TRIVIALITY OF THE SECOND COHOMOLOGY GROUP OF THE CONFORMAL ALGEBRAS Cend_n AND Cur_n

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ABSTRACT. It is proved that the second cohomology group of the conformal algebras Cend_n and Cur_n with coefficients in any bimodule is trivial. As a result, these algebras are segregated in any extension with a nilpotent kernel.

INTRODUCTION

A principal result of the theory of finite-dimensional algebras is the Wedderburn theorem on the structure of separable algebras.

Let A be a finite-dimensional algebra, and let $R = \text{Rad}(A)$. If $A/\text{Rad}(A)$ is a separable algebra, then there exists a subalgebra $S \subseteq A$ such that A is equal to the direct sum $S \oplus \text{Rad}(A)$ of spaces.

It is known that this statement is a consequence of the Hochschild theorem [1] on the triviality of the second cohomology group of a matrix algebra over a field. In [2], an analog of the Wedderburn theorem for a certain class of associative conformal algebras was proved.

The formal definition of a conformal algebra was introduced in [3] as an axiomatic description of the singular part of the operator product extension (OPE) of chiral fields in the conformal field theory. Another approach to the theory of conformal algebras is related to the notion of a pseudotensor category [4]: a conformal algebra is an algebra in the pseudotensor category $\mathcal{M}^*(H)$ associated with the polynomial algebra $H = \mathbb{k}[D]$ (see [5]). An object of that category is a left unital H -module, and C is an algebra in $\mathcal{M}^*(H)$ if $C \in \mathcal{M}^*(H)$ is endowed with an $(H \otimes H)$ -linear map $*$: $C \otimes C \rightarrow (H \otimes H) \otimes_H C$. An advantage of this language is that associativity, commutativity, and other identities admit a natural interpretation. Note that a usual algebra over a field $\mathbb{k}$ is an algebra in the pseudotensor category $\mathcal{M}^*(\mathbb{k})$.

Thus, the last approach seems the most natural for generalizing the notion of the algebra $\text{End } U$ of endomorphisms of a finite-dimensional linear space U . Namely, if V is a finitely generated H -module, then the set of all conformal endomorphisms (see [3, 5, 6]) forms an associative conformal algebra denoted by $\text{Cend } V$.

The notion of Hochschild cohomology of a conformal algebra was given in [7] by using the so-called λ -product. However, it is not clear how to establish the correspondence between cohomology and associative conformal algebra extensions. Another (equivalent) definition of Hochschild cohomology given in [8] involves the language of pseudoalgebras. In [8], triviality of the second cohomology group of the conformal Weyl algebra with coefficients in any bimodule was proved.

2000 *Mathematics Subject Classification.* Primary 13D03.

Key words and phrases. Associative conformal algebra, algebra of conformal endomorphisms, Hochschild cohomology.

In this paper we prove that the second cohomology group of the conformal algebra Cend_n of endomorphisms of a free finitely generated H -module with coefficients in any bimodule is trivial. This implies the main result of [2].

§1. PRELIMINARIES

1.1. Conformal algebras. Let $\mathbb{k}$ be a field of zero characteristic (for example, the field of complex numbers). By $H = \mathbb{k}[D]$ we denote a polynomial algebra in one variable over the field $\mathbb{k}$; $\mathbb{Z}_+$ is the set of nonnegative integers, $D^{(n)} = \frac{1}{n!}D^n$ for $n \in \mathbb{Z}_+$.

The algebra H is the Hopf algebra with the coproduct $\Delta: H \rightarrow H \otimes H$, $\Delta(D) = D \otimes 1 + 1 \otimes D$, the counit $\varepsilon: H \rightarrow \mathbb{k}$, $\varepsilon(D) = 0$, and the antipode $S: H \rightarrow H$, $S(D) = -D$, where Δ , ε , and S are homomorphisms of algebras. Letting $\Delta_1 = \Delta$, we inductively define $\Delta_n = (\Delta \otimes \text{id} \otimes \dots \otimes \text{id})\Delta_{n-1}$, where $n > 1$.

Definition 1 ([3]). A left H -module C is called a *conformal algebra* if C is endowed with bilinear operations $\circ_n: C \otimes C \rightarrow C$, $n \in \mathbb{Z}_+$, such that for any $a, b \in C$ and any $n \in \mathbb{Z}_+$ the following conditions are fulfilled:

$$\begin{aligned} (1.1) \quad & a \circ_n b = 0 \text{ for } n \text{ sufficiently large,} \\ (1.2) \quad & Da \circ_n b = -na \circ_{n-1} b, \\ (1.3) \quad & D(a \circ_n b) = Da \circ_n b + a \circ_n Db. \end{aligned}$$

Under condition (1.1), known as the *locality* axiom, the *locality function* $N: C \times C \rightarrow \mathbb{Z}_+$ on C is given by

$$N(a, b) = \min\{n \in \mathbb{Z}_+ \mid a \circ_k b = 0 \text{ for all } k \geq n\}, \quad a, b \in C.$$

A conformal algebra C is said to be *associative* if (see [9, 10])

$$(1.4) \quad (a \circ_n b) \circ_m c = \sum_{s \geq 0} (-1)^s \binom{n}{s} a \circ_{n-s} (b \circ_{m+s} c)$$

for all $a, b, c \in C$ and all $m, n \in \mathbb{Z}_+$.

Let C be a conformal algebra. Denote

$$\{a \circ_n b\} = \sum_{t \geq 0} (-1)^{n+t} D^{(t)}(a \circ_{n+t} b), \quad a, b \in C, \quad n \in \mathbb{Z}_+.$$

The axioms (1.2) and (1.3) of a conformal algebra imply the following identities:

$$\{a \circ_n Db\} = -n\{a \circ_{n-1} b\}, \quad \{Da \circ_n b\} = D\{a \circ_n b\} + n\{a \circ_{n-1} b\}.$$

Moreover, if an algebra C is associative, then (see [2, 9, 11])

$$(1.5) \quad a \circ_n \{b \circ_m c\} = \{(a \circ_n b) \circ_m c\},$$

$$(1.6) \quad \{a \circ_n (b \circ_m c)\} = \sum_{s \geq 0} (-1)^s \binom{m}{s} \{\{a \circ_{m-s} b\} \circ_{n+s} c\},$$

$$(1.7) \quad \{a \circ_n \{b \circ_m c\}\} = \sum_{s \geq 0} (-1)^s \binom{m}{s} \{\{a \circ_{n+s} b\} \circ_{m-s} c\},$$

$$(1.8) \quad \{a \circ_n b\} \circ_m c = \sum_{s \geq 0} (-1)^s \binom{n}{s} a \circ_{m+s} (b \circ_{n-s} c),$$

where $a, b, c \in C$, $n, m \in \mathbb{Z}_+$.

Let C be a conformal algebra, and let B_1, B_2 be two subsets in C . We use the notation

$$B_1 \circ_\omega B_2 = \left\{ \sum_i a_i \circ_{n_i} b_i \mid a_i \in B_1, b_i \in B_2, n_i \in \mathbb{Z}_+ \right\}.$$

It is clear how to define a subalgebra, a (left/right) ideal of a conformal algebra, and a homomorphism of conformal algebras, and what should be meant by a simple or nilpotent conformal algebra.

A conformal algebra is an example of a pseudoalgebra (see [5]). Therefore, the structure of a conformal algebra can be defined by using the language of a pseudoproduct.

Denote $H^{\otimes n} = H \otimes H \otimes \cdots \otimes H$. The identity

$$(h_1 \otimes h_2 \otimes \cdots \otimes h_n)h = (h_1 \otimes h_2 \otimes \cdots \otimes h_n)\Delta_{n-1}(h)$$

yields a right action of H on $H^{\otimes n}$. Then $H^{\otimes n}$ is a right H -module.

Let V_1 , V_2 , and V_3 be left H -modules. Then $V_1 \otimes V_2$ equipped with the operation

$$(h_1 \otimes h_2)(v_1 \otimes v_2) = h_1 v_1 \otimes h_2 v_2, \quad h_1, h_2 \in H, \quad v_1 \in V_1, \quad v_2 \in V_2,$$

is an $H^{\otimes 2}$ -module. Furthermore, $H^{\otimes 2} \otimes_H V_3$ is an $H^{\otimes 2}$ -module. Let $*$: $V_1 \otimes V_2 \rightarrow H^{\otimes 2} \otimes_H V_3$ be some $H^{\otimes 2}$ -linear operation. Note that $*$ extends naturally to an operation $*$: $(H^{\otimes n} \otimes_H V_1) \otimes (H^{\otimes m} \otimes_H V_2) \rightarrow H^{\otimes(n+m)} \otimes_H V_3$:

$$\begin{aligned} (h_1 \otimes \cdots \otimes h_n \otimes_H v_1) * (g_1 \otimes \cdots \otimes g_m \otimes_H v_2) \\ = (h_1 \otimes \cdots \otimes h_n \otimes g_1 \otimes \cdots \otimes g_m \otimes_H 1)(\Delta_{n-1} \otimes \Delta_{m-1} \otimes_H \text{id})(v_1 * v_2). \end{aligned}$$

If $V_1 = V_2 = V_3 = V$, then V is called an H -pseudoalgebra, and the operation $*$ is called a pseudoproduct.

A pseudoproduct $*$ is associative if it satisfies

$$u * (v * w) = (u * v) * w$$

for all $u, v, w \in V$.

Let C be a conformal algebra. Then C is an H -pseudoalgebra, and the correspondence between the pseudoproduct $*$ and the operations $\circ_n$, $n \in \mathbb{Z}_+$, is given by the formula

$$a * b = \sum_{n \geq 0} ((-D)^{(n)} \otimes 1) \otimes_H (a \circ_n b), \quad a, b \in C.$$

1.2. Conformal linear maps.

Definition 2 (see [5]). Let V and W be left H -modules. A linear map $a: V \rightarrow H^{\otimes 2} \otimes_H W$ is called a *conformal linear map* from V to W if it satisfies

$$a(fv) = ((1 \otimes f) \otimes_H 1)a(v)$$

for all $v \in V$, $f \in H$.

We denote by $\text{Chom}(V, W)$ the set of all conformal linear maps from V to W . If $a \in \text{Chom}(V, W)$ and $v \in V$, then (see [12])

$$a(v) = \sum_{s \geq 0} ((-D)^{(s)} \otimes 1) \otimes_H a(n)(v),$$

and $\{a(n)\}_{n=0}^{\infty}$ is the sequence of (usual) $\mathbb{k}$ -linear maps from V to W such that

for any $v \in V$, we have $a(n)(v) = 0$ for sufficiently large n ,

$$[a(n), D] = na(n-1), \quad n \in \mathbb{Z}_+.$$

We write $a \circ_n v$ for $a(n)(v)$, where $a \in \text{Chom}(V, W)$, $v \in V$, $n \in \mathbb{Z}_+$.

A structure of an H -module on $\text{Chom}(V, W)$ can be defined by

$$(1.9) \quad (Da)(v) = ((D \otimes 1) \otimes_H 1)a(v),$$

whence it follows that $(Da)(n) = -na(n-1)$ for any $a \in \text{Chom}(V, W)$, $n \in \mathbb{Z}_+$.

Let U , V , W be left H -modules. Then, for any $n \in \mathbb{Z}_+$, an operation

$$\circ_n: \text{Chom}(V, W) \otimes \text{Chom}(U, V) \rightarrow \text{Chom}(U, W)$$

can be introduced by the rule

$$(1.10) \quad (a \circ_n b)(m) = \sum_{s \geq 0} (-1)^s \binom{n}{s} a(n-s)b(m+s), \quad m \in \mathbb{Z}_+.$$

It is easy to check that conditions similar to (1.2) and (1.3) are fulfilled for the operations (1.9) and (1.10):

$$\begin{aligned} Da \circ_n b &= -na \circ_{n-1} b, \\ a \circ_n Db &= D(a \circ_n b) + na \circ_{n-1} b, \end{aligned}$$

where $a \in \text{Chom}(V, W)$, $b \in \text{Chom}(U, V)$, and $n \in \mathbb{Z}_+$. Moreover, if U is a finitely generated H -module, then for any $a \in \text{Chom}(V, W)$ and $b \in \text{Chom}(U, V)$ we have $a \circ_n b = 0$ for sufficiently large n (see, e.g., [5]).

If $V = W$, then $\text{Chom}(V, V) = \text{Cend } V$. From the foregoing it follows that if V is a finitely generated H -module, then $\text{Cend } V$ endowed with the operations (1.9) and (1.10) is an associative conformal algebra.

1.3. Examples of associative conformal algebras. Consider the free finitely generated H -module $V = H \otimes \mathbb{k}^n$. The corresponding algebra of conformal endomorphisms is denoted by Cend_n . There exists an isomorphism between Cend_n and $H \otimes M_n(\mathbb{k}[v])$ [9, 12, 13], where v is a formal variable. Therefore, every $a \in \text{Cend}_n$ can be expressed in the form

$$a = \sum_{s \geq 0} (-D)^{(s)} \otimes A_s(v) \in H \otimes M_n(\mathbb{k}[v]),$$

and the operations $\circ_m$, $m \in \mathbb{Z}_+$, are given by

$$(1 \otimes A) \circ_m (1 \otimes B) = 1 \otimes A \partial_v^m(B),$$

where $A, B \in M_n(\mathbb{k}[v])$ and $\partial_v^m = \partial^m / \partial v^m$. From now on we identify Cend_n and $M_n(\mathbb{k}[D, v])$.

Denote $x = 1 \otimes vE$, where E is the identity matrix of $M_n(\mathbb{k})$. As a conformal algebra, Cend_n is generated by e_{ij} , $i, j = 1, \dots, n$, and x , which satisfy the following relations:

$$(1.11) \quad e_{ij} \circ_m e_{kl} = \delta_{m,0} \delta_{j,k} e_{il}, \quad m \geq 0,$$

$$(1.12) \quad e_{ij} \circ_0 x = x \circ_0 e_{ij},$$

$$(1.13) \quad e_{ij} \circ_1 x = e_{ij}, \quad e_{ij} \circ_m x = 0, \quad m \geq 2,$$

$$(1.14) \quad x \circ_m e_{ij} = 0, \quad m \geq 1,$$

$$(1.15) \quad \sum_i e_{ii} \circ_0 x = \sum_i x \circ_0 e_{ii} = x.$$

Proposition 1. *Relations (1.11)–(1.15) form a full system of relations in Cend_n .*

Proof. Note that (1.4), (1.13), and (1.15) imply that

$$(1.16) \quad x \circ_1 x = \sum_i (x \circ_0 e_{ii}) \circ_1 x = x \circ_0 \sum_i (e_{ii} \circ_1 x) = \sum_i x \circ_0 e_i i = x,$$

$$(1.17) \quad x \circ_m x = 0, \quad m \geq 2.$$

Consider the free conformal algebra C_2 [14] generated by the elements x , e_{ij} , $i, j = 1, \dots, n$, with the locality function $N = 2$. Using (1.11)–(1.17), we can represent any element of C_2 as a linear combination of words of the form

$$u_{m,s,i,j} = D^s \underbrace{(x \circ_0 x \circ_0 \cdots \circ_0 x \circ_0 e_{ij})}_m.$$

The image of the word $u_{m,s,i,j}$ in the conformal algebra Cend_n is equal to the matrix $D^s v^m e_{ij}$. Therefore, the conformal polynomials corresponding to relations (1.11)–(1.15) generate an ideal I of C_2 such that $\text{Cend}_n \simeq C_2/I$. $\square$

The conformal subalgebra $H \otimes M_n(\mathbb{k}) \cong M_n(\mathbb{k}[D])$ of $M_n(\mathbb{k}[D, v]) \cong \text{Cend}_n$ generated by e_{ij} , $i, j = 1, \dots, n$, is denoted by Cur_n .

1.4. Modules over conformal algebras.

Definition 3 (see [6]). Any homomorphism $\rho: C \rightarrow \text{Cend } V$ of conformal algebras is called a *representation* of an associative conformal algebra C on a left H -module V . If C has a representation on V , then the H -module V is called a *module over a conformal algebra* C .

This definition is equivalent to the following: an H -module V is called a (*left*) *module* over an associative conformal algebra C if there exist maps $\circ_n: C \otimes V \rightarrow V$, $n \in \mathbb{Z}_+$, such that

$$\begin{aligned} a \circ_n v &= 0 \text{ for sufficiently large } n, \\ Da \circ_n v &= -na \circ_{n-1} v, \\ a \circ_n Dv &= D(a \circ_n v) + na \circ_{n-1} v, \\ (a \circ_m b) \circ_n v &= \sum_{t \geq 0} (-1)^t \binom{m}{t} a \circ_{m-t} (b \circ_{n+t} v) \end{aligned}$$

for all $a, b \in C$, $v \in V$, $m \in \mathbb{Z}_+$.

In what follows, by a left C -module we shall mean a left module over an associative conformal algebra C . As in the above definition, one can define a *right module* over an associative conformal algebra C .

A *bimodule* over a conformal algebra C is a right and a left C -module V satisfying the following additional axiom:

$$(a \circ_m v) \circ_n b = \sum_{t \geq 0} (-1)^t \binom{m}{t} a \circ_{m-t} (v \circ_{n+t} b),$$

where $a, b \in C$, $v \in V$, $m, n \in \mathbb{Z}_+$.

The definition of a (bi)module over an associative conformal algebra in the language of a pseudoproduct $*$ is similar to the corresponding definition of a (bi)module over a (usual) associative algebra.

§2. HOCHSCHILD COHOMOLOGY OF ASSOCIATIVE CONFORMAL ALGEBRAS

2.1. Main definitions. Let V be a bimodule over an associative conformal algebra C .

Definition 4 (see [8]). A map

$$\varphi: C^{\otimes n} \rightarrow (H^{\otimes n}) \otimes_H V$$

is called an n -*cochain* of C with coefficients in V if it is $H^{\otimes n}$ -linear:

$$\varphi(h_1 a_1 \otimes \dots \otimes h_n a_n) = (h_1 \otimes \dots \otimes h_n \otimes_H 1) \varphi(a_1 \otimes \dots \otimes a_n).$$

Let $C^n(C, V)$ be the set of all n -cochains of an algebra C with coefficients in a bimodule V . Then $C^n(C, V)$ is a left H -module, and H acts on $C^n(C, V)$ as follows:

$$(h\varphi)(a_1 \otimes \dots \otimes a_n) = (\Delta_{n-1}(h) \otimes_H 1) \varphi(a_1 \otimes \dots \otimes a_n),$$

where $h \in H$, $\varphi \in C^n(C, V)$, $a_i \in C$, $i = 1, 2, \dots, n$.

Recall (see [8]) that $C^0(C, V) \cong V/DV$ and $C^1(C, V) \cong \text{Hom}_H(C, V)$.

If $\varphi \in C^2(C, V)$, then $\varphi(a, b)$ can uniquely be expressed in the form

$$\varphi(a, b) = \sum_{s \geq 0} ((-D)^{(s)} \otimes 1) \otimes_H \varphi_s(a, b), \quad \varphi_s(a, b) \in V$$

for all $a, b \in C$.

For any set of integers $m_i \in \mathbb{Z}_+$, $m_i \geq 1$, $i = 1, \dots, n$, $m_1 + \dots + m_n = m$, an n -cochain φ extends to

$$\varphi: (H^{\otimes m_1} \otimes_H C) \otimes \dots \otimes (H^{\otimes m_n} \otimes_H C) \rightarrow (H^{\otimes m} \otimes_H C)$$

as follows:

$$\begin{aligned} \varphi(F_1 \otimes_H a_1, \dots, F_n \otimes_H a_n) \\ = (F_1 \otimes \dots \otimes F_n \otimes_H 1)(\Delta_{m_1-1} \otimes \Delta_{m_2-1} \otimes \dots \otimes_H \text{id})\varphi(a_1, \dots, a_n), \end{aligned}$$

$F_i \in H^{\otimes m_i}$, $a_i \in C$, $i = 1, \dots, n$.

Definition 5. A map $\delta_n: C^n(C, V) \rightarrow C^{n+1}(C, V)$,

$$\begin{aligned} (\delta_n \varphi)(a_1, \dots, a_{n+1}) &= a_1 * \varphi(a_2, \dots, a_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i \varphi(a_1, \dots, a_i * a_{i+1}, \dots, a_{n+1}) + (-1)^{n+1} \varphi(a_1, \dots, a_n) * a_{n+1} \end{aligned}$$

is called a *differential*.

A direct calculation (see [8]) shows that $\delta_{n+1}\delta_n = 0$.

If $\delta_n \varphi = 0$, then an n -cochain φ is called an *n -cocycle*. A cocycle $\varphi \in C^n(C, V)$ is called an *n -coboundary* if there exists an $(n-1)$ -cochain ψ such that $\varphi = \delta_{n-1}\psi$. Let $Z^n(C, V)$ denote the set of n -cocycles and $B^n(C, V)$ the set of n -coboundaries. Then $Z^n(C, V) = \ker \delta_n$, $B^n(C, V) = \text{Im } \delta_{n-1}$.

Definition 6. The H -bimodule $H^n(C, V) = Z^n(C, V)/B^n(C, V)$ is called the group of *Hochschild cohomology* of a conformal algebra C with coefficients in a C -bimodule V .

2.2. Extensions and the second cohomology group. An *extension* of a conformal algebra C is a pair (B, σ) , where B is a conformal algebra, and $\sigma: B \rightarrow C$ is a homomorphism of B onto C . A conformal algebra C is said to be *segregated* in the extension (B, σ) if $B = C' \oplus \ker \sigma$, where a subalgebra $C' \subseteq B$ is isomorphic to C . An extension (B, σ) is *singular* if $(\ker \sigma \circ_\omega \ker \sigma) = 0$.

We say that the extensions (B, σ) and (B^*, σ^*) are *isomorphic* if there exists an isomorphism I of conformal algebras from B onto B^* such that $\sigma^* I = \sigma$.

Suppose C is an associative conformal algebra that is a projective (in this case, a free) H -module, and (B, σ) is a singular extension of C with kernel $\ker \sigma = M$. Let $\rho: C \rightarrow B$ be an H -linear map such that

$$\sigma \rho = \text{id}.$$

We define an action of C on M as follows:

$$a * x = \rho(a) * x, \quad x * a = x * \rho(a),$$

where $x \in M$, $a \in C$. Then the kernel M is a C -bimodule. This action is independent of the choice of a map ρ . Thus, the C -bimodule $M = \ker \sigma$ is uniquely determined by the extension (B, σ) . Obviously, the kernels of isomorphic extensions are isomorphic as C -bimodules.

Theorem 1 (see [8]). *Let C be an associative conformal algebra, and let M be a C -bimodule. If C is a projective H -module, then there exists a one-to-one correspondence between the classes of isomorphic singular extensions of C with kernel isomorphic to M and the elements of the second cohomology group $H^2(C, M)$.*

Namely, let (B, σ) be a singular extension with kernel isomorphic to M , and let $\varphi_\rho: C \times C \rightarrow M$ be the map defined by the rule

$$\varphi_\rho(a, b) = \rho(a) * \rho(b) - (\text{id} \otimes \text{id} \otimes_H \rho)(a * b),$$

where $a, b \in C$. The proof of Theorem 1 shows that $\varphi_\rho \in Z^2(C, M)$.

Conversely, a cocycle $\varphi \in Z^2(C, M)$ determines an extension (B, σ) denoted by $B = (C; M, \varphi)$, $B = C \oplus M$, with the pseudoproduct

$$(a + x) * (b + y) = a * b + x * b + a * y + \varphi(a, b), \quad a, b \in C, \quad x, y \in M,$$

and $\sigma: B \rightarrow C$ is a projection from B to C parallel to M .

Theorem 2 (see [8]). *A conformal algebra C is segregated in a singular extension (B, σ) if and only if the cocycle φ_ρ is trivial in $H^2(C, \ker \sigma)$.*

Corollary 1 (see [8]). *If $H^2(C, M) = 0$ for any C -bimodule M , then a conformal algebra C is segregated in any extension with nilpotent kernel.*

§3. THE SECOND COHOMOLOGY GROUP OF THE CONFORMAL ALGEBRAS Cur_n AND Cend_n

Definition 7 (see [15]). Let C be a conformal algebra. An element $e \in C$ is called an *idempotent* of C if it satisfies the following conditions:

$$e \circ_0 e = e, \quad e \circ_n e = 0 \quad \text{for all } n \geq 1.$$

An idempotent $e \in C$ is called a (*conformal*) *unit* if $e \circ_0 a = a$ for all $a \in C$.

The proof of the following statements is similar to that of Lemmas 2 and 3 in [8].

Lemma 1. *Let C be a conformal algebra that is a projective H -module, let M be a C -bimodule, and let $\varphi \in Z^2(C, M)$. If $e' \in C$ is a (*conformal*) unit, then the extension $B = (C; M, \varphi)$ of C contains a (*conformal*) idempotent e such that $\sigma(e) = e'$.*

Lemma 2. *Let C be a conformal algebra that is a projective H -module, let M be a C -bimodule, and let $\varphi \in Z^2(C, M)$. Assume that the extension $B = (C; M, \varphi)$ of C contains a (*conformal*) unit e and that there exists $x' \in C$ such that $x' \circ_0 e' = x'$ and $e' \circ_1 x' = e'$, where $e' = \sigma(e)$. Then there exists $x \in B$ such that $\sigma(x) = x'$, $x \circ_0 e = x$, and $e \circ_1 x = e$.*

Note that the conformal algebras Cend_n and Cur_n contain the canonical (*conformal*) unit $e = \sum_{i=1}^n e_{ii}$, and

$$e_{ii} \circ_m e_{jj} = \{e_{ii} \circ_m e_{jj}\} = \delta_{m,0} \delta_{ij} e_{ii}.$$

Lemma 3. *Let C be an associative conformal algebra with unit e' , let M be a C -bimodule, and let $\varphi \in Z^2(C, M)$. Assume that $e'_1, \dots, e'_n$ is a family of pairwise orthogonal idempotents (i.e., $e'_i \circ_m e'_j = \delta_{m,0} \delta_{ij} e'_j$) in C satisfying $\{e'_i \circ_0 e'\} = e'_i$. Then there exist pairwise orthogonal idempotents $e_1, \dots, e_n$ in $B = (C; M, \varphi)$ such that $\sigma(e_i) = e'_i$.*

Proof. If e' is a unit of C , then, by Lemma 1, there exists an idempotent $e \in B$ such that $\sigma(e) = e'$.

Consider the subalgebra $B_e = e \circ_0 \{B \circ_0 e\} \subseteq B$. Note that e is a unit of B_e . Moreover, $C \subseteq B_e/M \cap B_e$, and $B_e/M \cap B_e$ contains pairwise mutually orthogonal idempotents

$e'_1, \dots, e'_n \in C$. We want to find idempotents $e_1, \dots, e_n$ in B_e such that $\{e_i \circ_0 e\} = e_i$ for all $i = 1, \dots, n$.

We use induction on n . Suppose that the claim is true for $n \geq 1$. Let $e'_1, \dots, e'_n, e'_{n+1}$ be a family of pairwise orthogonal idempotents of C such that $\{e'_i \circ_0 e'\} = e'_i$. Then, by the inductive hypothesis, there exist $e_1, \dots, e_n \in B$ satisfying the same conditions. Consider $f = e - e_1 - \dots - e_n \in B$ and the subalgebra $B_0 = f \circ_0 \{B \circ_0 f\}$, with the ideal $M_0 = M \cap B_0$, $M_0 \circ_\omega M_0 = 0$. Note that $\{f \circ_0 e\} = \{(e - e_1 - \dots - e_n) \circ_0 e\} = \{e \circ_0 e\} - \{e_1 \circ_0 e\} - \dots - \{e_n \circ_0 e\} = e \circ_0 e - e_1 - \dots - e_n = e - e_1 - \dots - e_n = f$, and for any $i \in \{1, \dots, n\}$ we have $f \circ_0 e_i = (e - e_1 - \dots - e_n) \circ_0 e_i = e \circ_0 e_i - e_i \circ_i e_i = e_i - e_i = 0$.

Since $e'_{n+1} \in B_0/M_0 \subseteq B/M$, by Lemma 1 there exists $e_{n+1} \in B_0$ such that $e_{n+1} \circ_0 e_{n+1} = e_{n+1}$. If we represent e_{n+1} in the form $f \circ_0 \{y \circ_0 f\}$ for some $y \in B$, then $\{e_{n+1} \circ_0 e\} = e_{n+1}$. Indeed, using (1.5) and (1.7), we obtain

$$\begin{aligned} \{e_{n+1} \circ_0 e\} &= \{(f \circ_0 \{y \circ_0 f\}) \circ_0 e\} = f \circ_0 \{\{y \circ_0 f\} \circ_0 e\} = f \circ_0 \{y \circ_0 \{f \circ_0 e\}\} \\ &= f \circ_0 \{y \circ_0 f\} = e_{n+1}. \end{aligned}$$

Also, $e_{n+1} \circ_0 e_i = 0$ for all $i = 1, \dots, n$. Indeed, by (1.5) and (1.8) we have

$$e_{n+1} \circ_0 e_i = (f \circ_0 \{y \circ_0 f\}) \circ_0 e_i = \{(f \circ_0 y) \circ_0 f\} \circ_0 e_i = (f \circ_0 y) \circ_0 (f \circ_0 e_i) = 0.$$

Next,

$$\begin{aligned} (e_i \circ_0 f) \circ_0 a &= (e_i \circ_0 (e - e_1 - \dots - e_n)) \circ_0 a = (e_i \circ_0 e - e_i) \circ_0 a \\ &= (e_i \circ_0 e) \circ_0 a - e_i \circ_0 a = e_i \circ_0 (e \circ_0 a) - e_i \circ_0 a \\ &= e_i \circ_0 a - e_i \circ_0 a = 0 \end{aligned}$$

for any $a \in B$. Therefore, $e_i \circ_0 e_{n+1} = e_i \circ_0 (f \circ_0 \{y \circ_0 f\}) = (e_i \circ_0 f) \circ_0 \{y \circ_0 f\} = 0$.

So, we have found a set of $f_i \in B$, $i = 1, \dots, n$, such that

$$\begin{aligned} f_i \circ_0 f_j &= \delta_{ij} f_j, \\ \{f_i \circ_0 e\} &= f_i, \\ \sigma(f_i) &= e'_i. \end{aligned}$$

Note that for all $i = 1, \dots, n$ the subalgebra $B_i = f_i \circ_0 \{B \circ_0 f_i\}$ contains a preimage of $e'_i \in C$. By Lemma 1, there exists an idempotent $e_i \in B_i$. We express e_i in the form $f_i \circ_0 \{\tilde{e}_i \circ_0 f_i\}$ for some $\tilde{e}_i \in B$, $i = 1, \dots, n$. Then $f_i \circ_0 e_j = f_i \circ_0 f_j \circ_0 \{\tilde{e}_j \circ_0 f_j\} = 0$ for $i \neq j$. Relation (1.8) implies that, for $i \neq j$,

$$\begin{aligned} e_i \circ_m e_j &= (f_i \circ_0 \{\tilde{e}_i \circ_0 f_i\}) \circ_m e_j = f_i \circ_0 (\{\tilde{e}_i \circ_0 f_i\} \circ_m e_j) \\ &= f_i \circ_0 (\tilde{e}_i \circ_m (f_i \circ_0 e_j)) = f_i \circ_0 (\tilde{e}_i \circ_m ((f_i \circ_0 f_j) \circ_0 \{\tilde{e}_j \circ_0 f_j\})) = 0. \quad \square \end{aligned}$$

Lemma 4. Let $C = \text{Cur}_n$ ($n > 1$), let M be a C -bimodule, and let $\varphi \in Z^2(C, M)$. Suppose that e'_{ij} , $i, j = 1, \dots, n$, is a family of (conformal) matrix units of C (i.e., $e'_{ij} \circ_m e'_{kl} = \delta_{m,0} \delta_{jk} e'_{il}$). Then there exists a family of pairwise orthogonal idempotents e_{ij} , $i, j = 1, \dots, n$, in $B = (C; M, \varphi)$ such that $\sigma(e_{ij}) = e'_{ij}$.

Proof. Let e' be a unit of the algebra C . By Lemma 1, there exists an idempotent $e \in B$ such that $\sigma(e) = e'$. Consider the set of $e'_i = e'_{ii} \in C$, $i = 1, \dots, n$, which forms a system of pairwise orthogonal idempotents of C . By Lemma 3, the algebra B contains pairwise orthogonal idempotents $e_1, \dots, e_n$.

Denote $B_e = e \circ_0 \{B \circ_0 e\} \subseteq B$; then e is a unit of B_e . We choose some preimages v_{1j} , $v_{i1} \in B_e$, $i, j = 2, \dots, n$, of the corresponding matrix units. We may assume that $e_1 \circ_0 v_{1j} \circ_0 e_j = v_{1j}$, $e_i \circ_0 v_{i1} \circ_0 e_1 = v_{i1}$ and

$$v_{1i} \circ_0 v_{i1} = e_1 + a_i,$$

where $a_i \in M \cap B_e$, $a_i \circ_m a_i = 0$ for all $m \in \mathbb{Z}_+$. Let $b_i = -a_i$. Then $e_1 \circ_0 b_i \circ_0 e_1 = -e_1 \circ_0 a_i \circ_0 e_1 = -a_i = b_i$. Consider the elements

$$f_{1j} = v_{1j}, \quad f_{i1} = v_{i1} + v_{i1} \circ_0 b_1, \quad i, j = 2, \dots, n.$$

They satisfy the following relations:

$$\begin{aligned} e_1 \circ_m e_1 &= \delta_{m,0} e_1, \\ e_1 \circ_0 f_{1j} &= f_{1j}, \\ f_{i1} \circ_0 e_1 &= f_{i1}, \\ f_{1j} \circ_m f_{i1} &= 0 & (i \neq j, m \geq 0), \\ f_{i1} \circ_0 f_{1i} &= e_1, \\ f_{1j} \circ_m f_{1i} &= f_{i1} \circ_m f_{j1} = 0 & (m \geq 0), \\ f_{1j} \circ_m e_1 &= f_{1j} \circ_m e_j = 0 & (m \geq 1). \end{aligned}$$

We check them:

$$\begin{aligned} e_1 \circ_0 f_{1j} &= e_1 \circ_0 v_{1j} = v_{1j} = f_{1j}; \\ f_{i1} \circ_0 e_1 &= (v_{i1} + v_{i1} \circ_0 b_i) \circ_0 e_1 = v_{i1} \circ_0 e_1 + v_{i1} \circ_0 (e_1 - v_{1i} \circ_0 v_{i1}) \circ_0 e_1 \\ &= v_{i1} + v_{i1} \circ_0 (e_1 - v_{1i} \circ_0 v_{i1}) = v_{i1} + v_{i1} \circ_0 b_i = f_{i1}; \\ f_{1j} \circ_m f_{i1} &= (f_{1j} \circ_0 e_j) \circ_m f_{i1} = f_{1j} \circ_0 (e_j \circ_m f_{i1}) = f_{1j} \circ_0 (e_j \circ_m (e_i \circ_0 f_{i1})) \\ &= f_{1j} \circ_0 \left(\sum_{s \geq 0} \binom{m}{s} (e_j \circ_{m-s} e_i) \circ_s f_{i1} \right) = 0, \quad \text{because } e_i \circ_k e_j = 0, k \in \mathbb{Z}_+; \\ f_{1i} \circ_0 f_{i1} &= v_{1i} \circ_0 (v_{i1} + v_{i1} \circ_0 b_i) = v_{1i} \circ_0 v_{i1} + v_{1i} \circ_0 v_{i1} \circ_0 b_i \\ &= e_1 + a_i + (e_1 + a_i) \circ_0 (-a_i) = e_1 + a_i - e_1 \circ_0 a_i - a_i \circ_0 a_i \\ &= e_1 + a_i - a_i = e_1, \quad \text{because } e_1 \circ_0 a_i = e_1 \circ_0 (v_{1i} \circ_0 v_{i1} - e_1) = a_i; \\ f_{1j} \circ_0 f_{1i} &= v_{1j} \circ_0 v_{1i} = v_{1j} \circ_0 (e_j \circ_0 e_1) \circ_0 v_{1i} = 0; \\ f_{1j} \circ_0 f_{j1} &= (v_{i1} + v_{i1} \circ_0 b_i) \circ_0 (v_{j1} + v_{j1} \circ_0 b_j) \\ &= v_{i1} \circ_0 v_{j1} + v_{i1} \circ_0 v_{j1} \circ_0 b_j + v_{i1} \circ_0 b_i \circ_0 v_{j1} + v_{i1} \circ_0 b_i \circ_0 v_{j1} \circ_0 b_j \\ &= 0 \quad \text{as in the preceding relation;} \\ f_{i1} \circ_m e_1 &= (v_{i1} + v_{i1} \circ_0 b_i) \circ_m e_1 = (v_{i1} + v_{i1} \circ_0 b_i) \circ_0 (e_1 \circ_m e_1) = 0, \quad m \geq 1; \\ f_{1j} \circ_m e_j &= v_{1j} \circ_m e_j = v_{1j} \circ_0 (e_j \circ_m e_j) = 0, \quad m \geq 1. \end{aligned}$$

For any $a \in B_e$, a regular representation $a(n): B_e \rightarrow B_e$, $n \in \mathbb{Z}_+$, of B_e is defined by $a(n)(x) = a \circ_n x$.

For fixed $j \in \{2, \dots, n\}$, consider the element

$$\begin{aligned} e_{1j} &= f_{1j} + (-D)^{(1)}(e_1 \circ_1 f_{1j}) + \dots + (-D)^{(k)}(\overbrace{e_1 \circ_1 (\dots (e_1 \circ_1 f_{1j}) \dots)}^k) + \dots \\ &= f_{1j} + (-D)^{(1)}(e_1 \circ_1 f_{1j}) + \dots + (-D)^{(k)}(e_1 \circ_k f_{1j}) + \dots = \{e_1 \circ_0 f_{1j}\}, \end{aligned}$$

$e_{1j} \in B_e$.

We claim that $e_1 \circ_m e_{1j} = \delta_{m,0} e_{1j}$ and $e_{1j} \circ_m f_{j1} = \delta_{m,0} e_1$. Indeed,

$$\begin{aligned} e_1 \circ_m \{e_1 \circ_0 f_{1j}\} &= \{(e_1 \circ_m e_1) \circ_0 f_{1j}\} = \delta_{m,0} \{e_1 \circ_0 f_{1j}\} = \delta_{m,0} e_{1j}; \\ e_{1j} \circ_m f_{j1} &= \{e_1 \circ_0 f_{1j}\} \circ_m f_{j1} = e \circ_m (f_{1j} \circ_0 f_{j1}) = e_1 \circ_m e_1 = \delta_{m,0} e_1. \end{aligned}$$

It is easily seen that the elements

$$e_{1j}, \quad e_{ij} = f_{i1} \circ_0 e_{1j}, \quad e_{i1} = f_{i1}, \quad i, j = 2, \dots, n,$$

form a system of matrix units of B . □

Theorem 3. *Let $C = \text{Cur}_n$ ($n \geq 1$), and let M be a C -bimodule. Then the second cohomology group $H^2(C, M)$ is trivial.*

Proof. Let $\varphi \in Z^2(C, M)$, and let $B = (C; M, \varphi)$. We denote by $\sigma: B \rightarrow C$ the homomorphism $\sigma: (a + m) \mapsto a$ of conformal algebras. Since C is free, C is a projective H -module. Hence, there exists an H -linear map $\rho: C \rightarrow B$ such that $\sigma\rho = \text{id}$.

Let e' be the canonical unit of the algebra C . Then by Lemma 1 there exists $e \in B$ with $\sigma(e) = e'$. If $n = 1$, then the subalgebra $C' \subset B$ generated by e is isomorphic to C .

Let $n > 1$. By Lemma 4, the algebra B contains a system of pairwise orthogonal (conformal) matrix units, i.e., a set of $e_{ij} \in B$, $i, j = 1, \dots, n$, such that

$$e_{ij} \circ_m e_{kl} = \delta_{m,0} \delta_{jk} e_{il}, \quad m \geq 0.$$

They generate a subalgebra $C' \subseteq B$ isomorphic to C .

We have a homomorphism $\tau: C \rightarrow B$ of conformal algebras,

$$\tau: e'_{ij} \mapsto e_{ij}.$$

Now suppose $\psi: C \rightarrow M$, $\psi(a) = \rho(a) - \tau(a)$; then ψ is an H -linear map. Identifying C with the subalgebra C' , we obtain $\tau(a) = a - \psi(a)$. Then

$$\begin{aligned} (a - \psi(a)) \circ_n (b - \psi(b)) &= a \circ_n b - \psi(a) \circ_n b - a \circ_n \psi(b) + \varphi_n(a, b) \\ &= \tau(a) \circ_n \tau(b) = \tau(a \circ_n b) = a \circ_n b - \psi(a \circ_n b) \end{aligned}$$

and $\varphi_n(a, b) = a \circ_n \psi(b) - \psi(a \circ_n b) + \psi(a) \circ_n b$. Therefore, $\varphi \in B^2(C, M)$. $\square$

Theorem 4. *Let $C = \text{Cend}_n$ ($n \geq 1$), and let M be a C -bimodule. Then the second cohomology group $H^2(C, M)$ is trivial.*

Proof. Let $\varphi \in Z^2(C, M)$, and let $B = (C; M, \varphi)$. Let $\sigma: B \rightarrow C$ be the homomorphism $\sigma: (a + m) \mapsto a$ of conformal algebras.

The fact that B contains a subalgebra $S \cong \text{Cur}_n \subseteq \text{Cend}_n$ follows from the proof of Theorem 3. Let $e_{11} \in S$ be a preimage of the corresponding matrix unit $E_{11} \in \text{Cend}_n$. The subalgebra $B_1 = e_{11} \circ_0 \{B \circ_0 e_{11}\}$ is unital and contains a preimage $x'_1 \in B/M$ of $vE_{11} \in \text{Cend}_n$. By Lemma 2, there exists a preimage $x_1 \in B_1$ such that $e_{11} \circ_1 x_1 = e_{11}$, $x_1 \circ_0 e_{11} = x_1$. We can find preimages e_{1i} , e_{i1} of the corresponding matrix units of Cend_n , as in the proof of Lemma 3. Denote

$$x = \sum_{i=1}^n e_{i1} \circ_0 x_1 \circ_0 e_{1i} \in B.$$

This is a preimage of $vE \in M_n(\mathbb{k}[D, v]) \cong \text{Cend}_n$, and it satisfies relations (1.12)–(1.15).

The subalgebra $C' \subseteq B$ generated by the e_{ij} , $i, j = 1, \dots, n$, and x is isomorphic to C .

Arguing as in the proof of Theorem 3, we obtain $\varphi \in B^2(C, M)$. $\square$

Corollary 2. *The conformal algebra Cend_n (Cur_n) is segregated in any extension with nilpotent kernel.*

Corollary 3. *Let C be an associative conformal algebra containing a nilpotent ideal M such that $C/M \cong C_1 \oplus \dots \oplus C_m$, where $C_i = \text{Cend}_{n_i}$ or $C_i = \text{Cur}_{n_i}$, $n_i \geq 1$, $i = 1, \dots, m$. Then $C \cong \bar{C} \oplus M$, where $\bar{C}$ is a subalgebra of C isomorphic to C/M .*

Proof. Observe that the algebra C_i ($i = 1, \dots, m$) contains a canonical unit e'_i . Then $e'_1, e'_2, \dots, e'_m$ are pairwise orthogonal idempotents of C/M , and $e' = \sum_{i=1}^m e'_i$ is a (conformal) unit of C/M such that $\{e'_i \circ_0 e'\} = \{e'_i \circ_0 e'_i\} = e'_i \circ_0 e'_i = e'_i$. By Lemma 3, the algebra C contains a system of pairwise orthogonal idempotents $e_1, \dots, e_m$.

We fix $i \in \{1, \dots, m\}$. We have a projection $C \rightarrow \bigoplus_{i=1}^m C_i \rightarrow C_i$. Consider the subalgebra $e_i \circ_0 \{C \circ_0 e_i\} \subset C$. Since C_i is a homomorphic image of $e_i \circ_0 \{C \circ_0 e_i\}$,

Theorems 3 and 4 imply that there exists a subalgebra $\bar{C}_i \subseteq e_i \circ_0 \{C \circ_0 e_i\}$ such that $\bar{C}_i \cong C_i$.

Let $\bar{C} = \bar{C}_1 + \cdots + \bar{C}_m$. Since $e_1, \dots, e_m$ are orthogonal idempotents, we have $\bar{C}_i \circ_\omega \bar{C}_j = 0$, $i \neq j$. Hence, $\bar{C} = \bar{C}_1 \oplus \cdots \oplus \bar{C}_m \subseteq C$ and $\bar{C} \cong C/M$. Therefore, $C \cong \bar{C} \oplus M$. $\square$

In conclusion, we apply the results obtained to the structure theory of conformal algebras.

Let $\mathbb{k}$ be an algebraically closed field, and let V be a left H -module with $\text{rank } V < \infty$. If $C \subseteq \text{Cend } V$, then C contains a maximal nilpotent ideal M ; see [12]. Then $C/M \subseteq \text{Cend}_N$ is semisimple; see [2]. By the results of [12], $C/M \cong C_1 \oplus \cdots \oplus C_m$, where $C_i \cong \text{Cur}_{n_i}$ or $C_i \cong \text{Cend}_{n_i, Q_i}$, $\det Q_i \neq 0$, $i = 1, \dots, m$. Obviously, C/M contains a unit if and only if $Q_i = E_{n_i}$ for all i , where E_{n_i} is a unit matrix of order n_i . Hence, by Corollary 3, we obtain the main result of [2].

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Received 5/FEB/2008

Translated by THE AUTHOR