ON θ -CENTRALIZERS OF SEMIPRIME RINGS (II)

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ABSTRACT. The following result is proved: Let R be a 2-torsion free semiprime ring, and let $T : R \rightarrow R$ be an additive mapping, related to a surjective homomorphism $\theta : R \rightarrow R$, such that $2T(x^2) = T(x)\theta(x) + \theta(x)T(x)$ for all $x \in R$. Then T is both a left and a right θ -centralizer.

§1. INTRODUCTION

This paper has been motivated by the work of Brešar [5], Vukman [10], and Zalar [11]. Throughout, R will represent an associative ring with center $Z(R)$. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and R is semiprime if $aRa = (0)$ implies $a = 0$. A ring R is 2-torsion free if $2x = 0$, $x \in R$ implies $x = 0$. As usual, the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use the commutator identities $[x, yz] = [x, y]z + y[x, z]$ and $[xy, z] = [x, z]y + x[y, z]$. An additive mapping $D : R \rightarrow R$ is called a *derivation* if $D(xy) = D(x)y + xD(y)$ for all pairs $x, y \in R$; D is called a *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ for all $x \in R$. A derivation D is *inner* if there exists an element $a \in R$ such that $D(x) = [a, x]$ for all $x \in R$. An additive mapping $D : R \rightarrow R$ related to a map $\theta : R \rightarrow R$ is called a (θ, θ) -*derivation* if $D(xy) = D(x)\theta(y) + \theta(x)D(y)$ for all pairs $x, y \in R$; D is called a Jordan (θ, θ) -derivation if $D(x^2) = D(x)\theta(x) + \theta(x)D(x)$ for all $x \in R$. A (θ, θ) -derivation D is *inner* if there exists $a \in R$ such that $D(x) = [a, \theta(x)]$ for all $x \in R$. It is clear that if θ is the identity map on R , then every (θ, θ) -derivation is an ordinary derivation.

Every derivation is a Jordan derivation; the converse is in general not true. A classical result of Herstein [7] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in [2]. Cusack [6] generalized Herstein's result to 2-torsion free semiprime rings (see also [4] for an alternative proof). Zalar [11] gave the following definition: An additive mapping $T : R \rightarrow R$ is called a left (right) centralizer if $T(xy) = T(x)y$ ($T(xy) = xT(y)$) for all $x, y \in R$. If R is a ring with involution $*$, then every additive mapping $E : R \rightarrow R$ that satisfies $E(x^2) = E(x)x^* + xE(x)$ for all $x \in R$ is called a *Jordan $*$ -derivation*. These mappings are closely related to the question of the representability of quadratic forms by bilinear forms. Some algebraic properties of Jordan $*$ -derivations were considered in [3], where further references can be found. For quadratic forms, see [9].

If the product in R is given by $x \circ y = xy + yx$, then a Jordan derivation is an additive mapping D satisfying $D(x \circ y) = D(x) \circ y + x \circ D(y)$ for all $x, y \in R$; a Jordan homomorphism is an additive mapping A satisfying $A(x \circ y) = A(x) \circ A(y)$ for all $x, y \in R$. Zalar [11] defined the Jordan centralizer to be an additive mapping T such that $T(x \circ y) = T(x) \circ y = x \circ T(y)$. Since the product $\circ$ is commutative, there is no difference between the left and the right Jordan centralizers.

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We follow Zalar [11] and call T a centralizer if T is both a left and a right centralizer. If $a \in R$, then $L_a(x) = ax$ is a left centralizer and $R_a(x) = xa$ is a right centralizer. An additive mapping $T : R \rightarrow R$ is called a *left (right) Jordan centralizer* if $T(x^2) = T(x)x$ ($T(x^2) = xT(x)$). Following the ideas of [4], Zalar [11] proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. In [1], the definition of a θ -centralizer was introduced. This is a generalization of the definition of a centralizer. Albash [1] followed the same direction as Zalar [11] and proved his result with this definition. In our paper, Vukman's result [10] will be proved for the θ -centralizer case under some additional conditions.

Definition 1.1 ([1]). An additive mapping $T : R \rightarrow R$ is called a *left (right) θ -centralizer associated with a function $\theta : R \rightarrow R$* if for all $x, y \in R$ we have

$$T(xy) = T(x)\theta(y) \quad (T(xy) = \theta(x)T(y)).$$

T is called a *left (right) Jordan θ -centralizer* if for all $x, y \in R$ we have

$$T(x^2) = T(x)\theta(x) \quad (T(x^2) = \theta(x)T(x)).$$

It is clear that if $T : R \rightarrow R$ is an additive mapping associated with a homomorphism $\theta : R \rightarrow R$, if $L_a(x) = a\theta(x)$ and $R_a(x) = \theta(x)a$ for all $x \in R$ and $a \in R$, then L_a is a left θ -centralizer and R_a is a right θ -centralizer.

Remark 1.1. Clearly, every centralizer is a special case of a θ -centralizer with $\theta = I_R$, and every Jordan centralizer is a special case of a Jordan θ -centralizer with $\theta = I_R$.

If $T : R \rightarrow R$ is a θ -centralizer associated with a function $\theta : R \rightarrow R$, where R is an arbitrary ring, then T satisfies the relation

$$2T(x^2) = T(x)\theta(x) + \theta(x)T(x) \quad \text{for all } x \in R.$$

It seems natural to ask, as Vukman [10] did in the centralizer case, whether the converse is true. More precisely, we ask whether an additive mapping T on a ring R satisfying the above relation is a θ -centralizer for all $x, y \in R$. It is our aim to prove that the answer is in the affirmative if R is a 2-torsion free semiprime ring and θ is a surjective endomorphism with $\theta(Z) = Z$.

Theorem 1.2. *Let R be a 2-torsion free semiprime ring with identity element, and let $T : R \rightarrow R$ be an additive mapping such that $2T(x^2) = T(x)\theta(x) + \theta(x)T(x)$ for all $x \in R$, where θ is a surjective endomorphism of R . Then T is both a left and a right θ -centralizer.*

Theorem 1.3. *Let R be a 2-torsion free semiprime ring with center Z , and let $T : R \rightarrow R$ be an additive mapping such that $2T(x^2) = T(x)\theta(x) + \theta(x)T(x)$ for all $x \in R$, where θ is a surjective endomorphism of R with $\theta(Z) = Z$. Then T is both a left and a right θ -centralizer.*

§2. PROOF OF THEOREM 1.2

We assume that R is noncommutative (the theorem is trivial when R is commutative). Replacing x by $x + 1$ in the equation

$$(1) \quad 2T(x^2) = T(x)\theta(x) + \theta(x)T(x), \quad x \in R,$$

after some calculations we obtain

$$(2) \quad 2T(x) = a\theta(x) + \theta(x)a, \quad x \in R,$$

where a denotes $T(1)$. We intend to prove that $a \in Z(R)$. Combining (1) and (2), we obtain

$$2(a\theta(x^2) + \theta(x^2)a) = (a\theta(x) + \theta(x)a)\theta(x) + \theta(x)(a\theta(x) + \theta(x)a).$$

This reduces to

$$(3) \quad [D(x), x]_\theta = 0, \quad x \in R.$$

Here $[m, n]_\theta$ always denotes $m\theta(n) - \theta(n)m$ and $D(x)$ stands for $[a, x]_\theta$. It is well known that if $f(x) = [x, a]$ for every $x \in R$, then f is a derivation on R , which is called an inner derivation. Similarly, $D(x)$ is a (θ, θ) -derivation; we call $D(x)$ an *inner (θ, θ) -derivation* on R . For an ordinary inner derivation f defined on a 2-torsion free semiprime ring R , Herstein [8, Lemma 1.1.9] proved that if $a \in R$ commutes with the inner derivation f defined by a (i.e., if $f(x) = [a, x]$ and $[a, f(x)] = 0$ for all $x \in R$), then $a \in Z(R)$, where $Z(R)$ is the center of R . We want to prove the same result in the case of an inner (θ, θ) -derivation. For this, we put $x + y$ for x in (3). We have

$$\begin{aligned} [D(x+y), x+y]_\theta &= 0 = [D(x), x]_\theta + [D(x), y]_\theta + [D(y), x]_\theta + [D(y), y]_\theta \\ &= [D(x), y]_\theta + [D(y), x]_\theta = 0, \quad x, y \in R. \end{aligned}$$

We rewrite the above equation in the form

$$(4) \quad [a, \theta(x)]\theta(y) + [a, \theta(y)]\theta(x) = \theta(y)[a, \theta(x)] + \theta(x)[a, \theta(y)].$$

In particular, since θ is surjective, there exists $y \in R$ such that $\theta(y) = a$, so that we can put $\theta(y) = a$ in (4). We obtain

$$[a, \theta(x)]a = a[a, \theta(x)].$$

Thus,

$$(5) \quad [D(x), a] = 0 = [[a, x]_\theta, a], \quad x \in R.$$

We substitute xy for x in (5); since D is a (θ, θ) -derivation, we get

$$\begin{aligned} 0 &= [a, [a, \theta(xy)]] = [a, [a, \theta(x)\theta(y)]] = [a, \theta(x)[a, \theta(y)] + [a, \theta(x)]\theta(y)] \\ &= [a, \theta(x)[a, \theta(y)]] + [a, [a, \theta(x)]\theta(y)] \\ &= \theta(x)[a, [a, \theta(y)]] + [a, \theta(x)][a, \theta(y)] + [a, \theta(x)][a, \theta(y)] + [a, [a, \theta(x)]\theta(y)]. \end{aligned}$$

By (5), the above relation reduces to

$$2[a, \theta(x)][a, \theta(y)] = 0, \quad x, y \in R.$$

In a 2-torsion free semiprime ring this gives

$$[a, \theta(x)][a, \theta(y)] = 0, \quad x, y \in R.$$

Substituting $y = rx$, $x, y, r \in R$, in the above relation, we get

$$\begin{aligned} [a, \theta(x)][a, \theta(rx)] &= [a, \theta(x)][a, \theta(r)\theta(x)] \\ &= [a, \theta(x)]\theta(r)[a, \theta(x)] + [a, \theta(x)][a, \theta(r)]\theta(x) \\ &= [a, \theta(x)]\theta(r)[a, \theta(x)] = 0, \quad x, y \in R. \end{aligned}$$

Since θ is surjective and R is a semiprime ring, it follows that

$$D(x) = [a, \theta(x)] = [a, x]_\theta = 0, \quad x \in R.$$

This means that $a\theta(x) = \theta(x)a$ for all $x \in R$. Now from (2) it follows that $T(x) = a\theta(x) = \theta(x)a$. Thus, T is both a left and a right θ -centralizer. This completes the proof of the theorem in the case where R has an identity element.

§3. PROOF OF THEOREM 1.3

We intend to prove that

$$(6) \quad [T(x), x]_\theta = 0, \quad x \in R.$$

For this, first we prove a weaker result: T satisfies the relation

$$(7) \quad [T(x), x^2]_\theta = 0, \quad x \in R.$$

Since the above relation can be written in the form

$$[T(x), x]_\theta \theta(x) + \theta(x)[T(x), x]_\theta = 0,$$

it is obvious that T satisfies (7) if T satisfies (6). Substituting $x + y$ for x in the relation

$$(8) \quad 2T(x^2) = T(x)\theta(x) + \theta(x)T(x), \quad x \in R,$$

we obtain

$$(9) \quad 2T(xy + yx) = T(x)\theta(y) + \theta(x)T(y) + T(y)\theta(x) + \theta(y)T(x), \quad x, y \in R.$$

Our next step is to prove the relation

$$(10) \quad \begin{aligned} 8T(xyx) &= T(x)\theta(xy + 3yx) + \theta(yx + 3xy)T(x) \\ &\quad + 2\theta(x)T(y)\theta(x) - \theta(x^2)T(y) - T(y)\theta(x^2), \quad x, y \in R. \end{aligned}$$

For this, we substitute $2(xy + yx)$ for y in (9). Using (9), we obtain

$$\begin{aligned} &4T(x(xy + yx) + (xy + yx)x) \\ &= 2T(x)\theta(xy + yx) + 2\theta(x)T(xy + yx) + 2T(xy + yx)\theta(x) + 2\theta(xy + yx)T(x) \\ &= 2T(x)\theta(xy + yx) + \theta(x)T(x)\theta(y) + \theta(x^2)T(y) + \theta(x)T(y)\theta(x) \\ &\quad + \theta(xy)T(x) + T(x)\theta(yx) + \theta(x)T(y)\theta(x) + T(y)\theta(x^2) \\ &\quad + \theta(y)T(x)\theta(x) + 2\theta(xy + yx)T(x). \end{aligned}$$

Then we have

$$(11) \quad \begin{aligned} &4T(x(xy + yx) + (xy + yx)x) \\ &= T(x)\theta(2xy + 3yx) + \theta(3xy + 2yx)T(x) + \theta(x)T(x)\theta(y) + \theta(y)T(x)\theta(x) \\ &\quad + 2\theta(x)T(y)\theta(x) + \theta(x^2)T(y) + T(y)\theta(x^2) \quad \forall x, y \in R. \end{aligned}$$

On the other hand,

$$\begin{aligned} &4T(x(xy + yx) + (xy + yx)x) = 4T(x^2y + yx^2) + 8T(xyx) \\ &= 2T(x^2)\theta(y) + 2T(y)\theta(x^2) + 2\theta(x^2)T(y) + 2\theta(y)T(x^2) + 8T(xyx) \\ &= T(x)\theta(xy) + \theta(x)T(x)\theta(y) + 2\theta(x^2)T(y) \\ &\quad + 2T(y)\theta(x^2) + \theta(y)T(x)\theta(x) + \theta(yx)T(x) + 8T(xyx). \end{aligned}$$

Therefore,

$$(12) \quad \begin{aligned} &4T(x(xy + yx) + (xy + yx)x) \\ &= T(x)\theta(xy) + \theta(yx)T(x) + \theta(x)T(x)\theta(y) + \theta(y)T(x)\theta(x) \\ &\quad + 2\theta(x^2)T(y) + 2T(y)\theta(x^2) + 8T(xyx), \quad x, y \in R. \end{aligned}$$

Comparing (11) and (12), we arrive at (10). Now, we prove the relation

$$(13) \quad \begin{aligned} &T(x)\theta(xyx - 2yx^2 - 2x^2y) + \theta(xyx - 2yx^2 - 2x^2y)T(x) \\ &\quad + \theta(x)T(x)\theta(xy + yx) + \theta(yx + xy)T(x)\theta(x) \\ &\quad + \theta(x^2)T(x)\theta(y) + \theta(y)T(x)\theta(x^2) = 0, \quad x, y \in R. \end{aligned}$$

Replacing y by $8xyx$ in (9) and using (10), we obtain

$$\begin{aligned}
 16T(x^2yx + xyx^2) &= 8T(x)\theta(xy x) + 8\theta(x)T(xy x) + 8T(xy x)\theta(x) + 8\theta(xy x)T(x) \\
 &= 8T(x)\theta(xy x) + \theta(x)T(x)\theta(xy + 3yx) + \theta(xy x + 3x^2y)T(x) \\
 &\quad + 2\theta(x^2)T(y)\theta(x) - \theta(x^3)T(y) - \theta(x)T(y)\theta(x^2) \\
 &\quad + T(x)\theta(xy x + 3yx^2) + \theta(yx + 3xy)T(x)\theta(x) + 2\theta(x)T(y)\theta(x^2) \\
 &\quad - \theta(x^2)T(y)\theta(x) - T(y)\theta(x^3) + 8\theta(xy x)T(x).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 16T(x^2yx + xyx^2) &= T(x)\theta(9xyx + 3yx^2) + \theta(9xyx + 3x^2y)T(x) \\
 (14) \quad &\quad + \theta(x)T(x)\theta(xy + 3yx) + \theta(yx + 3xy)T(x)\theta(x) + \theta(x^2)T(y)\theta(x) \\
 &\quad + \theta(x)T(y)\theta(x^2) - T(y)\theta(x^3) - \theta(x^3)T(y), \quad x, y \in R.
 \end{aligned}$$

On the other hand, first using (10), and after collecting some terms and using (9), we obtain

$$\begin{aligned}
 16T(x^2yx + xyx^2) &= 16T(x(xy)x) + 16T(x(yx)x) \\
 &= 2T(x)\theta(x^2y + 3xyx) + 2\theta(xy x + 3x^2y)T(x) + 4\theta(x)T(xy)\theta(x) \\
 &\quad - 2\theta(x^2)T(xy) - 2T(xy)\theta(x^2) + 2T(x)\theta(xy x + 3yx^2) \\
 &\quad + 2\theta(yx^2 + 3xyx)T(x) + 4\theta(x)T(yx)\theta(x) - 2\theta(x^2)T(yx) - 2T(yx)\theta(x^2) \\
 &= T(x)\theta(2x^2y + 6yx^2 + 8xyx) + \theta(8xyx + 2yx^2 + 6x^2y)T(x) \\
 &\quad + 4\theta(x)T(xy + yx)\theta(x) - 2\theta(x^2)T(xy + yx) - 2T(xy + yx)\theta(x^2) \\
 &= T(x)\theta(2x^2y + 6yx^2 + 8xyx) + \theta(8xyx + 2yx^2 + 6x^2y)T(x) \\
 &\quad + 2\theta(x)T(x)\theta(yx) + 2\theta(x^2)T(y)\theta(x) + 2\theta(x)T(y)\theta(x^2) + 2\theta(xy)T(x)\theta(x) \\
 &\quad - \theta(x^2)T(x)\theta(y) - \theta(x^3)T(y) - \theta(x^2)T(y)\theta(x) - \theta(x^2y)T(x) \\
 &\quad - T(x)\theta(yx^2) - \theta(x)T(y)\theta(x^2) - T(y)\theta(x^3) - \theta(y)T(x)\theta(x^2).
 \end{aligned}$$

Now we have

$$\begin{aligned}
 16T(x^2yx + xyx^2) &= T(x)\theta(2x^2y + 5yx^2 + 8xyx) \\
 (15) \quad &\quad + \theta(2yx^2 + 5x^2y + 8xyx)T(x) + 2\theta(x)T(x)\theta(yx) + 2\theta(xy)T(x)\theta(x) \\
 &\quad + \theta(x^2)T(y)\theta(x) + \theta(x)T(y)\theta(x^2) - \theta(x^2)T(x)\theta(y) - \theta(y)T(x)\theta(x^2) \\
 &\quad - \theta(x^3)T(y) - T(y)\theta(x^3), \quad x, y \in R.
 \end{aligned}$$

Comparing (14) and (15), we arrive at (13).

Replacing y by yx in (13), we obtain

$$\begin{aligned}
 (16) \quad &T(x)\theta(xy x^2 - 2yx^3 - 2x^2yx) + \theta(xy x^2 - 2x^2yx - 2yx^3)T(x) \\
 &\quad + \theta(x)T(x)\theta(xy x + yx^2) + \theta(xy x + yx^2)T(x)\theta(x) + \theta(x^2)T(x)\theta(yx) \\
 &\quad + \theta(yx)T(x)\theta(x^2) = 0 \quad \forall x, y \in R.
 \end{aligned}$$

Right multiplication of (13) by $\theta(x)$ yields

$$\begin{aligned}
 (17) \quad &T(x)\theta(xy x^2 - 2yx^3 - 2x^2yx) + \theta(xy x - 2x^2y - 2yx^2)T(x)\theta(x) \\
 &\quad + \theta(x)T(x)\theta(xy x + yx^2) + \theta(xy + yx)T(x)\theta(x^2) + \theta(x^2)T(x)\theta(yx) \\
 &\quad + \theta(y)T(x)\theta(x^3) = 0, \quad x, y \in R.
 \end{aligned}$$

Subtracting (17) from (16), we get

$$\begin{aligned} & \theta(xy)[\theta(x), T(x)] + \theta(2x^2y)[T(x), \theta(x)] + \theta(2yx^2)[T(x), \theta(x)] \\ & + \theta(xy)[\theta(x), T(x)]\theta(x) + \theta(yx)[\theta(x), T(x)]\theta(x) \\ & + \theta(y)[\theta(x), T(x)]\theta(x^2) = 0, \quad x, y \in R. \end{aligned}$$

After collecting the first and the fourth term together, this reduces to

$$(18) \quad \begin{aligned} & \theta(xy)[\theta(x^2), T(x)] + 2\theta(x^2y)[T(x), \theta(x)] + 2\theta(yx^2)[T(x), \theta(x)] \\ & + \theta(yx)[\theta(x), T(x)]\theta(x) + \theta(y)[\theta(x), T(x)]\theta(x^2) = 0, \quad x, y \in R. \end{aligned}$$

We substitute $T(x)\theta(y)$ for $\theta(y)$ in (18) to obtain

$$(19) \quad \begin{aligned} & \theta(x)T(x)\theta(y)[\theta(x^2), T(x)] + 2\theta(x^2)T(x)\theta(y)[T(x), \theta(x)] \\ & + 2T(x)\theta(y)\theta(x^2)[T(x), \theta(x)] + T(x)\theta(y)\theta(x)[\theta(x), T(x)]\theta(x) \\ & + T(x)\theta(y)[\theta(x), T(x)]\theta(x^2) = 0, \quad x, y \in R. \end{aligned}$$

Left multiplication of (18) by $T(x)$ leads to the relation

$$(20) \quad \begin{aligned} & T(x)\theta(xy)[\theta(x^2), T(x)] + 2T(x)\theta(x^2y)[T(x), \theta(x)] \\ & + 2T(x)\theta(yx^2)[T(x), \theta(x)] + T(x)\theta(yx)[\theta(x), T(x)]\theta(x) \\ & + T(x)\theta(y)[\theta(x), T(x)]\theta(x^2) = 0, \quad x, y \in R. \end{aligned}$$

Subtracting (20) from (19), we get

$$[T(x), \theta(x)]\theta(y)[T(x), \theta(x^2)] - 2[T(x), \theta(x^2)]\theta(y)[T(x), \theta(x)] = 0, \quad x, y \in R.$$

If we set

$$a = [T(x), \theta(x)], \quad b = [T(x), \theta(x^2)], \quad c = -2[T(x), \theta(x^2)],$$

the above relation becomes

$$(21) \quad a\theta(y)b + c\theta(y)a = 0, \quad y \in R.$$

Substituting yz for y in (21), we obtain

$$a\theta(yz)b + c\theta(yz)a = a\theta(y)\theta(z)b + c\theta(y)\theta(z)a = 0, \quad z, y \in R.$$

Next, substituting $\theta(y)a$ for $\theta(y)$ in the last equation, we see that

$$(22) \quad a\theta(y)a\theta(z)b + c\theta(y)a\theta(z)a = 0, \quad z, y \in R.$$

Left multiplication of (21) by $a\theta(y)$ gives

$$(23) \quad a\theta(y)a\theta(z)b + a\theta(y)c\theta(z)a = 0, \quad z, y \in R.$$

After subtracting (23) from (22), we obtain

$$(24) \quad (a\theta(y)c - c\theta(y)a)\theta(z)a = 0, \quad z, y \in R.$$

After replacing $\theta(z)$ by $\theta(z)c\theta(y)$ in (24), we obtain

$$(25) \quad (a\theta(y)c - c\theta(y)a)\theta(z)c\theta(y)a = 0, \quad z, y \in R.$$

Right multiplication of (24) by $\theta(y)c$ gives

$$(26) \quad (a\theta(y)c - c\theta(y)a)\theta(z)a\theta(y)c = 0, \quad z, y \in R.$$

After subtracting (25) from (26), we get

$$(a\theta(y)c - c\theta(y)a)\theta(z)(a\theta(y)c - c\theta(y)a) = 0, \quad x, y \in R.$$

Since R is semiprime and θ is surjective, it follows that

$$(27) \quad a\theta(y)c = c\theta(y)a, \quad y \in R.$$

Combining (21) with (27), we arrive at the relation

$$(28) \quad a\theta(y)(b+c) = 0, \quad y \in R.$$

In other words,

$$(29) \quad [T(x), \theta(x)]\theta(y)[T(x), \theta(x^2)] = 0, \quad x, y \in R.$$

Since

$$\begin{aligned} & [T(x), \theta(x^2)]\theta(y)[T(x), \theta(x^2)] \\ &= \theta(x)[T(x), \theta(x)]\theta(y)[T(x), \theta(x^2)] + [T(x), \theta(x)]\theta(x)\theta(y)[T(x), \theta(x^2)] \\ &= [T(x), \theta(x)]\theta(xy)[T(x), \theta(x^2)] = 0, \quad x, y \in R, \end{aligned}$$

we get

$$(30) \quad [T(x), \theta(x^2)]\theta(y)[T(x), \theta(x^2)] = 0, \quad x, y \in R.$$

Since R is semiprime and θ is surjective, we obtain

$$(31) \quad [T(x), \theta(x^2)] = 0, \quad x \in R.$$

After substituting $x+y$ for x in (31), we get

$$[T(x), \theta(y^2)] + [T(y), \theta(x^2)] + [T(x), \theta(xy+yx)] + [T(y), \theta(xy+yx)] = 0, \quad x, y \in R.$$

Substituting $-x$ for x and comparing the resulting relation obtained with the above, we obtain

$$(32) \quad [T(x), \theta(xy+yx)] + [T(y), \theta(x^2)] = 0, \quad x, y \in R,$$

because R is torsion free. Substituting $2(xy+yx)$ for y in (32) and using (9) and (31), we arrive at

$$\begin{aligned} 0 &= 2[T(x), \theta(x^2y+yx^2+2xyx)] \\ &\quad + [T(x)\theta(y) + \theta(x)T(y) + T(y)\theta(x) + \theta(y)T(x), \theta(x^2)] \\ &= 2\theta(x^2)[T(x), \theta(y)] + 2[T(x), \theta(y)]\theta(x^2) + 4[T(x), \theta(xy+x)] + T(x)[\theta(y), \theta(x^2)] \\ &\quad + \theta(x)[T(y), \theta(x^2)] + [T(y), \theta(x^2)]\theta(x) + [\theta(y), \theta(x^2)]T(x) = 0, \quad x, y \in R. \end{aligned}$$

Now we have

$$(33) \quad \begin{aligned} & 2\theta(x^2)[T(x), \theta(y)] + 2[T(x), \theta(y)]\theta(x^2) + 4[T(x), \theta(xy+x)] + T(x)[\theta(y), \theta(x^2)] \\ & + \theta(x)[T(y), \theta(x^2)] + [T(y), \theta(x^2)]\theta(x) + [\theta(y), \theta(x^2)]T(x) = 0, \quad x, y \in R. \end{aligned}$$

For $y=x$ the above relation reduces to

$$\theta(x^2)[T(x), \theta(x)] + [T(x), \theta(x)]\theta(x^2) + 2[T(x), \theta(x^3)] = 0, \quad x \in R.$$

Thus,

$$\theta(x^2)[T(x), \theta(x)] + 3[T(x), \theta(x)]\theta(x^2) = 0, \quad x \in R.$$

In accordance with the relation $\theta(x)[T(x), \theta(x)] + [T(x), \theta(x)]\theta(x) = 0$ (see (31)), in the above equation we replace $\theta(x^2)[T(x), \theta(x)]$ by $-[T(x), \theta(x)]\theta(x^2)$. This gives

$$(34) \quad [T(x), \theta(x)]\theta(x^2) = 0, \quad x, y \in R,$$

and

$$(35) \quad \theta(x^2)[T(x), \theta(x)] = 0, \quad x, y \in R.$$

We also have

$$(36) \quad \theta(x)[T(x), \theta(x)]\theta(x) = 0, \quad x, y \in R.$$

Because of (32), we can replace $[T(y), \theta(x^2)]$ by $-[T(x), \theta(xy + yx)]$ in (33), obtaining

$$\begin{aligned} 0 &= 2\theta(x^2)[T(x), \theta(y)] + 2[T(x), \theta(y)]\theta(x^2) + 4[T(x), \theta(xy + yx)] + T(x)[\theta(y), \theta(x^2)] \\ &\quad + [\theta(y), \theta(x^2)]T(x) - \theta(x)[T(x), \theta(xy + yx)] - [T(x), \theta(xy + yx)]\theta(x) \\ &= 2\theta(x^2)[T(x), \theta(y)] + 2[T(x), \theta(y)]\theta(x^2) + 4[T(x), \theta(x)]\theta(yx) \\ &\quad + 4\theta(x)[T(x), \theta(y)]\theta(x) + 4\theta(xy)[T(x), \theta(x)] + T(x)[\theta(y), \theta(x^2)] \\ &\quad + [\theta(y), \theta(x^2)]T(x) - \theta(x)[T(x), \theta(x)]\theta(y) - \theta(x^2)[T(x), \theta(y)] \\ &\quad - \theta(x)[T(x), \theta(y)]\theta(x) - \theta(xy)[T(x), \theta(x)] - [T(x), \theta(x)]\theta(yx) \\ &\quad - \theta(x)[T(x), \theta(y)]\theta(x) - [T(x), \theta(y)]\theta(x^2) - \theta(y)[T(x), \theta(x)]\theta(x). \end{aligned}$$

Now we have

$$\begin{aligned} &\theta(x^2)[T(x), \theta(y)] + [T(x), \theta(y)]\theta(x^2) + 3[T(x), \theta(x)]\theta(yx) \\ &\quad + 3\theta(xy)[T(x), \theta(x)] + 2\theta(x)[T(x), \theta(y)]\theta(x) + T(x)[\theta(y), \theta(x^2)] \\ (37) \quad &\quad + [\theta(y), \theta(x^2)]T(x) - \theta(x)[T(x), \theta(x)]\theta(y) \\ &\quad - \theta(y)[T(x), \theta(x)]\theta(x) = 0, \quad x, y \in R. \end{aligned}$$

The substitution of yx for y in (37) gives

$$\begin{aligned} 0 &= \theta(x^2)[T(x), \theta(yx)] + [T(x), \theta(yx)]\theta(x^2) + 3[T(x), \theta(x)]\theta(yx^2) \\ &\quad + 3\theta(xyx)[T(x), \theta(x)] + 2\theta(x)[T(x), \theta(yx)]\theta(x) + T(x)[\theta(yx), \theta(x^2)] \\ &\quad + [\theta(yx), \theta(x^2)]T(x) - \theta(x)[T(x), \theta(x)]\theta(yx) - \theta(yx)[T(x), \theta(x)]\theta(x) \\ &= \theta(x^2)[T(x), \theta(y)]\theta(x) + \theta(x^2y)[T(x), \theta(x)] + [T(x), \theta(y)]\theta(x^3) \\ &\quad + \theta(y)[T(x), \theta(x)]\theta(x^2) + 3[T(x), \theta(x)]\theta(yx^2) + 3\theta(xyx)[T(x), \theta(x)] \\ &\quad + 2\theta(x)[T(x), \theta(y)]\theta(x^2) + 2\theta(xy)[T(x), \theta(x)]\theta(x) + T(x)[\theta(y), \theta(x^2)]\theta(x) \\ &\quad + [\theta(y), \theta(x^2)]\theta(x)T(x) - \theta(x)[T(x), \theta(x)]\theta(yx) - \theta(yx)[T(x), \theta(x)]\theta(x), \\ &\hspace{25em} x, y \in R. \end{aligned}$$

By (34) and (36), this reduces to

$$\begin{aligned} &\theta(x^2)[T(x), \theta(y)]\theta(x) + \theta(x^2y)[T(x), \theta(x)] + [T(x), \theta(y)]\theta(x^3) \\ &\quad + 3[T(x), \theta(x)]\theta(yx^2) + 3\theta(xyx)[T(x), \theta(x)] + 2\theta(x)[T(x), \theta(y)]\theta(x^2) \\ (38) \quad &\quad + 2\theta(xy)[T(x), \theta(x)]\theta(x) + T(x)[\theta(y), \theta(x^2)]\theta(x) \\ &\quad + [\theta(y), \theta(x^2)]\theta(x)T(x) - \theta(x)[T(x), \theta(x)]\theta(yx) = 0, \quad x, y \in R. \end{aligned}$$

Right multiplication of (37) by $\theta(x)$ gives

$$\begin{aligned} &\theta(x^2)[T(x), \theta(y)]\theta(x) + [T(x), \theta(y)]\theta(x^3) + 3[T(x), \theta(x)]\theta(yx^2) \\ &\quad + 3\theta(xy)[T(x), \theta(x)]\theta(x) + 2\theta(x)[T(x), \theta(y)]\theta(x^2) \\ (39) \quad &\quad + T(x)[\theta(y), \theta(x^2)]\theta(x) + [\theta(y), \theta(x^2)]T(x)\theta(x) \\ &\quad - \theta(x)[T(x), \theta(x)]\theta(yx) - \theta(y)[T(x), \theta(x)]\theta(x^2) = 0, \quad x, y \in R. \end{aligned}$$

By (34) we have $[T(x), \theta(x)]\theta(x^2) = 0$, and (39) becomes

$$(40) \quad \begin{aligned} & \theta(x^2)[T(x), \theta(y)]\theta(x) + [T(x), \theta(y)]\theta(x^3) + 3[T(x), \theta(x)]\theta(yx^2) \\ & + 3\theta(xy)[T(x), \theta(x)]\theta(x) + 2\theta(x)[T(x), \theta(y)]\theta(x^2) + T(x)[\theta(y), \theta(x^2)]\theta(x) \\ & + [\theta(y), \theta(x^2)]T(x)\theta(x) - \theta(x)[T(x), \theta(x)]\theta(yx) = 0, \quad x, y \in R. \end{aligned}$$

After subtracting (40) from (38), we obtain

$$\begin{aligned} & \theta(x^2y)[T(x), \theta(x)] + 3\theta(xy)[\theta(x), [T(x), \theta(x)]] \\ & + 2\theta(xy)[T(x), \theta(x)]\theta(x) + [\theta(y), \theta(x^2)][\theta(x), T(x)] = 0. \end{aligned}$$

By (35), this reduces to

$$2\theta(x^2y)[T(x), \theta(x)] + 3\theta(xyx)[T(x), \theta(x)] - \theta(xy)[T(x), \theta(x)]\theta(x) = 0, \quad x, y \in R.$$

Replacing $-[T(x), \theta(x)]\theta(x)$ by $\theta(x)[T(x), \theta(x)]$ in the above relation, we obtain

$$\theta(x^2y)[T(x), \theta(x)] + 2\theta(xyx)[T(x), \theta(x)] = 0, \quad x, y \in R.$$

Using (31), (34), (35), and (36), we see that (18) reduces to

$$\theta(x^2y)[T(x), \theta(x)] = 0, \quad x, y \in R.$$

Combining this with the above relation, we get

$$\theta(x)\theta(y)\theta(x)[T(x), \theta(x)] = 0, \quad x, y \in R.$$

Since θ is surjective, we can replace $\theta(y)$ by $[T(x), \theta(x)]\theta(y)$; it follows that

$$\theta(x)[T(x), \theta(x)]\theta(y)\theta(x)[T(x), \theta(x)] = 0, \quad x, y \in R.$$

Since θ is surjective and R is semiprime, we have

$$(41) \quad \theta(x)[T(x), \theta(x)] = 0, \quad x \in R.$$

Also,

$$[T(x), \theta(x)]\theta(x) = -\theta(x)[T(x), \theta(x)] = 0, \quad x, y \in R.$$

Thus,

$$(42) \quad [T(x), \theta(x)]\theta(x) = 0, \quad x \in R.$$

Substituting $x + y$ for x in (41) and using (41), we get

$$\begin{aligned} & \theta(x + y)[T(x + y), \theta(x + y)] \\ & = \theta(x)[T(x), \theta(y)] + \theta(x)[T(y), \theta(x)] + \theta(x)[T(y), \theta(y)] \\ & \quad + \theta(y)[T(x), \theta(x)] + \theta(y)[T(x), \theta(y)] + \theta(y)[T(y), \theta(x)] = 0, \quad x, y \in R. \end{aligned}$$

Substituting $-x$ for x , comparing the resulting relation with the above, and using the fact that R is 2-torsion free, we obtain

$$\theta(y)[T(x), \theta(x)] + \theta(x)[T(x), \theta(y)] + \theta(x)[T(y), \theta(x)] = 0, \quad x, y \in R.$$

Multiplying the above relation by $[T(x), \theta(x)]$ and using (42), we arrive at the formula

$$[T(x), \theta(x)]\theta(y)[T(x), \theta(x)] = 0, \quad x, y \in R.$$

Since R is semiprime and θ is surjective, we get

$$(43) \quad [T(x), \theta(x)] = 0, \quad x \in R.$$

Combining (43) with (8), we see that

$$T(x^2) = \theta(x)T(x), \quad x \in R.$$

This means that T is a left and also a right Jordan θ -centralizer. It remains to use [1, Theorem 2]. The proof is complete.

Corollary 3.1 ([10, Theorem 1]). *Let R be a 2-torsion free semiprime ring, and let $T : R \rightarrow R$ be an additive mapping such that $2T(x^2) = T(x)x + xT(x)$ for all $x \in R$. Then T is both a left and a right centralizer.*

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