ON HOMOTOPIZATION OF THE UNITARY K_1 -FUNCTOR

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ABSTRACT. A unitary K_1 -analog of the Karoubi–Villamayor functor is constructed, which solves the problem of homotopization of the unitary K_1 -functor on the category of rings with involution.

§1. INTRODUCTION

In the late 1960s and early 1970s, several definitions of higher K -functors on the category of rings, which were compatible on the subcategory of regular rings, were suggested (Quillen, Nobile–Villamayor, Karoubi–Villamayor, Gersten, Swan, Volodin). A characteristic property of the Karoubi–Villamayor functors KV_n is their homotopy invariance on the category of rings, i.e., $KV_n(A[X]) \cong KV_n(A)$ for any ring A and any natural number n . At the present time, the following definition of the KV_* -functors, which is somewhat different from the original one [1] but compatible with it [2], is generally recognized.

For an arbitrary ring A , the coordinate rings of standard simplexes form the simplicial ring $A[\Delta^\bullet]$ with the natural maps of faces and singularities, where $A[\Delta^m] = A[X_0, X_1, \dots, X_m] / (\sum X_i - 1)$. For any natural number n , we set $KV_n(A) = \pi_{n-1}(GL) = \pi_n(BGL)$. It is not difficult to show that if $\text{Uni}(A)$ denotes the subgroup of $GL(A)$ generated by the unipotent matrices $1 + \alpha$, where α is nilpotent, then $KV_1(A) \cong GL(A)/\text{Uni}(A)$. This is precisely the original definition of $KV_1(A)$ given in [1]. Actually, KV_1 is the homotopization of the functor K_1 on the category of rings (for the definition, see §4 below). The functors KV_n were thought to be the homotopization of the Quillen K_n -functors for any natural number n . However, in [3] there is an example of a ring that illustrates the fact that even KV_2 is not the homotopization of the functor K_2 . The problem of constructing homotopy invariant K_n -functors for any integer n was solved in [4], both on the category of rings and on the category of schemes, as well as for the K -theory with coefficients. For any ring A and any natural number n , there is a chain of natural homomorphisms $K_n(A) \rightarrow KV_n(A) \rightarrow KH_n(A)$, which may fail to be isomorphisms even for $n = 1$. The functors KH_n also do not solve the problem of homotopization of the Quillen K_n -functors. One application of the KV - and KH -functors is related to the fact that the previously constructed Chern characters from algebraic K -theory to cyclic homology admit a factorization either through KV_* or through KH_* [4, 5].

The unitary analogs of the Karoubi–Villamayor functors on the category of rings with involution are constructed much as the above ones, with the replacement of the general linear group GL by the unitary group U^ϵ [6]. More precisely, let (A, Λ, ϵ) be a unitary ring, and let $U^\epsilon(A, \Lambda)$ be its unitary group (see §2). Consider the simplicial

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group $U^\epsilon = U^\epsilon(A[\Delta], \Lambda[\Delta])$; we set $KV_n U^\epsilon(A, \Lambda) = \pi_{n-1}(U^\epsilon) = \pi_n(BU^\epsilon)$ for natural numbers n .

Our aim in this paper is to give a simple description of the functor $KV_1 U^\epsilon$ that solves the problem of homotopization of the unitary K_1 -functor on the category of rings with involution. We note that if 2 is an invertible element of a ring A , then $\Lambda = \Lambda_{\max}^\epsilon$ and the unitary K -functors $K_n U^\epsilon$ reduce to the Hermitian K -functors ${}_\epsilon K_n^h$, the homotopy invariance of which on the subcategory of regular rings with involution was proved recently in [7]. The procedure of constructing (hyperbolic) rings such that the functor $K_1 U^\epsilon$ is not homotopy invariant is described below in §5.

In the paper we adhere to standard definitions and notation; all the rings considered are assumed to be associative with 1 and all the modules are right and unitary. For a ring A , we denote by $\mathbf{P}(A)$ the category of finitely generated projective (right) A -modules.

§2. THE HYPERBOLIC UNITARY GROUP

We recall some definitions and results pertaining to unitary K -theory [8, 9]. Let (A, Λ, ϵ) be a unitary ring, where A is a ring with involution $a \rightarrow a^*$, ϵ is a central element of the ring A satisfying $\epsilon \cdot \epsilon^* = 1$, and Λ is an additive subgroup of $\Lambda_{\max}^\epsilon = \{a \in A : a = -\epsilon \cdot a^*\}$ that contains the group $\Lambda_{\min}^\epsilon = \{a - \epsilon \cdot a^*, a \in A\}$ and satisfies the condition $a^* \cdot \Lambda \cdot a \subset \Lambda$ for arbitrary $a \in A$. Then $(A, \Lambda^*, \epsilon^*)$ is also a unitary ring. We extend the involution to the matrix ring $M_r(A) : (\alpha_{ij})^* = (\alpha_{ji}^*)$.

For $M \in \mathbf{P}(A)$, we denote by $\text{Sesq}(M)$ the set of biadditive maps $\phi : M \times M \rightarrow A$ such that $\phi(m_1 \cdot a_1, m_2 \cdot a_2) = a_1^* \cdot \phi(m_1, m_2) \cdot a_2$ for any $m_1, m_2 \in M$ and any $a_1, a_2 \in A$. Moreover, we set $\text{Sesq}^{-\epsilon}(M) = \{Q \in \text{Sesq}(M) : Q + \epsilon \cdot Q^* = 0\}$, where $Q^*(m, n) = Q(n, m)^*$ for any $m, n \in M$, and $\text{Sesq}_\Lambda^{-\epsilon}(M) = \{Q \in \text{Sesq}^{-\epsilon}(M) : Q(m, m) \in \Lambda, m \in M\}$. The forms in $\text{Sesq}^{-\epsilon}(M)$ are said to be $(-\epsilon)$ -Hermitian, and the forms in $\text{Sesq}_\Lambda^{-\epsilon}(M)$ are said to be Λ -Hermitian. In particular, if $M \cong A^r$ is a free A -module, then we obtain Λ -Hermitian matrices, i.e., $\alpha \in M_r(A)$ is Λ -Hermitian if and only if α is $(-\epsilon)$ -Hermitian and its diagonal elements α_{ii} belong to Λ .

The cosets $[q]$ of the quotient group $\text{Sesq}(M)/\text{Sesq}_\Lambda^{-\epsilon}(M)$ are called Λ -quadratic forms on M , and a pair $(M, [q])$, where $[q] = Q + \text{Sesq}_\Lambda^{-\epsilon}(M)$ is a Λ -quadratic form on M , is called a Λ -quadratic module. In the sequel, for brevity, the Λ -quadratic forms (modules) will be called simply *quadratic forms (modules)*, and the square brackets of quadratic forms will be omitted.

For an arbitrary quadratic module (M, q) , we introduce the length map $\|_q : M \rightarrow A/\Lambda : \|m\|_q = Q(m, m) + \Lambda$ and the ϵ -Hermitian form (scalar product) $(\cdot, \cdot)_q : M \times M \rightarrow A : (m, n)_q = Q(m, n) + \epsilon \cdot Q^*(m, n)$. The length map and the scalar product are well defined, i.e., they do not depend on the choice of the form $Q \in [q]$ and determine the form $[q]$ uniquely. If an ϵ -Hermitian form $(\cdot, \cdot)_q$ is nonsingular, then the quadratic module (M, q) is called a *quadratic space*. The isometry group of a quadratic Λ -space (M, q) , i.e., the group of A -automorphisms $\alpha : M \rightarrow M$ preserving the quadratic form $[q] : [q](\alpha(m)) = [q](m)$, $m \in M$, is denoted by $O(M, q)$ (or simply by $O(q)$) and is called the *orthogonal group*.

It is not difficult to show that an automorphism α belongs to $O(q)$ if and only if α preserves the lengths $\|\alpha(m)\|_q = \|m\|_q$ and the scalar products $(\alpha(m), \alpha(n))_q = (m, n)_q$. In particular, the orthogonal group $O(q)$ is a subgroup of the unitary group $U(Q + \epsilon \cdot Q^*)$, and if $\Lambda = \Lambda_{\max}^\epsilon$ (for example, if 2 is invertible in A), then $O(q) = U(Q + \epsilon \cdot Q^*)$ for all $Q \in [q]$. The orthogonal group of the hyperbolic space $(A^{2r}, \left[\begin{pmatrix} 0 & 1_r \\ 0 & 0 \end{pmatrix}\right])$ is denoted by $U_{2r}^\epsilon(A, \Lambda)$ and is called the (*hyperbolic*) Λ -unitary group.

The statement below and its consequences follow immediately from the definition.

Proposition 1. *Let $\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in GL_{2r}(A)$, where $\alpha_{ij} \in M_r(A)$. Then the following conditions are equivalent:*

- (1) *the matrix α is Λ -unitary;*
- (2) *the matrices $\alpha_{11}^* \cdot \alpha_{21}$ and $\alpha_{12}^* \cdot \alpha_{22}$ are Λ -Hermitian and $\alpha_{11}^* \cdot \alpha_{22} + \epsilon \cdot \alpha_{21}^* \cdot \alpha_{12} = 1_r$;*
- (3) *the matrix α^* is Λ -unitary;*
- (4) *the matrices $\alpha_{21} \cdot \alpha_{22}^*$ and $\alpha_{11} \cdot \alpha_{12}^*$ are Λ -Hermitian and $\alpha_{11} \cdot \alpha_{22}^* + \epsilon \cdot \alpha_{12} \cdot \alpha_{21}^* = 1_r$.*

Corollary 1. *If $\alpha \in GL_r(A)$, then $H(\alpha) = \text{diag}(\alpha, (\alpha^*)^{-1}) \in U_{2r}^\epsilon(A, \Lambda)$. The matrices of the form $H(\alpha)$ are said to be hyperbolic.*

Corollary 2. (a) *If β is Λ^* -Hermitian, then $T_{12}(\beta) = \begin{pmatrix} 1_r & \beta \\ 0 & 1_r \end{pmatrix} \in U_{2r}^\epsilon(A, \Lambda)$.*

(b) *If γ is Λ -Hermitian, then $T_{21}(\gamma) = \begin{pmatrix} 1_r & 0 \\ \gamma & 1_r \end{pmatrix} \in U_{2r}^\epsilon(A, \Lambda)$.*

Let $EU_{2r}^\epsilon(A, \Lambda)$ denote the subgroup of $U_{2r}^\epsilon(A, \Lambda)$ generated by the matrices of the form $H(\alpha)$, $T_{12}(\beta)$, and $T_{21}(\gamma)$, where $\alpha \in E_r(A)$ and β and γ satisfy the assumptions of Corollary 2.

Corollary 3. *Suppose $\alpha, \beta, \gamma \in M_r(A)$ satisfy the following conditions:*

- (a) *β is Λ^* -Hermitian and γ is Λ -Hermitian;*
- (b) *$\alpha \cdot \beta = \beta \cdot \alpha^*$ and the diagonal elements of the matrix $\alpha \cdot \beta$ lie in Λ^* ;*
- (c) *$\gamma \cdot \alpha = \alpha^* \cdot \gamma$ and the diagonal elements of the matrix $\gamma \cdot \alpha$ lie in Λ ;*
- (d) *$\beta \cdot \gamma = \alpha^{n+1}$ for some natural number n . Then*

$$\begin{pmatrix} 1_r - \alpha & \beta \\ -\gamma & 1_r + \alpha^* + (\alpha^*)^2 + \dots + (\alpha^*)^n \end{pmatrix} \in U_{2r}^\epsilon(A, \Lambda).$$

We denote by $HU_{2r}(A, \Lambda)$ the subgroup of $U_{2r}(A, \Lambda)$ generated by the matrices described in Corollary 3. Since the matrices in Corollaries 1 and 2 are special cases of those in Corollary 3, the group $EU_{2r}^\epsilon(A, \Lambda)$ is a subgroup of $HU_{2r}^\epsilon(A, \Lambda)$.

We define the embedding

$$U_{2r}^\epsilon(A, \Lambda) \rightarrow U_{2r+2}^\epsilon(A, \Lambda) : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and set $U^\epsilon(A, \Lambda) = \bigcup U_{2r}^\epsilon(A, \Lambda)$, $HU^\epsilon(A, \Lambda) = \bigcup HU_{2r}^\epsilon(A, \Lambda)$, and $EU^\epsilon(A, \Lambda) = \bigcup EU_{2r}^\epsilon(A, \Lambda)$. By the unitary analog of the Whitehead lemma [9], the group $EU^\epsilon(A, \Lambda)$ coincides with the derived subgroup of the group $U^\epsilon(A, \Lambda)$, so that the (Abelian) group $K_1U^\epsilon(A, \Lambda) = U^\epsilon(A, \Lambda)/EU^\epsilon(A, \Lambda)$ is well defined.

§3. THE UNITARY ANALOG OF HIGMAN'S TRICK

Let (A, Λ, ϵ) be a unitary ring, and let s be a central element of the ring A invariant with respect to the involution. The involution has a natural extension to the ring $A' = A/s : (a + s \cdot A)^* = a^* + s \cdot A$; let Λ' (ϵ') denote the image of Λ (respectively, of ϵ) under the canonical projection $f : A \rightarrow A'$. Then $(A', \Lambda', \epsilon')$ is a unitary ring, and f is a unitary surjection that induces a homomorphism $f^* : U_{2r}^\epsilon(A, \Lambda) \rightarrow U_{2r}^{\epsilon'}(A', \Lambda')$ for any natural number r (respectively, $f^* : U^\epsilon(A, \Lambda) \rightarrow U^{\epsilon'}(A', \Lambda')$ and $f^* : K_1U^\epsilon(A, \Lambda) \rightarrow K_1U^{\epsilon'}(A', \Lambda')$), the kernel of which is denoted by $U_{2r}^\epsilon(A, \Lambda, s)$ (respectively, $U^\epsilon(A, \Lambda, s)$ and $K_1U^\epsilon(A, \Lambda, s)$). As a result, we arrive at relative unitary groups.

We extend the involution to the polynomial ring $A[X]$ by setting $X^* = X$. Then $(A[X], \Lambda[X], \epsilon)$ is a unitary ring.

Proposition 2. *For an arbitrary matrix $\alpha(X) \in U^\epsilon(A[X], \Lambda[X], X)$, there are matrices $\delta_1(X), \delta_2(X) \in EU^\epsilon(A[X], \Lambda[X], X) \cap U^\epsilon(A[X], \Lambda[X], X)$ such that $\delta_1(X) \cdot \alpha(X) \cdot \delta_2(X) \in HU^\epsilon(A[X], \Lambda[X], X) \cap U^\epsilon(A[X], \Lambda[X], X)$.*

Proof. Suppose $\alpha(X) \in U_{2r}^\epsilon(A[X], \Lambda[X], X)$ for some natural number r , and let $\alpha_{11}(X)$ (in $M_r(A[X], X)$) be an upper left submatrix of $\alpha(X)$. Using Higman's trick (see [10, Chapter 12]), we find $\gamma_1(X), \gamma_2(X) \in E_s(A[X]) \cap GL_s(A[X], X)$ such that $\gamma_1(X) \cdot \alpha_{11}(X) \cdot \gamma_2(X) = 1_s - \alpha \cdot X$ for some $\alpha \in M_s(A)$ and some $s \geq r$. Then $H(\gamma_1(X)), H(\gamma_2(X)) \in EU_{2s}^\epsilon(A[X], \Lambda[X]) \cap U_{2s}^\epsilon(A[X], \Lambda[X], X)$, and the upper half of the matrix $\alpha_1(X) = H(\gamma_1(X)) \cdot \alpha(X) \cdot H(\gamma_2(X))$ has the form

$$(1_s - \alpha \cdot X \quad \beta_1 \cdot X^n + \beta_2 \cdot X^{n+1} + \cdots + \beta_{m-n+1} \cdot X^m),$$

where $\beta_i \in M_s(A)$ and $m \geq n \geq 1$. By Proposition 1, the matrix β_1 is Λ^* -Hermitian. If $m = n$, then we pass to the lower half of the matrix $\alpha_1(X)$. Otherwise (i.e., if $m - n \geq 1$), we multiply the matrix $\alpha_1(X)$ from the right by $T_{12}(-\beta_1 \cdot X^n) (\in EU_{2s}^\epsilon(A[X], \Lambda[X]) \cap U_{2s}^\epsilon(A[X], \Lambda[X], X))$ to obtain a matrix $\alpha_2(X)$, the upper half of which has the form

$$(1_s - \alpha \cdot X \quad \delta_1 \cdot X^{n_1} + \delta_2 \cdot X^{n_1+1} + \cdots + \delta_{m_1-n_1+1} \cdot X^{m_1}),$$

where $\delta_i \in M_s(A)$, $m \geq m_1 \geq n_1 > n \geq 1$. In particular, $m - n > m_1 - n_1 \geq 0$, and thus, by induction arguments, there is a matrix $\gamma_3(X) = T_{12}(\ast) (\in EU_{2s}^\epsilon(A[X], \Lambda[X]) \cap U_{2s}^\epsilon(A[X], \Lambda[X], X))$ such that the upper half of the matrix $\alpha_3(X) = \alpha_2(X) \cdot \gamma_3(X)$ has the form $(1_s - \alpha \cdot X \quad \beta \cdot X^k)$ for some $\beta \in M_s(A)$, $k \geq 1$. By Proposition 1, β is Λ^* -Hermitian; moreover, $\alpha \cdot \beta = \beta \cdot \alpha^*$ and the diagonal elements of $\alpha \cdot \beta$ lie in Λ^* . Similarly, with the help of the lower half of the matrix obtained, we find a matrix $\gamma_4(X) = T_{21}(\ast) (\in EU_{2s}^\epsilon(A[X], \Lambda[X]) \cap U_{2s}^\epsilon(A[X], \Lambda[X], X))$ such that the upper half of $\alpha_4(X) = \gamma_4(X) \cdot \alpha_3(X)$ coincides with the upper half of $\alpha_3(X)$, and the lower left submatrix has the form $-\gamma \cdot X^p$ for some $\gamma \in M_s(A)$, $p \geq 1$. By Proposition 1, γ is Λ -Hermitian; moreover, $\gamma \cdot \alpha = \alpha^* \cdot \gamma$ and the diagonal elements of $\gamma \cdot \alpha$ lie in Λ . Also, we see that the lower right submatrix of the matrix obtained has the form $1_s + \alpha^* \cdot X + (\alpha^* \cdot X)^2 + \cdots + (\alpha^* \cdot X)^l$ for some $l \geq 1$. Thus, if $k + p = l + 1$, then the matrix in question lies in $HU^\epsilon(A[X], \Lambda[X]) \cap U^\epsilon(A[X], \Lambda[X], X)$, which completes the proof of the proposition. Otherwise, i.e., if $k + p \neq l + 1$, the matrix α is nilpotent by Proposition 1. In particular, $1_s - \alpha \cdot X$ is unipotent. Multiplying $\alpha_4(X)$ from the left by $T_{21}(\gamma \cdot X^p \cdot (1_s - \alpha \cdot X)^{-1}) (\in EU_{2s}^\epsilon(A[X], \Lambda[X]) \cap U_{2s}^\epsilon(A[X], \Lambda[X], X))$ and from the right by $T_{12}(-(1_s - \alpha \cdot X)^{-1} \cdot \beta \cdot X^k) (\in EU_{2s}^\epsilon(A[X], \Lambda[X]) \cap U_{2s}^\epsilon(A[X], \Lambda[X], X))$, we obtain a hyperbolic matrix $H(1_s - \alpha \cdot X) (\in HU_{2s}^\epsilon(A[X], \Lambda[X]) \cap U_{2s}^\epsilon(A[X], \Lambda[X], X))$, which completes the proof of the proposition. $\square$

Remark 1. Similar arguments show that for any $\alpha(X) \in U^\epsilon(A[X, X^{-1}], \Lambda[X, X^{-1}], X - 1)$ there are matrices $\delta_1(X), \delta_2(X) \in EU^\epsilon(A[X, X^{-1}], \Lambda[X, X^{-1}])$ such that the product $\delta_1(X) \cdot \alpha(X) \cdot \delta_2(X)$ has the form

$$\begin{pmatrix} 1_r - \alpha \cdot T & \beta \cdot T^k \\ -\gamma \cdot T^p & 1_r + \alpha^* \cdot T + \cdots + (\alpha^* \cdot T)^{k+p-1} \end{pmatrix}$$

with $T = X - 1$, for some natural r, k , and p , and for $\alpha, \beta, \gamma (\in M_r(A))$ satisfying the assumptions of Corollary 3.

Since $EU^\epsilon(A[X], \Lambda[X])$ is a normal subgroup of $HU^\epsilon(A[X], \Lambda[X])$, we have the following consequences.

Corollary 1. *An arbitrary matrix $\alpha(X)$ in the group $U^\epsilon(A[X], \Lambda[X])$ can be written in the form $\alpha(X) = \alpha(0) \cdot \beta(X)$ for some*

$$\beta(X) \in HU^\epsilon(A[X], \Lambda[X]) \cap U^\epsilon(A[X], \Lambda[X], X).$$

Corollary 2. *The group $HU^\epsilon(A, \Lambda)$ is a normal subgroup of $U^\epsilon(A, \Lambda)$.*

Proof. It suffices to show that for an arbitrary generator

$$\delta_0 = \begin{pmatrix} 1_r - \alpha & \beta \\ -\gamma & 1_r + \alpha^* + (\alpha^*)^2 + \cdots + (\alpha^*)^n \end{pmatrix}$$

of the group $HU^\epsilon(A, \Lambda)$, where $\alpha, \beta, \gamma \in M_r(A)$ satisfy the assumptions of Corollary 3 in §2, and for any $\lambda \in U_{2r}^\epsilon(A, \Lambda)$, the matrix

$$\delta(X) = \begin{pmatrix} 1_r - \alpha \cdot X & \beta \cdot X \\ -\gamma \cdot X^n & 1_r + \alpha^* \cdot X + \cdots + (\alpha^* \cdot X)^n \end{pmatrix}$$

lies in $HU_{2r}^\epsilon(A[X], \Lambda[X]) \cap U_{2r}^\epsilon(A[X], \Lambda[X], X)$, and $\delta(1) = \delta_0$. By the preceding corollary, $\lambda^{-1} \cdot \delta(X) \cdot \lambda = \beta(X)$ for some $\beta(X) \in HU^\epsilon(A[X], \Lambda[X])$. To complete the proof of the corollary, it remains to set $X = 1$. $\square$

By Corollary 2, for any ring A with involution, the (Abelian) group $KV_1U^\epsilon(A, \Lambda) = U^\epsilon(A, \Lambda)/HU^\epsilon(A, \Lambda)$ is well defined, and the natural map $K_1U^\epsilon(A, \Lambda) \rightarrow KV_1U^\epsilon(A, \Lambda)$ is an epimorphism.

§4. HOMOTOPIZATION OF THE UNITARY K_1 -FUNCTOR

All the functors considered in what follows act from the category of rings to the category of groups. We recall that a functor F is said to be *homotopy invariant* if $F(A[X]) \cong F(A)$ for any ring A . For example, as was mentioned in the Introduction, the Karoubi–Villamayor functors KV_n are homotopy invariant on the category of rings, and the Quillen functors K_n are homotopy invariant on the subcategory of regular rings.

Corollary 3. *The functor $KV_1U^\epsilon(A, \Lambda)$ is homotopy invariant.*

Proof. It suffices to show that the canonical epimorphism

$$KV_1U^\epsilon(A[X], \Lambda[X]) \rightarrow KV_1U^\epsilon(A, \Lambda) : X \rightarrow 0$$

is a monomorphism. We take an arbitrary $\alpha(X) \in U^\epsilon(A[X], \Lambda[X])$ such that $\alpha(0) \in HU^\epsilon(A, \Lambda)$. By Corollary 1 to Proposition 2, we have $\alpha(X) = \alpha(0) \cdot \beta(X)$ for some $\beta(X) \in HU^\epsilon(A[X], \Lambda[X])$, which implies the claim. $\square$

We recall another definition. The homotopization of an arbitrary functor F is a homotopy invariant functor $[F]$, together with a natural mapping $h : F \rightarrow [F]$ that satisfies the following universality condition: for any homotopy invariant functor H and any natural mapping $f : F \rightarrow H$, there exists a unique natural mapping $g : [F] \rightarrow H$ with $f = g \cdot h$. For example, the Karoubi–Villamayor KV_1 -functor is the homotopization of the K_1 -functor. Now we state the main result of the paper.

Theorem. *The functor KV_1U^ϵ is the homotopization of the functor K_1U^ϵ .*

Proof. First, we prove the auxiliary result asserting that $[HU^\epsilon](A, \Lambda) = 1$. Indeed, we take an arbitrary generator

$$\delta_0 = \begin{pmatrix} 1_r - \alpha & \beta \\ -\gamma & 1_r + \alpha^* + (\alpha^*)^2 + \cdots + (\alpha^*)^n \end{pmatrix}$$

of the group $HU^\epsilon(A, \Lambda)$, where $\alpha, \beta, \gamma \in M_r(A)$ satisfy the assumptions of Corollary 3 in §2, and consider the matrix

$$\delta(X) = \begin{pmatrix} 1_r - \alpha \cdot X & \beta \cdot X \\ -\gamma \cdot X^n & 1_r + \alpha^* \cdot X + (\alpha^* \cdot X)^2 + \cdots + (\alpha^* \cdot X)^n \end{pmatrix}$$

in the group $HU_{2r}^\epsilon(A[X], \Lambda[X])$. We have $\partial_0(\delta(X)) = \delta(0) = 1_{2r}$, $\partial_1(\delta(X)) = \delta(1) = \delta_0$.

Since the images of ∂_0 and ∂_1 are identified under homotopization (see [6, Theorem 4.2]), and the matrices of the form δ_0 are generators of the group $HU^\epsilon(A, \Lambda)$, we have $[HU^\epsilon](A, \Lambda) = 1$.

To prove the theorem, we need to show that $[K_1U^\epsilon](A, \Lambda) \cong KV_1U^\epsilon(A, \Lambda)$ for any unitary ring (A, Λ, ϵ) . The composition $HU^\epsilon(A, \Lambda) \rightarrow U^\epsilon(A, \Lambda) \rightarrow [U^\epsilon](A, \Lambda)$ is trivial, because it factors through $[HU^\epsilon](A, \Lambda) = 1$. In particular, we have an epimorphism $\phi : KV_1U^\epsilon(A, \Lambda) \rightarrow [U^\epsilon](A, \Lambda) \rightarrow [K_1U^\epsilon](A, \Lambda)$. On the other hand, there is a canonical epimorphism $K_1U^\epsilon(A, \Lambda) \rightarrow KV_1U^\epsilon(A, \Lambda)$. Therefore, since the functor KV_1U^ϵ is homotopy invariant and the map $K_1U^\epsilon(A, \Lambda) \rightarrow [K_1U^\epsilon](A, \Lambda)$ is universal, we have a homomorphism $\psi : [K_1U^\epsilon](A, \Lambda) \rightarrow KV_1U^\epsilon(A, \Lambda)$. By the universality property, ϕ and ψ are mutually inverse isomorphisms, and this completes the proof of the theorem. $\square$

Remark 2. The above theorem and the universality property show that the functor KV_1U^ϵ constructed above is compatible with the definition in [6] (see the Introduction).

§5. HYPERBOLIC RINGS

In this section we describe the procedure of constructing rings that illustrate the fact that the functor K_1U^ϵ is not homotopy invariant. We recall a series of definitions and results. For an associative ring A with 1, we denote by A° the opposite ring, i.e., the operation of multiplication in A is replaced by the operation $a \circ b = b \cdot a$. We set $H(A) = A \times A^\circ$ and introduce the involution $(a, b)^* = (b, a)$ on the hyperbolic ring $H(A)$. As a result, we obtain a functor H from the category of rings to the category of rings with involution, which is right conjugate to the forgetful functor.

The matrix ring $M_n(H(A))$ consists of pairs of matrices (α, β) , where $\alpha, \beta \in M_n(A)$, and $(\alpha, \beta) \in GL_n(H(A))$ if and only if $\alpha, \beta \in GL_n(A)$. We extend the involution from $H(A)$ to $M_n(H(A)) : (\alpha, \beta)^* = (\beta^t, \alpha^t)$. Any central element ϵ of the ring $H(A)$ that satisfies $\epsilon \cdot \epsilon^* = 1 = (1, 1)$ has the form (a, a^{-1}) , where a is a central invertible element of the ring A . Since $\lambda + \lambda^* = 1$, where $\lambda = (1, 0)$, it follows that $\Lambda_{\max} = \Lambda_{\min}$, so that the unitary group of the ring $H(A)$ does not depend on the choice of the system of parameters Λ and coincides with the orthogonal group, and any unitary space over $H(A)$ is even Hermitian [10, Chapter 1].

It is not difficult to verify that a matrix $(\alpha, \beta) (\in GL_r(H(A)))$ is ϵ -unitary if and only if $\alpha = (I_r^a)^{-1} \cdot (\beta^t)^{-1} \cdot I_r^a$, where $I_r^a = \begin{pmatrix} 0 & 1_r \\ a \cdot 1_r & 0 \end{pmatrix}$. Consequently, the homomorphism $GL_{2r}(A) \rightarrow U_{2r}^\epsilon(H(A)) : \beta \mapsto ((I_r^a)^{-1} \cdot (\beta^t)^{-1} \cdot I_r^a, \beta)$ is an isomorphism, and, up to isomorphism, the unitary group of the ring $H(A)$ does not depend on the choice of ϵ . For this reason, in the sequel we can take $\epsilon = 1$, omitting the superscript ϵ . We note that the isomorphism obtained allows us to view the general linear group as a special case of a unitary (orthogonal) group. The isomorphism constructed above is compatible with embeddings and takes the elementary matrices over A to elementary unitary matrices over $H(A)$.

More precisely, let δ be a bijection of the set $\{1, 2, \dots, 2r\}$ onto itself that takes i to $i + r$ if $i \leq r$ and to $i - r$ if $i > r$. Then $E_{ij}(a)$ is mapped to $E_{ij}^1((-a, a))$ if $j = \delta i$ and to $E_{ij}^1((0, a)) = E_{\delta j \delta i}^1((-a, 0))$ otherwise. This results in a well-defined homomorphism $K_1(A) \rightarrow K_1U(H(A))$, which is in fact an isomorphism. Thus, the functor K_1 is a special case of the functor K_1U . However, for the functor K_1 , examples of rings A for which the group $K_1(A[X])$ is not isomorphic to $K_1(A)$ are well known (see [12]); therefore, for such rings the group $K_1U(H(A[X]))$ is not isomorphic to $K_1U(H((A)))$. Consequently, the functor K_1U^ϵ is not homotopy invariant.

Actually, in the paper [12], necessary and sufficient conditions were found on one-dimensional Noetherian reduced rings A with finite normalization, under which $K_1(A[X])$

is isomorphic to $K_1(A)$. One of these conditions is seminormality. We recall that a commutative reduced ring A is said to be *seminormal* if for any $b, c \in A$ satisfying $b^3 = c^2$, there is $a \in A$ such that $a^2 = b$ and $a^3 = c$. An example of a nonseminormal ring is the coordinate ring $A = k[X, Y]/(X^3 - Y^2)$ of the (plane) cuspidal curve over the field k ; thus, the group $K_1U(H(A[X]))$ is not isomorphic to $K_1U(H(A))$ for this ring.

As was noted in [13, p. 263], in the notation of [13], we have $K_0(A) \cong L_0(H(A))$ for an arbitrary ring A , and thus, by the isomorphism constructed above, the space of the algebraic K -theory $\mathbf{K}(A) = K_0(A) \times BGL(A)^+$ and that of the Hermitian K -theory $\mathbf{L}(H(A)) = L_0(H(A)) \times BO(H(A))^+$ are homotopy equivalent. Consequently, $K_n(A) \cong L_n(H(A))$ for all n , and the algebraic K -theory is part of the Hermitian (unitary) K -theory, which was mentioned previously by some authors.

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