u = \log |f_{\Lambda}|$ and $\log |g|$ in place of p), much as in the proof of Theorem 16.1, we find a holomorphic function $h \not\equiv 0$ in Ω satisfying a version of (16.5) in which the *balayage* of p on the right is replaced with that of $\log |g|$. But the assumption $\log |g| \leq p$ allows us to return precisely to (16.5). A word-for-word repetition of the arguments after (16.5) in the proof of the nonuniqueness Theorem 16.4 shows that Λ is a nonuniqueness sequence for $H_{p+\log}(\Omega)$. $\square$

§19. NONUNIQUENESS AND STABILITY THEOREMS

FOR CLASSES OF FUNCTIONS IN THE DISK WITH A SYSTEM OF RADIAL WEIGHTS

We introduce the notation for a “polar rectangle” in $\mathbb{C}$:

$$(19.1) \quad \begin{aligned} R(t_1, t_2; \theta_1, \theta_2) &\stackrel{\text{def}}{=} \{te^{i\theta} \in \mathbb{C} : t_1 \leq t < t_2, \theta_1 \leq \theta < \theta_2\}, \\ 0 &\leq t_1 < t_2, \quad \theta_1 < \theta_2 \leq \theta_1 + 2\pi. \end{aligned}$$

In particular, $R(t_1, t_2) \stackrel{\text{def}}{=} R(t_1, t_2; 0, 2\pi)$ is an annulus, $R(t; \theta_1, \theta_2) \stackrel{\text{def}}{=} R(0, t; \theta_1, \theta_2)$ is a sector, $R(t; 0, 2\pi) = D(t)$ is a disk, and $R(\infty; \theta_1, \theta_2) \stackrel{\text{def}}{=} R(0, +\infty; \theta_1, \theta_2)$ is an angle.

For a measure μ , we use the widely accepted notation

$$(19.2r) \quad \mu(t) \stackrel{\text{def}}{=} \mu(D(t)), \quad \mu(t; \theta_1, \theta_2) \stackrel{\text{def}}{=} \mu(R(t; \theta_1, \theta_2)),$$

$$(19.2a) \quad \mu(t_1, t_2) \stackrel{\text{def}}{=} \mu(R(t_1, t_2)), \quad \mu(t_1, t_2; \theta_1, \theta_2) \stackrel{\text{def}}{=} \mu(R(t_1, t_2; \theta_1, \theta_2)).$$

For simplicity, we consider only the families $\Sigma = \{R_k\}$ such that

(R) *the subsets $R_k \subseteq \mathbb{D}$, $k = 1, 2, \dots$, are mutually disjoint and are polar rectangles, i.e., have the form*

$$(19.3) \quad R_k = R(t_k, t_{k+1}; \theta_k, \theta_{k+1}), \quad 0 \leq t_k < t_{k+1} < 1, \quad \theta_k < \theta_{k+1} \leq \theta_k + 2\pi.$$

The special family of polar rectangles defined by (see [12])

$$(19.4) \quad \begin{aligned} R_{k,l} &= R(1 - 2^{-k}, 1 - 2^{-k-1}; \pi 2^{-k}l, \pi 2^{-k}(l+1)), \\ k &= 0, 1, \dots, \quad l = 0, 1, \dots, 2^{k+1} - 1, \end{aligned}$$

is called the *dyadic family* and is denoted by $\Sigma_2 = \{R_{k,l}\}$.

A function $\varphi: I \rightarrow \mathbb{R}$, $I \subset \mathbb{R}$, is *monotone increasing* if for every $x_1, x_2 \in I$ with $x_1 \leq x_2$ we have the *nonstrict inequality* $\varphi(x_1) \leq \varphi(x_2)$.

For classes such as $H_{\mathcal{P}}^{\uparrow}(\mathbb{D})$ and $A_{\mathcal{P}}^{\uparrow}(\mathbb{D})$ with a system of radial weights, it is traditional to represent weights $p \in \mathcal{P}$ in the form

$$(19.5) \quad p(z) = p(|z|) = \varphi\left(\frac{1}{1-|z|}\right), \quad z \in \mathbb{D},$$

where there is no loss of generality in assuming that $p: [0, 1) \rightarrow [0, +\infty)$ (accordingly, $\varphi: [1, +\infty) \rightarrow [0, +\infty)$) is a *positive monotone increasing* function. For simplicity, and also for convenience of comparison with known results, we additionally assume that $p(t)$ (respectively, $\varphi(x)$) is *continuously differentiable* on $[0, 1)$ (on $[1, +\infty)$). The derivatives p' and φ' are related by the formulas

$$(19.6) \quad \begin{aligned} p'(t) &= x^2 \varphi'(x), \quad \varphi'(x) = (1-t)^2 p'(t), \quad tp'(t) = x(x-1)\varphi'(x), \\ 1 \leq x &= \frac{1}{1-t}, \quad 0 \leq t = 1 - \frac{1}{x} < 1. \end{aligned}$$

For such a p to be subharmonic, it is necessary and sufficient that $tp'(t)$ (respectively, $x(x-1)\varphi'(x)$) is monotone increasing. If p is differentiable and subharmonic, the value of the Riesz measure ν_p on a polar rectangle in (19.1) admits a fairly simple expression in terms of p' or φ' :

$$(19.7) \quad \begin{aligned} \nu_p(t_1, t_2; \theta_1, \theta_2) &= (t_2 p'(t_2) - t_1 p'(t_1)) \frac{\theta_2 - \theta_1}{2\pi} \\ &\stackrel{(19.6)}{=} (x_2(x_2-1)\varphi'(x_2) - x_1(x_1-1)\varphi'(x_1)) \frac{\theta_2 - \theta_1}{2\pi}, \\ x_j &= \frac{1}{1-t_j}, \quad j = 1, 2. \end{aligned}$$

Since the further conditions imposed on p and φ will mostly be of an asymptotic nature, we may assume (unless otherwise stipulated) that they are fulfilled only for t sufficiently close to 1 (respectively, for sufficiently large $x \geq 1$).

In particular, in terms of φ , the functions that are convex with respect to $\log x$ satisfy such conditions, and so do the functions

$$(19.8) \quad \varphi(x) = \underbrace{\log \cdots \log x}_{k \text{ times}}, \quad k \geq 1, \quad \varphi(x) = (\log x)^\alpha, \quad \alpha \geq 0,$$

for x sufficiently large (see, e.g., [8, III.1]). However, already condition (L) in the Introduction eliminates some weight function of this type from our study. For instance, within (19.8), only functions φ with $k = 1$ (respectively, $\alpha \geq 1$) remain admissible. In a sense, a minimal requirement ensuring that a system of weights of the form (0.2) on $\mathbb{D}$ satisfy (0.3) and (L) is one of the following two equivalent conditions:

- (L $_{\mathbb{D}}$) the functions $p \geq 0$ and $(1-t)p'(t)$ are monotone increasing on $[t_0, 1)$, and $\lim_{t \rightarrow 1-0} p(t) = +\infty$;
- (L $_{\mathbb{D}}$) the functions $\varphi \geq 0$ and $x\varphi'(x)$ are monotone increasing on $[x_0, +\infty)$, and $\lim_{x \rightarrow +\infty} \varphi(x) = +\infty$.

Indeed, if $(L_{\mathbb{D}})$ is fulfilled, then, by [13, Theorem 2.14], we have

$$(19.9) \quad \lim_{x \rightarrow +\infty} \frac{\varphi(x)}{\log x} > 0 \quad \stackrel{(19.5)}{\Longleftrightarrow} \quad \lim_{t \rightarrow 1-0} \frac{p(t)}{-\log(1-t)} > 0,$$

which implies (L) for the system of weights as in (19.5). For such p and φ , the nonuniqueness Theorem 13.1 and the stability Theorem 18.1 are meaningful for the algebras $A_p^\infty(\mathbb{D})$, and Theorems 16.1 and 18.4 are meaningful for the spaces $H_{p+\log}(\mathbb{D})$.

If, along with $(L_{\mathbb{D}})$, we also have

$$(19.10) \quad \lim_{x \rightarrow +\infty} \frac{\varphi(x)}{\log x} = +\infty \quad \stackrel{(19.5)}{\Longleftrightarrow} \quad \lim_{t \rightarrow 1-0} \frac{p(t)}{-\log(1-t)} = +\infty.$$

Then, for p as in (19.5), Theorem 15.1, the nonuniqueness Theorem 17.1, and the stability Theorem 18.3 make sense for $H_p^1(\mathbb{D})$ and $H_{p+\log}(\mathbb{D})$.

1. The algebras $A_p^\infty(\mathbb{D})$. Under the agreements (19.5) and $(L_{\mathbb{D}})$, and by (19.9), condition (LD_0) of the nonuniqueness Theorem 13.1 for the system (0.2) of weights turns into any of the following equivalent conditions:

$$\begin{aligned} (LD_0) \quad & \text{there exists } \varepsilon, 0 < \varepsilon < 1, \text{ with } \limsup_{t \rightarrow 1-0} \frac{p(t+\varepsilon(1-t))}{p(t)} < +\infty; \\ (LD_0) \quad & \text{there exists } a > 1 \text{ with } \limsup_{x \rightarrow +\infty} \frac{\varphi(ax)}{\varphi(x)} < +\infty. \end{aligned}$$

In terms of φ , among the weights satisfying (LD_0) , we can name, for instance, the powers of the logarithm in (19.8) with $\alpha \geq 1$, the functions $\exp(\log x)^\alpha$, and the power functions x^α for $\alpha > 0$ as well as their derivatives, etc. Condition (LD_0) easily follows from the asymptotic formulas

$$(19.11) \quad \limsup_{x \rightarrow +\infty} \frac{x\varphi'(x)}{\varphi(x)} < +\infty \quad \stackrel{(19.6)}{\Longleftrightarrow} \quad \limsup_{t \rightarrow 1-0} \frac{(1-t)p'(t)}{p(t)} < +\infty.$$

Let Λ be a sequence in $\mathbb{D}$. In the nonuniqueness theorems below, we assume the following condition:

$$(R_\Lambda) \quad \text{for some } t_0 < 1, \text{ the family } \Sigma = \{R_k\} \text{ from (R) covers the intersection of the annulus } R(t_0, 1) \text{ with the support of } \Lambda.$$

The following statement is a consequence of the nonuniqueness Theorem 13.1, because $\mathbb{D}$ is convex.

Theorem 19.1. *Let Σ satisfy (R_Λ) . Under the conditions $(L_{\mathbb{D}})$ and (LD_0) , if*

$$(19.12s) \quad \delta(\Sigma) \stackrel{\text{def}}{=} \limsup_{k \rightarrow \infty} \frac{(t_{k+1} - t_k) + (\theta_{k+1} - \theta_k)}{1 - t_{k+1}} < +\infty,$$

$$(19.12l) \quad Q_p^\Lambda(\Sigma) \stackrel{\text{def}}{=} \limsup_{k \rightarrow \infty} \frac{n_\Lambda(t_k, t_{k+1}; \theta_k, \theta_{k+1})}{(t_{k+1}p'(t_{k+1}) - t_kp'(t_k))(\theta_{k+1} - \theta_k)} < +\infty,$$

then Λ is a nonuniqueness sequence for $A_p^\infty(\mathbb{D})$.

Here (19.12s) is equivalent to (Sd2) in Proposition 11.4, and, by (19.7), condition (19.12l) is equivalent to (13.4). From $(L_{\mathbb{D}})$, we obtain

$$\begin{aligned} (19.13) \quad & t_{k+1}p'(t_{k+1}) - t_kp'(t_k) \stackrel{(19.7)}{=} x_{k+1}(x_{k+1} - 1)\varphi'(x_{k+1}) - x_k(x_k - 1)\varphi'(x_k) \\ & = x_{k+1}(x_{k+1} - x_k)\varphi'(x_{k+1}) + (x_k - 1)(x_{k+1}\varphi'(x_{k+1}) - x_k\varphi'(x_k)) \\ & \stackrel{(L_{\mathbb{D}})}{\geq} x_{k+1}(x_{k+1} - x_k)\varphi'(x_{k+1}) \stackrel{(19.6)}{=} \frac{t_{k+1} - t_k}{1 - t_k} p'(t_{k+1}), \quad x_k = \frac{1}{1 - t_k}; \end{aligned}$$

that is, in (19.12l), the first expression in parentheses in the denominator can be replaced with the rightmost term in (19.13). Moreover, if (cf. (19.11))

$$(19.14) \quad 0 < a < \liminf_{x \rightarrow +\infty} \frac{x\varphi'(x)}{\varphi(x)} \stackrel{(19.6)}{\iff} 0 < a < \liminf_{t \rightarrow 1-0} \frac{(1-t)p'(t)}{p(t)},$$

then (19.12l) in Theorem 19.1 can be replaced by a relation with no derivative, namely,

$$(19.12l') \quad \limsup_{k \rightarrow \infty} \frac{n_{\Lambda}(t_k, t_{k+1}; \theta_k, \theta_{k+1})}{\frac{t_{k+1}-t_k}{(1-t_k)(1-t_{k+1})} p(t_{k+1}) (\theta_{k+1} - \theta_k)} < +\infty.$$

If, moreover, we assume that

$$(19.15) \quad \liminf_{k \rightarrow \infty} \frac{t_{k+1} - t_k}{1 - t_{k+1}} > 0 \iff \liminf_{k \rightarrow \infty} \frac{1 - t_k}{1 - t_{k+1}} > 1 \stackrel{(19.13)}{\iff} \liminf_{k \rightarrow \infty} \frac{x_{k+1}}{x_k} > 1,$$

then (19.12l') simplifies to

$$(19.12l'') \quad \limsup_{k \rightarrow \infty} \frac{n_{\Lambda}(t_k, t_{k+1}; \theta_k, \theta_{k+1})}{p(t_{k+1})} \frac{1 - t_k}{\theta_{k+1} - \theta_k} < +\infty.$$

Finally, if we supplement (19.14) and (19.15) with yet another condition

$$(19.16) \quad \limsup_{k \rightarrow \infty} \frac{1 - t_k}{\theta_{k+1} - \theta_k} < +\infty,$$

we arrive at the final simplification of (19.12l)–(19.12l''):

$$(19.12l''') \quad \limsup_{k \rightarrow \infty} \frac{n_{\Lambda}(t_k, t_{k+1}; \theta_k, \theta_{k+1})}{p(t_{k+1})} < +\infty.$$

Thus, Theorem 19.1 implies the following statement.

Corollary 19.1. *Suppose Σ is as in (R_Λ) and, along with (19.12s), conditions (19.15) and (19.16) hold true. If (19.11), (19.14), and (19.12l''') are fulfilled, then Λ is a nonuniqueness sequence for $A_p^{\infty}(\mathbb{D})$.*

Obviously, conditions (19.12s), (19.15), and (19.16) are fulfilled for the dyadic family Σ_2 (see (19.4)).

However, in this case there is a much subtler result by Shamoyan (see [12, Theorem 2.2]). Within a certain framework, it gives a definitive description of zero sequences for the algebra $A_p^{\infty}(\mathbb{D})$ if p satisfies (19.11) and (19.14): *for a sequence Λ in $\mathbb{D}$ to be a sequence of zeros for $A_p^{\infty}(\mathbb{D})$, it is sufficient, and also necessary if $a > 1$ in (19.14), that $n_{\Lambda}(R_{k,l}) \leq \text{const } \varphi(2^k)$ for all k and l as in (19.4).* It is easily seen that, for the family Σ_2 , the last restriction is equivalent to (19.12l'''). Shamoyan's criterion implies that, for the algebras to which it is applicable, every nonuniqueness sequence is a sequence of zeros (this happens also for many other spaces determined by a system of radial weights; see [14]). Thus, in fact, Λ must be a sequence of zeros under the assumptions of Corollary 19.1 provided $a > 1$ in (19.14). In spite of the fact that the above comparison is not in favor of Corollary 19.1, this corollary is interesting in some other respect. First, the above analysis shows that, in terms of coverings, Corollary 19.1, and therefore the nonuniqueness Theorem 13.1, are sharp for algebras. Second, it demonstrates that the nonuniqueness Theorems 19.1 and 13.1, and also statements (U₁) and (U₂) of Theorem 0.1 are direct partial extensions of Shamoyan's result [12, Theorem 2.2] to arbitrary bounded domains $\Omega \Subset \mathbb{C}$ and general coverings Σ , and to a broad class of weight systems $\mathcal{P}$ on Ω ; weights may be of variable sign and need not be radial in the case of the disk.

A lower bound of nontriviality for Theorem 19.1 is the algebra $A_p^{\infty}(\mathbb{D})$, where $\varphi(x) = \log x$ in (19.5). Traditionally, this algebra is denoted by $A^{-\infty}$ (or $\mathcal{A}^{-\infty}$) and is often viewed as the union of the spaces $A^{-\alpha} = H_p(\mathbb{D})$ with $p(z) = -\alpha \log(1 - |z|)$, $z \in \mathbb{D}$, over

all $\alpha \geq 0$, or the union of all (nonweighted) Bergman spaces in $\mathbb{D}$; see [5, 14] and [6]–[10]. It is known that, for $0 \leq \alpha \leq \infty$, a nonuniqueness sequence for $A^{-\alpha}$ is also a sequence of zeros for $A^{-\alpha}$ (for $\alpha = 0$, this is a classical result of Nevanlinna; for $\alpha = \infty$, see [5, Corollary 2], and for the remaining α see [14, Corollary 1]). Thus, we have the following consequence of Theorem 19.1.

Corollary 19.2. *Suppose Σ is as in (R_Λ) , and (19.12s) is fulfilled. It (cf. (19.12l'))*

$$\limsup_{k \rightarrow \infty} \frac{n_\Lambda(t_k, t_{k+1}; \theta_k, \theta_{k+1})}{\frac{t_{k+1} - t_k}{(1-t_k)(1-t_{k+1})} (\theta_{k+1} - \theta_k)} < +\infty,$$

then Λ is a sequence of zeros for $A^{-\infty}$.

Against the background of Korenblum's deep result [5] (giving a definitive description of the zero sets for $A^{-\infty}$) and its subsequent developments (see, in particular, [6]–[8]) Corollary 19.2 is modest and can only be viewed as a fairly simple test for finding sequences of zeros for $A^{-\infty}$.

It should be noted that condition (LD_0) in Theorem 13.1 and, *a fortiori*, condition (LD_1) in Theorem 14.1 do not allow weights $p \in \mathcal{P}$ to grow fast without a substantial increase of gaps between weights. This restriction can be overcome, for instance, by replacing $d_\Omega(z)$ in (13.1) and (14.1) (see conditions (LD_0) and (LD_1)) with a function of $d_\Omega(z)$ that tends to zero as $z \rightarrow \partial\Omega$ faster than $d_\Omega(z)$, followed by an appropriate choice of the values $t = t(z)$ in (7.4) in the preparatory Theorem 7.1 for algebras. A realization of this plan for the disk was presented in Cherednikova's paper [15]. For arbitrary domains $\Omega \Subset \mathbb{C}$, as has turned out, such a version of a nonuniqueness theorem requires a laborious and extensive study, similar to that in Chapter III, but pertaining to the case of functions of d_Ω (in place of d_Ω itself). This is one reason for which we did not touch upon this version in our nonuniqueness theorems. Another, more substantial reason is that in most cases (at least for weighted algebras in the disk with rapidly growing radial weights), sufficiency tests for being a (sub)sequence of zeros are detected already at the level of traditional functions, specifically, $n_\Lambda(r)$ and the Nevanlinna characteristic $N_\Lambda(r) = \int_0^r \frac{n_\Lambda(t)}{t} dt$, $0 \leq r < 1$, and fairly often such sufficient conditions turn out to be also necessary. Such is the case for Samoyan's theorem [16, Theorem 2], though this is somewhat implicit. That theorem provides a complete description of the sequences of zeros for the algebras $A_\mathcal{P}^\infty(\mathbb{D})$ in the case where in (19.11) the usual (rather than the upper) limit is equal to $+\infty$, and under a supplementary condition (see [16, formula (2.3)]) saying that the convergence to this limit is sufficiently regular and fast. The sufficient conditions for being a sequence of zeros for $A_\mathcal{P}^1(\mathbb{D})$, where $\mathcal{P}$ is a rapidly growing system of weights, in terms of the Nevanlinna characteristic $N_\Lambda(r)$ can easily be extracted from [17, Theorem 2] and [18, Theorem 1]. In many cases, these conditions can be made necessary without any regularity assumptions about weights in $\mathcal{P}$.

Consider, for instance, the weighted algebra

$$(19.17) \quad A^\infty[\varphi] \stackrel{\text{def}}{=} \left\{ f \in H(\mathbb{D}) : \log |f(z)| \leq A_f \varphi\left(\frac{B_f}{1-|z|}\right), z \in \mathbb{D} \right\},$$

of holomorphic functions in $\mathbb{D}$, where $\varphi: (0, +\infty) \rightarrow (0, +\infty)$ is an *arbitrary monotone increasing* function, and A_f and B_f are constants depending only on f .

Theorem 19.2 (cf. [17, Corollary]). *Suppose a function φ satisfies*

$$(19.18) \quad \limsup_{b \rightarrow +\infty} \limsup_{x \rightarrow +\infty} \frac{x\varphi(x)}{\varphi(bx)} < +\infty.$$

A sequence $\Lambda \subset \mathbb{D}$ is a nonuniqueness sequence for $A^\infty[\varphi]$ if and only if

$$(19.19) \quad N_\Lambda(r) \leq A\varphi\left(\frac{B}{1-r}\right) \quad \text{for all } r < 1$$

with some constants A and B .

Proof. The necessity of (19.19) is an easy consequence of the classical Poisson–Jensen formula. Conversely, if (19.19) is true, then, by [18, Theorem 1], for the linear function $d(r) \equiv (r+1)/2$ there exists a constant A' and a function $f \in H(\mathbb{D})$ such that $f \not\equiv 0$ on $\mathbb{D}$, $\Lambda \subset \text{Zero}_f$, and

$$\begin{aligned} \log |f(z)| &\leq \frac{A'}{d(r)-r} N_\Lambda(d(r)) \\ &= \frac{2A'}{1-r} N_\Lambda\left(\frac{1+r}{2}\right) \leq \frac{2A'A}{1-r} \varphi\left(\frac{2B}{1-r}\right), \quad r > 0. \end{aligned}$$

By (19.18) and the definition (19.17), this shows that $f \in A^\infty[\varphi]$. $\square$

In conclusion of this subsection, we give a corollary of the stability Theorem 18.1.

Corollary 19.3. *Two sequences Λ and Γ in $\mathbb{D}$ satisfying $\limsup_{k \rightarrow \infty} \frac{|\lambda_k - \gamma_k|}{1 - \max\{|\lambda_k|, |\gamma_k|\}} < +\infty$ can be uniqueness sequences for $A_p^\infty(\mathbb{D})$ only simultaneously, provided p satisfies $(L_\mathbb{D})$ and (LD_0) .*

If (19.11) and (19.14) with $a > 1$ are fulfilled, the words “uniqueness sequences” can be replaced with “sequences of zeros” (surely, the same applies to any other condition under which nonuniqueness sequences are the same as sequences of zeros).

2. Spaces $H_p^1(\mathbb{D})$. For monotone increasing functions of type (19.5) and under the assumption (19.10), condition (LD_0^1) in Theorem 15.1 can be rewritten in any of the following equivalent forms:

$$(LD_0^1) \quad \lim_{\varepsilon \rightarrow +0} \limsup_{t \rightarrow 1-0} \frac{p(t + \varepsilon(1-t))}{p(t)} \leq 1 \quad \stackrel{(19.5)}{\iff} \quad \lim_{a \rightarrow 1+0} \limsup_{x \rightarrow +\infty} \frac{\varphi(ax)}{\varphi(x)} \leq 1.$$

Furthermore (see (15.1)),

$$H_p^1(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \limsup_{t \rightarrow 1-0} \frac{\log |f(te^{i\theta})|}{p(t)} < 1 \right\}.$$

In the notation (19.12), the nonuniqueness Theorem 15.1, or statement (U_1) of Theorem 0.1 for the disk and a family Σ of type (R), can be stated as follows.

Theorem 19.3. *Suppose $\delta(\Sigma) = 0$ for Σ from (R_Λ) , and conditions (19.10) , $(L_\mathbb{D})$, (LD_0^1) are satisfied. If $Q_p^\Lambda(\Sigma) < 1$, then Λ is a nonuniqueness sequence for $H_p^1(\mathbb{D})$.*

In the case of the logarithmic function $p(t) = -\alpha \log(1-t)$, it is natural to denote $H_p^1(\mathbb{D})$ by $A^{-\alpha+0} = \bigcup_{\alpha' < \alpha} A^{-\alpha'}$ (see the commentary before Corollary 19.2 about the spaces A^α). In the case of such spaces, a criterion for being a sequence of zeros (consequently, a nonuniqueness sequence) was established by Seip [6, Theorem 10]: Λ is a sequence of zeros for $A^{-\alpha+0}$ if and only if the Korenblum density of Λ is smaller than α .

In conclusion of this subsection, we present a corollary of the stability theorem 18.3 (this is (S_3) in Theorem 0.2).

Corollary 19.4. *Two sequences $\Lambda = (\lambda_k)$ and $\Gamma = (\gamma_k)$ in $\mathbb{D}$ satisfying*

$$\limsup_{k \rightarrow \infty} \frac{|\lambda_k - \gamma_k|}{1 - \max\{|\lambda_k|, |\gamma_k|\}} = 0$$

can be uniqueness sequences for $H_p^1(\mathbb{D})$ only simultaneously, provided p has properties (19.10), $(L_{\mathbb{D}})$, and (LD_0^1) .

3. Spaces $H_{p+\log}(\mathbb{D})$. For monotone increasing functions of type (19.5), condition (LD_0^0) of Theorem 16.1 can be rewritten in any of the following equivalent forms:

$$(LD_0^0) \quad \lim_{\varepsilon \rightarrow +0} \limsup_{t \rightarrow 1-0} \frac{p(t + \varepsilon(1-t)) - p(t)}{-\log(1-t)} < +\infty$$

$$\stackrel{(19.5)}{\Longleftrightarrow} \lim_{a \rightarrow +0} \limsup_{x \rightarrow +\infty} \frac{\varphi(ax) - \varphi(x)}{\log x} < +\infty.$$

Furthermore (see (16.1)),

$$H_{p+\log}(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \limsup_{t \rightarrow 1-0} \frac{\log|f(te^{i\theta})| - p(t)}{-\log(1-t)} < +\infty \right\}.$$

The nonuniqueness Theorem 16.1 or statement (U_4) of Theorem 0.1 can be formulated for families Σ of type (R) as follows.

Theorem 19.4. *In the notation (19.12s), let $\delta(\Sigma) = 0$ for Σ from (R_{Λ}) , and let conditions $(L_{\mathbb{D}})$ and (LD_0^0) be fulfilled. If*

$$\sum_{k=1}^{\infty} \frac{(t_{k+1} - t_k) + (\theta_{k+1} - \theta_k)}{1 - t_{k+1}} (t_{k+1}p'(t_{k+1}) - t_kp'(t_k)) < +\infty$$

and $n_{\Lambda}(t_k, t_{k+1}; \theta_k, \theta_{k+1}) \leq t_{k+1}p'(t_{k+1}) - t_kp'(t_k)$ for all sufficiently large k , then Λ is a nonuniqueness sequence for $H_{p+\log}(\Omega)$.

The stability Theorem 18.12 or (S_4) in Theorem 0.2 yields the following statement.

Corollary 19.5. *Two sequences $\Lambda = (\lambda_k)$ and $\Gamma = (\gamma_k)$ satisfying*

$$\sum_{k=1}^{\infty} \frac{|\lambda_k - \gamma_k|}{1 - \max\{|\lambda_k|, |\gamma_k|\}} < +\infty$$

can be uniqueness sequences for $H_{p+\log}(\mathbb{D})$ only simultaneously, provided p has properties $(L_{\mathbb{D}})$ and (LD_0^0) .

If $p(t) = -\log(1-t)$, then, clearly, $H_{p+\log}(\mathbb{D}) = A^{-\infty}$, and even Corollaries 19.2 and 19.3 are much stronger in this case. In general, the range of applicability of Theorem 19.4 and Corollary 19.5 is fairly narrow. By (LD_0^0) , this range is, roughly speaking, between $\log x$ and $(\log x)^2$ if expressed in terms of φ . Other cases can be covered by the spaces $H_{p+s}(\mathbb{D})$ in the next subsection.

4. Spaces $H_{p+s}(\mathbb{D})$. We consider the case of a system $\mathcal{S}$ of weights of type (0.2) generated by one monotone increasing differentiable function $s \geq 0$, i.e., $\mathcal{S} = \{cs : 0 < c < +\infty\}$. Assume that $s(t)$ and p satisfy a condition of type $L_{\mathbb{D}}$, i.e., $ts'(t)$ is a monotone increasing function. In this case, condition (LDS) in the nonuniqueness Theorem 17.1 can be written as follows:

$$(LDS) \quad \lim_{\varepsilon \rightarrow +0} \limsup_{t \rightarrow 1-0} \frac{p(t + \varepsilon(1-t)) - p(t)}{s(t)} < +\infty.$$

Furthermore (see (17.1)),

$$H_{p+s}(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \limsup_{t \rightarrow 1-0} \frac{\log |f(te^{i\theta})| - p(t)}{s(t)} < +\infty \right\}.$$

For the disk, the nonuniqueness Theorem 17.1 can be stated for the family Σ of type (R) as follows.

Theorem 19.5. *In the notation (19.12s), suppose $\delta(\Sigma) = 0$ for Σ from (R_Λ) ; let p and s satisfy $(L_{\mathbb{D}})$ and (LDS) , and let $(L\mathbb{D}_0)$ be fulfilled with s in place of p . If*

$$\limsup_{k \rightarrow \infty} \frac{(t_{k+1} - t_k) + (\theta_{k+1} - \theta_k)}{1 - t_{k+1}} \frac{t_{k+1}p'(t_{k+1}) - t_kp'(t_k)}{t_{k+1}s'(t_{k+1}) - t_k s'(t_k)} < +\infty$$

and $n_\Lambda(t_k, t_{k+1}; \theta_k, \theta_{k+1}) \leq t_{k+1}p'(t_{k+1}) - t_kp'(t_k)$ for all sufficiently large k , then Λ is a nonuniqueness sequence for $H_{p+s}(\Omega)$.

REFERENCES

- [1] B. N. Khabibullin, F. B. Khabibullin, and L. Yu. Cherednikova, *Subsequences of zeros for classes of holomorphic functions, their stability, and the entropy of arcwise connectedness*. I, Algebra i Analiz **20** (2008), no. 1, 146–189; English transl., St. Petersburg Math. J. **20** (2009), no. 1, 101–129. MR2411972
- [2] V. S. Azarin, *Asymptotic behavior of subharmonic functions of finite order*, Mat. Sb. (N.S.) **108** (**150**) (1979), no. 2, 147–167; English transl., Math. USSR-Sb. **36** (1979), no. 2, 135–154 (1980). MR0525835 (81e:31001)
- [3] K. Leichtweiss, *Konvexe Mengen*, Hochschultext, Springer-Verlag, Berlin–New York, 1980. MR0586235 (81j:52001)
- [4] T. J. Ransford, *Potential theory in the complex plane*, London Math. Soc. Student Texts, vol. 28, Cambridge Univ. Press, Cambridge, 1995. MR1334766 (96e:31001)
- [5] B. Korenblum, *An extension of the Nevanlinna theory*, Acta Math. **135** (1975), no. 3–4, 187–219. MR0425124 (54:13081)
- [6] K. Seip, *On a theorem of Korenblum*, Ark. Mat. **32** (1994), 237–243. MR1277927 (95f:30054)
- [7] ———, *On Korenblum’s density condition for the zero sequences of $A^{-\alpha}$* , J. Anal. Math. **67** (1995), 307–322. MR1383499 (97c:30044)
- [8] J. Bruna and X. Massaneda, *Zero sets of holomorphic functions in the unit ball with slow growth*, J. Anal. Math. **66** (1995), 217–252. MR1370351 (97f:32006)
- [9] D. Luecking, *Zero sequences for Bergman spaces*, Complex Variables Theory Appl. **30** (1996), 345–362. MR1413164 (97g:30007)
- [10] L. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman spaces*, Grad. Texts in Math., vol. 199, Springer-Verlag, New York, 2000. MR1758653 (2001c:46043)
- [11] B. N. Khabibullin, *Zero (sub)sets for spaces of holomorphic functions and (sub)harmonic minors*, Electronic Archive at LANL, 18 Dec 2004, 42 pp., <http://arxiv.org/abs/math.CV/0412359>.
- [12] F. A. Shamoyan, *A factorization theorem of M. M. Dzhrbashyan and the characteristic of zeros of functions analytic in the circle with a majorant of finite growth*, Izv. Akad. Nauk Armyan. SSR Ser. Mat. **13** (1978), no. 5–6, 405–422; English transl. in Soviet J. Contemporary Math. Anal. **13** (1978), no. 5–6. MR0541789 (80i:30054)
- [13] W. K. Hayman and P. B. Kennedy, *Subharmonic functions*. Vol. 1, London Math. Soc. Monogr., No. 9, Academic Press, London–New York, 1976. MR0460672 (57:665)
- [14] E. Beller, *Factorization for non-Nevanlinna classes of analytic functions*, Israel J. Math. **27** (1977), no. 3–4, 320–330. MR0442234 (56:620)
- [15] L. Yu. Cherednikova, *Nonuniqueness sequences for weighted algebras of holomorphic functions in the unit disk*, Mat. Zametki **77** (2005), no. 5, 775–787; English transl., Math. Notes **77** (2005), no. 5–6, 715–725. MR2178847 (2006i:30074)
- [16] F. A. Shamoyan, *Zeros of functions analytic in the disk and growing near the boundary*, Izv. Akad. Nauk Armyan. SSR Ser. Mat. **18** (1983), no. 1, 15–27; English transl., Soviet J. Contemporary Math. Anal. **18** (1983), no. 1, 13–25. MR0705983 (84g:30032)

- [17] B. N. Khabibullin, *Growth of entire functions with prescribed zeros and representation of meromorphic functions*, Mat. Zametki **73** (2003), no. 1, 120–134; English transl., Math. Notes **73** (2003), no. 1–2, 110–124. MR1993545 (2004i:30023)
- [18] ———, *Zero subsets, representation of meromorphic functions, and Nevanlinna characteristics in a disk*, Mat. Sb. **197** (2006), no. 2, 117–136; English transl., Sb. Math. **197** (2006), no. 1–2, 259–279. MR2230094 (2007b:30034)

BASHKIR STATE UNIVERSITY, INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER, URALS SCIENTIFIC CENTER, RUSSIAN ACADEMY OF SCIENCES, UFA, BASHKORTOSTAN, RUSSIA
E-mail address: `khabib-bulat@mail.ru`

BASHKIR STATE UNIVERSITY, INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER, URALS SCIENTIFIC CENTER, RUSSIAN ACADEMY OF SCIENCES, UFA, BASHKORTOSTAN, RUSSIA

BASHKIR STATE UNIVERSITY, INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER, URALS SCIENTIFIC CENTER, RUSSIAN ACADEMY OF SCIENCES, UFA, BASHKORTOSTAN, RUSSIA

Received 8/NOV/2006

Translated by S. KISLYAKOV