THE SPECTRUM OF SOME COMPRESSIONS OF UNILATERAL SHIFTS

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ABSTRACT. Let E be a star-shaped Banach space of analytic functions on the open unit disk $\mathbb{D}$. It is assumed that the unilateral shift $S : z \rightarrow zf$ and the backward shift $T : f \rightarrow \frac{f-f(0)}{z}$ are bounded on E and that their spectrum is the closed unit disk.

Let M be a closed z -invariant subspace of E such that $\dim(M/zM) = 1$, and let $g \in M$. The main result of the paper states that if g has an analytic extension to $\mathbb{D} \cup D(\zeta, r)$ for some $r > 0$, with $g(\zeta) \neq 0$, and if S and T satisfy the “nonquasianalytic condition”

$$\sum_{n \geq 0} \frac{\log \|S^n\| + \log \|T^n\|}{1 + n^2} < +\infty,$$

then ζ does not belong to the spectrum of the compression $S_M : f + M \rightarrow zf + M$ of the unilateral shift to the quotient space E/M . This shows in particular that $\text{Spec}(S_M) = \{1\}$ for some z -invariant subspaces M of weighted Hardy spaces that were constructed by N. K. Nikol'skiĭ in the 1970s by using the Keldysh method.

§1. INTRODUCTION

Consider the weighted Hardy space

$$H_\sigma^2(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}), \|f\| := \left(\sum_{n \geq 0} |\hat{f}(n)|^2 \sigma(n)^2 \right)^{1/2} < +\infty \right\},$$

where $\mathcal{H}(\mathbb{D})$ denotes the set of holomorphic functions on the open unit disk $\mathbb{D}$, and where $\hat{f}(n)$ denotes the n th Taylor coefficient of f at the origin. We assume that the weight $\sigma : \mathbb{N} \rightarrow (0, +\infty)$ satisfies the conditions $0 < \inf_{n \geq 0} \frac{\sigma(n+1)}{\sigma(n)} \leq \sup_{n \geq 0} \frac{\sigma(n+1)}{\sigma(n)} < +\infty$ and $\lim_{n \rightarrow +\infty} \tilde{\sigma}(n)^{1/n} = \lim_{n \rightarrow +\infty} \bar{\sigma}(n)^{1/n} = 1$, where $\tilde{\sigma}(n) := \sup_{p \geq 0} \frac{\sigma(n+p)}{\sigma(p)}$ and $\bar{\sigma}(n) := \sup_{p \geq 0} \frac{\sigma(p)}{\sigma(n+p)}$. These weighted Hardy spaces are standard examples of “star-shaped spaces of analytic functions on the unit disk $\mathbb{D}$ ”, to be defined below.

The spectrum of an operator $R : E \rightarrow E$ acting on a Banach space E will be denoted by $\text{Spec}_E(R)$, or simply by $\text{Spec}(R)$ if no confusion can occur. It follows from N. Nikol'skiĭ's work (see [20]) that if $\sigma \leq 1$ satisfies some regularity conditions, then the functions $\varphi_c = e^{c \frac{z+1}{z-1}}$ ($c > 0$) generate a nontrivial z -invariant subspace of $H_\sigma^2(\mathbb{D})$ if and only if σ satisfies the “one-sided nonquasianalytic condition”:

$$(1) \quad \sum_{n \geq 1} \frac{\log 1/\sigma(n)}{n^{3/2}} < +\infty.$$

2000 *Mathematics Subject Classification*. Primary 47B37.

Key words and phrases. Weighted Hardy space, z -invariant subspace, one-sided nonquasianalytic condition.

A. Beurling had previously obtained analogous results in [5] in the (easier) case of some classes of locally convex spaces of holomorphic functions. A. Atzmon later showed in [1] that the nontrivial z -invariant subspaces M_c generated by the functions φ_c are the only z -invariant subspaces M of $H^2_\sigma(\mathbb{D})$ satisfying $\text{Spec}(S_M) = \{1\}$, where $S_M(f + M) = Sf + M$ for $f \in H^2_\sigma(\mathbb{D})$. In the case where

$$\sum_{n \geq 0} \frac{\log 1/\sigma(n)}{n^{3/2}} = +\infty,$$

N. Nikol'skiĭ gave in [20] a method, based on the so-called “Keldysh lemma”, to construct functions $f \in H^2_\sigma(\mathbb{D})$ that have no zeros in $\mathbb{D}$ and generate a nontrivial z -invariant subspace for decreasing weights that tend to zero arbitrarily fast. In the case of the weights $\sigma_\alpha : n \rightarrow e^{-n^\alpha}$, $\alpha \in (1/2, 1)$, Nikol'skiĭ's method becomes explicit, and he constructs a function ψ_α of the type

$$\psi_\alpha(z) = e^{\sum_{k=0}^{p_\alpha} \frac{b_k}{(1-z)^\alpha/(1-\alpha)-k}}, \quad z \in \mathbb{D},$$

where $b_0, \dots, b_n \in \mathbb{R}$ and $p_\alpha = \lfloor \frac{\alpha}{1-\alpha} \rfloor$, such that if we set

$$\psi_{\alpha,c}(z) = e^{\sum_{k=0}^{p_\alpha} \frac{b_k}{(1-z)^\alpha/(1-\alpha)-k} - \frac{1}{(1-z)^c}}$$

with $c \in (0, 1)$, then the function $\psi_{\alpha,c}$ generates a nontrivial z -invariant subspace of $H^2_{\sigma_\alpha}(\mathbb{D})$. Our aim in this paper is to show that if the shift $S : f \mapsto zf$ and the backward shift $T : f \mapsto \frac{f-f(0)}{z}$, acting on a “star-shaped space E of analytic functions on $\mathbb{D}$ ”, satisfy

$$(2) \quad \sum_{n \geq 0} \frac{\log^+ \|S^n\| + \log^+ \|T^n\|}{1 + n^2} < +\infty,$$

and if $f \in E$ admits an analytic extension to $\mathbb{D} \cup D(\zeta, r)$ with $|\zeta| = 1$, $f(\zeta) \neq 0$, then $\zeta \notin \text{Spec}(S_M)$ for all z -invariant subspaces M containing f and such that $\dim(M/zM) = 1$. From this result, it follows immediately that the z -invariant subspace $M_{\alpha,c} \subset H^2_{\sigma_\alpha}(\mathbb{D})$ generated by N. Nikol'skiĭ's function $\psi_{\alpha,c}$ satisfies $\text{Spec}(S_{M_{\alpha,c}}) = \{1\}$.

No classification of z -invariant subspaces M of $H^2_{\sigma_\alpha}(\mathbb{D})$ such that $\text{Spec}(S_M) = \{1\}$ has been obtained so far. More recently such subspaces were constructed by A. Atzmon [2], by a very different method based on entire functions of zero exponential type, for log-convex weights satisfying a regularity condition. Also A. Borichev, H. Hedenmalm, and A. Volberg constructed in [6] functions f without zeros and generating a nontrivial z -invariant subspace for all log-convex weights decreasing fast enough. We do not have a description of $\text{Spec}(S_{[f]})$, where $[f]$ is the z -invariant subspace generated by these functions.

Now we describe the content of the paper more precisely. Note that if condition (i) below is satisfied, and if $S(E) \subset E$, then, by the closed graph theorem, the restriction of S to E is a bounded linear map from E into itself, and a similar remark applies to the backward shift T . We shall say that a Banach space E is a *Banach space of analytic functions on $\mathbb{D}$* if the following conditions are satisfied:

- (i) $E \subset \mathcal{H}(\mathbb{D})$, and the evaluation map $f \rightarrow f(\lambda)$ is continuous on E for every $\lambda \in \mathbb{D}$;
- (ii) $S(E) \cup T(E) \subset E$, and $\text{Spec}_E(S) = \text{Spec}_E(T) = \overline{\mathbb{D}}$.

A closed subspace M of a Banach space E of analytic functions on $\mathbb{D}$ is said to be z -invariant if the function $Sf : z \rightarrow zf(z)$ belongs to M for every $f \in M$. The “compression” $S_M : E/M \rightarrow E/M$ of the unilateral shift is defined (as above) by the formula

$$(3) \quad S_M(f + M) = Sf + M, \quad f \in E.$$

We denote by $Z(M) := \bigcap_{f \in M} Z(f)$, where $Z(f) = \{z \in \mathbb{D} : f(z) = 0\}$, the set of common zeros of elements of M in the open unit disk.

For $f \in \mathcal{H}(\mathbb{D})$, we shall denote by f_λ the function defined by $f_\lambda(z) = \frac{f(z)-f(\lambda)}{z-\lambda}$ for $z \in \mathbb{D}$, $z \neq \lambda$, and $f_\lambda(\lambda) = f'(\lambda)$. A well-known elementary computation shows that if E is a space of analytic functions on $\mathbb{D}$, then $f_\lambda = T(I - \lambda T)^{-1}$ for every $f \in E$ and every $\lambda \in \mathbb{D}$, so that $f_\lambda \in E$.

In §2 we quote preliminary results, which are essentially contained in Richter's thesis and can be found in [24], where they were proved in a more general context. Let E be a space of analytic functions on $\mathbb{D}$, and let M be a z -invariant subspace of E . Then $\dim(M/(z-\lambda)M) = \dim(M/zM)$ for every $\lambda \in \mathbb{D}$. If $\dim(M/zM) = 1$, then $\text{Spec}(S_M) \cap \mathbb{D} = Z(M)$, and if $\dim(M/zM) > 1$, then $\text{Spec}(S_M) = \mathbb{D}$. We also give a (certainly well-known) formula for the resolvent $(S - \lambda I)^{-1}$ when $\lambda \in \mathbb{D} \setminus Z(M)$, $\dim(M/zM) = 1$.

For $f \in \mathcal{H}(\mathbb{D})$, $r \in [0, 1]$, we define $f^{[r]} \in \mathcal{H}(\mathbb{D})$ by the formula $f^{[r]}(z) = f(rz)$, $z \in \mathbb{D}$. Let E be a Banach space of analytic functions on $\mathbb{D}$. Then $f^{[r]} = (\sum_{n=0}^{+\infty} \hat{f}(n)r^n S^n) \cdot 1 \in E$ for every $f \in E$. We shall say that the Banach space E of analytic functions is *star-shaped* if the following condition is satisfied:

$$(4) \quad \lim_{r \rightarrow 1^-} \|f^{[r]} - f\| = 0 \quad \text{for every } f \in E.$$

The main result of the paper is the following theorem.

Theorem 4.3. *Let E be a star-shaped Banach space of analytic functions on $\mathbb{D}$ satisfying (2), and let M be a z -invariant subspace of E such that $\dim(M/zM) = 1$. Let $\zeta \in \mathbb{T}$ and $r > 0$; if there exists $f \in M$ having an analytic extension to $\mathbb{D} \cup D(\zeta, r)$ and such that $f(\zeta) \neq 0$, then*

$$\zeta \notin \text{Spec}(S_M).$$

The first step of the proof of Theorem 4.3 is based on the following result.

Lemma 4.1. *Let E be a star-shaped space of analytic functions on $\mathbb{D}$ satisfying (2), and let $f \in E$; assume that there exists $\zeta \in \mathbb{T}$ and $r_0 > 0$ such that f has an analytic extension to $\mathbb{D} \cup D(\zeta, r_0)$. Then the E -valued map $\lambda \mapsto f_\lambda$ has an analytic extension to $\mathbb{D} \cup D(\zeta, r_0)$.*

Of course, this result is based on the nonquasianalytic condition (2). First, we show that the function $f_\lambda^{[r]}$ has an analytic extension to $\mathbb{D} \cup D(\zeta, r_0)$, where $f^{[r]}(z) = f(rz)$ for $z \in \mathbb{D}$. Then, using a normal family argument based on the classical Levinson–Cartwright theorem, see §3, we show that the function $\lambda \mapsto f_\lambda$ admits an E -valued analytic extension to $\mathbb{D} \cup D(\zeta, r_0)$.

From Lemma 4.1 it follows that if f and M satisfy the conditions of Theorem 4.3, then there exists $r_1 > 0$ such that the map

$$\lambda \mapsto (S_M - \lambda I)^{-1} p$$

admits an analytic extension to $D(\zeta, r_1)$ for every polynomial p . The second step of the proof of Theorem 4.3 consists in applying again the Levinson–Cartwright theorem to show that $\|(S_M - \lambda I)^{-1}\|$ is bounded on the set $\{z \in D(\zeta, r) : |z| \neq 1\}$ for every $r \in (0, r_1)$, which shows that $S_M - \zeta I$ is invertible.

It would be interesting to see if we can get results in the reverse direction, that is to say, if $\zeta \in \mathbb{T}$ and $\zeta \notin \text{Spec}(S_M)$, does there exist necessarily a function $f \in M$ such that f has an analytic extension to a neighborhood of ζ ? We do not investigate this question here. It would also be interesting to try to use Domar's results [8] to try to extend Theorem 4.3 to “quasianalytic” situations. The existence of nontrivial z -invariant subspaces M such that $\dim(M/zM) = 1$ for general weighted Hardy spaces on the unit disk remains

an open problem, and a counterexample to this problem would allow us to construct an invertible operator a on separable Hilbert space having no common nontrivial invariant subspace with its inverse; see [13].

§2. PRELIMINARY RESULTS

Let E be a Banach space of analytic functions on $\mathbb{D}$, let M be a z -invariant subspace of E , and set $Z(M) := \{z \in \mathbb{D} : f(z) = 0 \text{ for all } f \in M\}$. We denote by S_M the compression of the unilateral shift S to the quotient space E/M , defined by formula (3). We say that $\lambda \in \mathbb{D}$ is a *division point* for M if $f_\lambda \in M$ for every $f \in M$ such that $f(\lambda) = 0$. Clearly, the set of division points for M is disjoint from $Z(M)$. Notice that if $\lambda \in Z(M)$, then $1 \notin (z - \lambda)E + M$, and so $S_M - \lambda I$ is not onto. In the following lemma, we quote some other basic elementary facts concerning the compression S_M . Similar results can be found in Richter's paper [24] in a more general context of Banach spaces of analytic functions on open subsets of $\mathbb{C}$, but here we give a proof for the sake of completeness.

Lemma 2.1. *Let M be a z -invariant subspace of a space E of analytic functions on $\mathbb{D}$.*

(i) *If $\lambda \in \mathbb{D} \setminus Z(M)$, then $S_M - \lambda I : E/M \rightarrow E/M$ is onto, and if $f \in M$, $f(\lambda) \neq 0$, then a right inverse $R_{M,\lambda,f}$ of $S_M - \lambda I$ is given by the formula*

$$(5) \quad R_{M,\lambda,f}(g + M) = g_\lambda - \frac{g(\lambda)}{f(\lambda)} f_\lambda + M, \quad g \in E.$$

(ii) *Let $\lambda \in \mathbb{D}$. Then $S_M - \lambda I$ is one-to-one on E/M if and only if λ is a division point for M .*

(iii) *Let $\lambda \in \mathbb{D} \setminus Z(M)$. Then λ is a division point for M if and only if*

$$\dim(M/(z - \lambda)M) = 1.$$

Proof. Denote by $\pi : E \rightarrow E/M$ the canonical surjection, let $\lambda \in \mathbb{D} \setminus Z(M)$, and let $f \in M$ be such that $f(\lambda) \neq 0$. For $g \in E$, we have

$$(S_M - \lambda I)\pi(g_\lambda) = \pi(g) - g(\lambda)\pi(1),$$

whence $(S_M - \lambda I)\pi(f_\lambda) = -f(\lambda)\pi(1)$, so that $(S_M - \lambda I)\pi(g_\lambda) = \pi(g) + \frac{g(\lambda)}{f(\lambda)}(S_M - \lambda I)\pi(f_\lambda)$. Hence $(S_M - \lambda I)$ is onto, and $[(S_M - \lambda I) \circ R_{M,\lambda,f}]\pi(g) = \pi(g)$, which proves (i).

Now let $\lambda \in \mathbb{D}$, and let $g \in E$. Then $g = [(z - \lambda)g]_\lambda$, and $(S_M - \lambda I)\pi(g) = 0$ if and only if $(z - \lambda)g \in M$. Thus, we see that $S_M - \lambda I$ is one-to-one on E/M if and only if λ is a division point for M , which proves (ii).

Let $\lambda \in \mathbb{D} \setminus Z(M)$ be a division point for M , let $u \in M$ be such that $u(\lambda) = 1$, and let $f \in M$. Set $g = f - f(\lambda)u$. Then $g \in M$ and $g(\lambda) = 0$, and so $g_\lambda \in M$. Hence $f = f(\lambda)u + (z - \lambda)g_\lambda \in \mathbb{C}u + (z - \lambda)M$, and $\dim(M/(z - \lambda)M) = 1$.

Conversely, assume that $\lambda \in \mathbb{D} \setminus Z(M)$ and $\dim(M/(z - \lambda)M) = 1$. Again, consider $u \in M$ with $u(\lambda) = 1$, so that $M = \mathbb{C}u + (z - \lambda)M$. Let $f \in M$ be such that $f(\lambda) = 0$. There exists an element $g \in M$ and $\alpha \in \mathbb{C}$ such that $f = \alpha u + (z - \lambda)g$. Then $\alpha = 0$, and $f_\lambda = g \in M$, so that λ is a division point for M , which proves (iii). $\square$

It was observed in Corollaries 2–3 of [14] that, in the case of weighted Hardy spaces on the disk, either the set of division points for a z -invariant subspace is empty, or it is the whole set $\mathbb{D} \setminus Z(M)$. Since a Banach space of analytic functions E on $\mathbb{D}$ satisfies the condition $\text{Spec}_E(T) = \mathbb{D}$, we have $f_\lambda = T(T - \lambda I)^{-1}f \in E$ for every $f \in E$ and every $\lambda \in \mathbb{D}$. In particular, if $f(\lambda) = 0$, then $f = (z - \lambda)f_\lambda \in (z - \lambda)E$, and we can apply Richter's general results valid for Banach spaces of analytic functions on open sets of $\mathbb{C}$ having this “division property”. In particular, standard results concerning semi-Fredholm operators show that we have the following property, which we state as a lemma.

Lemma 2.2 ([24]). *Let E be a Banach space of analytic functions on $\mathbb{D}$, and let $M \neq \{0\}$ be a z -invariant subspace of M . Then $\dim(M/(z-\lambda)M) = \dim(M/zM)$ for every $\lambda \in \mathbb{D}$.*

The point spectrum of an operator R will be denoted by $\text{Spec}_p(R)$. We obtain the following result.

Proposition 2.3. *Let E be a star-shaped space of analytic functions on $\mathbb{D}$, and let $M \neq \{0\}$ be a z -invariant subspace of E .*

- (i) *If $\dim(M/zM) > 1$, then $\text{Spec}(S_M) \cap \mathbb{D} = \text{Spec}_p(S_M) \cap \mathbb{D} = \mathbb{D}$, and $\text{Spec}(S_M) = \overline{\mathbb{D}}$.*
- (ii) *If $\dim(M/zM) = 1$, then $\text{Spec}(S_M) \cap \mathbb{D} = \text{Spec}_p(S_M) \cap \mathbb{D} = Z(M)$. Also, if $f \in M \setminus \{0\}$, then all singularities of the E/M -valued meromorphic function $\lambda \mapsto \frac{f_\lambda}{f(\lambda)} + M$ on the open set $\mathbb{D} \setminus Z(M)$ are removable, and for $\lambda \in \mathbb{D} \setminus Z(M)$ we have*

$$(6) \quad (S_M - \lambda I)^{-1}(g + M) = g_\lambda - \frac{g(\lambda)}{f(\lambda)}f_\lambda + M, \quad g \in E.$$

Proof. Property (i) and the statement that $\text{Spec}(S_M) \cap \mathbb{D} = \text{Spec}_p(S_M) \cap \mathbb{D} = Z(M)$ whenever $\dim(M/zM) = 1$ follow immediately from Lemmas 2.1 and 2.2. Now assume that $\dim(M/zM) = 1$, and let $f \in M \setminus \{0\}$. Since $\text{Spec}(S_M \cap \mathbb{D}) = \mathbb{D} \setminus Z(M)$, Lemma 2.1 implies that for $\lambda \in \mathbb{D} \setminus Z(f) \subset \mathbb{D} \setminus Z(M)$, we have

$$(S_M - \lambda I)^{-1}(g + M) = g_\lambda - \frac{g(\lambda)}{f(\lambda)}f_\lambda + M, \quad g \in E.$$

In particular, $\frac{f_\lambda}{f(\lambda)} = -(S_M - \lambda I)^{-1}(1 + M)$ for $\lambda \in \mathbb{D} \setminus Z(f)$, which shows that the E/M -valued function $\lambda \mapsto \frac{f_\lambda}{f(\lambda)}$ has an analytic extension to $\mathbb{D} \setminus Z(M)$, and proves formula (6). $\square$

We shall say that a z -invariant subspace M of a star-shaped space E of analytic functions on $\mathbb{D}$ is *singly generated* if there exists $f \in M$ such that the set $\{pf\}_{p \in \mathbb{C}[x]}$ is dense in M . In this case we shall say that f is a *generator* of M , and we shall write $M = [f]$. The following standard observation gives a simple sufficient condition on M ensuring that $\dim(M/zM) = 1$.

Proposition 2.4. *Let E be a star-shaped space of analytic functions on $\mathbb{D}$, and let $M \neq \{0\}$ be a z -invariant subspace of M . If M is singly generated, then $\dim(M/zM) = 1$.*

Proof. Let f be a generator of M , let $\lambda \in \mathbb{D} \setminus Z(f)$, let $g \in M$ such that $g(\lambda) = 0$, and let $(p_n)_{n \geq 1}$ be a sequence of polynomial functions such that $g = \lim_{n \rightarrow \infty} p_n f$. Then $\lim_{n \rightarrow \infty} p_n(\lambda_n) = 0$, and so $g = \lim_{n \rightarrow \infty} q_n f$, where $q_n = p_n - p_n(\lambda_n)$.

We have $(q_n f)_{\lambda_n} = \frac{q_n}{z - \lambda_n} f \in M$, whence $g_\lambda = \lim_{n \rightarrow \infty} (q_n f)_{\lambda_n} \in M$. This shows that λ is a division point for M , and Lemma 2.1 implies that $\dim(M/(z - \lambda)M) = 1$. Then Lemma 2.2 shows that $\dim(M/zM) = 1$. $\square$

§3. THE LEVINSON–CARTWRIGHT THEOREM

Let Ω be an open set of $\mathbb{C}$ and X a Banach space. We shall say that a map $f : \Omega \rightarrow X$ is *analytic* if $\lim_{z \rightarrow w} \frac{f(z) - f(w)}{z - w}$ exists for all $w \in \Omega$. It is well known [22, Theorem 3.3, p. 79] that $f : \Omega \rightarrow X$ is analytic if and only if $\phi \circ f : \Omega \rightarrow \mathbb{C}$ is analytic for every $\phi \in X^*$. We denote by $\mathcal{H}(\Omega, X)$ the space of all analytic functions $f : \Omega \rightarrow X$, equipped with the topology of uniform convergence on compact subsets of Ω . If Ω is connected and if $\Omega_1 \subset \Omega$, the space $\mathcal{H}(\Omega, X)$ will be identified with a subset of $\mathcal{H}(\Omega_1, X)$. A set $\Delta \subset \mathcal{H}(\Omega, X)$ is said to be *normal* on Ω if it is bounded on every compact subset of Ω .

The Levinson–Cartwright theorem (see [4, 7, 9, 10, 12, 15], [16, p. 376], [17, 18, 19], and [23]) gives a growth condition on a rectangle R deprived of one of its axes of symmetry

I , which guarantees that a family of analytic functions on R that satisfies this condition is normal on R (even though the condition gives no information on the behavior of the functions on I). A vector-valued version of this result can be stated as follows.

Theorem 3.1. *Let $\delta > 0$, let I an open interval of $\mathbb{R}$, and set $R := \{z \in \mathbb{C}, \operatorname{Re}(z) \in I, \operatorname{Im}(z) \in (-\delta, \delta)\}$. Let $\rho : (0, \delta] \rightarrow (0, +\infty)$ be a monotone nonincreasing function satisfying the condition*

$$(7) \quad \int_0^\delta \log(\log(\max(1, \rho(x)))) dx < +\infty.$$

Then the set $V(X) := \{f \in \mathcal{H}(R, X) : |f(z)| \leq \rho(|\operatorname{Im}(z)|) \forall z \in R \setminus I\}$ is a closed subset of $\mathcal{H}(R \setminus I, X)$ normal on R .

Proof. When $X = \mathbb{C}$, this result is the standard Levinson–Cartwright theorem. Now, let X be a Banach space, and let B be the unit ball of X^* . Since $\phi(V(X)) \subset V(\mathbb{C})$ for $\phi \in B$, and since $V(\mathbb{C})$ is normal on R , from the Hahn–Banach theorem it follows that $V(X)$ is normal on R . Let $f \in \mathcal{H}(R \setminus I, X)$ and assume that there exists a sequence $(f_n)_{n \geq 1}$ of elements of $V(X)$ that converges uniformly to f on compact subsets of $R \setminus I$. We have $|f(z)| \leq \rho(|\operatorname{Im}(z)|)$ for all $z \in R \setminus I$. Let $\zeta \in I$, let $\alpha > 0$, and assume that $D := \overline{D}(\zeta, \alpha) \subset R$. Choose $\eta > \alpha$ such that the circle $C := \{z \in \mathbb{C} : |z - \zeta| = \eta\}$ is contained in R . Since $V(X)$ is normal on R , there exists $M > 0$ such that $\max_{z \in C} \|f_n(z)\| \leq M$ for every $n \geq 1$. We have

$$f_n(z) - f_m(z) = \frac{1}{2i\pi} \int_C \frac{f_n(u) - f_m(u)}{u - z} du$$

for $m \geq 1, n \geq 1, z \in D$.

Set $C_\epsilon = \{z \in C : |\operatorname{Im}(z)| \geq \epsilon |\operatorname{Re}(z - \eta)|\}$. We obtain

$$\max_{z \in D} \|f_n(z) - f_m(z)\| \leq \frac{4\eta M}{\eta - \alpha} \arctan(\epsilon) + \frac{\eta}{\eta - \alpha} \max_{z \in C_\epsilon} \|f_n(z) - f_m(z)\|$$

for $\epsilon \in (0, \eta)$, and this inequality shows that $(f_n)_{n \geq 1}$ is a Cauchy sequence with respect to the norm of uniform convergence on D . Hence, $(f_n)_{n \geq 1}$ is a Cauchy sequence of elements of the Fréchet space $\mathcal{H}(R, X)$, which shows that $f \in \mathcal{H}(R, X)$. $\square$

For $r > 0$, set $C_r = \{z \in \mathbb{C} : |z| = r\}$. We now state a consequence of the Levinson–Cartwright theorem, which will be used to establish the main results of the paper.

Corollary 3.2. *Let X be a Banach space, let $\delta \in (0, 1)$, and let $\rho : (0, \delta] \rightarrow (0, +\infty)$ satisfy the conditions of Theorem 3.1. Let $\eta > 0$ and $\zeta \in \mathbb{T}$, let $\alpha = \min(\delta, \eta)$, and put $L = D(\zeta, \eta) \cap \mathbb{T}$.*

Then the set

$$W(X) := \left\{ f \in \mathcal{H}(D(\zeta, \eta) \setminus L, X) \mid \max_{z \in (C_r \cup (C_{1/r} \cap D(\zeta, \eta)))} \|f(z)\| \leq \rho(1 - r) \right. \\ \left. \text{for all } r \in (1 - \alpha, 1) \right\}$$

is a closed subset of $\mathcal{H}(D(\zeta, \eta) \setminus L, X)$ which is normal on $D(\zeta, \eta)$.

Proof. By conformal mapping, we deduce immediately from Theorem 3.1 that the restriction of $W(X)$ to all subsets of $D(\zeta, \eta)$ of the form $V = V_{\beta, \gamma, \theta} := \{z \in \mathbb{C}, \beta < |z| < 1/\beta, |\arg(z) - \theta| < \gamma\}$, where $\theta \in L$, is a closed subset of $\mathcal{H}(V \setminus V \cap \mathbb{T})$ which is normal on V , and the corollary follows. $\square$

In the next section, we shall consider vector-valued analytic functions h on a neighborhood of a point of the unit circle $\mathbb{T}$ for which the growth of $\|h(\lambda)\|$ is controlled by $\|(S - \lambda I)^{-1}\|$ for $|\lambda| > 1$ and by $\|(I - \lambda T)^{-1}\|$ for $|\lambda| < 1$, where S is a bounded operator on a space of analytic functions on $\mathbb{D}$ and T is a left inverse of S .

In this situation, the following result is known (see [25] and [11]).

Proposition 3.3. *Let A and B be bounded operators on a Banach space X . Suppose that $BA = I$ and $\text{Spec}(A) = \text{Spec}(B) = \overline{\mathbb{D}}$. For $r \in [0, 1)$, put*

$$\begin{aligned}\mu_1(r) &= \sup_{|\lambda|=1/r} \|(A - \lambda I)^{-1}\|, \\ \mu_2(r) &= \sup_{|\lambda|=r} \|(B - \lambda I)^{-1}\|, \\ \mu(r) &= \max(1, \mu_1(r), \mu_2(r)).\end{aligned}$$

If A and B satisfy the nonquasianalyticity condition

$$\sum_{n \geq 0} \frac{\log^+ \|A^n\| + \log^+ \|B^n\|}{1 + n^2} < +\infty,$$

then

$$\int_1^\delta \log \log(\mu(1-r)) dr < +\infty.$$

§4. SPECTRUM OF COMPRESSIONS OF UNILATERAL SHIFTS IN A NONQUASIANALYTIC SETTING

In this section we consider star-shaped Banach spaces E of analytic functions on the open unit disk $\mathbb{D}$. Then the dual space E^* of E can be identified with a subspace of the space $\mathcal{H}_0(\mathbb{D}_e) := \{g \in \mathcal{H}(\mathbb{D}_e) : \lim_{|z| \rightarrow \infty} g(z) = 0\}$, where $\mathbb{D}_e := \mathbb{C} \setminus \overline{\mathbb{D}}$. Indeed, for $g \in E^*$, set $\tilde{g}(\lambda) = \langle (S - \lambda I)^{-1} 1, g \rangle$, so that $\tilde{g} \in \mathcal{H}_0(\mathbb{D}_e)$; if $\tilde{g}(\lambda) = 0$ for $|\lambda| > 1$, then, since

$$\tilde{g}(\lambda) = - \sum_{n \geq 0} \langle S^n 1, g \rangle \lambda^{-n-1},$$

we have $\langle S^n 1, g \rangle = 0$ for all $n \geq 0$, and the fact that the polynomials are dense in E implies that $g = 0$.

If M is a (closed) z -invariant subspace of E , set

$$M^\perp = \{g \in E^*, \langle f, g \rangle = 0, f \in M\}.$$

We shall denote by $\pi : f \rightarrow f + M$ the standard surjection from E onto E/M , so that formula (3), which defines the compression S_M of the unilateral shift S to the quotient space E/M , can be written in the form $S_M(\pi(f)) = \pi(Sf)$ ($f \in E$).

We say that E satisfies the nonquasianalyticity condition if the unilateral shift S and the backward shift T satisfy condition (2) on E :

$$\sum_{n \geq 0} \frac{\log^+ \|S^n\| + \log^+ \|T^n\|}{1 + n^2} < +\infty.$$

Let E be a space of analytic functions on $\mathbb{D}$, let $\lambda \in \mathbb{D}$, and let $f \in E$; again set $f_\lambda(z) = \frac{f(z) - f(\lambda)}{z - \lambda}$ for $z \in \mathbb{D} \setminus \{\lambda\}$, with the convention $f_\lambda(\lambda) = f'(\lambda)$. Then $f_\lambda \in E$, and for $\lambda \in \mathbb{D}$ we have (see, for example, [14]):

$$(8) \quad f_\lambda = T(I - \lambda T)^{-1} f.$$

In particular, $f_\lambda \in E$ and the map $\lambda \mapsto f_\lambda$ is analytic from $\mathbb{D}$ into E .

Similarly, let U be a connected open set such that $U \cap \mathbb{T} \neq \emptyset$, and assume that $f \in E$ admits an analytic extension to $\mathbb{D} \cup U$, which we also denote by f . For $\lambda \in U$, $|\lambda| > 1$, we define $f_\lambda \in \mathcal{H}(\mathbb{D})$ by the formula

$$f_\lambda(z) = \frac{f(z) - f(\lambda)}{z - \lambda}.$$

Again, we have $f_\lambda \in E$ and

$$(9) \quad f_\lambda = (S - \lambda I)^{-1}(f - f(\lambda)),$$

so that the map $f \rightarrow f_\lambda$ is analytic on $U \setminus U \cap \overline{\mathbb{D}}$.

Lemma 4.1. *Let E be a star-shaped space of analytic functions on $\mathbb{D}$ satisfying (2), and let $f \in E$; assume that there exists a point $\zeta \in \mathbb{T}$ and $r_0 > 0$ such that f has an analytic extension to $\mathbb{D} \cup D(\zeta, r_0)$. Then the E -valued map $\lambda \mapsto f_\lambda$ has an analytic extension to $\mathbb{D} \cup D(\zeta, r_0)$, which is defined by formula (9) for $z \in D(\zeta, r_0)$, $|z| > 1$.*

Proof. Let $r \in [0, 1)$, and let $f \in E$. From formula (8), it follows that the map $\lambda \mapsto f_\lambda^{[r]} := [f^{[r]}]_\lambda$ is analytic from $\mathbb{D}$ into E . Since $f_\lambda^{[r]} = r(f_{r\lambda})^{[r]}$, this map is the composition of the map $\lambda \mapsto f_{r\lambda}$, which is analytic from $\frac{1}{r}\mathbb{D}$ into E , and the map $g \mapsto rg^{[r]}$, which is linear and continuous from E into itself. Hence, the E -valued map $\lambda \mapsto f_\lambda^{[r]}$ extends analytically to $r^{-1}\mathbb{D}$.

For $1 < |\lambda| < r^{-1}$, we also have

$$f_\lambda^{[r]} = (S - \lambda I)^{-1}(f^{[r]} - f^{[r]}(\lambda)),$$

with $\text{Spec}(S) \subset \overline{\mathbb{D}}$, so that the functions $\lambda \mapsto (S - \lambda I)^{-1}f^{[r]}$ and $\lambda \mapsto (S - \lambda I)^{-1}1$ are analytic from $\mathbb{C} \setminus \overline{\mathbb{D}}$ into E . The map $\lambda \mapsto f^{[r]}(\lambda) = f(r\lambda)$ is analytic on $r^{-1}D(\zeta, r_0)$. Hence, the E -valued map $\lambda \mapsto f_\lambda^{[r]}$ extends analytically to $r^{-1}\mathbb{D} \cup r^{-1}D(\zeta, r_0)$.

Now, assume that $r \in [\frac{1}{\sqrt{1+r_0^2}}, 1)$. A simple geometric argument shows that the set $r^{-1}\mathbb{D} \cup r^{-1}D(\zeta, r_0)$ is star-shaped with respect to the origin, and so $\mathbb{D} \cup D(\zeta, r_0) \subset r^{-1}\mathbb{D} \cup r^{-1}D(\zeta, r_0)$. Formulas (8) and (9) imply that $f_\lambda^{[r]}$ converges uniformly to f_λ on compact subsets of $\mathbb{D} \cup (D(\zeta, r_0) \cap \mathbb{D}_e)$ as $r \rightarrow 1^-$, and the lemma follows from Corollary 3.2 and Proposition 3.3. $\square$

Corollary 4.2. *Let M be a z -invariant subspace of a star-shaped space E of analytic functions on $\mathbb{D}$ satisfying (2). If there exists $f \in M$ having an analytic extension to $\mathbb{D} \cup D(\zeta, r_0)$, where $\zeta \in \mathbb{T}$, $r_0 > 0$, then*

$$f(\lambda)g(\lambda) = -\langle f_\lambda, g \rangle \quad (|\lambda| > 1, \lambda \in \mathbb{D} \cup D(\zeta, r_0))$$

for $g \in M^\perp$, so that each function $g \in M^\perp$ has a meromorphic extension to $(\mathbb{C} \setminus \overline{\mathbb{D}}) \cup D(\zeta, r_0)$. In particular, if $f(\zeta) \neq 0$, then there exists $r_1 \in]0, r_0]$ such that each function $g \in M^\perp$ has an analytic extension to $(\mathbb{C} \setminus \overline{\mathbb{D}}) \cup D(\zeta, r_1)$.

Proof. Let $f \in M$ and $g \in M^\perp$; denote by φ the function defined by

$$\varphi(\lambda) = -\langle f_\lambda, g \rangle \quad (|\lambda| > 1, \lambda \in D(\zeta, r_0)).$$

Notice that $(S - \lambda I)^{-1}f = -\sum_{n \leq 0} \lambda^{-n-1} S^n f \in M$ for $|\lambda| > 1$. For $\lambda \in D(\zeta, r_0)$, $|\lambda| > 1$, we have

$$\begin{aligned}\varphi(\lambda) &= -\langle f_\lambda, g \rangle = -\langle (S - \lambda I)^{-1}(f - f(\lambda)), g \rangle \\ &= f(\lambda) \langle (S - \lambda I)^{-1}1, g \rangle = f(\lambda)g(\lambda).\end{aligned}$$

Hence, if f has no zeros in $D(\zeta, r_1)$, then

$$g(\lambda) = -\frac{\langle f_\lambda, g \rangle}{f(\lambda)},$$

and since the E -valued map $\lambda \mapsto f_\lambda$ is analytic on $D(\zeta, r_0)$, we see that g admits an analytic extension to $(\mathbb{C} \setminus \overline{\mathbb{D}}) \cup D(\zeta, r_1)$. $\square$

Notice that, in the case where $f(\zeta) = 0$ with $f \neq 0$, Corollary 4.2 shows that the elements of $M^\perp$ admit a meromorphic extension to $(\mathbb{C} \setminus \overline{\mathbb{D}}) \cup D(\zeta, r_0)$. The following example shows that, in general, this pole at ζ is not a pseudosingularity.

Example 1. Consider the Wiener algebra

$$W^+ = \left\{ f \in \mathcal{H}(\mathbb{D}), \sum_{n \geq 0} |\widehat{f}(n)| < +\infty \right\},$$

which is a star-shaped space of analytic functions on $\mathbb{D}$ satisfying (2). Set

$$M = \{f \in W^+, f(1) = 0\}.$$

Identify the dual of W^+ with the space $\{g \in \mathcal{H}_0(\mathbb{D}_e), (\widehat{g}(n))_{n < 0} \in l^\infty(\mathbb{Z}^-)\}$, where $\widehat{g}(n)$ denotes the n th Laurent coefficient of g at ∞ . Then

$$f(1) = \sum_{n=0}^{+\infty} \widehat{f}(n) = \left\langle f, \frac{1}{z-1} \right\rangle = 0$$

for $f \in M^+$, so that $M^\perp = \mathbb{C} \frac{1}{z-1}$. Hence, all nonzero functions $g \in M^\perp$ have a pole at 1, whereas $z-1 \in M$ has an analytic extension to $\mathbb{C}$.

We saw in §2 that if M is a z -invariant subspace of a star-shaped space E of analytic functions on $\mathbb{D}$, and if $\dim(M/zM) = 1$, then $\text{Spec}(S_M) \cap \mathbb{D} = Z(M)$, where $S_M : f + M \rightarrow zf + M$ is the compression of the unilateral shift $S : f \rightarrow zf$ to the quotient space E/M . Using Lemma 4.1, we now obtain the following theorem.

Theorem 4.3. *Let E be a star-shaped space of analytic functions on $\mathbb{D}$ satisfying the condition of nonquasianalyticity*

$$\sum_{n \geq 0} \frac{\log^+ \|S^n\| + \log^+ \|T^n\|}{1 + n^2} < +\infty.$$

Let M be a z -invariant subspace of E such that $\dim(M/zM) = 1$, and let $\zeta \in \mathbb{T}$, $r > 0$. If there exists $f \in M$ having an analytic extension to $\mathbb{D} \cup D(\zeta, r)$ and such that $f(\zeta) \neq 0$, then $\zeta \notin \text{Spec}(S_M)$.

Proof. Take $f \in M \setminus \{0\}$ having an analytic extension to $\mathbb{D} \cup D(\zeta, r)$, and denote by π the canonical surjection from E onto E/M . Since f extends analytically to $D(\zeta, r)$, there exists $r_1 \in (0, r)$ such that $\inf_{z \in D(\zeta, r_1)} |f(z)| > 0$, and since $\text{Spec}(S_M) \cap \mathbb{D} \subset Z(f)$, we obtain

$$\text{Spec}(S_M) \cap \mathbb{D} \cap D(\zeta, r_1) = \emptyset.$$

From Lemma 4.1 we know that the E -valued map $\lambda \rightarrow f_\lambda$ extends analytically to $\mathbb{D} \cup D(\zeta, r_1)$, and the identity $(S - \lambda I)f_\lambda = f - f(\lambda)1$ is fulfilled on $D(\zeta, r_1)$.

Let $p \in \mathbb{C}[x]$ be a polynomial; for $\lambda \in D(\zeta, r_1)$, $|\lambda| \neq 1$, we have

$$(S - \lambda I)f_\lambda p = fp - f(\lambda)p,$$

so that

$$(S_M - \lambda I)\pi(f_\lambda p) = -f(\lambda)\pi(p),$$

because $fp = p(S)f \in M$. We have

$$(S_M - \lambda I)\pi(f_\lambda p) = (S_M - \lambda I)p(S_M)\pi(f_\lambda),$$

whence

$$(10) \quad (S_M - \lambda I)^{-1}\pi(p) = -p(S_M)\frac{\pi(f_\lambda)}{f(\lambda)}.$$

Formula (10) and Lemma 4.1 show that the map $\phi_p : \lambda \mapsto (S_M - \lambda I)^{-1}\pi(p)$ extends analytically to $D(\zeta, r_1)$. Set

$$\mu_1(r) = \max_{|\lambda|=r^{-1}} \|(S - \lambda I)^{-1}\|, \quad \mu_2(r) = \max_{|\lambda|=r} \|(I - \lambda T)^{-1}\|.$$

For $\lambda \in D(\zeta, r_1)$, $|\lambda| > 1$, we have

$$(11) \quad \|\phi_p(\lambda)\| \leq \|\pi(p)\|\mu_1(1/|\lambda|).$$

Now, from Proposition 2.3 (ii) it follows that

$$(S_M - \lambda I)^{-1}\pi(p) = \pi(p_\lambda) - p(\lambda)\frac{\pi(f_\lambda)}{f(\lambda)}$$

for $\lambda \in \mathbb{D} \setminus Z(M)$. We have $g(\lambda) = g + (S - \lambda)g_\lambda$; consequently,

$$|g(\lambda)| \leq (\|S - \lambda\| \|T\| \|(T - \lambda I)^{-1}\| + 1)\|g\|$$

for every polynomial p and all $\lambda \in \mathbb{D} \setminus Z(M)$. Since $\inf_{z \in D(\zeta, r_1)} |f(z)| > 0$ (again by (8)), we see that there exist two positive constants a, b independent of the polynomial p such that

$$(12) \quad \|\phi_p(\lambda)\| = \|(S_M - \lambda I)^{-1}\pi(p)\| \leq (a\mu_2(|\lambda|) + b\mu_2(|\lambda|^2))\|\pi(p)\|.$$

Using Corollary 3.2 and Proposition 3.3, from (11) and (12) we deduce that the family $\{\phi_p : \|\pi(p)\| \leq 1\}$ is normal on $D(\zeta, r_1)$. Since the polynomials are dense in E , we have

$$\sup_{\|\pi(p)\| \leq 1} \|\varphi_p\| = \|(S_M - \lambda I)^{-1}\| \quad \text{for } \lambda \in D(\zeta, r_1), \quad \lambda \neq 1,$$

and we obtain

$$\sup_{|\lambda - \zeta| < r, |\lambda| \neq 1} \|(S_M - \lambda I)^{-1}\| < +\infty$$

for every $r \in (0, r_1)$.

Therefore, for any $\lambda \in D(\zeta, r_1) \cap \mathbb{T}$ there exists a sequence $(R_{n,\lambda})_{n \geq 1}$ of bounded invertible operators on E/M with $\lim_{n \rightarrow \infty} \|S_M - \lambda I - R_{n,\lambda}\| = 0$, $\sup_{n \geq 1} \|R_{n,\lambda}^{-1}\| < +\infty$. Then a standard elementary result from the theory of Banach algebras, see [21], implies that $S_M - \lambda I$ is invertible for $|\lambda| = 1$, $|\lambda - \zeta| < r_1$. In particular, $\zeta \notin \text{Spec}(S_M)$. $\square$

The weighted Hardy spaces $H_\sigma^2(\mathbb{D})$, where the weight σ satisfies the conditions given at the beginning of the Introduction, are standard examples of star-shaped Banach spaces of analytic functions on the unit disk $\mathbb{D}$. In the case of the weights $\sigma_\alpha : n \rightarrow e^{-n^\alpha}$, $\alpha \in (1/2, 1)$, Nikol'skiĭ's method (mentioned in the Introduction) to produce functions without zeros in $\mathbb{D}$ that generate nontrivial z -invariant subspaces becomes explicit. Let p_α be the natural integer satisfying $p_\alpha \leq \frac{\alpha}{1-\alpha} < p_\alpha + 1$. In [20], Nikol'skiĭ constructed a function ψ_α of the type

$$\psi_\alpha(z) = e^{\sum_{k=0}^{p_\alpha} \frac{b_k}{(1-z)^\alpha/(1-\alpha)^{-k}}}, \quad z \in \mathbb{D},$$

where $b_0, \dots, b_n \in \mathbb{R}$, such that if we set

$$\psi_{\alpha,c}(z) = e^{\sum_{k=0}^{p_\alpha} \frac{b_k}{(1-z)^\alpha/(1-\alpha)-k} - \frac{1}{(1-z)^c}}$$

with $c \in (0, 1)$, then the function $\psi_{\alpha,c}$ generates a nontrivial z -invariant subspace of $H_{\sigma_\alpha}^2(\mathbb{D})$. The shift S on the weighted Hardy space $H_{\sigma_\alpha}^2(\mathbb{D})$ is a contraction, and so

$$\sum_{n=0}^{\infty} \frac{\log^+(\|S^n\|)}{1+n^2} = 0.$$

Also, the backward shift T on $H_{\sigma_\alpha}^2(\mathbb{D})$ satisfies

$$\|T^n\| = \sup_{p \geq 0} \frac{\sigma_\alpha(p)}{\sigma_\alpha(n+p)} = \sup_{p \geq 0} e^{(n+p)^\alpha - p^\alpha} = e^{n^\alpha},$$

so that

$$\sum_{n \geq 0} \frac{\log^+ \|T^n\|}{1+n^2} < \infty.$$

Thus, Theorem 4.3 immediately implies the following result.

Corollary 4.4. *Let $\alpha \in (1/2, 1)$, $c > 0$, and let*

$$\psi_{\alpha,c}(z) = e^{\sum_{k=0}^{p_\alpha} \frac{b_k}{(1-z)^\alpha/(1-\alpha)-k} - \frac{1}{(1-z)^c}}$$

be the functions constructed by Nikol'skiĭ in [20], which generate a nontrivial z -invariant subspace $M_{\alpha,c}$ for the weighted Hardy space $H_{\sigma_\alpha}^2(\mathbb{D})$. Then

$$\text{Spec}(S_{M_{\alpha,c}}) = \{1\}.$$

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Received 12/AUG/2006

Originally published in English