SPECTRAL ANALYSIS OF LINEARIZED STATIONARY EQUATIONS OF VISCOUS COMPRESSIBLE FLUID IN $\mathbb{R}^3$, WITH PERIODIC BOUNDARY CONDITIONS

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ABSTRACT. The operator whose spectrum is studied corresponds to linearized stationary equations of viscous compressible fluid in $\mathbb{R}^3$, with periodic boundary conditions. The equations are obtained by linearization of the nonlinear model equations of viscous compressible fluid near an arbitrary solution depending on the variable x . It is proved that the operator in question is sectorial and that its spectrum is discrete. Also, a subset of the complex plane that contains the spectrum is described. The resolvent is estimated off a sector in the complex plane that is symmetric with respect to the real axis.

INTRODUCTION

In the paper, we investigate the spectrum of the operator of linearized equations of viscous compressible fluid; also, we estimate the corresponding resolvent.

The study of the spectrum of such operators and estimation of the resolvent are needed if we want to know the long-term behavior of the solution of the corresponding nonstationary equations, in particular, if we are interested in the stability of solutions. As far as I know, in the publications devoted to the long-term behavior of solutions of such equations [1, 2, 3], linearization was done at the zero velocity and the unit specific volume. Such linearization yields equations with constant coefficients. The spectrum of the operator (with constant coefficients) corresponding to linearized stationary equations of viscous compressible fluid was studied in [4].

In [5], the object of study was a similar operator on the 2-dimensional torus $\mathbb{R}^2/2\pi\mathbb{Z}^2$ and with variable coefficients. In that paper, it was proved that the spectrum is discrete and lies in some complex sector symmetric relative to the real axis.

In the present paper we study the spectrum of the operator with variable coefficients on the 3-dimensional torus $\mathbb{R}^3/2\pi\mathbb{Z}^3$. Linearization is done at a fixed solution $(\bar{u}_1(x), \bar{u}_2(x), \bar{u}_3(x), \bar{v}(x))$ depending on x . Here $\bar{u}(x) = (\bar{u}_1(x), \bar{u}_2(x), \bar{u}_3(x))$ is the velocity and $\bar{v}(x) > 0$ is the specific volume of the liquid. Our aim is to prove that the operator in question is sectorial and that its spectrum is discrete. Namely, we prove that the spectrum is discrete and lies in some complex sector symmetric relative to the real axis. However, the core of the paper is in the proof of a certain bound for the resolvent provided the spectral parameter is off the sector indicated. This issue was not touched upon in [5]. Since the operator under study generates a system with variable coefficients, investigation of the spectrum and estimation of the resolvent encounter considerable difficulties. In particular, some components of the operator act continuously in one and the same Sobolev space, which makes it impossible to prove the compactness of the inverse

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operator by standard methods.¹ Nevertheless, in the present paper we manage to use the theory of pseudodifferential operators as in [5] in order to prove that the spectrum of the operator in question is discrete and to give a satisfactory description of a subset of $\mathbb{C}$ that contains the spectrum. It turns out that only finitely many points of the spectrum are located in the unstable half-plane.

We expect that the results of the present paper will find application in a series of problems. In particular, we hope to generalize the stabilization method for parabolic equations and the Navier–Stokes system, suggested in [6, 7], to the case of equations of viscous compressible fluid.

The paper is organized as follows. §1 is devoted to formulation of the problem. In §2, we consider the operator with constant coefficients and prove that it is sectorial and has discrete spectrum. For the future estimation of the resolvent, we prove some necessary estimates for λ off a sector in $\mathbb{C}$ that contains the spectrum. In §3 we prove our main result: the operator with variable coefficients that corresponds to linearized stationary equations of viscous compressible fluid, is sectorial and its spectrum is discrete. We give a satisfactory description of a set on the complex plane that contains the spectrum. Also, like in the case of constant coefficients, we prove an estimate of the resolvent for the spectral parameter lying off a sector in $\mathbb{C}$ that contains the spectrum of the operator.

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§1. SETTING OF THE PROBLEM

Consider the model stationary equations of a viscous compressible fluid

$$(1.1) \quad \mu \sum_{j=1}^3 \frac{\partial}{\partial x_j} \left(\frac{1}{v(x)} \frac{\partial u_k(x)}{\partial x_j} \right) - \frac{\partial p(v)}{\partial x_k} = f_k(x), \quad k = 1, 2, 3,$$

$$(1.2) \quad \operatorname{div} u(x) = 0,$$

where $x = (x_1, x_2, x_3)$, $u(x) = (u_1(x), u_2(x), u_3(x))$ is the velocity of the fluid, $v(x) > 0$ is its specific volume, $\mu = \text{const} > 0$ is the viscosity coefficient, $p(v) \in C^1(0, \infty)$ is the pressure, and $f(x) = (f_1(x), f_2(x), f_3(x))$ is the external force. All functions in (1.1), (1.2) satisfy periodic boundary conditions with period 2π , i.e., it is assumed that x runs through the torus $T = \mathbb{R}^3/2\pi\mathbb{Z}^3$.

Assume for simplicity that $\mu = 1$. Let $(\bar{u}(x), \bar{v}(x))$ be an arbitrary solution of (1.1), (1.2). Linearizing the system at that solution and ignoring the lower order terms, we obtain

$$(1.3) \quad \frac{1}{\bar{v}(x)} \Delta u_k(x) + \sum_{j=1}^3 b_{kj}(x) \frac{\partial v(x)}{\partial x_j} = 0, \quad k = 1, 2, 3,$$

$$(1.4) \quad \operatorname{div} u(x) = 0,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, and for every $k, j = 1, 2, 3$ the coefficients $b_{kj}(x)$ are determined by the formula

$$b_{kj}(x) = \begin{cases} - \left(\frac{1}{\bar{v}(x)} \frac{\partial \bar{u}_k(x)}{\partial x_k} + p'(\bar{v}) \right) & \text{if } k = j, \\ - \frac{1}{\bar{v}(x)} \frac{\partial \bar{u}_k(x)}{\partial x_j} & \text{if } k \neq j. \end{cases}$$

¹Recall that the Laplace operator in a bounded domain in Ω acts from $H^2(\Omega) \cap H_0^1(\Omega)$ into $L_2(\Omega)$, so that its inverse is compact on $L_2(\Omega)$.

In what follows we use the Sobolev function spaces, which are defined as the completion of the space $C_0^\infty(T)$ in the norm

$$(1.5) \quad \|y\|_{H^k} = \left(\sum_{\xi \in \mathbb{Z}^3} (1 + |\xi|^2)^k |(Fy)(\xi)|^2 \right)^{\frac{1}{2}},$$

where $(Fy)(\xi)$ are the Fourier coefficients of the 2π -periodic function $y(x)$. It k is a positive integer, then, by the Parseval identity, this norm is equivalent to the norm

$$\left(\int_T \sum_{|\alpha| \leq k} |D^\alpha y(x)|^2 dx \right)^{\frac{1}{2}}.$$

Here $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}$, $D_j = -i \frac{\partial}{\partial x_j}$, $j = 1, 2, 3$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

We introduce the space $H = \mathbb{L}_2(T) \times \mathbb{L}_2(T) \times \mathbb{L}_2(T) \times H^1(T)$ with the norm

$$\|y\|_H^2 = \sum_{k=1}^3 \|y_k\|_{\mathbb{L}_2}^2 + \|y_4\|_{H^1}^2,$$

and the space $G = H^2(T) \times H^2(T) \times H^2(T) \times H^1(T)$ with the norm

$$\|z\|_G^2 = \sum_{k=1}^3 \|z_k\|_{H^2}^2 + \|z_4\|_{H^1}^2.$$

Let $A(x, D) = \{a_{kj}(x, D)\}_{k,j=1}^4$ denote the operator given by the left-hand side of system (1.3), (1.4); this means that the elements $a_{kj}(x, D)$ are defined by the formulas

$$(1.6) \quad a_{kk}(x, D) = \frac{1}{\bar{v}(x)} \Delta, \quad a_{k4}(x, D) = \sum_{j=1}^3 b_{kj}(x) \frac{\partial}{\partial x_j}, \quad k = 1, 2, 3,$$

$$a_{kj}(x, D) = a_{44}(x, D) = 0, \quad k \neq j, \quad k, j = 1, 2, 3,$$

$$(1.7) \quad a_{4j}(x, D) = \frac{\partial}{\partial x_j}, \quad j = 1, 2, 3.$$

The domain of the operator $A(x, D) : H \rightarrow H$ is $D(A(x, D)) = G$, and the range of it is $R(A(x, D)) \subset H$. Consider the following spectral problem:

$$(1.8) \quad (A(x, D) - \lambda E)U(x) = 0,$$

where $\lambda \in \mathbb{C}$ is the spectral parameter, $U(x) = (u_1(x), u_2(x), u_3(x), v(x))$, and E is the identity operator. The set of all numbers $\lambda \in \mathbb{C}$ such that $R(A(x, D) - \lambda E) = H$ and the inverse operator $(A(x, D) - \lambda E)^{-1} : H \rightarrow H$ is defined is called the *resolvent set* for the operator $A(x, D)$ and is denoted by $\rho(A(x, D))$. The set $\sigma(A(x, D)) = \mathbb{C} \setminus \rho(A(x, D))$ is the *spectrum* of $A(x, D)$.

Definition 1 (see [8]). A closed operator $A(x, D) : H \rightarrow H$ is called sectorial if there exist $\varphi \in (\frac{\pi}{2}, \pi)$, $a_0 \in \mathbb{R}$, and $M > 0$ such that the sector $S_{a_0, \varphi} = \{\lambda : 0 \leq |\arg(\lambda - a_0)| \leq \varphi, \lambda \neq a_0\}$ lies in $\rho(A(x, D))$ and $\|(A(x, D) - \lambda E)^{-1}\| \leq \frac{M}{|\lambda - a_0|}$ for any $\lambda \in S_{a_0, \varphi}$.

Our aim in the present paper is to prove that the operator $A(x, D) = \{a_{kj}(x, D)\}_{k,j=1}^4$, with elements defined in (1.6), (1.7) is sectorial and that its spectrum is discrete.

§2. THE CASE OF CONSTANT COEFFICIENTS

First, we consider equation (1.8) for the operator with constant coefficients,

$$(2.1) \quad (A(D) - \lambda E)U(x) = 0,$$

where the entries of the (matrix) operator $A(D)$ are of the form (1.6), (1.7) with coefficients independent of x , i.e., $a_{kj}(x, D) = a_{kj}(D)$.

To find the eigenvalues of $A(D)$, we shall seek the eigenvectors in the form

$$(2.2) \quad (FU)(\xi)e^{i(x,\xi)}, \quad \xi \in \mathbb{Z}^3,$$

where

$$(x, \xi) = \sum_{j=1}^3 x_j \xi_j, \quad (FU)(\xi) = ((Fu_1)(\xi), (Fu_2)(\xi), (Fu_3)(\xi), (Fv)(\xi)),$$

and

$$(Fw)(\xi) = \frac{1}{(2\pi)^3} \int_T w(x) e^{-i(x,\xi)} dx$$

are the Fourier coefficients of a 2π -periodic function $w(x)$. Plugging (2.2) in (2.1) and using the formula $\frac{\partial}{\partial x_j}(e^{i(x,\xi)}) = i\xi_j e^{i(x,\xi)}$, we arrive at the system of equations

$$(2.3) \quad (A(\xi) - \lambda E)(FU)(\xi) = 0, \quad \xi \in \mathbb{Z}^3.$$

The entries of the matrix $A(\xi) = \{a_{kj}(\xi)\}_{k,j=1}^4$ are determined by the formulas

$$(2.4) \quad a_{kk}(\xi) = -\frac{|\xi|^2}{v}, \quad a_{k4}(\xi) = i \sum_{j=1}^3 b_{kj} \xi_j, \quad k = 1, 2, 3,$$

$$(2.5) \quad a_{kj}(\xi) = a_{44}(\xi) = 0, \quad k \neq j, \quad k, j = 1, 2, 3, \quad a_{4j}(\xi) = i\xi_j, \quad j = 1, 2, 3,$$

where $|\xi| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$. We calculate the determinant of the matrix $A(\xi) - \lambda E$:

$$\det(A(\xi) - \lambda E) = \left(\frac{|\xi|^2}{v} + \lambda\right)^2 \left(\lambda^2 + \lambda \frac{|\xi|^2}{v} + \sum_{k=1}^3 \xi_k \sum_{j=1}^3 b_{kj} \xi_j\right).$$

The equation $\det(A(\xi) - \lambda E) = 0$ yields

$$(2.6) \quad \lambda_1(\xi) = \lambda_2(\xi) = -\frac{|\xi|^2}{v}, \quad \xi \in \mathbb{Z}^3,$$

$$(2.7) \quad \lambda_{3,4}(\xi) = -\frac{|\xi|^2}{2v} \pm \frac{1}{2} \sqrt{\tilde{b}(\xi)}, \quad \xi \in \mathbb{Z}^3,$$

where

$$(2.8) \quad \tilde{b}(\xi) = \frac{|\xi|^4}{v^2} - 4 \sum_{k=1}^3 \xi_k \sum_{j=1}^3 b_{kj} \xi_j.$$

Since $\lambda_1(\xi) = \lambda_2(\xi)$, in what follows we denote both roots by $\lambda_1(\xi)$.

Lemma 2.1. *For any $\xi \in \mathbb{Z}^3$ there are four linearly independent eigenvectors. The two of them that correspond to the eigenvalue (2.6) are as follows:*

$$(-\xi_3, 0, \xi_1, 0)e^{i(x,\xi)}, \quad (\xi_2, -\xi_1, 0, 0)e^{i(x,\xi)}$$

for $\xi_1 \neq 0$ and any ξ_2, ξ_3 ;

$$(-\xi_2, \xi_1, 0, 0)e^{i(x,\xi)}, \quad (0, -\xi_3, \xi_2, 0)e^{i(x,\xi)}$$

for $\xi_2 \neq 0$ and any ξ_1, ξ_3 ; and

$$(-\xi_3, 0, \xi_1, 0)e^{i(x,\xi)}, \quad (0, -\xi_3, \xi_2, 0)e^{i(x,\xi)}$$

for $\xi_3 \neq 0$ and any ξ_1, ξ_2 . The two other eigenvectors corresponding to the eigenvalues (2.7) are given by the formula

$$\left(i \sum_{j=1}^3 b_{1j} \xi_j, i \sum_{j=1}^3 b_{2j} \xi_j, i \sum_{j=1}^3 b_{3j} \xi_j, \frac{|\xi|^2}{\bar{v}} + \lambda_{3,4}(\xi) \right) e^{i(x, \xi)}.$$

Proof. It suffices to substitute the numbers $\lambda_j(\xi)$ in equation (2.3). $\square$

Consider the set

$$(2.9) \quad M = \{\lambda \in \mathbb{C} : \lambda = \lambda(\xi) \text{ defined as in (2.6), (2.7)}\}.$$

Lemma 2.2. *There exist numbers $C_0, C_1, C_2 > 0$ such that M is included in the set*

$$(2.10) \quad \begin{aligned} M' = \{ \lambda \in \mathbb{C} : \operatorname{Im} \lambda = 0, \operatorname{Re} \lambda \in (-\infty, -C_2] \cup (0, C_0] \} \\ \cup \{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda \in [-C_2, 0], \operatorname{Im} \lambda \in [-C_1, C_1] \}. \end{aligned}$$

Proof. Recall that in the case of a 2-dimensional torus we have three eigenvalues: $\lambda_1(\xi)$ coincides with the number (2.6), and $\lambda_{2,3}(\xi)$ differ from the numbers (2.7) only by the function $\tilde{b}(\xi)$. In the 2-dimensional case

$$\tilde{b}(\xi) = \frac{|\xi|^4}{\bar{v}^2} - 4 \sum_{k=1}^2 \xi_k \sum_{j=1}^2 b_{kj} \xi_j.$$

Therefore, the proof of Lemma 2.2 is similar to the proof of the corresponding statement in [5]. $\square$

Consider the sector

$$(2.11) \quad S = \left\{ \lambda \in \mathbb{C} : 0 \leq |\arg(\lambda - C_0)| < \pi - \arctan \frac{C_1}{C_0}, \lambda \neq C_0 \right\},$$

where C_0 and C_1 are the numbers occurring in (2.10). Assuming that $\alpha > 0$ is small, we consider the sector

$$(2.12) \quad S_\alpha = \left\{ \lambda \in \mathbb{C} : 0 \leq |\arg(\lambda - C_0)| < \pi - \arctan \frac{C_1}{C_0} - \alpha, \lambda \neq C_0 \right\}.$$

Observe that $M' \cap S_\alpha = \emptyset$. Let $r(\lambda, M')$ be the distance from $\lambda \in S_\alpha$ to the set M' , and let $r(\lambda, \mathbb{C} \setminus S)$ be the distance from $\lambda \in S_\alpha$ to $\mathbb{C} \setminus S$.

Lemma 2.3. *Suppose $\lambda \in S_\alpha$. There exists $c \in \mathbb{R}$, $c > 0$ such that $r(\lambda, M') \geq c|\lambda - C_0|$.*

Proof. If the point C_0 is the nearest to $\lambda \in S_\alpha$ among the points of M' , then $r(\lambda, M') \geq |\lambda - C_0|$. Otherwise, we have $r(\lambda, M') \geq r(\lambda, \mathbb{C} \setminus S) \geq |\lambda - C_0| \sin \alpha$, where α is the angle by which the sector S is larger than the sector S_α . $\square$

Lemma 2.4. *Suppose $\lambda \in S_\alpha$. There exists $b > 0$ independent of λ and ξ and such that $|\lambda - \lambda_j(\xi)| \geq b|\lambda_j(\xi)|$ for all real $\lambda_j(\xi)$, $j = 1, \dots, 4$.*

Proof. Observe that $\lambda_j(\xi)$, as defined in (2.6), belongs to $\mathbb{R}_-$. If $\operatorname{Re} \lambda \geq C_0$, then

$$|\lambda - \lambda_j(\xi)| = |\operatorname{Re} \lambda - \lambda_j(\xi)|^2 + |\operatorname{Im} \lambda|^2 \geq |\operatorname{Re} \lambda - \lambda_j(\xi)|^2 \geq |\lambda_j(\xi)|^2.$$

If $\operatorname{Re} \lambda < C_0$, then

$$\begin{aligned} |\lambda - \lambda_j(\xi)| &= |\operatorname{Re} \lambda - \lambda_j(\xi)|^2 + |\operatorname{Im} \lambda|^2 \\ &\geq |\operatorname{Re} \lambda - \lambda_j(\xi)|^2 + |\operatorname{Re} \lambda + C_0|^2 \tan^2 \left(\alpha + \arctan \frac{C_1}{C_0} \right), \end{aligned}$$

where $\pi - (\arctan \frac{C_1}{C_0} + \alpha)$ is the angle of the sector S_α defined in (2.12). If $|\operatorname{Re} \lambda - \lambda_j(\xi)|^2 \geq \delta |\lambda_j(\xi)|^2$ with $1 > \delta > 0$, then

$$|\operatorname{Re} \lambda - \lambda_j(\xi)|^2 + ||\operatorname{Re} \lambda| + C_0|^2 \tan^2 \left(\alpha + \arctan \frac{C_1}{C_0} \right) \geq \delta |\lambda_j(\xi)|^2.$$

If $|\operatorname{Re} \lambda - \lambda_j(\xi)|^2 < \delta |\lambda_j(\xi)|^2$ with $1 > \delta > 0$, then

$$\begin{aligned} & |\operatorname{Re} \lambda - \lambda_j(\xi)|^2 + ||\operatorname{Re} \lambda| + C_0|^2 \tan^2 \left(\alpha + \arctan \frac{C_1}{C_0} \right) \\ & \geq |\operatorname{Re} \lambda|^2 \tan^2 \left(\alpha + \arctan \frac{C_1}{C_0} \right) \\ & \geq ||\lambda_j(\xi)| - |\operatorname{Re} \lambda - \lambda_j(\xi)||^2 \tan^2 \left(\alpha + \arctan \frac{C_1}{C_0} \right) \\ & \geq (1 - \sqrt{\delta})^2 \tan^2 \left(\alpha + \arctan \frac{C_1}{C_0} \right) |\lambda_j(\xi)|^2. \quad \square \end{aligned}$$

Lemma 2.5. *Suppose $\lambda \in S_\alpha$. There exists $a > 0$ independent of λ and ξ and such that*

$$\frac{|\lambda - \lambda_1(\xi)|}{|\lambda - \lambda_3(\xi)| \cdot |\lambda - \lambda_4(\xi)|} \leq \frac{a}{|\lambda - C_0|}.$$

Proof. If $|\lambda - \lambda_4(\xi)| \geq |\lambda - \lambda_1(\xi)|$, then by Lemma 2.3 we have

$$\frac{|\lambda - \lambda_1(\xi)|}{|\lambda - \lambda_3(\xi)| \cdot |\lambda - \lambda_4(\xi)|} \leq \frac{1}{|\lambda - \lambda_3(\xi)|} \leq \frac{1}{r(\lambda, M')} \leq \frac{1}{c|\lambda - C_0|}.$$

If $|\lambda - \lambda_1(\xi)| > |\lambda - \lambda_4(\xi)|$, we consider two cases. Suppose $|\xi|$ is sufficiently large. Then $\lambda_4(\xi)$, as defined in (2.7), is real. Using Lemma 2.4, we see that, in this case, there exists $c_1 > 0$ independent of λ and ξ and such that $|\lambda - \lambda_4(\xi)| \geq b|\lambda_4(\xi)| > c_1$. Then we can find a constant $c_2 > 0$ such that

$$\begin{aligned} |\lambda - \lambda_1(\xi)| & \leq |\lambda - \lambda_4(\xi)| + |\lambda_4(\xi) - \lambda_1(\xi)| \\ & \leq |\lambda - \lambda_4(\xi)| + \left| -\frac{|\xi|^2}{2\bar{v}} - \frac{1}{2}\sqrt{\bar{b}(\xi)} + \frac{|\xi|^2}{\bar{v}} \right| \\ & \leq |\lambda - \lambda_4(\xi)| + \left| \frac{|\xi|^2}{2\bar{v}} \left(1 - \left(1 - \frac{4\bar{v}^2}{|\xi|^4} \sum_{k=1}^3 \xi_k \sum_{j=1}^3 b_{kj} \xi_j \right)^{1/2} \right) \right| \\ & \leq |\lambda - \lambda_4(\xi)| \\ & \quad + \left| \frac{|\xi|^2}{2\bar{v}} \left(\frac{2\bar{v}^2}{|\xi|^4} \sum_{k=1}^3 \xi_k \sum_{j=1}^3 b_{kj} \xi_j + 2 \left(\frac{\bar{v}^2}{|\xi|^4} \sum_{k=1}^3 \xi_k \sum_{j=1}^3 b_{kj} \xi_j \right)^2 + \dots \right) \right| \\ & \quad \times |\lambda - \lambda_4(\xi)| + c_2 \leq (1 + c_2/c_1) |\lambda - \lambda_4(\xi)|. \end{aligned}$$

Now, suppose that $|\xi|$ is small. Put $c_3 = \max_\xi |\lambda_4(\xi) - \lambda_1(\xi)|$. Lemma 2.3 yields $|\lambda - \lambda_4(\xi)| \geq r(\lambda, M') \geq c|\lambda - C_0| \geq c \min_\lambda |\lambda - C_0|$. Then there exists a constant $c_4 > 0$ such that

$$\begin{aligned} |\lambda - \lambda_1(\xi)| & \leq |\lambda - \lambda_4(\xi)| + |\lambda_4(\xi) - \lambda_1(\xi)| \\ & \leq |\lambda - \lambda_4(\xi)| + c_3 \leq \left(1 + c_3 / \left(c \min_\lambda |\lambda - C_0| \right) \right) |\lambda - \lambda_4(\xi)| \\ & \leq c_4 |\lambda - \lambda_4(\xi)|. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{|\lambda - \lambda_1(\xi)|}{|\lambda - \lambda_3(\xi)| |\lambda - \lambda_4(\xi)|} &\leq \frac{\max\{1 + c_2/c_1, c_4\} |\lambda - \lambda_4(\xi)|}{|\lambda - \lambda_3(\xi)| |\lambda - \lambda_4(\xi)|} \\ &\leq \frac{\max\{1 + c_2/c_1, c_4\}}{r(\lambda, M')} \leq \frac{\max\{1 + c_2/c_1, c_4\}}{c|\lambda - C_0|}. \end{aligned} \quad \square$$

We use elementary algebraic operations to find the matrix inverse to $A(\xi) - \lambda E$, where $A(\xi) = \{a_{kj}(\xi)\}_{k,j=1}^4$ is the matrix with entries (2.4), (2.5). Then, for $\lambda \notin M$, the entries of the matrix $(A(\xi) - \lambda E)^{-1} = \{\theta_{kj}(\xi, \lambda)\}_{k,j=1}^4$ are as follows:

$$(2.13) \quad \theta_{kk}(\xi, \lambda) = \frac{1}{\left(\frac{|\xi|^2}{v} + \lambda\right) d(\xi, \lambda)} \left(\xi_k \sum_{j=1}^3 b_{kj} \xi_j - d(\xi, \lambda) \right), \quad k = 1, 2, 3,$$

$$(2.14) \quad \theta_{kj}(\xi, \lambda) = \frac{1}{\left(\frac{|\xi|^2}{v} + \lambda\right) d(\xi, \lambda)} \xi_k \sum_{j=1}^3 b_{kj} \xi_j, \quad k \neq j, \quad k, j = 1, 2, 3,$$

$$(2.15) \quad \theta_{44}(\xi, \lambda) = -\frac{\frac{|\xi|^2}{v} + \lambda}{d(\xi, \lambda)}, \quad \theta_{4k}(\xi, \lambda) = -\frac{i \xi_k}{d(\xi, \lambda)}, \quad k = 1, 2, 3,$$

$$(2.16) \quad \theta_{k4}(\xi, \lambda) = -\frac{i}{d(\xi, \lambda)} \sum_{j=1}^3 b_{kj} \xi_j, \quad k = 1, 2, 3,$$

where

$$(2.17) \quad d(\xi, \lambda) = \lambda^2 + \lambda \frac{|\xi|^2}{v} + \sum_{k=1}^3 \xi_k \sum_{j=1}^3 b_{kj} \xi_j = (\lambda - \lambda_3(\xi))(\lambda - \lambda_4(\xi)).$$

We shall prove that the numbers (2.6), (2.7) are eigenvalues of the operators $A(D)$ and that there are no other eigenvalues. Also, we estimate the resolvent for $\lambda \in S_\alpha$.

Lemma 2.6. 1) The operator $(A(D) - \lambda E)^{-1} : H \rightarrow H$ is well defined and continuous whenever $\lambda \notin M$, i.e., $\lambda \in \rho(A(D))$.

2) There exists $m > 0$ such that for $\lambda \in S_\alpha$ we have $\|(A(D) - \lambda E)^{-1}\| \leq \frac{m}{|\lambda - C_0|}$, where C_0 is the vertex of the sector S_α .

Proof. The theory of pseudodifferential operators (see [9]) says that, for an arbitrary infinitely smooth complex-valued function $J(x, \xi)$ satisfying

$$|\partial_x^p \partial_\xi^q J(x, \xi)| \leq C_{p,q} (1 + |\xi|)^{m-|q|} \quad \text{for any } p, q, \text{ and } m \in \mathbb{R},$$

a pseudodifferential operator of the form

$$J(x, D)y(x) = \sum_{\xi \in \mathbb{Z}^3} J(x, \xi)(Fy)(\xi) e^{i(x, \xi)}$$

is well defined; $J(x, \xi)$ is called the *symbol* of the operator $J(x, D)$ (we denote by $(Fy)(\xi)$ the Fourier coefficients of a 2π -periodic function $y(x)$).

Let $\theta_{kj}(D, \lambda)$ denote the elements of the operator matrix $(A(D) - \lambda E)^{-1}$, and let $\Theta_k(D, \lambda) = (\theta_{k1}(D, \lambda), \theta_{k2}(D, \lambda), \theta_{k3}(D, \lambda), \theta_{k4}(D, \lambda))$ be the rows of that matrix. For

any $y \in H$, we have

$$\begin{aligned}
 \|(A(D) - \lambda E)^{-1}y\|_H^2 &= \sum_{k=1}^3 \|\Theta_k(D, \lambda)y\|_{\mathbb{L}_2}^2 + \|\Theta_4(D, \lambda)y\|_{H^1}^2 \\
 (2.18) \quad &= \sum_{k=1}^3 \left\| \sum_{j=1}^4 \theta_{kj}(D, \lambda)y_j \right\|_{\mathbb{L}_2}^2 + \left\| \sum_{j=1}^4 \theta_{4j}(D, \lambda)y_j \right\|_{H^1}^2 \\
 &\leq 4 \sum_{j=1}^4 \left(\sum_{k=1}^3 \|\theta_{kj}(D, \lambda)y_j\|_{\mathbb{L}_2}^2 + \|\theta_{4j}(D, \lambda)y_j\|_{H^1}^2 \right).
 \end{aligned}$$

Observe that the matrix entries $\theta_{kj}(\xi, \lambda)$ defined in (2.13)–(2.16) are the symbols of the pseudodifferential operators $\theta_{kj}(D, \lambda)$, $k, j = 1, \dots, 4$. Denoting

$$c_{kk}(\lambda) = \max_{\xi \in \mathbb{Z}^3} \{|\theta_{kk}(\xi, \lambda)|^2\},$$

and using (2.13), we obtain that for any $\lambda \notin M$,

$$\begin{aligned}
 \|\theta_{kk}(D, \lambda)y_k\|_{\mathbb{L}_2}^2 &= \left\| \sum_{\xi \in \mathbb{Z}^3} \theta_{kk}(\xi, \lambda)(Fy_k)(\xi)e^{i(x, \xi)} \right\|_{\mathbb{L}_2}^2 \\
 (2.19) \quad &= \sum_{\xi \in \mathbb{Z}^3} |\theta_{kk}(\xi, \lambda)(Fy_k)(\xi)|^2 \leq \max_{\xi \in \mathbb{Z}^3} \{|\theta_{kk}(\xi, \lambda)|^2\} \sum_{\xi \in \mathbb{Z}^3} |(Fy_k)(\xi)|^2 \\
 &\leq c_{kk}(\lambda) \sum_{\xi \in \mathbb{Z}^3} |(Fy_k)(\xi)|^2 = c_{kk}(\lambda) \|y_k\|_{\mathbb{L}_2}^2, \quad k = 1, 2, 3.
 \end{aligned}$$

On the other hand, denoting $c_1 = \max_{k,j} |b_{kj}|$ and using Lemmas 2.3, 2.4, and 2.5, and also the inequality $2\xi_i\xi_j \leq \xi_i^2 + \xi_j^2$, we get for $\lambda \in S_\alpha$,

$$\begin{aligned}
 |\theta_{kk}(\xi, \lambda)| &= \left| \frac{1}{\left(\frac{|\xi|^2}{\bar{v}} + \lambda\right) d(\xi, \lambda)} \left(\xi_k \sum_{j=1}^3 b_{kj}\xi_j - d(\xi, \lambda) \right) \right| \\
 &\leq \frac{2c_1|\xi|^2}{\left|\frac{|\xi|^2}{\bar{v}} + \lambda\right| |d(\xi, \lambda)|} + \frac{1}{\left|\frac{|\xi|^2}{\bar{v}} + \lambda\right|} \leq \frac{2c_1|\xi|^2}{\left|\frac{|\xi|^2}{\bar{v}} + \lambda\right| |d(\xi, \lambda)|} + \frac{1}{r(\lambda, M')} \\
 &\leq \frac{2c_1\bar{v}^2 \left|\frac{|\xi|^2}{\bar{v}}\right|^2}{|\lambda - \lambda_1(\xi)| |d(\xi, \lambda)|} + \frac{1}{c|\lambda - C_0|} \leq \frac{2c_1\bar{v}^2 |\lambda_1(\xi)|^2}{|\lambda - \lambda_1(\xi)| |d(\xi, \lambda)|} + \frac{1}{c|\lambda - C_0|} \\
 &\leq \frac{2c_1\bar{v}^2}{b^2} \frac{|\lambda - \lambda_1(\xi)|}{|\lambda - \lambda_3(\xi)| |\lambda - \lambda_4(\xi)|} + \frac{1}{c|\lambda - C_0|} \leq \left(\frac{2c_1\bar{v}^2 a}{b^2} + \frac{1}{c} \right) \frac{1}{|\lambda - C_0|}.
 \end{aligned}$$

Consequently, there exists $m_1 > 0$ such that

$$(2.20) \quad \|\theta_{kk}(D, \lambda)y_k\|_{\mathbb{L}_2} \leq \frac{m_1}{|\lambda - C_0|} \|y_k\|_{\mathbb{L}_2}, \quad k = 1, 2, 3.$$

Similarly, by (2.14), we obtain for $\lambda \notin M$ the inequality

$$(2.21) \quad \|\theta_{kj}(D, \lambda)y_j\|_{\mathbb{L}_2}^2 \leq c_{kj}(\lambda) \|y_j\|_{\mathbb{L}_2}^2, \quad k, j = 1, 2, 3,$$

where $c_{kj}(\lambda) = \max_{\xi \in \mathbb{Z}^3} \{|\theta_{kj}(\xi, \lambda)|^2\}$, $k \neq j$.

Since

$$\begin{aligned} |\theta_{kj}(\xi, \lambda)| &= \left| \frac{1}{\left(\frac{|\xi|^2}{v} + \lambda\right)d(\xi, \lambda)} \xi_k \sum_{j=1}^3 b_{kj} \xi_j \right| \leq \frac{2c_1 |\xi|^2}{\left|\frac{|\xi|^2}{v} + \lambda\right| |d(\xi, \lambda)|} + \frac{1}{\left|\frac{|\xi|^2}{v} + \lambda\right|} \\ &\leq \left(\frac{2c_1 \bar{v}^2 a}{b^2} + \frac{1}{c} \right) \frac{1}{|\lambda - C_0|}, \end{aligned}$$

for $k \neq j$ we have

$$(2.22) \quad \|\theta_{kj}(D, \lambda)y_j\|_{\mathbb{L}_2} \leq \frac{m_1}{|\lambda - C_0|} \|y_j\|_{\mathbb{L}_2}, \quad k, j = 1, 2, 3.$$

By (2.16), denoting $c_{k4}(\lambda) = \max_{\xi \in \mathbb{Z}^3} \{|\theta_{k4}(\xi, \lambda)|^2\}$, for $\lambda \notin M$ we have

$$\begin{aligned} \|\theta_{k4}(D, \lambda)y_4\|_{\mathbb{L}_2}^2 &\leq c_{k4}(\lambda) \sum_{\xi \in \mathbb{Z}^3} |(Fy_4)(\xi)|^2 \\ (2.23) \quad &\leq c_{k4}(\lambda) \sum_{\xi \in \mathbb{Z}^3} (1 + |\xi|^2) |(Fy_4)(\xi)|^2 \\ &\leq c_{k4}(\lambda) \|y_4\|_{H^1}^2, \quad k = 1, 2, 3. \end{aligned}$$

If $\lambda \in S_\alpha$ and $k = 1, 2, 3$, then

$$\begin{aligned} |\theta_{k4}(\xi, \lambda)| &= \left| \frac{i}{d(\xi, \lambda)} \sum_{j=1}^3 b_{kj} \xi_j \right| \leq \frac{1}{|d(\xi, \lambda)|} \sum_{j=1}^3 |b_{kj} \xi_j| \leq \frac{c_1}{|d(\xi, \lambda)|} \sum_{j=1}^3 |\xi_j| \\ &\leq \frac{c_1 |\xi|^2}{|d(\xi, \lambda)|} \leq \frac{\frac{c_1 \bar{v}}{b} |\lambda - \lambda_1(\xi)|}{|\lambda - \lambda_3(\xi)| |\lambda - \lambda_4(\xi)|} \leq \frac{c_1 \bar{v} a}{b} \frac{1}{|\lambda - C_0|}. \end{aligned}$$

Consequently, there exists $m_2 > 0$ such that

$$(2.24) \quad \|\theta_{k4}(D, \lambda)y_4\|_{\mathbb{L}_2} \leq \frac{m_2}{|\lambda - C_0|} \|y_4\|_{H^1}, \quad k = 1, 2, 3.$$

By (2.14), denoting $c_{4j}(\lambda) = \max_{\xi \in \mathbb{Z}^3} \{(1 + |\xi|^2) |\theta_{4j}(\xi, \lambda)|^2\}$ with $j = 1, 2, 3$, we obtain for $\lambda \notin M$ the inequality

$$(2.25) \quad \|\theta_{4j}(D, \lambda)y_j\|_{H^1}^2 \leq \sum_{\xi \in \mathbb{Z}^3} (1 + |\xi|^2) |\theta_{4j}(\xi, \lambda)|^2 |(Fy_j)(\xi)|^2 \leq c_{4j}(\lambda) \|y_j\|_{\mathbb{L}_2}^2.$$

If $\lambda \in S_\alpha$, then

$$\begin{aligned} (1 + |\xi|^2) (\theta_{4j}(\xi, \lambda))^2 &= \frac{(1 + |\xi|^2) |\xi_j|^2}{|d(\xi, \lambda)|^2} \leq \frac{(1 + |\xi|^2) |\xi|^2}{|d(\xi, \lambda)|^2} \leq \frac{2|\xi|^4}{|d(\xi, \lambda)|^2} \\ &\leq \frac{2\bar{v}^2 \left(\frac{|\xi|^2}{v}\right)^2}{|d(\xi, \lambda)|^2} \leq \frac{\frac{2\bar{v}^2}{b^2} |\lambda - \lambda_1(\xi)|^2}{|\lambda - \lambda_3(\xi)|^2 |\lambda - \lambda_4(\xi)|^2} \leq \frac{2\bar{v}^2 a^2}{b^2} \frac{1}{|\lambda - C_0|^2}. \end{aligned}$$

Consequently, there exists $m_3 > 0$ such that

$$(2.26) \quad \|\theta_{4j}(D, \lambda)y_j\|_{H^1}^2 \leq \frac{m_3}{|\lambda - C_0|^2} \|y_j\|_{\mathbb{L}_2}^2, \quad j = 1, 2, 3.$$

Finally, denoting $c_{44}(\lambda) = \max_{\xi \in \mathbb{Z}^3} \{|\theta_{44}(\xi, \lambda)|^2\}$, we obtain that for $\lambda \notin M$,

$$(2.27) \quad \|\theta_{44}(D, \lambda)y_4\|_{H^1}^2 \leq c_{44}(\lambda) \sum_{\xi \in \mathbb{Z}^3} (1 + |\xi|^2) |(Fy_4)(\xi)|^2 \leq c_{44}(\lambda) \|y_4\|_{H^1}^2.$$

If $\lambda \in S_\alpha$, then

$$|\theta_{44}(\xi, \lambda)| = \left| \frac{\frac{|\xi|^2}{v} + \lambda}{d(\xi, \lambda)} \right| = \frac{|\lambda - \lambda_1(\xi)|}{|\lambda - \lambda_3(\xi)| |\lambda - \lambda_4(\xi)|} \leq \frac{a}{|\lambda - C_0|}.$$

Therefore,

$$(2.28) \quad \|\theta_{44}(D, \lambda)y_4\|_{H^1} \leq \frac{a}{|\lambda - C_0|} \|y_4\|_{H^1}.$$

Inequalities (2.19), (2.21), (2.23), (2.25), and (2.27) imply the estimate

$$\begin{aligned} \|(A(D) - \lambda E)^{-1}y\|_H^2 &\leq 4 \sum_{j=1}^3 \sum_{k=1}^4 c_{kj}(\lambda) \|y_j\|_{\mathbb{L}_2}^2 + 4 \sum_{k=1}^4 c_{k4}(\lambda) \|y_4\|_{H^1}^2 \\ &\leq 4 \max_{k,j} \{c_{kj}(\lambda)\} \left(\sum_{j=1}^3 \|y_j\|_{\mathbb{L}_2}^2 + \|y_4\|_{H^1}^2 \right) \leq c(\lambda) \|y\|_H^2. \end{aligned}$$

Moreover, inequalities (2.20), (2.22), (2.24), (2.26), and (2.28) show that there exists $m > 0$ such that $\|(A(D) - \lambda E)^{-1}y\|_H \leq \frac{m}{|\lambda - C_0|} \|y\|_H$. $\square$

Corollary 2.1. *We have $M = \sigma(A(D))$, where M is the set (2.9).*

Proof. By statement 1 of Lemma 2.6, we have $\sigma(A(D)) \subset M$, and the reverse inclusion $M \subset \sigma(A(D))$ follows from Lemma 2.1. $\square$

The main statement of this section is a consequence of Lemmas 2.1, 2.2, and 2.6.

Theorem 2.1. *The operator $A(D)$ is sectorial, and its spectrum is discrete.*

Proof. Lemma 2.1 and Corollary 2.1 show that the spectrum of $A(D)$ is discrete. Lemma 2.6 implies that, for $\lambda \in S_\alpha$, we have

$$\|(A(D) - \lambda E)^{-1}\| \leq \frac{m}{|\lambda - C_0|}. \quad \square$$

§3. THE CASE OF VARIABLE COEFFICIENTS

In this section we prove a result similar to Theorem 2.1 in the case of the operator $A(x, D)$ with variable coefficients.

3.1. Composition formula. Consider equation (1.8) for the operator

$$A(x, D) = \{a_{kj}(x, D)\}_{k,j=1}^4$$

with variable coefficients, where the entries $a_{kj}(x, D)$ are defined by (1.6), (1.7). Consider the symbol of the operator $A(x, D)$, i.e., the function $A(x, \xi) = \{a_{kj}(x, \xi)\}_{k,j=1}^4$ determined by the entries (2.4), (2.5) with $\bar{v} = \bar{v}(x)$, $b_{kj} = b_{kj}(x)$. Plugging $\bar{v} = \bar{v}(x)$, $b_{kj} = b_{kj}(x)$ in (2.6)–(2.8), we obtain expressions for $\lambda_j(x, \xi)$, $j = 1, \dots, 4$. We introduce the set

$$(3.1) \quad M(x) = \{\lambda(x, \xi) \text{ as defined in (2.6), (2.7) with } \bar{v} = \bar{v}(x), b_{kj} = b_{kj}(x)\}.$$

If $a_{kj}(x, \xi) = a_{kj}(\xi)$, then the set $M(x)$ coincides with the set M defined in (2.9). We introduce the following constants: $N_0 = \max_{x \in T} C_0(x)$, $N_1 = \max_{x \in T} C_1(x)$, $N_2 = \max_{x \in T} C_2(x)$, where $C_0 = C_0(x)$, $C_1 = C_1(x)$, and $C_2 = C_2(x)$ are as defined in (2.10) for every $x \in T$. Then the union $\bigcup_{x \in T} M(x)$ is included in the set

$$(3.2) \quad \begin{aligned} M_1 &= \{\lambda \in \mathbb{C} : \operatorname{Im} \lambda = 0, \operatorname{Re} \lambda \in (-\infty, -N_2) \cup (0, N_0)\} \\ &\cup \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \in [-N_2, 0], \operatorname{Im} \lambda \in [-N_1, N_1]\}. \end{aligned}$$

We put

$$(3.3) \quad S_1 = \left\{ \lambda \in \mathbb{C} : 0 \leq |\arg(\lambda - N_0)| < \pi - \arctan \frac{N_1}{N_0}, \lambda \neq N_0 \right\}.$$

Observe that the sector S_1 does not contain the set M_1 . From formulas (2.13)–(2.16) with $\bar{v} = \bar{v}(x)$, $b_{kj} = b_{kj}(x)$, we deduce expressions for $\{\theta_{kj}(x, \xi, \lambda)\}_{k,j=1}^4 = (A(x, \xi) - \lambda E)^{-1}$, $\lambda \in S_1$.

Recall that the solution $U(x) = (u_1(x), u_2(x), u_3(x), v(x))$ of equation (1.8) is defined on the torus T . For any $h \in H$, we seek a solution of the equation

$$(3.4) \quad (A(x, D) - \lambda E)U(x) = h(x)$$

in the form of the following series:

$$(3.5) \quad \begin{aligned} U(x) &= R_\lambda(x, D)h^1(x) \\ &\equiv \sum_{\xi \in \mathbb{Z}^3} \psi(\xi)(A(x, \xi) - \lambda E)^{-1}(E - Q(x, \xi, \lambda))(Fh^1)(\xi)e^{i(x, \xi)}, \end{aligned}$$

where $h^1(x)$ is a function to be determined, and

$$\psi(\xi) = \begin{cases} 1 & \text{if } |\xi| > \xi_0, \\ 0 & \text{if } |\xi| \leq \xi_0 \end{cases}$$

($\xi_0 > 1$ will be chosen later on). Finally, the matrix $Q(x, \xi, \lambda)$ has the form

$$Q(x, \xi, \lambda) = \begin{pmatrix} 0 & 0 & q_1(x, \xi, \lambda) \\ 0 & 0 & q_2(x, \xi, \lambda) \\ 0 & 0 & q_3(x, \xi, \lambda) \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda \in S_1.$$

The elements $q_k(x, \xi, \lambda)$ of the matrix $Q(x, \xi, \lambda)$ arise under the action of the operator $A(x, D) - \lambda E$ on the matrix $(A(x, \xi) - \lambda E)^{-1}e^{i(x, \xi)}$ and look like this:

$$(3.6) \quad \begin{aligned} q_k(x, \xi, \lambda) &= a_{k4}(x, D)\theta_{44}(x, \xi, \lambda) \\ &+ \frac{2i}{\bar{v}(x)} \sum_{j=1}^3 \xi_j \frac{\partial}{\partial x_j} \theta_{k4}(x, \xi, \lambda), \quad k = 1, 2, 3, \quad \lambda \in S_1, \end{aligned}$$

where the $a_{kj}(x, D)$ are the entries of $A(x, D)$ defined in (1.6), (1.7). The form (3.5) of a solution U of (3.4) is suggested by the commutation formula for pseudodifferential operators and is chosen with the purpose to obtain the operator inverse to $A(x, D) - \lambda E$ up to compact operators. Plugging (3.5) in (3.4) and using the composition formula for pseudodifferential operators (see [9]), we get

$$(3.7) \quad \begin{aligned} &(A(x, D) - \lambda E) \sum_{\xi \in \mathbb{Z}^3} \psi(\xi)(A(x, \xi) - \lambda E)^{-1}(E - Q(x, \xi, \lambda))e^{i(x, \xi)}(Fh^1)(\xi) \\ &= (A(x, D) - \lambda E) \sum_{|\xi| > \xi_0} (A(x, \xi) - \lambda E)^{-1}e^{i(x, \xi)}(Fh^1)(\xi) \\ &\quad - (A(x, D) - \lambda E) \sum_{|\xi| > \xi_0} (A(x, \xi) - \lambda E)^{-1}Q(x, \xi, \lambda)e^{i(x, \xi)}(Fh^1)(\xi) \\ &= \sum_{|\xi| > \xi_0} (E + K_1(x, \xi, \lambda) + Q(x, \xi, \lambda))e^{i(x, \xi)}(Fh^1)(\xi) \\ &\quad - \sum_{|\xi| > \xi_0} (K_2(x, \xi, \lambda) + Q(x, \xi, \lambda))e^{i(x, \xi)}(Fh^1)(\xi) \\ &= \sum_{|\xi| > \xi_0} (E + K(x, \xi, \lambda))e^{i(x, \xi)}(Fh^1)(\xi) = h(x). \end{aligned}$$

The elements of the matrix $K(x, \xi, \lambda) = \{k_{nj}(x, \xi, \lambda)\}_{n,j=1}^4$ are sums of those of the matrices $K_1(x, \xi, \lambda)$ and $-K_2(x, \xi, \lambda)$, namely, for $\lambda \in S_1$ we have

$$(3.8) \quad k_{nj}(x, \xi, \lambda) = \left(a_{nn}(x, D) + \frac{2i}{\bar{v}(x)} \sum_{j=1}^3 \xi_j a_{4j}(x, D) \right) \theta_{nj}(x, \xi, \lambda) + a_{n4}(x, D) \theta_{4n}(x, \xi, \lambda), \quad n, j = 1, 2, 3,$$

$$(3.9) \quad k_{4n}(x, \xi, \lambda) = a_{4n}(x, D) \theta_{nn}(x, \xi, \lambda) + a_{44-n}(x, D) \theta_{4-nn}(x, \xi, \lambda), \quad n = 1, 2, 3, \\ k_{n4}(x, \xi, \lambda) = a_{nn}(x, D) \theta_{n4}(x, \xi, \lambda)$$

$$(3.10) \quad - \left(a_{nn}(x, D) + \frac{2i}{\bar{v}(x)} \sum_{j=1}^3 \xi_j a_{4j}(x, D) \right) \cdot \sum_{j=1}^3 \theta_{nj}(x, \xi, \lambda) q_j(x, \xi, \lambda) + a_{n4}(x, D) \sum_{j=1}^3 \theta_{4j}(x, \xi, \lambda) q_j(x, \xi, \lambda), \quad n = 1, 2, 3,$$

where the dot denotes the action of an operator. Also,

$$(3.11) \quad k_{44}(x, \xi, \lambda) = \sum_{j=1}^3 a_{4j}(x, D) \theta_{j4}(x, \xi, \lambda) - \sum_{j=1}^3 (i\xi_j + a_{4j}(x, D)) \sum_{n=1}^3 (\theta_{jn}(x, \xi, \lambda) q_n(x, \xi, \lambda) - \lambda \sum_{j=1}^3 \theta_{4j}(x, \xi, \lambda) q_j(x, \xi, \lambda)),$$

where the $\theta_{nj}(x, \xi, \lambda)$ are the entries of the matrix $(A(x, \xi) - \lambda E)^{-1}$, as defined in (2.13)–(2.16) with $\bar{v} = \bar{v}(x)$, $b_{nm} = b_{nm}(x)$, and the $a_{nj}(x, D)$ are the entries of the operator $A(x, D)$ defined in (1.6), (1.7); recall that the numbers $q_n(x, \xi, \lambda)$ were defined in (3.6).

We introduce the operator $K(x, D, \lambda) : H \rightarrow H$,

$$(3.12) \quad K(x, D, \lambda)y(x) = \sum_{|\xi| > \xi_0} K(x, \xi, \lambda)(Fy)(\xi) e^{i(x, \xi)},$$

with the symbol $K(x, \xi, \lambda) = \{k_{nj}(x, \xi, \lambda)\}_{n,j=1}^4$ given in (3.8)–(3.11).

Observe that the entries (3.8)–(3.11) differ from the corresponding entries for the case of the 2-dimensional torus only in that in (3.8)–(3.11) the sums are over j and n running from 1 to 3, while in the 2-dimensional case j and n run from 1 to 2. Therefore, in essence, the proof of Lemmas 3.1–3.3 (see below) does not differ from that of similar statements in [5].

Let $r(\lambda, M_1)$ denote the distance from $\lambda \in S_1$ to the set M_1 defined in (3.2). We put $\Lambda = \max\{2N_0, 4N_1/\sin \varphi\}$, where $\varphi = \arctan \frac{N_1}{N_0} > 0$ is the angle defined in (3.3) (the quantities N_1 and N_2 are also defined in (3.3)).

Lemma 3.1. *Suppose $\lambda \in S_1$. There exists a number $C > 0$ such that $r(\lambda, M_1) \geq \frac{|\lambda|}{C}$ for any $|\lambda| > \Lambda$.*

Lemma 3.2. *Suppose $\lambda \in S_1$. There exists $\lambda_0 \in \mathbb{R}$, $\lambda_0 > \Lambda$, such that $\|K(x, D, \lambda)\| < 1$ for any $|\lambda| \geq \lambda_0$.*

Observe that λ_0 chosen in Lemma 3.2 does not depend on the number ξ_0 occurring in (3.12).

Lemma 3.3. *The operator $K(x, D, \lambda)$ is compact for any $\lambda \in S_1$.*

3.2. Proof of the main theorem. In this section we prove the following statement.

Theorem 3.1. *The operator $A(x, D)$ is sectorial, and its spectrum is discrete.*

We put

$$(3.13) \quad \xi_0 = \max \left\{ \sqrt{2\bar{v}^* \lambda_0}, \sqrt{N_1^2 + N_2^2}, N_0, 1 \right\},$$

where N_0 , N_1 , and N_2 are as defined in (3.2), $\bar{v}^* = \max_{x \in T} \bar{v}(x)$, and λ_0 is as chosen in Lemma 3.2. We recall that the following quantities are well defined for all $\lambda \in S_1$: the entries of the matrix $K(x, \xi, \lambda) = \{k_{nj}(x, \xi, \lambda)\}_{n,j=1}^4$, defined in (3.8)–(3.11), those of the matrix $Q(x, \xi, \lambda)$, defined in (3.6), and those of the matrix $(A(x, \xi) - \lambda E)^{-1} = \{\theta_{nj}(x, \xi, \lambda)\}_{n,j=1}^3$, defined in (2.13)–(2.16) with $\bar{v} = \bar{v}(x)$, $b_{nm} = b_{nm}(x)$. Note that in (3.13) the number ξ_0 is chosen so large that for $|\lambda| < \lambda_0$ and $|\xi| > \xi_0$ the matrices $K(x, \xi, \lambda)$, $Q(x, \xi, \lambda)$, and $(A(x, \xi) - \lambda E)^{-1}$ are defined for all $\lambda \in \mathbb{C}$ with $|\lambda| < \lambda_0$, and for $|\lambda| \geq \lambda_0$ and $|\xi| > \xi_0$ they are defined for all $\lambda \in S_1$.

Lemmas 3.2 and 3.3 imply the following theorems, which are proved in the same way as the corresponding statements in [5].

Theorem 3.2. *Suppose $\lambda \in S_1$. The operator*

$$(A(x, D) - \lambda E)^{-1} = R_\lambda(x, D)(E + K(x, D, \lambda))^{-1}$$

is well defined for all $\lambda \in \mathbb{C}$ except for a discrete set. Here $R_\lambda(x, D)$ is defined in (3.5) and $K(x, D, \lambda)$ in (3.12).

Theorem 3.3. *For $|\lambda| \geq \lambda_0$, where λ_0 is as in Lemma 3.2, the operator*

$$(A(x, D) - \lambda E)^{-1} = R_\lambda(x, D)(E + K(x, D, \lambda))^{-1}$$

is well defined for all $\lambda \in S_1$.

Consider the set

$$(3.14) \quad M_2 = (\mathbb{C} \setminus S_1) \cup \{\lambda \in \mathbb{C} : |\lambda| < \lambda_0\},$$

where λ_0 is as in Lemma 3.2.

Theorem 3.2 shows that for $|\lambda| < \lambda_0$ the spectrum of $A(x, D)$ is discrete. Theorems 3.2 and 3.3 imply that for $|\lambda| > \lambda_0$ the spectrum of $A(x, D)$ is discrete and is contained in the set $\mathbb{C} \setminus S_1$, where S_1 is the sector (3.3). Thus, the entire spectrum of $A(x, D)$ is discrete and is contained in the set (3.14). For a small $\gamma > 0$, we introduce the sector

$$S_\gamma = \left\{ \lambda \in \mathbb{C} : 0 \leq |\arg(\lambda - \sqrt{2\lambda_0})| \leq \frac{3\pi}{4} - \gamma \right\}.$$

The proof of Lemma 3.2 presented in [5], shows that $S_\gamma \cap M_2 = \emptyset$. Our next aim is to estimate the norm of $(A(x, D) - \lambda E)^{-1}$ for $\lambda \in S_\gamma$.

Consider the symbol of the operator $R_\lambda(x, D) : H \rightarrow H$ defined in (3.5), i.e., $R_\lambda(x, \xi) = \{r_{kj}(x, \xi, \lambda)\}_{k,j=1}^4$, where

$$(3.15) \quad r_{kj}(x, \xi, \lambda) = \theta_{kj}(x, \xi, \lambda), \quad k = 1, \dots, 4, \quad j = 1, 2, 3,$$

$$(3.16) \quad r_{k4}(x, \xi, \lambda) = - \sum_{j=1}^3 \theta_{kj}(x, \xi, \lambda) q_j(x, \xi, \lambda) + \theta_{k4}(x, \xi, \lambda), \quad k = 1, \dots, 4.$$

Lemma 3.4. *Suppose $\lambda \in S_\gamma$. There exists $l > 0$ such that none of the following quantities related to the matrix $R_\lambda(x, \xi)$ exceeds $\frac{l}{|\lambda - \sqrt{2\lambda_0}|}$:*

- 1) $|D_x^\alpha r_{kj}(x, \xi, \lambda)|$, $k = 1, 2, 3$, $j = 1, \dots, 4$, $|\alpha| \leq 4$;
- 2) $|\xi| |D_x^\alpha r_{4j}(x, \xi, \lambda)|$, $j = 1, 2, 3$, $|\alpha| \leq 6$;
- 3) $|D_x^\alpha r_{44}(x, \xi, \lambda)|$, $|\alpha| \leq 6$.

Proof. Consider the sets

$$(3.17) \quad L_n = \left\{ \frac{i\xi_j \prod_{l=1}^n g_l(x, \xi, \lambda)}{(d(x, \xi, \lambda))^{n+1}} : g_l(x, \xi, \lambda) = p_0^l(x)\lambda|\xi|^2 \right. \\ \left. + \sum_{k=1}^3 \xi_k \sum_{j=1}^3 p_{kj}^l(x)\xi_j, \lambda \in S_\gamma, \xi \in \mathbb{Z}^3, j = 1, 2, 3 \right\}, \quad n = 0, \dots, 6,$$

where $d(x, \xi, \lambda)$ is the number defined in (2.17) with $\bar{v} = \bar{v}(x)$, $b_{kj} = b_{kj}(x)$, and the functions $p_0^l(x), p_{kj}^l(x)$ depend on x . Observe that the entries $r_{4j}(x, \xi, \lambda)$, $j = 1, 2, 3$, defined in (3.15), belong to the set L_0 . We shall show that if $f(x, \xi, \lambda) \in L_0$, then

$$(3.18) \quad \frac{\partial^n}{\partial x_j^n} f(x, \xi, \lambda) = \sum_{l=1}^n f_l(x, \xi, \lambda), \quad n = 1, \dots, 6, \quad \text{where } f_l(x, \xi, \lambda) \in L_l.$$

Suppose $f(x, \xi, \lambda) \in L_0$; then

$$\frac{\partial}{\partial x_j} f(x, \xi, \lambda) = -\frac{i\xi_j \frac{\partial}{\partial x_j} d(x, \xi, \lambda)}{d^2(x, \xi, \lambda)} \equiv f_1(x, \xi, \lambda).$$

Relation (2.17) shows that $\frac{\partial}{\partial x_j} d(x, \xi, \lambda)$ has the form of a function $g_l(x, \xi, \lambda)$, as defined in (3.17); we denote it by $g_1(x, \xi, \lambda)$. Next, we have

$$\begin{aligned} \frac{\partial^2}{\partial x_j^2} f(x, \xi, \lambda) &= \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} f(x, \xi, \lambda) = \frac{\partial}{\partial x_j} f_1(x, \xi, \lambda) = \frac{\partial}{\partial x_j} \left(\frac{i\xi_j g_1(x, \xi, \lambda)}{(d(x, \xi, \lambda))^2} \right) \\ &= \frac{i\xi_j \frac{\partial}{\partial x_j} g_1(x, \xi, \lambda)}{d^2(x, \xi, \lambda)} - \frac{2i\xi_j g_1(x, \xi, \lambda) \frac{\partial}{\partial x_j} d(x, \xi, \lambda)}{d^3(x, \xi, \lambda)} \\ &\equiv f_1(x, \xi, \lambda) + f_2(x, \xi, \lambda). \end{aligned}$$

Again, the entries $\frac{\partial}{\partial x_j} d(x, \xi, \lambda)$ and $\frac{\partial}{\partial x_j} g_1(x, \xi, \lambda)$ have the form of a function $g_l(x, \xi, \lambda)$ as defined in (3.17); we denote them by $g_1(x, \xi, \lambda)$ and $g_2(x, \xi, \lambda)$, respectively. Consequently, $f_1(x, \xi, \lambda) \in L_1$, $f_2(x, \xi, \lambda) \in L_2$. We proceed by induction. We assume that (3.18) is proved for $n = 5$, and we prove it for $n = 6$. We have

$$\begin{aligned} \frac{\partial^6}{\partial x_j^6} f(x, \xi, \lambda) &= \frac{\partial}{\partial x_j} \frac{\partial^5}{\partial x_j^5} f(x, \xi, \lambda) = \frac{\partial}{\partial x_j} \sum_{n=1}^5 f_n(x, \xi, \lambda) \\ &= \frac{\partial}{\partial x_j} \left(\sum_{n=1}^5 \frac{i\xi_j \prod_{l=1}^n g_l(x, \xi, \lambda)}{(d(x, \xi, \lambda))^{n+1}} \right) \\ &= \sum_{n=1}^5 \frac{i\xi_j \sum_{m=1}^n \frac{\partial}{\partial x_j} g_m(x, \xi, \lambda) \prod_{l \neq m, l=1}^n g_l(x, \xi, \lambda)}{(d(x, \xi, \lambda))^{n+1}} \\ &\quad - \sum_{n=1}^5 \frac{(n+1)i\xi_j \prod_{l=1}^n g_l(x, \xi, \lambda) \frac{\partial}{\partial x_j} d(x, \xi, \lambda)}{(d(x, \xi, \lambda))^{n+2}} \\ &\equiv \sum_{n=1}^6 f_n(x, \xi, \lambda). \end{aligned}$$

Since the entries $\frac{\partial}{\partial x_j} d(x, \xi, \lambda)$ and $\frac{\partial}{\partial x_j} g_m(x, \xi, \lambda)$ have the form of a function $g_l(x, \xi, \lambda)$ as defined in (3.17), we see that (3.18) is true for $n = 6$ and, with this, for all $n = 1, \dots, 6$.

Now we show that if $h(x, \xi, \lambda) \in L_n$, then there exists $c_n > 0$ such that $|\xi| |h(x, \xi, \lambda)| \leq \frac{c_n}{|\lambda - \sqrt{2\lambda_0}|}$, $n = 0, \dots, 6$.

Let $r(\lambda, M_2)$ denote the distance from $\lambda \in S_\gamma$ to the set M_2 defined in (3.2), and let

$$a_1 = \max_l \left\{ \max_{x \in T} |p_0^l(x)|, \max_{k,j} \max_{x \in T} |p_{kj}^l(x)| \right\}, \quad a_2 = \max_{x \in T} |\bar{v}(x)|.$$

The proof of Lemma 2.5, which is valid for $\lambda_j(\xi) = \lambda_j(x, \xi)$, shows that there exists $a_3 > 0$ such that $|\lambda - \lambda_1(x, \xi)| \leq a_3 |\lambda - \lambda_4(x, \xi)|$. In the calculation below we use the inequality $|\lambda - \lambda_j(x, \xi)| \geq r(\lambda, M_2)$, Lemmas 2.3–2.5 with the set M_2 and $\lambda_j(\xi) = \lambda_j(x, \xi)$, and also Lemma 3.1. We have

$$\begin{aligned} |\xi| |h(x, \xi, \lambda)| &= |\xi| \left| \frac{i\xi_j \prod_{l=1}^n g(x, \xi, \lambda)}{(d(x, \xi, \lambda))^{n+1}} \right| \\ &\leq \frac{|\xi|^2 \prod_{l=1}^n |p_0^l(x) \lambda| |\xi|^2 + \sum_{k=1}^3 \xi_k \sum_{j=1}^3 |p_{kj}^l(x) \xi_j|}{|d(x, \xi, \lambda)|^{n+1}} \\ &\leq \frac{|\xi|^2 \prod_{l=1}^n |p_0^l(x) \lambda| |\xi|^2 + \sum_{k=1}^3 \xi_k \sum_{j=1}^3 |p_{kj}^l(x) \xi_j|}{|\lambda - \lambda_3(x, \xi)|^{n+1} |\lambda - \lambda_4(x, \xi)|^{n+1}} \\ &\leq \frac{|\xi|^2 \prod_{l=1}^n (\max_{x \in T} |p_0^l(x)| |\lambda| |\xi|^2 + \max_{k,j} \max_{x \in T} |p_{kj}^l(x)| |\xi|^2)}{|\lambda - \lambda_3(x, \xi)|^{n+1} |\lambda - \lambda_4(x, \xi)|^{n+1}} \\ &\leq |\xi|^2 a_1^n \left(\frac{|\lambda| |\xi|^2 + |\xi|^2}{|\lambda - \lambda_3(x, \xi)| |\lambda - \lambda_4(x, \xi)|} \right)^n \frac{1}{|\lambda - \lambda_3(x, \xi)| |\lambda - \lambda_4(x, \xi)|} \\ &\leq \left(\frac{|\lambda| |\xi|^2}{r(\lambda, M_2) |\lambda - \lambda_4(x, \xi)|} + \frac{|\xi|^2}{|\lambda - \lambda_3(x, \xi)| |\lambda - \lambda_4(x, \xi)|} \right)^n \\ &\quad \times \frac{|\xi|^2 a_1^n}{|\lambda - \lambda_3(x, \xi)| |\lambda - \lambda_4(x, \xi)|} \\ &\leq \left(\frac{C |\xi|^2}{|\lambda - \lambda_4(x, \xi)|} + \frac{|\xi|^2}{|\lambda - \lambda_3(x, \xi)| |\lambda - \lambda_4(x, \xi)|} \right)^n \\ &\quad \times \frac{a_1^n a_2 |\lambda - \lambda_1(x, \xi)|}{b |\lambda - \lambda_3(x, \xi)| |\lambda - \lambda_4(x, \xi)|} \\ &\leq \left(\frac{a_2 C |\lambda + \frac{|\xi|^2}{\bar{v}(x)}|}{b |\lambda - \lambda_4(x, \xi)|} + \frac{a_2 |\lambda + \frac{|\xi|^2}{\bar{v}(x)}|}{b |\lambda - \lambda_3(x, \xi)| |\lambda - \lambda_4(x, \xi)|} \right)^n \frac{a_1^n a_2 a}{b |\lambda - \sqrt{2\lambda_0}|} \\ &\leq \left(\frac{a_2 C |\lambda - \lambda_1(x, \xi)|}{b |\lambda - \lambda_4(x, \xi)|} + \frac{a_2 |\lambda - \lambda_1(x, \xi)|}{b |\lambda - \lambda_3(x, \xi)| |\lambda - \lambda_4(x, \xi)|} \right)^n \frac{a_1^n a_2 a}{b |\lambda - \sqrt{2\lambda_0}|} \\ &\leq \left(\frac{a_2 a_3 C}{b} + \frac{a_2 a}{r(\lambda, M_2) b} \right)^n \frac{a_1^n a_2 a}{b |\lambda - \sqrt{2\lambda_0}|} \\ &\leq \left(\frac{a_2 a_3 C}{b} + \frac{a_2 a C}{b |\lambda|} \right)^n \frac{a_1^n a_2 a}{b |\lambda - \sqrt{2\lambda_0}|} \\ &\leq \left(\frac{a_2 a_3 C}{b} + \frac{a_2 a C}{b \min_{S_\gamma} |\lambda|} \right)^n \frac{a_1^n a_2 a}{b |\lambda - \sqrt{2\lambda_0}|} \leq \frac{c_n}{|\lambda - \sqrt{2\lambda_0}|}, \quad n = 0, \dots, 6. \end{aligned}$$

Since the function $r_{4j}(x, \xi, \lambda)$ belongs to L_0 , the above material shows that

$$D_x^\alpha r_{4j}(x, \xi, \lambda) = \sum_{n=1}^{|\alpha|} f_n(x, \xi, \lambda),$$

where $f_n(x, \xi, \lambda) \in L_n$, $|\alpha| \leq 6$. This proves the claim for the elements of the form 2) (see the formulation of the lemma).

Consider the following family of sets:

$$(3.19) \quad \begin{aligned} M_n = & \left\{ \eta_n(x, \xi) \sum_{j=0}^n \left(\frac{\prod_{k=1}^j a_k(x) |\xi|^2}{(h(x, \xi, \lambda))^{j+1}} \right) \left(\frac{\prod_{l=1}^{n-j} g_l(x, \xi, \lambda)}{(d(x, \xi, \lambda))^{n-j+1}} \right) + \frac{\prod_{i=1}^n a_i(x) |\xi|^2}{(h(x, \xi, \lambda))^{n+1}}, \right. \\ & h(x, \xi, \lambda) = \frac{|\xi|^2}{\bar{v}(x)} + \lambda, \quad \eta_n(x, \xi) = \xi_j \sum_{i=1}^3 b_{ji}^n(x) \xi_i, \lambda \in S_\gamma, \xi \in \mathbb{Z}^3 \Big\}, \\ & n = 0, \dots, 4. \end{aligned}$$

Here the functions $g_l(x, \xi, \lambda)$ are as defined in (3.17), $d(x, \xi, \lambda)$ is defined in (2.17) with $\bar{v} = \bar{v}(x)$, $b_{kj} = b_{kj}(x)$, and $a_k(x)$, $b_{ji}^n(x)$ depend on x . Observe that the elements $r_{kj}(x, \xi, \lambda)$, $j, k = 1, 2, 3$, belong to M_0 . For the sets M_n we can prove analogs of the statements about L_n proved before, obtaining the claim for the elements of the form 1) with $k, j = 1, 2, 3$.

Finally, we consider the family of sets

$$(3.20) \quad \begin{aligned} K_n = & \left\{ c_n(x) \eta_n(x, \xi) \sum_{j=0}^n \left(\frac{\prod_{k=1}^j a_k(x) |\xi|^4}{(h(x, \xi, \lambda))^{j+1}} \right) \left(\frac{\prod_{l=1}^{n-j} g_l(x, \xi, \lambda)}{(d(x, \xi, \lambda))^{n-j+2}} \right) \right. \\ & + b_{ki}(x) q_n(x, \xi) \frac{\prod_{l=1}^{n+1} g_l(x, \xi, \lambda) |\xi|^2}{(d(x, \xi, \lambda))^{n+3}} + w_n(x, \xi) \frac{\prod_{l=1}^n g_l(x, \xi, \lambda)}{(d(x, \xi, \lambda))^{n+1}}, \\ & \left. \lambda \in S_\gamma, \xi \in \mathbb{Z}^3, k, i = 1, 2, 3 \right\}, \\ & n = 0, \dots, 6, \end{aligned}$$

where $w_n(x, \xi)$ is either $\sum_{j=1}^3 b_{ij}^n(x) \xi_j$ or $h(x, \xi, \lambda)$ (see (3.19)). The functions $g_l(x, \xi, \lambda)$ and $q_n(x, \xi)$ are defined in (3.17) and (3.19), respectively, and $c_n(x)$ depends on x . Observe that the elements $r_{k4}(x, \xi, \lambda)$, $k = 1, \dots, 4$, belong to K_0 . For the sets K_n we can prove analogs of the statements about L_n proved before, obtaining the claim for the elements of the form 1) and 3) with $k = 1, \dots, 4$, $j = 4$. $\square$

Theorem 3.4. *There exists $L > 0$ such that*

$$\|R_\lambda(x, D)\| \leq \frac{L}{|\lambda - \sqrt{2\lambda_0}|}.$$

Proof. We estimate the norm of the operator $R_\lambda(x, D) : H \rightarrow H$ defined in (3.5). Let $r_{kj}(x, D, \lambda)$ be the entries of the operator-matrix $R_\lambda(x, D)$, and let $R_k(x, D, \lambda) = (r_{k1}(x, D, \lambda), r_{k2}(x, D, \lambda), r_{k3}(x, D, \lambda), r_{k4}(x, D, \lambda))$ be the corresponding rows. For any $h^1 \in H$ we have

$$(3.21) \quad \begin{aligned} \|R_\lambda(x, D)h^1\|_H^2 &= \sum_{k=1}^3 \|R_k(x, D, \lambda)h^1\|_{\mathbb{L}_2}^2 + \|R_4(x, D, \lambda)h^1\|_{H^1}^2, \\ \|R_k(\cdot, D, \lambda)h^1\|_{\mathbb{L}_2}^2 &= \sum_{\eta \in \mathbb{Z}^3} |F(R_k(x, D, \lambda)h^1(x))(\eta)|^2, \quad k = 1, 2, 3, \end{aligned}$$

$$(3.22) \quad \|R_4(\cdot, D, \lambda)h^1\|_{\mathbb{L}_2}^2 = \sum_{\eta \in \mathbb{Z}^3} (1 + |\xi|^2) |F(R_4(x, D, \lambda)h^1(x))(\eta)|^2,$$

where $F(R_k(x, D, \lambda)h^1(x))(\eta)$ are the coefficients of $R_k(x, D, \lambda)h^1(x)$. Note that the entries $r_{kj}(x, \xi, \lambda)$ defined in (3.15), (3.16) are the symbols of the pseudodifferential operators $r_{kj}(x, D, \lambda)$ for any $k, j = 1, \dots, 4$. We have

$$\begin{aligned}
 (3.23) \quad F(R_k(x, D, \lambda)h^1(x))(\eta) &= F\left(\sum_{j=1}^4 r_{kj}(x, D, \lambda)h_j^1(x)\right)(\eta) \\
 &= (2\pi)^{-3} \int_T e^{-i(x, \eta)} \sum_{j=1}^4 r_{kj}(x, D, \lambda)h_j^1(x) dx \\
 &= (2\pi)^{-3} \int_T e^{-i(x, \eta)} \sum_{j=1}^4 \sum_{|\xi| > \xi_0} r_{kj}(x, \xi, \lambda)(Fh_j^1)(\xi)e^{i(x, \xi)} dx \\
 &= \sum_{j=1}^4 \sum_{|\xi| > \xi_0} d_{kj}(\eta - \xi, \xi, \lambda)(Fh_j^1)(\xi), \quad k = 1, 2, 3,
 \end{aligned}$$

where

$$(3.24) \quad d_{kj}(\eta - \xi, \xi, \lambda) = (2\pi)^{-3} \int_T e^{-i(x, \eta - \xi)} r_{kj}(x, \xi, \lambda) dx, \quad k, j = 1, \dots, 4.$$

Integrating the right-hand side of (3.24) by parts, and using the periodicity of the entries $r_{kj}(x, \xi, \lambda)$ in x , we obtain

$$\begin{aligned}
 (3.25) \quad &(1 + |\eta - \xi|^2)^2 d_{kj}(\eta - \xi, \xi, \lambda) \\
 &= (2\pi)^{-3} \int_T r_{kj}(x, \xi, \lambda)(\Delta - 1)^2 e^{-i(x, \eta - \xi)} dx \\
 &= (2\pi)^{-3} \int_T e^{-i(x, \eta - \xi)} (\Delta - 1)^2 r_{kj}(x, \xi, \lambda) dx.
 \end{aligned}$$

By Lemma 3.4, there is $l_1 > 0$ such that, for any $\xi \in \mathbb{Z}^3$ and any $\lambda \in S_\gamma$,

$$(3.26) \quad |(\Delta - 1)^2 r_{kj}(x, \xi, \lambda)| < \frac{l_1}{|\lambda - \sqrt{2\lambda_0}|}.$$

Therefore, by (3.25)

$$(3.27) \quad |d_{kj}(\eta - \xi, \xi, \lambda)| \leq \frac{l_1}{|\lambda - \sqrt{2\lambda_0}|} (1 + |\eta - \xi|^2)^{-2}, \quad k = 1, 2, 3, \quad j = 1, 2, 3, 4.$$

Observe that there exists $c_1 > 0$ such that

$$(3.28) \quad \int_{\mathbb{R}^3} (1 + |\eta - \xi|^2)^{-2} d\xi = c_1, \quad \int_{\mathbb{R}^3} (1 + |\eta - \xi|^2)^{-2} d\eta = c_1.$$

These integrals are calculated by passing to spherical coordinates

$$\begin{aligned}\eta_1 - \xi_1 &= \rho \sin \varphi \cos \theta, \\ \eta_2 - \xi_2 &= \rho \cos \varphi \cos \theta, \\ \eta_3 - \xi_3 &= \rho \sin \theta\end{aligned}$$

(the corresponding Jacobian is $\rho^2 \cos \theta$). Plugging (3.23) in (3.21) and using (3.27), (3.28), and the Hölder inequality, we see that

$$\begin{aligned}(3.29) \quad \|R_k(\cdot, D, \lambda)h^1\|_{\mathbb{L}_2}^2 &= \sum_{\eta \in Z^3} \left| \sum_{j=1}^4 \sum_{|\xi| > \xi_0} d_{kj}(\eta - \xi, \xi, \lambda)(Fh_j^1)(\xi) \right|^2 \\ &\leq 4 \sum_{j=1}^4 \sum_{\eta \in Z^3} \left(\sum_{|\xi| > \xi_0} |d_{kj}(\eta - \xi, \xi, \lambda)| |(Fh_j^1)(\xi)|^2 \right) \left(\sum_{|\xi| > \xi_0} |d_{kj}(\eta - \xi, \xi, \lambda)| \right) \\ &\leq 4 \sum_{j=1}^4 \sum_{\eta \in Z^3} \left(\sum_{|\xi| > \xi_0} |d_{kj}(\eta - \xi, \xi, \lambda)| |(Fh_j^1)(\xi)|^2 \right) \int_{\mathbb{R}^3} \frac{l_1(1 + |\eta - \xi|^2)^{-2}}{|\lambda - \sqrt{2\lambda_0}|} d\xi \\ &\leq \frac{4l_1c_1}{|\lambda - \sqrt{2\lambda_0}|} \sum_{j=1}^4 \sum_{|\xi| > \xi_0} |(Fh_j^1)(\xi)|^2 \sum_{\eta \in Z^3} |d_{kj}(\eta - \xi, \xi, \lambda)| \\ &\leq \frac{4l_1^2c_1}{|\lambda - \sqrt{2\lambda_0}|^2} \sum_{j=1}^4 \sum_{|\xi| > \xi_0} |(Fh_j^1)(\xi)|^2 \int_{\mathbb{R}^3} (1 + |\eta - \xi|^2)^{-2} d\eta \\ &\leq \frac{4l_1^2c_1^2}{|\lambda - \sqrt{2\lambda_0}|^2} \sum_{j=1}^4 \sum_{|\xi| > \xi_0} |(Fh_j^1)(\xi)|^2 \leq \frac{4l_1^2c_1^2}{|\lambda - \sqrt{2\lambda_0}|^2} \sum_{j=1}^4 \|h_j^1\|_{\mathbb{L}_2}^2 \\ &\leq \frac{4l_1^2c_1^2}{|\lambda - \sqrt{2\lambda_0}|^2} \|h^1\|_H^2, \quad k = 1, 2, 3.\end{aligned}$$

We estimate the norm of $R_4(\cdot, D, \lambda)$. For the Fourier coefficients of $R_4(x, D, \lambda)y(x)$ we have

$$(3.30) \quad F(R_4(x, D, \lambda)y(x))(\eta) = \sum_{j=1}^4 \sum_{|\xi| > \xi_0} d_{4j}(\eta - \xi, \xi, \lambda)(Fh_j^1)(\xi),$$

where $d_{4j}(\eta - \xi, \xi, \lambda)$ are the functions defined in (3.24). There exist constants $M_1 > 0$ and $M_2 > 0$ such that

$$(3.31) \quad \max_{\eta \in \mathbb{R}^3} \int_{|\xi| > \xi_0} \frac{\sqrt{1 + |\eta|^2}}{|\xi|} (1 + |\eta - \xi|^2)^{-3} d\xi < M_1,$$

$$(3.32) \quad \max_{|\xi| > \xi_0} \int_{\mathbb{R}^3} \frac{\sqrt{1 + |\eta|^2}}{|\xi|} (1 + |\eta - \xi|^2)^{-3} d\eta < M_2.$$

We prove (3.31):

$$\begin{aligned}
& \max_{\eta \in \mathbb{R}^3} \int_{|\xi| > \xi_0} \frac{\sqrt{1+|\eta|^2}}{|\xi|} (1+|\eta-\xi|^2)^{-3} d\xi \\
&= \max_{|\eta| > 1} \frac{\sqrt{1+|\eta|^2}}{|\eta|} \int_{|\xi| > \xi_0} \frac{|\eta|}{|\xi|} (1+|\eta-\xi|^2)^{-3} d\xi \\
&\quad + \max_{|\eta| \leq 1} \int_{|\xi| > \xi_0} \frac{\sqrt{1+|\eta|^2}}{|\xi|} (1+|\eta-\xi|^2)^{-3} d\xi \\
&\leq \sqrt{2} \max_{|\eta| > 1} \int_{|\xi| > \xi_0} \frac{|\eta|}{|\xi|} (1+|\eta-\xi|^2)^{-3} d\xi \\
&\quad + \int_{|\xi| > \xi_0} \frac{\sqrt{2}}{\xi_0} (1+|\eta-\xi|^2)^{-3} d\xi \\
&\leq \sqrt{2} \max_{|\eta| > 1} \int_{|\xi| > \xi_0} (1+|\eta-\xi|^2)^{-3} d\xi \\
&\quad + \sqrt{2} \max_{|\eta| > 1} \int_{|\xi| > \xi_0} \frac{|\eta-\xi|}{\xi_0(1+|\eta-\xi|^2)^3} d\xi \\
&\quad + \int_{|\xi| > \xi_0} \frac{\sqrt{2}}{\xi_0} (1+|\eta-\xi|^2)^{-3} d\xi.
\end{aligned}$$

As in the proof of (3.28), the last integrals are calculated by passing to spherical coordinates, which results in inequality (3.31). Now, we prove (3.32):

$$\begin{aligned}
& \int_{\mathbb{R}^3} \frac{\sqrt{1+|\eta|^2}}{|\xi|} (1+|\eta-\xi|^2)^{-3} d\eta \\
&\leq \int_{|\eta| < 1} \frac{\sqrt{2}}{|\xi|} (1+|\eta-\xi|^2)^{-3} d\eta + \int_{|\eta| \geq 1} \frac{|\eta|\sqrt{2}}{|\xi|} (1+|\eta-\xi|^2)^{-3} d\eta \\
&\leq \int_{|\eta| < 1} \frac{\sqrt{2}}{|\xi|} (1+|\eta-\xi|^2)^{-3} d\eta \\
&\quad + \sqrt{2} \int_{|\eta| \geq 1} \left(1 + \frac{|\eta-\xi|}{|\xi|}\right) (1+|\eta-\xi|^2)^{-3} d\eta.
\end{aligned}$$

Here, the first integral is taken over a bounded set, and in the second integral we pass to the spherical coordinates. Next, we integrate by parts and take the maximum over $\xi : |\xi| > \xi_0$; as a result, we obtain inequality (3.32).

By Lemma 3.4, there exist $l_2 > 0$ such that for any $\lambda \in S_\gamma$, any $\xi \in \mathbb{Z}^3$, and $j = 1, 2, 3$ we have

$$(3.33) \quad |\xi| |(\Delta - 1)^3 r_{4j}(x, \xi, \lambda)| < \frac{l_2}{|\lambda - \sqrt{2\lambda_0}|}.$$

Therefore, by (3.25),

$$(3.34) \quad |d_{4j}(\eta - \xi, \xi, \lambda)| \leq \frac{l_2}{|\lambda - \sqrt{2\lambda_0}| |\xi|} (1+|\eta-\xi|^2)^{-3}, \quad j = 1, 2, 3.$$

Using (3.30)–(3.34) and the Hölder inequality, we deduce the following estimate for $j = 1, 2, 3$:

$$\begin{aligned}
\|r_{4j}(\cdot, D, \lambda)h_j^1\|_{H^1}^2 &= \sum_{\eta \in \mathbb{Z}^3} (1 + |\eta|^2) |F(r_{4j}(x, D, \lambda)h_j^1(x))(\eta)|^2 \\
&= \sum_{\eta \in \mathbb{Z}^2} (1 + |\eta|^2) \left| \sum_{|\xi| > \xi_0} d_{4j}(\eta - \xi, \xi, \lambda) (Fh_j^1)(\xi) \right|^2 \\
&= \sum_{\eta \in \mathbb{Z}^3} \left| \sum_{|\xi| > \xi_0} \sqrt{1 + |\eta|^2} d_{4j}(\eta - \xi, \xi, \lambda) (Fh_j^1)(\xi) \right|^2 \\
&\leq \sum_{\eta \in \mathbb{Z}^3} \left(\sum_{|\xi| > \xi_0} \sqrt{1 + |\eta|^2} |d_{4j}(\eta - \xi, \xi, \lambda)| |(Fh_j^1)(\xi)|^2 \right) \\
&\quad \times \left(\sum_{|\xi| > \xi_0} \sqrt{1 + |\eta|^2} |d_{4j}(\eta - \xi, \xi, \lambda)| \right) \\
(3.35) \quad &\leq \sum_{\eta \in \mathbb{Z}^3} \left(\sum_{|\xi| > \xi_0} \sqrt{1 + |\eta|^2} |d_{4j}(\eta - \xi, \xi, \lambda)| |(Fh_j^1)(\xi)|^2 \right) \\
&\quad \times \left(\max_{\eta \in \mathbb{R}^3} \int_{|\xi| > \xi_0} \frac{l_2 \sqrt{1 + |\eta|^2}}{|\lambda - \sqrt{2\lambda_0}| |\xi|} (1 + |\eta - \xi|^2)^{-2} d\xi \right) \\
&\leq \frac{l_2 M_1}{|\lambda - \sqrt{2\lambda_0}|} \sum_{|\xi| > \xi_0} |(Fh_j^1)(\xi)|^2 \sum_{\eta \in \mathbb{Z}^3} \left(\sqrt{1 + |\eta|^2} |d_{4j}(\eta - \xi, \xi, \lambda)| \right) \\
&\leq \frac{l_2^2 M_1}{|\lambda - \sqrt{2\lambda_0}|^2} \sum_{|\xi| > \xi_0} |(Fh_j^1)(\xi)|^2 \max_{|\xi| > \xi_0} \int_{\mathbb{R}^3} \frac{\sqrt{1 + |\eta|^2}}{|\xi|} (1 + |\eta - \xi|^2)^{-2} d\eta \\
&\leq \frac{l_2^2 M_1 M_2}{|\lambda - \sqrt{2\lambda_0}|^2} \sum_{|\xi| > \xi_0} |(Fh_j^1)(\xi)|^2 \leq \frac{l_2^2 M_1 M_2}{|\lambda - \sqrt{2\lambda_0}|^2} \|h_j^1\|_{L^2}^2.
\end{aligned}$$

By Lemma 3.4, there exists $l_3 > 0$ such that for any $\lambda \in S_\gamma$ and any $\xi \in \mathbb{Z}^3$ we have

$$(3.36) \quad |(\Delta - 1)^3 r_{44}(x, \xi, \lambda)| < \frac{l_3}{|\lambda - \sqrt{2\lambda_0}|},$$

whence

$$(3.37) \quad |d_{44}(\eta - \xi, \xi, \lambda)| \leq \frac{l_3}{|\lambda - \sqrt{2\lambda_0}|} (1 + |\eta - \xi|^2)^{-3}.$$

Observe that there exist $c_2, c_3 > 0$ such that

$$(3.38) \quad \int_{\mathbb{R}^3} (1 + |\eta - \xi|^2)^{-3} d\xi = c_2, \quad \int_{\mathbb{R}^3} \frac{(1 + |\eta|^2)}{(1 + |\eta - \xi|^2)^3} d\eta \leq c_3 (1 + |\xi|^2).$$

We check the inequality in (3.38).

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{(1 + |\eta|^2)}{(1 + |\eta - \xi|^2)^3} d\eta &= \int_{\mathbb{R}^3} \frac{1}{(1 + |\eta - \xi|^2)^3} d\eta \\
&\quad + \int_{\mathbb{R}^3} \frac{2|\eta - \xi|^2}{(1 + |\eta - \xi|^2)^3} d\eta + \int_{\mathbb{R}^3} \frac{2|\xi|^2}{(1 + |\eta - \xi|^2)^3} d\eta.
\end{aligned}$$

Passing to spherical coordinates in the last-written integrals and integrating by parts, we deduce the required inequality. Using (3.36)–(3.38) and the Hölder inequality, we obtain

$$\begin{aligned}
 \|r_{44}(\cdot, D, \lambda)h_4^1\|_{H^1}^2 &= \sum_{\eta \in \mathbb{R}^3} (1 + |\eta|^2) |F(r_{44}(x, D, \lambda)h_4^1(x))(\eta)|^2 \\
 &\leq \sum_{\eta \in \mathbb{R}^3} (1 + |\eta|^2) \left(\sum_{|\xi| > \xi_0} |d_{44}(\eta - \xi, \xi, \lambda)| |(Fh_4^1)(\xi)|^2 \right) \\
 &\quad \times \left(\sum_{|\xi| > \xi_0} |d_{44}(\eta - \xi, \xi, \lambda)| \right) \\
 &\leq \sum_{\eta \in \mathbb{R}^3} (1 + |\eta|^2) \left(\sum_{|\xi| > \xi_0} |d_{44}(\eta - \xi, \xi, \lambda)| |(Fh_4^1)(\xi)|^2 \right) \\
 (3.39) \quad &\quad \times \left(\int_{\mathbb{R}^3} \frac{l_3}{|\lambda - \sqrt{2\lambda_0}|} (1 + |\eta - \xi|^2)^{-3} d\xi \right) \\
 &\leq \frac{l_3 c_2}{|\lambda - \sqrt{2\lambda_0}|} \sum_{|\xi| > \xi_0} |(Fh_4^1)(\xi)|^2 \sum_{\eta \in \mathbb{R}^3} (1 + |\eta|^2) (|d_{4j}(\eta - \xi, \xi, \lambda)|) \\
 &\leq \frac{l_3^2 c_2}{|\lambda - \sqrt{2\lambda_0}|^2} \sum_{|\xi| > \xi_0} |(Fh_4^1)(\xi)|^2 \int_{\mathbb{R}^3} (1 + |\eta|^2) (1 + |\eta - \xi|^2)^{-3} d\eta \\
 &\leq \frac{l_3^2 c_2 c_3}{|\lambda - \sqrt{2\lambda_0}|^2} \sum_{|\xi| > \xi_0} |(Fh_4^1)(\xi)|^2 (1 + |\xi|^2) \leq \frac{l_3^2 c_2 c_3}{|\lambda - \sqrt{2\lambda_0}|^2} \|h_4^1\|_{H^1}^2.
 \end{aligned}$$

By (3.29), (3.35), and (3.39), for some $L > 0$ we have $\|R_\lambda(\cdot, D, \lambda)h^1\|_H \leq \frac{L}{|\lambda - \sqrt{2\lambda_0}|} \|h^1\|_H$ for $\lambda \in S_\gamma$. $\square$

Now, we show that there exists $B > 0$ such that

$$\|(A(x, D) - \lambda E)^{-1}\| \leq \frac{B}{|\lambda - \sqrt{2\lambda_0}|}$$

for all $\lambda \in S_\gamma$, where $\sqrt{2\lambda_0}$ is the vertex of the sector S_γ . The definition of S_γ implies that all $\lambda \in S_\gamma$ satisfy $|\lambda| > \lambda_0$. Consequently, by Lemma 3.2, we can find $N > 0$ such that $\|(E + K(x, D, \lambda))^{-1}\| \leq N$. By Theorem 3.3,

$$\begin{aligned}
 \|(A(\cdot, D) - \lambda E)^{-1}h\|_H^2 &= \|R_\lambda(\cdot, D)(E + K(\cdot, D, \lambda))^{-1}h\|_H^2 \\
 &\leq \|R_\lambda(x, D)\|^2 \|(E + K(x, D, \lambda))^{-1}\|^2 \|h\|_H^2 \leq \frac{L^2 N^2}{|\lambda - \sqrt{2\lambda_0}|^2} \|h\|_H^2.
 \end{aligned}$$

Thus, Theorem 3.1 is proved.

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