ASYMPTOTICS FOR THE SOLUTIONS OF ELLIPTIC SYSTEMS WITH RAPIDLY OSCILLATING COEFFICIENTS

D. BORISOV

ABSTRACT. A singularly perturbed second order elliptic system in the entire space is treated. The coefficients of the systems oscillate rapidly and depend on both slow and fast variables. The homogenized operator is obtained and, in the uniform norm sense, the leading terms of the asymptotic expansion are constructed for the resolvent of the operator described by the system. The convergence of the spectrum is established, and examples are given.

INTRODUCTION

There are many publications devoted to homogenization of differential operators with rapidly oscillating coefficients in bounded domains (see, e.g., [1]–[6]). Similar questions for operators in unbounded domains attracted considerably less attention. On the other hand, during recent years the case of an unbounded domain has been studied intensively. In the series of papers [6]–[13], Birman and Suslina developed a new original technique, which allowed them to prove convergence theorems, to obtain order precise estimates for the rates of convergence, and to construct the first terms in the expansion for the resolvent of a wide class of differential operators with rapidly oscillating coefficients in unbounded domains. It should be emphasized that these results were obtained for the uniform norm, while usually results for bounded domains are formulated in the sense of strong or weak convergence. The approach of Birman and Suslina is based on spectral theory and treats homogenization as a threshold phenomenon. It applies to the operators that admit factorization, and at the same time their coefficients must depend on the fast variable x/ε only; no dependence on the slow variable x is allowed. We should also note the paper [14] by Zhikov, where, by employing another technique, he obtained order precise estimates for the rate of convergence for the resolvent of a scalar operator as well as for the case of the operator of elasticity theory. Again, it was assumed that the coefficients are periodic and depend only on the fast variable.

The one-dimensional scalar operators with coefficients depending on both fast and slow variables were studied in [15]–[17]. In [15], the Schrödinger operator with rapidly oscillating and compactly supported potential was considered. The object of study was the phenomenon of a new eigenvalue emerging from the threshold of the continuous spectrum. The case of a periodic operator (independent of the small parameter) perturbed by a rapidly oscillating compactly supported potential with an increasing amplitude was

2000 *Mathematics Subject Classification.* Primary 35B27.

Key words and phrases. Homogenization of differentiable operators, unbounded domain, fast and slow variables.

This work was supported in part by RFBR (07-01-00037) and by the Czech Academy of Sciences and Ministry of Education, Youth and Sports (LC06002). The author was also supported by *Marie Curie International Fellowship* within 6th European Community Framework Program (MIF1-CT-2005-006254), by a grant from the 2004 Balzan prize in mathematics, awarded to Pierre Deligne, and by a grant from the Bashkortostan Republic Program for supporting young scientists.

studied in the paper [16], where the structure and the behavior of the spectrum were investigated in detail. In [17], the Schrödinger operator with compactly supported potential independent of the small parameter was considered; perturbation was by a rapidly oscillating periodic potential. In that paper, the asymptotic behavior of the spectrum was described. We note that homogenization of resolvents was not treated in [15]–[17]. At the same time, the technique employed there allows one to study this question and to obtain results similar to those of [7].

In the present paper we consider a quite general second order elliptic system in the entire space. Compared to the operators treated in [6]–[13], the main difference is the presence of lower order terms. More precisely, the second order part of our operator is written in the divergence form, as in the papers cited. The lower terms are quite arbitrary; the only restriction is that the operator is selfadjoint and lower semibounded uniformly in the small parameter. A certain smoothness of the coefficients is also assumed. Yet another difference as compared to [6]–[14] is that in our case the coefficients depend on both slow and fast variables. The dependence on fast variables is periodic. The coefficients are assumed to be uniformly bounded with respect to slow variables; the same is assumed for certain derivatives of the coefficients.

In this paper, we construct the homogenized operator and obtain the first terms of the asymptotic expansion for the resolvent of the perturbed operator for all values of the spectral parameter separated *a priori* from the spectrum of the homogenized operator. These asymptotic expansions are obtained for the resolvent treated as an operator in L_2 as well as an operator from L_2 into W_2^1 . We borrow the main ideas of [14] to obtain these results. Moreover, we assume the coefficients to be smoother than in [6]–[14], which allows us to simplify certain details in the arguments. In particular, this makes it possible to avoid the smoothing used in the papers cited. At the end, we give examples of some operators to which our results can be applied.

§1. FORMULATION OF THE PROBLEM AND THE MAIN RESULTS

Let $x = (x_1, \dots, x_d)$ be Cartesian coordinates in $\mathbb{R}^d$, $d \geq 1$, and let $B = B(\zeta)$ be a matrix-valued function,

$$B(\zeta) = \sum_{i=1}^d B_i \zeta_i,$$

where $\zeta = (\zeta_1, \dots, \zeta_d)$, the B_i are constant complex-valued matrices of size $m \times n$, and $m \geq n$. We assume that $\text{rank } B(\zeta) = n$, $\zeta \neq 0$.

Let Y be a Banach space. By $W_\infty^k(\mathbb{R}^d; Y)$ we denote the Sobolev space of functions defined on $\mathbb{R}^d$, having values in Y , and such that

$$\|\mathbf{u}\|_{W_\infty^k(\mathbb{R}^d; Y)} := \max_{|\alpha| \leq k} \text{ess sup}_{x \in \mathbb{R}^d} \left\| \frac{\partial^{|\alpha|} \mathbf{u}}{\partial x^\alpha} \right\|_Y < \infty.$$

If $k = 0$, we use the notation $L_\infty(\mathbb{R}^d; Y)$.

In the space $\mathbb{R}^d$ we fix a lattice; its elementary cell is denoted by $\square$. We let $C_{\text{per}}^\gamma(\overline{\square})$ denote the space of $\square$ -periodic functions with finite Hölder norm $\|\cdot\|_{C^\gamma(\overline{\square})}$. The norm in this space coincides with the norm of $C^\gamma(\overline{\square})$.

We shall often treat a vector-function $\mathbf{f} = \mathbf{f}(x, \xi)$, $\square$ -periodic with respect to ξ , as a mapping that takes points $x \in \mathbb{R}^d$ to a function depending on ξ , defined by the rule $x \mapsto \mathbf{f}(x, \cdot)$. This will allow us to talk of a function $\mathbf{f}(x, \xi)$ as belonging to the space $W_2^k(Q; C_{\text{per}}^\gamma(\overline{\square}))$ or to the space $W_\infty^k(Q; C_{\text{per}}^\gamma(\overline{\square}))$.

Let $A = A(x, \xi)$ be a matrix-valued function of size $m \times m$. We suppose that the matrix A is Hermitian and $\square$ -periodic with respect to ξ , and that the estimate

$$(1.1) \quad c_1 E_m \leq A(x, \xi) \leq c_2 E_m$$

is valid uniformly in $(x, \xi) \in \mathbb{R}^{2d}$, where E_m is the unit $m \times m$ matrix. We also assume that $A \in W_\infty^1(\mathbb{R}^d; C_{\text{per}}^{1+\beta}(\square)) \cap W_\infty^2(\mathbb{R}^d; C_{\text{per}}^\beta(\square))$ for some $\beta \in (0, 1)$. By $V = V(x, \xi)$ and $a_i = a_i(x, \xi)$ we denote matrix-valued functions of size $n \times n$, $\square$ -periodic with respect to ξ . It is assumed that $a_i \in W_\infty^1(\mathbb{R}^d; C_{\text{per}}^{1+\beta}(\square)) \cap W_\infty^2(\mathbb{R}^d; C_{\text{per}}^\beta(\square))$, $V \in W_\infty^1(\mathbb{R}^d; C_{\text{per}}^\beta(\square))$, that the matrix V is Hermitian, and that the matrices a_j and B_j are complex-valued. Let $b_i = b_i(x) \in W_\infty^2(\mathbb{R}^d)$ be complex-matrix-valued functions of size $n \times n$.

We denote by ε a small positive parameter. Given a function $f(x, \xi)$, we put $f_\varepsilon(x) = f(x, \frac{x}{\varepsilon})$; for instance, $A_\varepsilon(x) := A(x, \frac{x}{\varepsilon})$.

Our aim in the present paper is to study the spectral properties of the operator

$$(1.2) \quad \mathcal{H}_\varepsilon := B(\partial)^* A_\varepsilon B(\partial) + a_\varepsilon(x, \partial) + V_\varepsilon,$$

with

$$a_\varepsilon(x, \partial) := a\left(x, \frac{x}{\varepsilon}, \partial\right), \quad a(x, \xi, \zeta) := \sum_{i=1}^d (a_i(x, \xi) \zeta_i b_i(x) - b_i^*(x) \zeta_i a_i^*(x, \xi)),$$

in $L_2(\mathbb{R}^d; \mathbb{C}^n)$, with $W_2^2(\mathbb{R}^d; \mathbb{C}^n)$ as the domain. Here $\partial = (\partial_1, \dots, \partial_d)$, ∂_i is the derivative with respect to x_i , the superscript $*$ means conjugation, and

$$B(\partial) := \sum_{i=1}^d B_i \partial_i, \quad B(\partial)^* := - \sum_{i=1}^d B_i^* \partial_i.$$

We shall show that the operator $\mathcal{H}_\varepsilon$ is selfadjoint and lower semibounded uniformly in ε (see Lemma 2.2).

Let $\Lambda_0 = \Lambda_0(x, \xi)$ and $\Lambda_1 = \Lambda_1(x, \xi)$ be matrices of size $n \times n$ and $n \times m$, respectively, that are $\square$ -periodic with respect to ξ and solve the equations

$$(1.3) \quad \begin{aligned} B(\partial_\xi)^* A(x, \xi) B(\partial_\xi) \Lambda_0(x, \xi) - \sum_{i=1}^d b_i^*(x) \frac{\partial a_i^*}{\partial \xi_i}(x, \xi) &= 0, \quad (x, \xi) \in \mathbb{R}^{2d}, \\ B(\partial_\xi)^* A(x, \xi) (B(\partial_\xi) \Lambda_1(x, \xi) + E_m) &= 0, \quad (x, \xi) \in \mathbb{R}^{2d}. \end{aligned}$$

It is assumed that the Λ_i satisfy the conditions

$$(1.4) \quad \int_{\square} \Lambda_i(x, \xi) d\xi = 0, \quad x \in \mathbb{R}^d, \quad i = 1, 2.$$

Here $\partial_\xi = \left(\frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_d}\right)$. In what follows, we shall show that such solutions of (1.3), (1.4) exist and are unique, and that $\Lambda_i \in W_\infty^1(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\square))$ (see the proof of Lemma 2.4).

Let $\mathcal{H}_0$ be the operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ defined as

$$(1.5) \quad \mathcal{H}_0 = B(\partial)^* A_2 B(\partial) + A_1(x, \partial) + A_0,$$

with

$$\begin{aligned}
 A_2(x) &:= \frac{1}{|\square|} \int_{\square} A(x, \xi) (B(\partial_{\xi}) \Lambda_1(x, \xi) + E_m) d\xi, \\
 A_1(x, \partial) &:= \frac{1}{|\square|} B(\partial)^* \int_{\square} A(x, \xi) B(\partial_{\xi}) \Lambda_0(x, \xi) d\xi \\
 &\quad + \left(\frac{1}{|\square|} \int_{\square} (B(\partial_{\xi}) \Lambda_0(x, \xi))^* A(x, \xi) d\xi \right) B(\partial) \\
 &\quad + \frac{1}{|\square|} \int_{\square} a(x, \xi, \partial) d\xi, \\
 A_0(x) &:= -\frac{1}{|\square|} \int_{\square} (B(\partial_{\xi}) \Lambda_0(x, \xi))^* A(x, \xi) B(\partial_{\xi}) \Lambda_0(x, \xi) d\xi \\
 &\quad + \frac{1}{|\square|} \int_{\square} V(x, \xi) d\xi,
 \end{aligned}
 \tag{1.6}$$

on the domain $W_2^2(\mathbb{R}^d; \mathbb{C}^n)$. In Lemma 2.4 we show that this operator is selfadjoint and lower semibounded, and that its coefficients are sufficiently smooth. We denote by $\mathfrak{h}_0$ the lower bound of $\mathcal{H}_0$.

Let $G = G(x, \xi) \in W_{\infty}^1(\mathbb{R}^d; C_{\text{per}}^{\beta}(\overline{\square}))$ be a positive Hermitian matrix of size $n \times n$. We also assume that the inverse matrix G^{-1} is uniformly bounded. Let $\mathfrak{g}_i > 0$, $i = 1, 2$, denote constants independent of ε , x , and ξ , such that

$$\mathfrak{g}_1 E_n \leq G(x, \xi) \leq \mathfrak{g}_2 E_n. \tag{1.7}$$

We put

$$G_0(x) := \frac{1}{|\square|} \int_{\square} G(x, \xi) d\xi.$$

We introduce the operator

$$\mathcal{L}_{\varepsilon} := \left(\Lambda_1 \left(x, \frac{x}{\varepsilon} \right) B(\partial) + \Lambda_0 \left(x, \frac{x}{\varepsilon} \right) \right). \tag{1.8}$$

It will be shown that for each ε the operator $\mathcal{L}_{\varepsilon}$ acts boundedly from $W_2^1(\mathbb{R}^d; \mathbb{C}^n)$ to $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and from $W_2^2(\mathbb{R}^d; \mathbb{C}^n)$ to $W_2^1(\mathbb{R}^d; \mathbb{C}^n)$ (see Lemma 3.1).

Our first result describes an approximation for the generalized resolvent of $\mathcal{H}_{\varepsilon}$.

Theorem 1.1. *Suppose $\lambda \in \mathbb{C} \setminus [\mu_0, +\infty)$, $\mu_0 := \min \left\{ \frac{\mathfrak{h}_0}{\mathfrak{g}_1}, \frac{\mathfrak{h}_0}{\mathfrak{g}_2} \right\}$. Then for all small ε the inequalities*

$$\begin{aligned}
 &\|(\mathcal{H}_{\varepsilon} - \lambda G_{\varepsilon})^{-1} - (\mathcal{H}_0 - \lambda G_0)^{-1}\|_{L_2 \rightarrow L_2} \leq C\varepsilon, \\
 &\|(\mathcal{H}_{\varepsilon} - \lambda G_{\varepsilon})^{-1} - (\mathbf{I} + \varepsilon \mathcal{L}_{\varepsilon})(\mathcal{H}_0 - \lambda G_0)^{-1}\|_{L_2 \rightarrow W_2^1} \leq C\varepsilon,
 \end{aligned}
 \tag{1.9}$$

hold true, where $\mathbf{I}$ is the identity operator, the constants C are independent of ε , and the norms are regarded as those for operators from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and to $W_2^1(\mathbb{R}^d; \mathbb{C}^n)$, respectively.

In what follows, $\sigma(\cdot)$ denotes the spectrum.

Corollary 1.2. *The spectrum of $\mathcal{H}_{\varepsilon}$ converges to the spectrum of $\mathcal{H}_0$. Namely, if $\lambda \notin \sigma(\mathcal{H}_0)$, then $\lambda \notin \sigma(\mathcal{H}_{\varepsilon})$ for all sufficiently small ε , and if $\lambda \in \sigma(\mathcal{H}_0)$, then there exists $\lambda_{\varepsilon} \in \sigma(\mathcal{H}_{\varepsilon})$ such that $\lambda_{\varepsilon} \rightarrow \lambda_0$ as $\varepsilon \rightarrow +0$. If $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \sigma(\mathcal{H}_0)$, then for the spectral projectors of $\mathcal{H}_{\varepsilon}$ and $\mathcal{H}_0$ we have $\mathcal{P}_{(\alpha_1, \alpha_2)}(\mathcal{H}_{\varepsilon}) \rightarrow \mathcal{P}_{(\alpha_1, \alpha_2)}(\mathcal{H}_0)$, $\varepsilon \rightarrow +0$.*

It should be mentioned that, in the papers [10, 13, 14], a particular case of the operator $\mathcal{H}_{\varepsilon}$ corresponding to $a_i = 0$, $b_i = 0$, and $V = 0$ was considered, and it was assumed that $A = A(\xi)$. In this case, estimates similar to (1.9) were obtained. It should be emphasized

that in the papers cited the matrices A and G were not assumed to be smooth, but bounded only. Moreover, the constants C in the above-mentioned estimates depend only on the L_∞ -norm of the matrices A , G , G^{-1} and on the lattice. In our case, these constants depend on λ , on the lattice, and on the norms of the coefficients in the spaces to which they belong.

§2. AUXILIARY STATEMENTS

In the present section we prove a series of auxiliary statements needed in the proofs of Theorem 1.1 and Corollary 1.2.

Lemma 2.1. *For any $\mathbf{u} \in W_2^1(\mathbb{R}^d; \mathbb{C}^n)$, the estimate*

$$C_1 \|\nabla \mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2 \leq (A_\varepsilon B(\partial) \mathbf{u}, B(\partial) \mathbf{u})_{L_2(\mathbb{R}^d; \mathbb{C}^m)} \leq C_2 \|\nabla \mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2$$

holds true uniformly in ε .

Proof. By (1.1), we have

$$c_1 \|B(\partial) \mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2 \leq (A_\varepsilon B(\partial) \mathbf{u}, B(\partial) \mathbf{u})_{L_2(\mathbb{R}^d; \mathbb{C}^m)} \leq c_2 \|B(\partial) \mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2.$$

Now the desired inequality follows from [7, Chapter 2, §1, estimate (1.11)]. $\square$

Lemma 2.2. *The operator $\mathcal{H}_\varepsilon$ is selfadjoint and lower semibounded uniformly in ε .*

Proof. Semiboundedness follows easily from the properties of the coefficients of $\mathcal{H}_\varepsilon$ and the identity

$$\begin{aligned} (\mathcal{H}_\varepsilon \mathbf{u}, \mathbf{u})_{L_2(\mathbb{R}^d; \mathbb{C}^n)} &= h_\varepsilon[\mathbf{u}] := (A_\varepsilon B(\partial) \mathbf{u}, B(\partial) \mathbf{u})_{L_2(\mathbb{R}^d; \mathbb{C}^m)} \\ &\quad + 2 \operatorname{Re} \sum_{i=1}^d (a_{i,\varepsilon} \partial_i b_i \mathbf{u}, \mathbf{u})_{L_2(\mathbb{R}^d; \mathbb{C}^n)} + (V_\varepsilon \mathbf{u}, \mathbf{u})_{L_2(\mathbb{R}^d; \mathbb{C}^n)}. \end{aligned}$$

It is clear that the operator $\mathcal{H}_\varepsilon$ is symmetric; to prove that it is selfadjoint, it suffices to show that $\mathcal{D}(\mathcal{H}_\varepsilon^*) = \mathcal{D}(\mathcal{H}_\varepsilon)$. In turn, this identity can be established easily if a weak solution of the equation

$$(2.1) \quad (B(\partial)^* A_\varepsilon B(\partial) + a_\varepsilon(x, \partial) + V_\varepsilon) \mathbf{u} = \mathbf{f}, \quad x \in \mathbb{R}, \quad \mathbf{f} \in L_2(\mathbb{R}^d; \mathbb{C}^n),$$

belongs to $W_2^2(\mathbb{R}^d; \mathbb{C}^n)$. We prove this.

Clearly, a weak solution of (2.1) is also a weak solution of

$$(2.2) \quad \begin{aligned} B(\partial)^* A_\varepsilon B(\partial) \mathbf{u} + \mathbf{u} &= \mathbf{g}, \quad x \in \mathbb{R}^d, \\ \mathbf{g} &:= \mathbf{f} - a_\varepsilon(x, \partial) \mathbf{u} - V_\varepsilon \mathbf{u} + \mathbf{u}, \quad \|\mathbf{g}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C (\|\mathbf{f}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} + \|\mathbf{u}\|_{W_2^1(\mathbb{R}^d; \mathbb{C}^n)}). \end{aligned}$$

Let $\delta \neq 0$ be a small fixed number, and let $\mathbf{e}_i^{(d)}$, $i = 1, \dots, d$, be the standard basis in $\mathbb{R}^d$. We denote

$$\mathbf{u}_\delta^{(i)}(x) := \frac{1}{\delta} \left(\mathbf{u}(x + \delta \mathbf{e}_i^{(d)}) - \mathbf{u}(x) \right).$$

This function is a weak solution of (2.2) with the right-hand side

$$\mathbf{g}_\delta^{(i)} - B(\partial)^* A_{\varepsilon, \delta}^{(i)} B(\partial) \mathbf{u}(x + \delta \mathbf{e}_i^{(d)}),$$

where $\mathbf{g}_\delta^{(i)}$ and $A_{\varepsilon, \delta}^{(i)}$ are defined via $\mathbf{g}$ and A_ε as in the formula for $\mathbf{u}_\delta^{(i)}$. The integral identity corresponding to the equation for $\mathbf{u}_\delta^{(i)}$ reads as follows:

$$\begin{aligned} & (A_\varepsilon B(\partial) \mathbf{u}_\delta^{(i)}, B(\partial) \varphi)_{L_2(\mathbb{R}^d; \mathbb{C}^m)} + (\mathbf{u}_\delta^{(i)}, \varphi)_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \\ &= - (\mathbf{g}, \varphi_{-\delta}^{(i)})_{L_2(\mathbb{R}^d; \mathbb{C}^n)} - (A_{\varepsilon, \delta}^{(i)} B(\partial) \mathbf{u}(\cdot + \delta \mathbf{e}_i^{(d)}), B(\partial) \varphi)_{L_2(\mathbb{R}^d; \mathbb{C}^n)}, \end{aligned}$$

where $\varphi \in W_2^1(\mathbb{R}^d; \mathbb{C}^n)$. We also observe that inequality (11) in the proof of item a) of Theorem 3 in [18, Chapter III, § 3.4] implies that

$$\|\varphi_{-\delta}^{(i)}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq \|\varphi\|_{W_2^1(\mathbb{R}^d; \mathbb{C}^n)}$$

for each $\varphi \in W_2^1(\mathbb{R}^d; \mathbb{C}^n)$. Letting $\varphi := \mathbf{u}_\delta^{(i)}$ in two last inequalities and taking the smoothness of A and Lemma 2.1 into account, we see that

$$\|\mathbf{u}_\delta^{(i)}\|_{W_2^1(\mathbb{R}^d; \mathbb{C}^n)} \leq C (\|\mathbf{g}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} + \|\mathbf{u}\|_{W_2^1(\mathbb{R}^d; \mathbb{C}^n)})$$

uniformly in δ . Using this estimate and repeating the arguments of the proof of item b) of Theorem 3 in [18, Chapter III, §3.4], we easily check that the second generalized derivatives of the function $\mathbf{u}$ exist, and that

$$(2.3) \quad \|\mathbf{u}\|_{W_2^2(\mathbb{R}^d; \mathbb{C}^n)} \leq C (\|\mathbf{g}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} + \|\mathbf{u}\|_{W_2^1(\mathbb{R}^d; \mathbb{C}^n)}). \quad \square$$

Lemma 2.3. *Let $\mathbf{f}(x, \cdot) \in C_{\text{per}}^\beta(\overline{\square})$ for all $x \in \mathbb{R}^d$. The system*

$$(2.4) \quad B(\partial_\xi)^* A(x, \xi) B(\partial_\xi) \mathbf{v}(x, \xi) = \mathbf{f}(x, \xi), \quad \xi \in \mathbb{R}^d,$$

has a solution $\mathbf{v}(x, \cdot) \in C_{\text{per}}^{2+\beta}(\overline{\square})$, $\square$ -periodic with respect to ξ , if and only if

$$(2.5) \quad \int_{\square} \mathbf{f}(x, \xi) d\xi = 0, \quad x \in \mathbb{R}^d.$$

If this solvability condition is fulfilled, then a solution of (2.4) is unique up to a vector independent of ξ . There exists a unique solution of (2.4) such that

$$(2.6) \quad \int_{\square} \mathbf{v}(x, \xi) d\xi = 0, \quad x \in \mathbb{R}^d.$$

This solution satisfies the estimate

$$(2.7) \quad \|\mathbf{v}(x, \cdot)\|_{C_{\text{per}}^{2+\beta}(\overline{\square})} \leq C \|\mathbf{f}(x, \cdot)\|_{C_{\text{per}}^\beta(\overline{\square})},$$

where the constant C is independent of $x \in \mathbb{R}^d$ and $\mathbf{f}$. If $\mathbf{f} \in W_\infty^k(\mathbb{R}^d; C_{\text{per}}^\beta(\overline{\square}))$, $k = 0, 1$, then $\mathbf{v} \in W_\infty^k(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))$, and we have

$$(2.8) \quad \|\mathbf{v}\|_{W_\infty^k(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))} \leq C \|\mathbf{f}\|_{W_\infty^k(\mathbb{R}^d; C_{\text{per}}^\beta(\overline{\square}))},$$

where the constant C is independent of $\mathbf{f}$.

Proof. The existence of a $\square$ -periodic weak solution of (2.4) in the class $W_2^1(\square; \mathbb{C}^n)$, the solvability condition (2.5), and the uniqueness of the solution satisfying (2.6) are implied by Theorem 1 in [2, Appendix]. Moreover, the proof of that theorem shows that

$$(2.9) \quad \|\mathbf{v}(x, \cdot)\|_{W_2^1(\square; \mathbb{C}^n)} \leq C \|\mathbf{f}(x, \cdot)\|_{L_2(\square; \mathbb{C}^n)}.$$

Throughout the proof, we denote by C inessential constants independent of $\mathbf{f}$ and $x \in \mathbb{R}^d$.

In the same way as for equation (2.1), we can show that $\mathbf{v}(x, \cdot) \in H_{\text{loc}}^2(\mathbb{R}^d)$. By Theorem 10.7 and Remark 1 in [19, Chapter IV, §10.3], the periodicity of $\mathbf{f}$ and $\mathbf{v}$ implies that

$$\|\mathbf{v}(x, \cdot)\|_{C^{2+\beta}(\overline{\square})} \leq C (\|\mathbf{f}(x, \cdot)\|_{C^\beta(\overline{\square})} + \|\mathbf{v}(x, \cdot)\|_{L_1(\square)}).$$

This estimate and (2.9) yield (2.7).

Assume that $\mathbf{f} \in W_\infty^k(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))$; we prove the claimed smoothness of $\mathbf{v}$. If $k = 0$ and $\mathbf{f} \in L_\infty(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))$, the fact that $\mathbf{v} \in L_\infty(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))$ and estimate (2.8) follow immediately from (2.7). Let $k = 1$. We take a small number $\delta \neq 0$, choose a point $x \in Q$, and put

$$\mathbf{v}_\delta^{(i)}(x, \xi) := \frac{1}{\delta} (\mathbf{v}(x + \delta \mathbf{e}_i^{(d)}, \xi) - \mathbf{v}(x, \xi)).$$

This function solves system (2.4) at x with the right-hand side

$$\mathbf{f}_\delta^{(i)} - B(\partial_\xi)A_{\varepsilon,\delta}^{(i)}B(\partial_\xi)\mathbf{v}(x + \delta\mathbf{e}_i^{(d)}, \xi),$$

where $\mathbf{f}_\delta^{(i)}$ and $A_{\varepsilon,\delta}^{(i)}$ are determined by analogy with $\mathbf{v}_\delta^{(i)}$. This right-hand side satisfies (2.5) and belongs to $L_\infty(\mathbb{R}^d; C_{\text{per}}^\beta(\overline{\square}))$. Now from (2.8) for $k = 0$ it follows that

$$(2.10) \quad \begin{aligned} \|\mathbf{v}_\delta^{(i)}\|_{L_\infty(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))} &\leq C\|\mathbf{f}_\delta^{(i)} - B(\partial_\xi)A_{\varepsilon,\delta}^{(i)}B(\partial_\xi)\mathbf{v}(\cdot + \delta, \cdot)\|_{L_\infty(\mathbb{R}^d; C_{\text{per}}^\beta(\overline{\square}))} \\ &\leq C\|\mathbf{f}\|_{C_1^*(\mathbb{R}^d; C_{\text{per}}^\beta(\overline{\square}))}, \end{aligned}$$

where C is independent of $\mathbf{f}$ and δ .

Let $\mathbf{v}_0^{(i)}$ be a solution of system (2.4) at x with the right-hand side

$$\frac{\partial \mathbf{f}}{\partial x_i}(x, \xi) - B(\partial_\xi)^* \frac{\partial A}{\partial x_i}(x, \xi)B(\partial_\xi)\mathbf{v}$$

satisfying (2.6). Clearly, the right-hand side satisfies (2.5) and belongs to the space $L_\infty(\mathbb{R}^d; C_{\text{per}}^\beta(\overline{\square}))$. The function $\mathbf{v}_\delta^{(i)} - \mathbf{v}_0^{(i)}$ solves system (2.4) at x with the right-hand side

$$\mathbf{f}_\delta^{(i)} - \frac{\partial \mathbf{f}}{\partial x_i} + B(\partial_\xi)^* \left(\frac{\partial A}{\partial x_i} - A_\delta^{(i)} \right) B(\partial_\xi)\mathbf{v} - \delta B(\partial_\xi)^* A_\delta^{(i)} B(\partial_\xi)\mathbf{v}_\delta^{(i)}.$$

Using (2.7) and (2.10), we see that

$$\|\mathbf{v}_\delta^{(i)} - \mathbf{v}_0^{(i)}\|_{L_\infty(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))} \xrightarrow{\delta \rightarrow 0} 0.$$

Hence, the derivative $\frac{\partial \mathbf{v}}{\partial x_i}$ exists and $\frac{\partial \mathbf{v}}{\partial x_i} = \mathbf{v}_0^{(i)} \in L_\infty(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))$. Also, inequality (2.7) implies inequality (2.8) with $k = 1$. $\square$

Lemma 2.4. *The operator $\mathcal{H}_0$ is selfadjoint and lower semibounded. The matrices A_2 , A_0 and the coefficients of $A_1(x, \partial)$ belong to $W_\infty^1(\mathbb{R}^d)$. We have*

$$(2.11) \quad \|\mathbf{u}\|_{W_2^2(\mathbb{R}^d; \mathbb{C}^n)} \leq C \left(\|\mathcal{H}_0 \mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} + \|\mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \right).$$

Proof. We begin with the proof of the solvability of (1.3), (1.4). Let $\mathbf{\Lambda}_0^{(j)} = \mathbf{\Lambda}_0^{(j)}(x, \xi) \in \mathbb{C}^n$, $j = 1, \dots, n$, and $\mathbf{\Lambda}_1^{(j)} = \mathbf{\Lambda}_1^{(j)}(x, \xi) \in \mathbb{C}^n$, $j = 1, \dots, m$, be solutions of the problems

$$B(\partial_\xi)^* A B(\partial_\xi) \mathbf{\Lambda}_0^{(j)} - \sum_{i=1}^d b_i^* \frac{\partial a_i^*}{\partial \xi_i} \mathbf{e}_j^{(n)} = 0, \quad (x, \xi) \in \mathbb{R}^{2d},$$

$$B(\partial_\xi)^* A (B(\partial_\xi) \mathbf{\Lambda}_1^{(j)} + \mathbf{e}_j^{(m)}) = 0, \quad (x, \xi) \in \mathbb{R}^{2d},$$

$\square$ -periodic with respect to ξ and satisfying (2.6). Here $A = A(x, \xi)$, $a_i = a_i(x, \xi)$, and $\mathbf{e}_j^{(n)}$, $j = 1, \dots, n$, are the elements of the standard basis in $\mathbb{C}^n$. Lemma 2.3 implies that these problems are uniquely solvable and that their solutions belong to $W_\infty^1(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))$. The vectors $\mathbf{\Lambda}_i^{(j)}$ are columns of the matrices Λ_i . Hence, the matrices A_2 and A_0 , as well as the coefficients of the operator $A_1(x, \partial)$, belong to $W_\infty^1(\mathbb{R}^d; \mathbb{C}^n)$.

We denote $X := (B(\partial_\xi) \Lambda_1 + E_m)$. Integrating by parts and using equations (1.3), we obtain

$$\begin{aligned} &\int_{\square} (B(\partial_\xi) \Lambda_1(x, \xi))^* A(x, \xi) X(x, \xi) d\xi \\ &= \int_{\square} \Lambda_1^*(x, \xi) B(\partial_\xi)^* A(x, \xi) (B(\partial_\xi) \Lambda_1(x, \xi) + E_m) d\xi = 0, \end{aligned}$$

which implies that

$$(2.12) \quad A_2(x) = \frac{1}{|\square|} \int_{\square} X^*(x, \xi) A(x, \xi) X(x, \xi) d\xi.$$

This identity and the definition of the lower terms of $\mathcal{H}_0$ show that this operator is symmetric. We prove that it is lower semibounded. With the help of the last identity and (1.1), it is easy to check that

$$\begin{aligned} (A_2(x)\mathbf{w}, \mathbf{w})_{\mathbb{C}^n} &= \left(A(x, \cdot) X(x, \cdot) \mathbf{w}, X(x, \cdot) \mathbf{w} \right)_{L_2(\square; \mathbb{C}^n)} \\ &\geq c_1 \left(\|\mathbf{w}\|_{\mathbb{C}^n}^2 |\square| + 2 \operatorname{Re} \left(B(\partial_\xi) \Lambda_1(x, \cdot) \mathbf{w}, \mathbf{w} \right)_{L_2(\square; \mathbb{C}^n)} + \|B(\partial_\xi) \Lambda_1(x, \cdot) \mathbf{w}\|_{L_2(\square; \mathbb{C}^n)}^2 \right) \\ &= c_1 \left(\|\mathbf{w}\|_{\mathbb{C}^n}^2 |\square| + \|B(\partial_\xi) \Lambda_1(x, \cdot) \mathbf{w}\|_{L_2(\square; \mathbb{C}^n)}^2 \right) \geq c_1 |\square| \|\mathbf{w}\|_{\mathbb{C}^n}^2 \end{aligned}$$

for all $x \in \mathbb{R}^d$ and $\mathbf{w} \in \mathbb{C}^n$. Combined with inequality (1.11) in [7, Chapter 2, §1], this yields

$$\|\nabla \mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2 \leq C (A_2 B(\partial) \mathbf{u}, B(\partial) \mathbf{u})_{L_2(\mathbb{R}^d; \mathbb{C}^n)}.$$

Using this estimate as in Lemma 2.2, we can easily check that $\mathcal{H}_0$ is selfadjoint. Estimate (2.11) follows from the corresponding analog of (2.3) and the obvious estimate

$$\|\mathbf{u}\|_{W_2^1(\mathbb{R}^d; \mathbb{C}^n)} \leq C (\|\mathcal{H}_0 \mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} + \|\mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}). \quad \square$$

§3. THE RESOLVENT ASYMPTOTICS

In this section we prove Theorem 1.1 and Corollary 1.2. For the proofs, we need four lemmas.

Lemma 3.1. *For each $\varepsilon > 0$ the operator $\mathcal{L}_\varepsilon$ defined by (1.8) is bounded as an operator from $W_2^2(\mathbb{R}^d; \mathbb{C}^n)$ to $W_2^1(\mathbb{R}^d; \mathbb{C}^n)$. The operator $\mathcal{L}_\varepsilon$ is bounded uniformly in ε as an operator from $W_2^1(\mathbb{R}^d; \mathbb{C}^n)$ to $L_2(\mathbb{R}^d; \mathbb{C}^n)$.*

The lemma follows from the fact that $\Lambda_i \in W_\infty^1(\mathbb{R}^d; C^{2+\beta}(\overline{\square}))$, proved in Lemma 2.4.

In what follows, we denote by $\frac{\partial u}{\partial x_i}$ the partial derivatives with respect to x_i of $u = u(x, \frac{x}{\varepsilon})$ treated as functions of the independent variables x and $\xi = \frac{x}{\varepsilon}$. In the same way we regard the partial derivatives $\frac{\partial u}{\partial \xi_i}$. We also recall that the symbols ∂_i denote the full derivatives with respect to x_i , i.e.,

$$(3.1) \quad \partial_i u \left(x, \frac{x}{\varepsilon} \right) = \frac{\partial u}{\partial x_i} \left(x, \frac{x}{\varepsilon} \right) + \varepsilon^{-1} \frac{\partial u}{\partial \xi_i} \left(x, \frac{x}{\varepsilon} \right),$$

and $\partial = (\partial_1, \dots, \partial_d)$.

Lemma 3.2. *Let $M = M(x, \xi)$ be a matrix of size $n \times n$, $\square$ -periodic with respect to ξ and such that*

$$(3.2) \quad M \in W_\infty^1(\mathbb{R}^d; C_{\text{per}}^\beta(\overline{\square})), \quad \int_{\square} M(x, \xi) d\xi = 0, \quad x \in \mathbb{R}^d,$$

and let $\mathbf{u}(x) \in W_2^1(\mathbb{R}^d; \mathbb{C}^n)$ be a vector-valued function. Then there exist vector-valued functions $\mathbf{v}_i^{(\varepsilon)}(x) \in L_2(\mathbb{R}^d; \mathbb{C}^n)$, $i = 0, \dots, d$, such that

$$M \left(x, \frac{x}{\varepsilon} \right) \mathbf{u}(x) = \varepsilon \sum_{i=1}^d \partial_i \mathbf{v}_i^{(\varepsilon)}(x) + \varepsilon \mathbf{v}_0^{(\varepsilon)}(x), \quad \|\mathbf{v}_i^{(\varepsilon)}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C \|\mathbf{u}\|_{W_2^1(\mathbb{R}^d; \mathbb{C}^n)},$$

where the constant C is independent of ε and $\mathbf{u}$.

Proof. Let $P = P(x, \xi)$ be a matrix of size $n \times n$, $\square$ -periodic with respect to ξ and satisfying the equation

$$\Delta_\xi P(x, \xi) = M(x, \xi), \quad (x, \xi) \in \mathbb{R}^{2d},$$

along with condition (2.6) for all $x \in \mathbb{R}^d$. By Lemma 2.3 this equation is solvable, the matrix P is determined uniquely, and $\|P\|_{W_\infty^1(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))} < \infty$. Using this estimate, we can easily check that the lemma is valid for

$$\mathbf{v}_i^{(\varepsilon)} := \frac{\partial P}{\partial \xi_i} \mathbf{u}, \quad \mathbf{v}_0^{(\varepsilon)} := - \sum_{i=1}^d \frac{\partial^2 P \mathbf{u}}{\partial x_i \partial \xi_i}, \quad P = P\left(x, \frac{x}{\varepsilon}\right). \quad \square$$

We denote

$$(3.3) \quad \widehat{A}_1(x, \xi) := A(x, \xi)(B(\partial_\xi) \Lambda_1(x, \xi) + E_m) - A_2(x).$$

Lemma 3.3. *Suppose $\mathbf{u} \in W_2^2(\mathbb{R}^d; \mathbb{C}^n)$. There exist vector-valued functions $\mathbf{v}_i^{(\varepsilon)} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$, $i = 1, \dots, d$, such that*

$$B(\partial_x)^* \widehat{A}_1\left(x, \frac{x}{\varepsilon}\right) \mathbf{v}(x) = \varepsilon \sum_{i=1}^d \partial_i \mathbf{v}_i^{(\varepsilon)}(x), \quad \|\mathbf{v}_i^{(\varepsilon)}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C \|\mathbf{v}\|_{W_2^1(\mathbb{R}^d; \mathbb{C}^n)},$$

where the constant C is independent of ε and $\mathbf{u}$.

Proof. Let $P^{(i)} = P^{(i)}(x, \xi)$ be matrices of size $n \times m$, $\square$ -periodic with respect to ξ and satisfying the equations

$$(3.4) \quad \Delta_\xi P^{(i)}(x, \xi) = -B_i^* \widehat{A}_1(x, \xi), \quad (x, \xi) \in \mathbb{R}^{2d},$$

and condition (2.6). By Lemma 2.3, these equations are solvable, the matrices $P^{(i)}$ are determined uniquely, and $P^{(i)} \in W_\infty^1(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))$. Equations (1.3) and the definition of $\widehat{A}$ imply that this matrix is $\square$ -periodic with respect to ξ and satisfies the equation

$$(3.5) \quad B(\partial_\xi)^* \widehat{A}_1(x, \xi) = 0, \quad (x, \xi) \in \mathbb{R}^{2d}.$$

This and (3.4) yield

$$(3.6) \quad \Delta_\xi \sum_{i=1}^d \frac{\partial P^{(i)}}{\partial \xi_i} = 0, \quad (x, \xi) \in \mathbb{R}^d,$$

and by the unique solvability of this equation we obtain

$$\sum_{i=1}^d \frac{\partial P^{(i)}}{\partial \xi_i} = 0.$$

Together with (3.5), this implies that

$$(3.7) \quad -B_i^* \widehat{A}_1 = \sum_{j=1}^d \frac{\partial M_{ij}}{\partial \xi_j}, \quad M_{ij} := \frac{\partial P^{(i)}}{\partial \xi_j} - \frac{\partial P^{(j)}}{\partial \xi_i}.$$

Thus, we arrive at the formula

$$(3.8) \quad B(\partial_x)^* \widehat{A}_1 \mathbf{v} = \sum_{i,j=1}^d \frac{\partial^2 M_{ij} \mathbf{v}}{\partial x_i \partial \xi_j} = \varepsilon \sum_{i,j=1}^d \partial_j \frac{\partial M_{ij} \mathbf{v}}{\partial x_i} - \varepsilon \sum_{i,j=1}^d \frac{\partial^2 M_{ij} \mathbf{v}}{\partial x_i \partial x_j},$$

where $\widehat{A}_1 = \widehat{A}_1(x, \frac{x}{\varepsilon})$ and $M_{ij} = M_{ij}(x, \frac{x}{\varepsilon})$. The second term on the right-hand side of this identity is zero because $M_{ij} = -M_{ji}$. Now the claim follows from the fact that $P^{(i)} \in W_\infty^1(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))$ if we put

$$\mathbf{v}_i^{(\varepsilon)}(x) := \sum_{j=1}^d \frac{\partial}{\partial x_j} M_{ji} \left(x, \frac{x}{\varepsilon} \right) \mathbf{v}(x). \quad \square$$

We denote

$$(3.9) \quad \widehat{A}_0(x, \xi) := A(x, \xi) B(\partial_\xi) \Lambda_0(x, \xi) - \frac{1}{|\square|} \int_{\square} A(x, \xi) B(\partial_\xi) \Lambda_0(x, \xi) d\xi.$$

Lemma 3.4. *Suppose $\mathbf{v} \in W_2^2(\mathbb{R}^d; \mathbb{C}^n)$. There exist vector-valued functions $\mathbf{v}_i^{(\varepsilon)} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$, $i = 0, \dots, d$, such that*

$$B(\partial_x)^* \widehat{A}_0 \left(x, \frac{x}{\varepsilon} \right) \mathbf{v}(x) = \varepsilon \sum_{i=1}^d \partial_i \mathbf{v}_i^{(\varepsilon)}(x) + \varepsilon \mathbf{v}_0^{(\varepsilon)}(x), \quad \|\mathbf{v}_i^{(\varepsilon)}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C \|\mathbf{v}\|_{W_2^2(\mathbb{R}^d; \mathbb{C}^n)},$$

where the constant C is independent of ε and $\mathbf{u}$.

Proof. We argue much as in the proof of Lemma 3.3; only minor modifications are needed. We introduce the matrices $P^{(i)}$ as the solutions of (3.4), (2.6) with $\widehat{A}_1$ replaced by $\widehat{A}_0$. Then $P^{(i)} \in W_\infty^1(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))$. By the first equation in (1.3), analogs of relations (3.5), (3.6), and (3.7) look like this:

$$\begin{aligned} B(\partial_\xi)^* \widehat{A}_0 &= \sum_{i=1}^d b_i^* \frac{\partial a_i^*}{\partial \xi_i}, & Q &:= \sum_{i=1}^d \frac{\partial P^{(i)}}{\partial \xi_i}, \\ -B_i^* \widehat{A}_0 &= \sum_{j=1}^d \frac{\partial M_{ij}}{\partial \xi_j} + \frac{\partial Q}{\partial \xi_i}, & \Delta_\xi Q &= \sum_{i=1}^d b_i^* \frac{\partial a_i^*}{\partial \xi_i}. \end{aligned}$$

Employing these identities, by analogy with (3.8) we obtain

$$\begin{aligned} B(\partial_x)^* \widehat{A}_0 \mathbf{v} &= \varepsilon \sum_{i,j=1}^d \partial_j \frac{\partial M_{ij} \mathbf{v}}{\partial x_i} + \sum_{i=1}^d \frac{\partial^2 Q \mathbf{v}}{\partial x_i \partial \xi_i} \\ &= \varepsilon \sum_{i,j=1}^d \partial_j \frac{\partial M_{ij} \mathbf{v}}{\partial x_i} + \varepsilon \sum_{i=1}^d \partial_i \frac{\partial Q \mathbf{v}}{\partial x_i} - \varepsilon \sum_{i=1}^d \frac{\partial^2 Q \mathbf{v}}{\partial x_i^2}. \end{aligned}$$

Now we put

$$\begin{aligned} \mathbf{v}_i^{(\varepsilon)}(x) &:= \sum_{j=1}^d \frac{\partial}{\partial x_j} M_{ji} \left(x, \frac{x}{\varepsilon} \right) \mathbf{v}(x) + \frac{\partial}{\partial x_i} Q \left(x, \frac{x}{\varepsilon} \right) \mathbf{v}(x), \\ \mathbf{v}_0^{(\varepsilon)}(x) &:= - \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} Q \left(x, \frac{x}{\varepsilon} \right) \mathbf{v}(x). \end{aligned}$$

The properties of $P^{(i)}$ mentioned above show that the vector-valued functions $\mathbf{v}_i^{(\varepsilon)}$ belong to $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and satisfy the required estimate. To complete the proof, it remains to establish the same for $\mathbf{v}_0^{(\varepsilon)}$. For this, it suffices to check that $Q \in W_\infty^2(\mathbb{R}^d; C_{\text{per}}^\beta(\overline{\square}))$.

Let $Q^{(i)}$ be the solutions of the equations

$$\Delta_\xi Q^{(i)} = b_i^* \left(a_i^* - \frac{1}{|\square|} \int_{\square} a_i^*(\cdot, \xi) d\xi \right), \quad (x, \xi) \in \mathbb{R}^{2d},$$

satisfying condition (2.6). Applying Lemma 2.3 and differentiating these equations with respect to x_j , it is easy to check that $Q^{(i)} \in W_\infty^2(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))$. Clearly,

$$Q = \sum_{i=1}^d \frac{\partial Q^{(i)}}{\partial \xi_i},$$

which implies $Q \in W_\infty^2(\mathbb{R}^d; C_{\text{per}}^\beta(\overline{\square}))$, as claimed. $\square$

Let $\mathfrak{h}_\varepsilon$ be the lower bound of $\mathcal{H}_\varepsilon$, and let $\mu_\varepsilon := \min \left\{ \frac{\mathfrak{h}_\varepsilon}{\mathfrak{g}_1}, \frac{\mathfrak{h}_\varepsilon}{\mathfrak{g}_2} \right\}$.

Lemma 3.5. *Suppose $\lambda \in \mathbb{C} \setminus [\mu, +\infty)$, where $\mu_\varepsilon - \mu \geq c > 0$, and the constant c is independent of ε . Then the weak solution $\mathbf{u} \in W_2^1(\mathbb{R}^d; \mathbb{C}^n)$ of the equation*

$$(B(\partial)^* A_\varepsilon B(\partial) + a_\varepsilon(x, \partial) + V_\varepsilon - \lambda G_\varepsilon) \mathbf{u} = \mathbf{f}_0 + \sum_{i=1}^d \partial_i \mathbf{f}_i, \quad \mathbf{f}_i \in L_2(\mathbb{R}^d; \mathbb{C}^n),$$

satisfies the estimate

$$\|\mathbf{u}\|_{W_2^1(\mathbb{R}^d; \mathbb{C}^n)} \leq C(\lambda) \sum_{i=0}^d \|\mathbf{f}_i\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)},$$

where the constant $C(\lambda)$ is independent of ε and $\mathbf{f}_i$.

Proof. Using Lemma 2.1, the identity

$$(3.10) \quad h_\varepsilon[\mathbf{u}] - \lambda(G_\varepsilon \mathbf{u}, \mathbf{u})_{L_2(\mathbb{R}^d; \mathbb{C}^n)} = (\mathbf{f}_0, \mathbf{u})_{L_2(\mathbb{R}^d; \mathbb{C}^n)} - \sum_{i=1}^d (\mathbf{f}_i, \partial_i \mathbf{u})_{L_2(\mathbb{R}^d; \mathbb{C}^n)},$$

and (1.7), we can prove that

$$(3.11) \quad \|\mathbf{u}\|_{W_2^1(\mathbb{R}^d; \mathbb{C}^n)} \leq C(\lambda) \left(\sum_{i=0}^d \|\mathbf{f}_i\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} + \|\mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \right),$$

where the constant C is independent of ε and $\mathbf{f}_i$. Since the first term on the left-hand side of (3.10) is real, identity (3.10) implies that

$$-\text{Im } \lambda(G_\varepsilon \mathbf{u}, \mathbf{u})_{L_2(\mathbb{R}^d; \mathbb{C}^n)} = \text{Im}(\mathbf{f}_0, \mathbf{u})_{L_2(\mathbb{R}^d; \mathbb{C}^n)} - \text{Im} \sum_{i=1}^d (\mathbf{f}_i, \partial_i \mathbf{u})_{L_2(\mathbb{R}^d; \mathbb{C}^n)}.$$

If $\text{Im } \lambda \neq 0$, this and (1.7) yield

$$(3.12) \quad \|\mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2 \leq \delta \|\mathbf{u}\|_{W_2^1(\mathbb{R}^d; \mathbb{C}^n)}^2 + C(\delta, \lambda) \sum_{i=0}^d \|\mathbf{f}_i\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2,$$

where the number δ can be chosen arbitrarily small, and the constant C is independent of ε and $\mathbf{f}_i$. If $\lambda \in (-\infty, \mu)$, the last estimate is also valid, which follows from (3.10) and the inequality

$$\begin{aligned} h_\varepsilon[\mathbf{u}] - \lambda(G_\varepsilon \mathbf{u}, \mathbf{u})_{L_2(\mathbb{R}^d; \mathbb{C}^n)} &\geq \mathfrak{h}_\varepsilon \|\mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2 - \mu(G_\varepsilon \mathbf{u}, \mathbf{u})_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \\ &\geq (\mathfrak{h}_\varepsilon - \mu \mathfrak{g}) \|\mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2 \geq (\mu_\varepsilon - \mu) \mathfrak{g} \|\mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2 \geq c \mathfrak{g} \|\mathbf{u}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2, \end{aligned}$$

where

$$\mathfrak{g} = \begin{cases} \mathfrak{g}_2 & \text{if } \mu \geq 0, \\ \mathfrak{g}_1 & \text{if } \mu < 0. \end{cases}$$

Now the lemma follows from estimates (3.11) and (3.12). $\square$

Proof of Theorem 1.1 for λ nonreal. Let $\mathbf{f} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$, and let

$$\begin{aligned} \mathbf{u}^{(\varepsilon)} &:= (\mathcal{H}_\varepsilon - \lambda G_\varepsilon)^{-1} \mathbf{f}, \quad \mathbf{u}^{(0)} := (\mathcal{H}_0 - \lambda G_0)^{-1} \mathbf{f}, \\ \mathbf{u}^{(1)}(x, \xi) &:= (\Lambda_1(x, \xi) B(\partial) + \Lambda_0(x, \xi)) \mathbf{u}^{(0)}(x), \quad \widehat{\mathbf{u}}^{(\varepsilon)}(x) := \mathbf{u}^{(0)}(x) + \varepsilon \mathbf{u}^{(1)}\left(x, \frac{x}{\varepsilon}\right). \end{aligned}$$

Obviously,

$$\begin{aligned} (3.13) \quad & (B(\partial)^* A_\varepsilon B(\partial) + a_\varepsilon(x, \partial) + V_\varepsilon - \lambda G_\varepsilon) (\mathbf{u}^{(\varepsilon)} - \widehat{\mathbf{u}}^{(\varepsilon)}) \\ &= \mathbf{f} - (B(\partial)^* A_\varepsilon B(\partial) + a_\varepsilon(x, \partial) + V_\varepsilon - \lambda G_\varepsilon) \widehat{\mathbf{u}}^{(\varepsilon)} \\ &= (\mathcal{H}_0 - \lambda G_0) \mathbf{u}^{(0)} - (B(\partial)^* A_\varepsilon B(\partial) + a_\varepsilon(x, \partial) + V_\varepsilon - \lambda G_\varepsilon) \widehat{\mathbf{u}}^{(\varepsilon)} =: \mathbf{F}^{(\varepsilon)}. \end{aligned}$$

We evaluate the function $\mathbf{F}^{(\varepsilon)}$. Taking identities (3.1) and the second equation in (1.3) into account, we obtain

$$\begin{aligned} (3.14) \quad & B(\partial)^* AB(\partial) \widehat{\mathbf{u}}^{(\varepsilon)} = B(\partial_x)^* AB(\partial) \mathbf{u}^{(0)} + B(\partial_x)^* AB(\partial_\xi) \mathbf{u}^{(1)} \\ & + \varepsilon B(\partial)^* AB(\partial_x) \mathbf{u}^{(1)} + \varepsilon^{-1} B(\partial_\xi)^* AB(\partial) \mathbf{u}^{(0)} \\ & + \varepsilon^{-1} B(\partial_\xi)^* AB(\partial_\xi) \mathbf{u}^{(1)} \\ & = B(\partial_x)^* (A + B(\partial_\xi) \Lambda_1) B(\partial) \mathbf{u}^{(0)} + B(\partial_x)^* AB(\partial_\xi) \Lambda_0 \mathbf{u}^{(0)} \\ & + \varepsilon B(\partial)^* AB(\partial_x) \mathbf{u}^{(1)} + \varepsilon^{-1} B(\partial_\xi)^* AB(\partial_\xi) \Lambda_0 \mathbf{u}^{(0)}, \\ & a_\varepsilon(x, \partial) \widehat{\mathbf{u}}^{(\varepsilon)} = \sum_{i=1}^d \left(a_i \frac{\partial}{\partial x_i} b_i \mathbf{u}^{(0)} - b_i^* \frac{\partial}{\partial x_i} a_i^* \mathbf{u}^{(0)} \right) - \varepsilon^{-1} \sum_{i=1}^d b_i^* \frac{\partial a_i^*}{\partial \xi_i} \mathbf{u}^{(0)} \\ & + \varepsilon \sum_{i=1}^d \left(a_i \frac{\partial}{\partial x_i} b_i \mathbf{u}^{(1)} + \frac{\partial b_i^*}{\partial x_i} a_i^* \mathbf{u}^{(1)} - \partial_i b_i^* a_i^* \mathbf{u}^{(1)} \right) \\ & + \sum_{i=1}^d a_i b_i \frac{\partial \mathbf{u}^{(1)}}{\partial \xi_i}. \end{aligned}$$

Here the arguments of all functions except $\mathbf{u}^{(0)}(x)$, $\mathbf{u}^{(\varepsilon)}(x)$, and $\widehat{\mathbf{u}}^{(\varepsilon)}(x)$ are $(x, \frac{x}{\varepsilon})$. Integrating by parts and using (1.3), we see that

$$\begin{aligned} (3.15) \quad & \int_{\square} (B(\partial_\xi) \Lambda_0)^* A \, d\xi = \int_{\square} \Lambda_0^* B(\partial_\xi)^* A \, d\xi \\ & = - \int_{\square} \Lambda_0^* B(\partial_\xi)^* AB(\partial_\xi) \Lambda_1 \, d\xi \\ & = - \int_{\square} (B(\partial_\xi)^* AB(\partial_\xi) \Lambda_0)^* \Lambda_1 \, d\xi \\ & = - \sum_{i=1}^d \int_{\square} \frac{\partial a_i}{\partial \xi_i} b_i \Lambda_1 \, d\xi = \sum_{i=1}^d \int_{\square} a_i b_i \frac{\partial \Lambda_1}{\partial \xi_i} \, d\xi, \\ & \int_{\square} (B(\partial_\xi) \Lambda_0)^* AB(\partial_\xi) \Lambda_0 \, d\xi = \int_{\square} (AB(\partial_\xi) \Lambda_0)^* B(\partial_\xi) \Lambda_0 \, d\xi \\ & = \int_{\square} (B(\partial_\xi)^* AB(\partial_\xi) \Lambda_0)^* \Lambda_0 \, d\xi \\ & = \sum_{i=1}^d \int_{\square} \frac{\partial a_i}{\partial \xi_i} b_i \Lambda_0 \, d\xi = - \sum_{i=1}^d \int_{\square} a_i b_i \frac{\partial \Lambda_0}{\partial \xi_i} \, d\xi. \end{aligned}$$

Here the arguments of all matrices are (x, ξ) . These identities and (1.6) yield

$$\begin{aligned} A_1(x, \partial) &:= \frac{1}{|\square|} B(\partial)^* \int_{\square} A(x, \xi) B(\partial_{\xi}) \Lambda_0(x, \xi) d\xi \\ &\quad + \left(\frac{1}{|\square|} \int_{\square} a_i(x, \xi) b_i(x) \frac{\partial \Lambda_1}{\partial \xi_i}(x, \xi) d\xi \right) B(\partial) + \frac{1}{|\square|} \int_{\square} a(x, \xi, \partial) d\xi, \\ A_0(x) &:= \frac{1}{|\square|} \sum_{i=1}^d \int_{\square} a_i(x, \xi) b_i(x) \frac{\partial \Lambda_0}{\partial \xi_i}(x, \xi) d\xi + \frac{1}{|\square|} \int_{\square} V(x, \xi) d\xi. \end{aligned}$$

Now we use these relations, (3.14), (1.3), and the definition of $\mathbf{u}^{(1)}$ to obtain

$$\begin{aligned} \mathbf{F}^{(\varepsilon)} &= \mathbf{F}_1^{(\varepsilon)} + \mathbf{F}_2^{(\varepsilon)} + \mathbf{F}_3^{(\varepsilon)}, \quad \mathbf{F}_1^{(\varepsilon)} = -B(\partial_x)^* \hat{A}_1 B(\partial) \mathbf{u}^{(0)} - B(\partial_x)^* \hat{A}_0 \mathbf{u}^{(0)}, \\ \mathbf{F}_2^{(\varepsilon)} &= \sum_{i=1}^d \left(\frac{1}{|\square|} \int_{\square} a_i(\cdot, \xi) d\xi - a_i \right) \frac{\partial}{\partial x_i} b_i \mathbf{u}^{(0)} \\ &\quad - \sum_{i=1}^d b_i^* \frac{\partial}{\partial x_i} \left(\frac{1}{|\square|} \int_{\square} a_i^*(\cdot, \xi) d\xi - a_i^* \right) \mathbf{u}^{(0)} \\ &\quad + \sum_{i=1}^d \left(\frac{1}{|\square|} \int_{\square} a_i(\cdot, \xi) b_i(\cdot) \frac{\partial \Lambda_1}{\partial \xi_i}(\cdot, \xi) d\xi - a_i b_i \frac{\partial \Lambda_1}{\partial \xi_i} \right) B(\partial) \mathbf{u}^{(0)} \\ &\quad + \sum_{i=1}^d \left(\frac{1}{|\square|} \int_{\square} a_i(\cdot, \xi) b_i(\cdot) \frac{\partial \Lambda_0}{\partial \xi_i}(\cdot, \xi) d\xi - a_i b_i \frac{\partial \Lambda_0}{\partial \xi_i} \right) \mathbf{u}^{(0)} \\ &\quad + \left(\frac{1}{|\square|} \int_{\square} V(\cdot, \xi) d\xi - V \right) \mathbf{u}^{(0)} + \lambda(G - G_0) \mathbf{u}^{(0)}, \\ \mathbf{F}_3^{(\varepsilon)} &= -\varepsilon B(\partial)^* A B(\partial_x) \mathbf{u}^{(1)} - \varepsilon \sum_{i=1}^d \left(a_i \frac{\partial}{\partial x_i} b_i + \frac{\partial b_i^*}{\partial x_i} a_i^* - \partial_i b_i^* a_i^* \right) \mathbf{u}^{(1)} - \varepsilon(V - \lambda G) \mathbf{u}^{(1)}, \end{aligned}$$

where the arguments of the functions are $(x, \frac{x}{\varepsilon})$. The relation $\Lambda_i \in W_{\infty}^1(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))$ and the inequality

$$(3.16) \quad \|\mathbf{u}^{(0)}\|_{W_2^2(\mathbb{R}^d; \mathbb{C}^n)} \leq C \|\mathbf{f}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}$$

imply that

$$\left\| B(\partial_x) \mathbf{u}^{(1)} \left(x, \frac{x}{\varepsilon} \right) \right\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C \|\mathbf{f}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}.$$

From this point on, till the end of the proof, we denote by C inessential constants independent of ε and $\mathbf{f}$. The estimate obtained above shows that the function $\mathbf{F}_3^{(\varepsilon)}$ can be represented as

$$(3.17) \quad \mathbf{F}_3^{(\varepsilon)} = \varepsilon \sum_{i=1}^d \frac{\partial \mathbf{f}_{3,i}^{(\varepsilon)}}{\partial x_i} + \varepsilon \mathbf{f}_{3,0}^{(\varepsilon)}, \quad \|\mathbf{f}_{3,i}^{(\varepsilon)}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C \|\mathbf{f}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}, \quad i = 0, \dots, d.$$

The formula for $\mathbf{F}_2^{(\varepsilon)}$ immediately implies that this function is a sum of terms satisfying the assumptions of Lemma 3.2. Hence, by that lemma, we have

$$\mathbf{F}_2^{(\varepsilon)} = \varepsilon \sum_{i=1}^d \frac{\partial \mathbf{f}_{2,i}^{(\varepsilon)}}{\partial x_i} + \varepsilon \mathbf{f}_{2,0}^{(\varepsilon)}, \quad \|\mathbf{f}_{2,i}^{(\varepsilon)}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C \|\mathbf{f}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}, \quad i = 0, \dots, d,$$

where the constant C is independent of ε and $\mathbf{f}$. These identities, Lemma 3.3, and relations (3.16), (3.17) yield

$$\mathbf{F}^{(\varepsilon)} = \varepsilon \sum_{i=1}^d \frac{\partial \mathbf{f}_i^{(\varepsilon)}}{\partial x_i} + \varepsilon \mathbf{f}_0^{(\varepsilon)}, \quad \|\mathbf{f}_i^{(\varepsilon)}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C \|\mathbf{f}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}, \quad i = 0, \dots, d.$$

We substitute this representation in (3.13); by Lemma 3.5, we arrive at the estimate

$$\|\mathbf{u}^{(\varepsilon)} - \hat{\mathbf{u}}^{(\varepsilon)}\|_{W_2^1(\mathbb{R}^d; \mathbb{C}^n)} \leq C \sum_{i=0}^d \|\mathbf{f}_i^{(\varepsilon)}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C\varepsilon \|\mathbf{f}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}.$$

This leads us immediately to the second estimate in (1.9). Using this estimate and Lemma 3.1, we obtain

$$\begin{aligned} & \|\mathbf{u}^{(\varepsilon)} - \mathbf{u}^{(0)}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \\ & \leq C \|\mathbf{u}^{(\varepsilon)} - \hat{\mathbf{u}}^{(\varepsilon)}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} + C\varepsilon \|\mathcal{L}_\varepsilon(\mathcal{H}_0 - \lambda G_0)^{-1} \mathbf{f}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C\varepsilon \|\mathbf{f}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}. \end{aligned}$$

The first estimate in (1.9) is proved. $\square$

Corollary 1.2 is implied by the first estimate in (1.9) with $G = G_0 = E_n$ and Theorems VIII.23 and VIII.24 in [20, Chapter VIII, §7].

Proof of Theorem 1.1 for $\lambda \in (-\infty, \mu_0)$. By Corollary 1.2, the lower bound of $\mathcal{H}_\varepsilon$ converges to that of $\mathcal{H}_0$. Hence, $\mu_\varepsilon \rightarrow \mu_0$ as $\varepsilon \rightarrow +0$, and thus, for sufficiently small ε the number $\lambda \in (-\infty, \mu_0)$ satisfies the assumptions of Lemma 3.5. From (1.7) it follows that this estimate holds true also for G_0 . It is also easy to check that estimate (3.16) is valid. Therefore, it is clear that all the arguments in the proof of the theorem for $\text{Im } \lambda \neq 0$ remain valid in the case where $\lambda \in (-\infty, \mu_0)$ provided ε is sufficiently small. This proves estimates (1.9) in the case in question. $\square$

§4. EXAMPLES

In this section we give examples of operators to which the results of the preceding sections can be applied.

Our first example is

$$(4.1) \quad \mathcal{H}_\varepsilon := \sum_{i,j=1}^d (-\partial_i + \mathbf{a}_{i,\varepsilon}^*) \mathbf{g}_\varepsilon^{ij} (\partial_j + \mathbf{a}_{j,\varepsilon}) + \mathbf{v}_\varepsilon,$$

where

$$\begin{aligned} \mathbf{g}^{ij} &= \mathbf{g}^{ij}(x, \xi) \in W_\infty^1(\mathbb{R}^d; C_{\text{per}}^{1+\beta}(\overline{\square})) \cap C_*^2(\mathbb{R}^d; C_{\text{per}}^\beta(\overline{\square})), \\ \mathbf{a}_i &= \mathbf{a}_i(x, \xi) \in W_\infty^1(\mathbb{R}^d; C_{\text{per}}^{1+\beta}(\overline{\square})) \cap C_*^2(\mathbb{R}^d; C_{\text{per}}^\beta(\overline{\square})), \\ \mathbf{v} &= \mathbf{v}(x, \xi) \in W_\infty^1(\mathbb{R}^d; C_{\text{per}}^\beta(\overline{\square})) \end{aligned}$$

are $\square$ -periodic matrices of size $n \times n$. Moreover, we assume that $\mathbf{v} = \mathbf{v}^*$, $(\mathbf{g}^{ij})^* = \mathbf{g}^{ji}$, and

$$c_1 \sum_{i=1}^d \|\mathbf{w}_i\|_{\mathbb{C}^n}^2 \leq \sum_{i,j=1}^d (\mathbf{g}^{ij} \mathbf{w}_j, \mathbf{w}_i) \leq c_2 \sum_{i=1}^d \|\mathbf{w}_i\|_{\mathbb{C}^n}^2$$

for all $\mathbf{w}_i \in \mathbb{C}^n$, $(x, \xi) \in \mathbb{R}^{2d}$, where c_1 and c_2 are constants. The operator (4.1) can be written as in (1.2); we specify the corresponding choice of A , a_i , b_i , and V .

We put $m = nd$ and choose the matrices A and $B(\zeta)$ as follows:

$$B(\zeta) := \begin{pmatrix} \zeta_1 E_n \\ \zeta_2 E_n \\ \vdots \\ \zeta_d E_n \end{pmatrix}, \quad A := \begin{pmatrix} \mathbf{g}^{11} & \mathbf{g}^{12} & \dots & \mathbf{g}^{1d} \\ \mathbf{g}^{21} & \mathbf{g}^{22} & \dots & \mathbf{g}^{2d} \\ \vdots & \vdots & \dots & \vdots \\ \mathbf{g}^{d1} & \mathbf{g}^{d2} & \dots & \mathbf{g}^{dd} \end{pmatrix}.$$

The matrices a_i , b_i , and V are introduced as

$$a_i := \sum_{j=1}^d \mathbf{a}_j^* \mathbf{g}^{ji}, \quad b_i := E_n, \quad V := \mathbf{v} + \sum_{i,j=1}^d \mathbf{a}_i^* \mathbf{g}^{ij} \mathbf{a}_j.$$

It is easy to check that in this case the operator in (1.2) coincides with that in (4.1). Many operators of mathematical physics are special cases of (4.1); we mention some of them.

If we put $\mathbf{a}_i := 0$, $\mathbf{g}^{ij} := E_n$, then the operator (4.1) becomes a matrix Schrödinger operator. The case where $\mathbf{g}^{ij} \neq E_n$ can be viewed as the matrix Schrödinger operator with a metric. If, moreover, $\mathbf{v} = 0$, we arrive at the elasticity theory operator; we must only assume additional symmetry conditions for the coefficients of the matrix $\mathbf{g}^{ij}$ (see, for instance, [5, Chapter 3]).

In the case where $n = 1$ and $\mathbf{a}_i := iA_i$, where the A_i are real-valued functions, the operator (4.1) describes the magnetic Schrödinger operator. The components of the magnetic potential are the functions A_i ; the function $\mathbf{v}$ is the electric potential. As above, the functions $\mathbf{g}^{ij}$ correspond to the metric.

Yet another example is the two- and three-dimensional Pauli operator. We deal with this operator if $d = 2$ or $d = 3$, $n = 2$, and $\mathbf{a}_i := iA_i E_n$, where the A_i are real-valued functions,

$$\begin{aligned} \mathbf{v} &:= \sigma_3 B, \quad B = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \quad \text{if } d = 2, \\ \mathbf{v} &:= \sigma_1 B_1 + \sigma_2 B_2 + \sigma_3 B_3, \quad (B_1, B_2, B_3) = \text{curl}(A_1, A_2, A_3) \quad \text{if } d = 3, \\ \sigma_1 &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

The case of $\mathbf{g}^{ij} = E_n$ corresponds to the usual Pauli operator; if $\mathbf{g}^{ij} \neq E_n$, we obtain the Pauli operator with metric. One can add an additional term to the potential $\mathbf{v}$ given above. In this case we have the Pauli operator with potential.

In all the examples above, the results of Theorem 1.1 and Corollary 1.2 are applicable. The homogenized operator is given by the general formulas (1.5), (1.6). For this reason, we do not repeat these formulas for the particular cases described.

The presence of the matrix G_ε in estimates (1.9) allow us to widen the class of the examples. For this, we employ the ideas of the papers [7, 10, 13].

Let $\mathbf{f} = \mathbf{f}(x, \xi) \in W_\infty^1(\mathbb{R}^d; C_{\text{per}}^{2+\beta}(\overline{\square}))$ be a positive matrix of size $n \times n$ such that the inverse matrix is uniformly bounded. We consider the operator $\tilde{\mathcal{H}}_\varepsilon := \mathbf{f}_\varepsilon^* \mathcal{H}_\varepsilon \mathbf{f}_\varepsilon$, where $\mathcal{H}_\varepsilon$ is as in (1.2). It is clear that

$$\mathbf{f}_\varepsilon(\tilde{\mathcal{H}}_\varepsilon - \lambda G_\varepsilon)^{-1} \mathbf{f}_\varepsilon^* = (\mathcal{H}_\varepsilon - \lambda \tilde{G}_\varepsilon)^{-1}, \quad \tilde{G} = (\mathbf{f}^*)^{-1} G \mathbf{f}^{-1}.$$

This allows us to approximate the generalized resolvent of the operator $\tilde{\mathcal{H}}_\varepsilon$:

$$\begin{aligned} \|(\tilde{\mathcal{H}}_\varepsilon - \lambda G_\varepsilon)^{-1} - \mathbf{f}_\varepsilon^{-1}(\mathcal{H}_0 - \lambda G_0)^{-1}(\mathbf{f}_\varepsilon^*)^{-1}\|_{L_2 \rightarrow L_2} &\leq C\varepsilon, \\ \|\mathbf{f}_\varepsilon(\tilde{\mathcal{H}}_\varepsilon - \lambda G_\varepsilon)^{-1} - (\mathbf{I} + \varepsilon \mathcal{L}_\varepsilon)(\mathcal{H}_0 - \lambda G_0)^{-1}(\mathbf{f}_\varepsilon^*)^{-1}\|_{L_2 \rightarrow W_2^1} &\leq C\varepsilon. \end{aligned}$$

Now, we introduce the operator $\mathcal{H}_\varepsilon$ by (4.1); the corresponding operator $\tilde{\mathcal{H}}_\varepsilon$ is determined by the same formula but with the coefficients replaced by

$$\tilde{\mathbf{g}}^{ij} := \mathbf{f}^* \mathbf{g}^{ij} \mathbf{f}, \quad \tilde{\mathbf{a}}_i := \mathbf{f}^{-1} \left(\frac{\partial \mathbf{f}}{\partial x_i} + \varepsilon^{-1} \frac{\partial \mathbf{f}}{\partial \xi_i} \right) + \mathbf{f}^{-1} \mathbf{a}_i \mathbf{f}, \quad \tilde{\mathbf{v}} := \mathbf{f}^* \mathbf{v} \mathbf{f}.$$

These formulas show that the coefficients of the operator $\tilde{\mathcal{H}}_\varepsilon$ can grow as $\varepsilon \rightarrow +0$. This allows us to extend the results of the paper to a certain class of operators with rapidly oscillating coefficients that grow as $\varepsilon \rightarrow +0$.

In conclusion, it should also be observed that, in the case where $\mathbf{a}_i = 0$ and $v = 0$, the operator $\tilde{\mathcal{H}}_\varepsilon$ was the main object of study in [7]–[13]; as was mentioned at the beginning of the paper, essentially weaker conditions were imposed on the coefficients. In the papers cited, a great many interesting examples of such operators were presented. Our results can be extended to these examples as well. The novelty will be in the dependence of the coefficients on the slow variable and in the asymptotics expansions for the eigenlements.

ACKNOWLEDGMENTS

I am very grateful to T. A. Suslina for her attention to the paper, discussion of the results, and many valuable remarks.

REFERENCES

- [1] E. Sanchez-Palencia, *Nonhomogeneous media and vibration theory*, Lecture Notes in Phys., vol. 127, Springer-Verlag, Berlin–New York, 1980. MR0578345 (82j:35010)
- [2] N. S. Bakhvalov and G. P. Panasenko, *Homogenization: averaging processes in periodic media*, “Nauka”, Moscow, 1984; English transl., Math. Appl. (Soviet Ser.), vol. 36, Kluwer Acad. Publ. Group, Dordrecht, 1989. MR0797571 (86m:73049); MR1112788 (92d:73002)
- [3] V. V. Zhikov, S. M. Kozlov, and O. A. Oleĭnik, *Homogenization of differential operators*, “Nauka”, Moscow, 1993; English transl., Springer-Verlag, Berlin, 1994. MR1318242 (96h:35003a); MR1329546 (96h:35003b)
- [4] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, Stud. Math. Appl., vol. 5, North-Holland Publishing Co., Amsterdam–New York, 1978. MR0503330 (82h:35001)
- [5] O. A. Oleĭnik, G. A. Iosif’yan, and A. S. Shamaev, *Mathematical problems in the theory of strongly inhomogeneous elastic media*, Moskov. Gos. Univ., Moscow, 1990. (Russian) MR1115306 (92i:73009)
- [6] M. Sh. Birman, *On homogenization procedure for periodic operators near the edge of an internal gap*, Algebra i Analiz **15** (2003), no. 4, 61–71; English transl., St. Petersburg Math. J. **15** (2004), no. 4, 507–513. MR2068979 (2006i:35010)
- [7] M. Sh. Birman and T. A. Suslina, *Second order periodic differential operators. Threshold properties and homogenization*, Algebra i Analiz **15** (2003), no. 5, 1–108; English transl., St. Petersburg Math. J. **15** (2004), no. 5, 639–714. MR2068790 (2005k:47097)
- [8] T. A. Suslina, *Homogenization of a stationary periodic Maxwell system*, Algebra i Analiz **16** (2004), no. 5, 162–244; English transl., St. Petersburg Math. J. **16** (2005), no. 5, 863–922. MR2106671 (2005h:35019)
- [9] ———, *On homogenization for a periodic elliptic operator in a strip*, Algebra i Analiz **16** (2004), no. 1, 269–292; English transl., St. Petersburg Math. J. **16** (2005), no. 1, 237–257. MR2068354 (2005c:35024)
- [10] M. Sh. Birman and T. A. Suslina, *Homogenization with corrector term for periodic elliptic differential operators*, Algebra i Analiz **17** (2005), no. 6, 1–104; English transl., St. Petersburg Math. J. **17** (2006), no. 6, 897–974. MR2202045 (2006k:35011)
- [11] ———, *Threshold approximations with corrector for the resolvent of a factorized selfadjoint operator family*, Algebra i Analiz **17** (2005), no. 5, 69–90; English transl., St. Petersburg Math. J. **17** (2006), no. 5, 745–762. MR2241423 (2008d:47047)
- [12] ———, *Homogenization of a multidimensional periodic elliptic operator in a neighborhood of the edge of an internal gap*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **318** (2004), 60–74; English transl., J. Math. Sci. (N.Y.) **136** (2006), no. 2, 3682–3690. MR2120232 (2005j:35006)

- [13] ———, *Homogenization with corrector for periodic differential operators. Approximation of solutions in the Sobolev class $H^1(\mathbb{R}^d)$* , Algebra i Analiz **18** (2006), no. 6, 1–130; English transl., St. Petersburg Math. J. **18** (2007), no. 6, 857–955. MR2307356 (2008d:35008)
- [14] V. V. Zhikov, *On operator estimates in homogenization theory*, Dokl. Akad. Nauk **403** (2005), no. 3, 305–308; English transl. in Dokl. Math. MR2164541
- [15] D. I. Borisov and R. R. Gadyl'shin, *On the spectrum of the Schrödinger operator with a rapidly oscillating compactly supported potential*, Teoret. Mat. Fiz. **147** (2006), no. 1, 58–63; English transl., Theoret. and Math. Phys. **147** (2006), no. 1, 496–500. MR2254715 (2007f:34158)
- [16] D. I. Borisov, *On some singular perturbations of periodic operators*, Teoret. Mat. Fiz. **151** (2007), no. 2, 207–218; English transl., Theoret. and Math. Phys. **151** (2007), no. 2, 614–624. MR2338076
- [17] ———, *On the spectrum of the Schrödinger operator perturbed by a rapidly oscillating potential*, Probl. Mat. Anal., No. 33, “T. Rozhkovskaya”, Novosibirsk, 2005, pp. 13–78; English transl., J. Math. Sci. (N.Y.) **139** (2006), no. 1, 6243–6322. MR2278906 (2007k:34286)
- [18] V. P. Mikhailov, *Partial differential equations*, “Nauka”, Moscow, 1976; English transl., “Mir”, Moscow, 1978. MR0481380 (58:1497); MR0601389 (82a:35003a)
- [19] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II*, Comm. Pure Appl. Math. **17** (1964), 35–92. MR0162050 (28:5252)
- [20] M. Reed and B. Simon, *Methods of modern mathematical physics. I. Functional analysis*, Acad. Press, New York–London, 1972. MR0493419 (58:12429a)

NUCLEAR PHYSICS INSTITUTE, ACADEMY OF SCIENCES, 25068 ŘEŽ NEAR PRAGUE, CZECH REPUBLIC,
 AND BASHKIR STATE PEDAGOGICAL UNIVERSITY, OCTOBER REVOLUTION STREET 3A, 450000 UFA, RUSSIA
E-mail address: borisovdi@yandex.ru

Received 30/NOV/2006

Translated by THE AUTHOR