HOCHSCHILD COHOMOLOGY OF ALGEBRAS OF SEMIDIHEDRAL TYPE. I. GROUP ALGEBRAS OF SEMIDIHEDRAL GROUPS

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ABSTRACT. For a family of local algebras of semidihedral type over an algebraically closed field of characteristic 2, the Hochschild cohomology algebra is described in terms of generators and relations. The calculations are based on the construction of a bimodule resolution for the algebras in question. As a consequence, the Hochschild cohomology algebra is described for the group algebras of semidihedral groups.

INTRODUCTION

In recent years, interest has grown considerably in the investigation of the Hochschild cohomology algebra, and appreciable success has been made for the case of finite-dimensional algebras over fields. In [1], a description was obtained of the Hochschild cohomology algebra for the symmetric group S_3 over the field $\mathbb{F}_3$, and also for the alternating group A_4 and the dihedral 2-groups over the field $\mathbb{F}_2$. In [2], the Hochschild cohomology algebra $\mathrm{HH}^*(R)$ was described in the case where R is a self-injective Nakayama algebra. In [3], a description of the Hochschild cohomology algebra was given for algebras of dihedral type in the family $D(3\mathcal{K})$ over an algebraically closed ground field of characteristic two. In that paper, unlike [1], we applied a direct method of constructing a bimodule resolution for the corresponding algebras, with subsequent use of this resolution in the calculation of Hochschild cohomology groups and multiplication in the cohomology algebra. We recall that algebras of dihedral, semidihedral, and quaternion types appeared in the work of K. Erdmann on the classification of group blocks of tame representation type (see [4]).

Later, the approach of [3] was applied to several families of algebras of quaternion type. Namely, in [5], the Hochschild cohomology algebras were calculated for a family of local algebras of quaternion type, and in [6, 7], the algebra $\mathrm{HH}^*(R)$ was described for the family $Q(2\mathcal{B})_1$ over an algebraically closed field of characteristic 2. We notice that bimodule resolutions for algebras of quaternion type were constructed also in [8]. Moreover, the approach of [3] was used for the calculation of the algebra $\mathrm{HH}^*(R)$ for Liu-Schulz algebras (see [9, 10]) and for the integer group ring $\mathbb{Z}[D_{4m}]$ of the “even” dihedral group (i.e., of order $4m$) [11]. Note that an independent description of the algebra $\mathrm{HH}^*(\mathbb{Z}[D_{2m}])$ for all dihedral groups was obtained in [12], where the methods of the paper [1] were used.

Some particular results were obtained in the description of the algebra $\mathrm{HH}^*(R)$ for the so-called Möbius algebra (see [13, 15]), for a family of self-injective algebras of finite representation type with tree class D_n (see [16]), and for group blocks of tame representation type that have one or three simple modules (see [17]).

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In the present paper, we use the technique of [3] to compute the Hochschild cohomology algebra for a family of local algebras of semidihedral type over an algebraically closed field of characteristic two. Since this family contains the group algebras of semidihedral groups (see [4]), this result can be employed for the description of the algebra $\mathrm{HH}^*(R)$ for such group algebras.

We outline the structure of the paper. In §1, we formulate the main result of the paper, Theorem 1.1. In it, for the family of local algebras of semidihedral type mentioned above, the algebra $\mathrm{HH}^*(R)$ is determined in terms of generators and relations. In §2, we build some bicomplex and use the spectral sequence of it to prove that the total complex of that bicomplex is the minimal projective resolution of the algebra R viewed as a module over its enveloping algebra $\Lambda = R^e = R \otimes_K R$. With the help of this resolution, in §3 we compute the groups $\mathrm{HH}^n(R)$. Finally, in §4 we pick a set of (homogeneous) generators for the algebra $\mathrm{HH}^*(R)$ and find relations satisfied by these generators.

It should be noted that the bimodule resolution mentioned above is constructed for algebras under consideration over the ground field of arbitrary characteristic.

§1. STATEMENT OF THE MAIN RESULT

Let K be an algebraically closed field with arbitrary characteristic, R a finite-dimensional K -algebra, $\Lambda = R^e = R \otimes_K R^{\mathrm{op}}$ its enveloping algebra, and $\mathrm{HH}^n(R) = \mathrm{Ext}_{\Lambda}^n(R, R)$ the n th Hochschild cohomology group of R (with coefficients in the R -bimodule R). The *Hochschild cohomology algebra* is the vector space

$$\mathrm{HH}^*(R) = \bigoplus_{n \geq 0} \mathrm{HH}^n(R) = \bigoplus_{n \geq 0} \mathrm{Ext}_{\Lambda}^n(R, R)$$

endowed with the $\smile$ -product (see [18, §5], [19, Chapter XI], and [20]). The algebra $\mathrm{HH}^*(R)$ is a graded commutative algebra [20]; moreover, the $\smile$ -product coincides with the Yoneda product on the Ext-algebra $\bigoplus_{n \geq 0} \mathrm{Ext}_{\Lambda}^n(R, R)$ of the Λ -module R ; see [21, p. 120].

For any $k \in \mathbb{N}$, $k \geq 2$, we introduce the K -algebra $R_k = K\langle X, Y \rangle / I$, where I is an ideal of the free algebra $K\langle X, Y \rangle$; I is generated by the elements

$$X^2 - Y(XY)^{k-1}, Y^2, (XY)^k - (YX)^k, X(YX)^k.$$

The images of X and Y under the canonical homomorphism from $K\langle X, Y \rangle$ into R_k will be denoted by x and y , respectively. The algebra R_k is a symmetric local algebra of tame representation type [4, III.1.2]; moreover, the algebra R_k is an algebra of semidihedral type in terms of [4, Chapter VIII]. If G is the semidihedral group of order 2^n ($n \geq 4$) and $\mathrm{char} K = 2$, then the group algebra KG is isomorphic to R_k with $k = 2^{n-2}$.

Below in this section, we assume that the ground field has characteristic two.

To describe the Hochschild cohomology algebra $\mathrm{HH}^*(R_k)$ for the algebras R_k ($k \geq 2$), we construct several graded algebras. Put

$$(1.1) \quad \mathcal{X}_1 = \{p_1, p_2, p_3, p_4, u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, t\}.$$

We introduce a grading on the algebra $K[\mathcal{X}_1]$ so that

$$(1.2) \quad \begin{aligned} \deg p_1 = \deg p_2 = \deg p_3 = \deg p_4 = 0, & \quad \deg u_1 = \deg u_2 = \deg u_3 = 1, \\ \deg v_1 = \deg v_2 = \deg v_3 = 2, & \quad \deg w_1 = \deg w_2 = 3, \quad \deg t = 4. \end{aligned}$$

Then we define a graded K -algebra $\mathcal{A}_1 = K[\mathcal{X}_1] / I_1$, where the ideal I_1 is generated by the following elements:

— of degree 0:

$$(1.3) \quad \begin{aligned} & p_1^k, p_2^2, p_3^2, p_4^2, \\ & p_i p_j \quad \text{for } 1 \leq i < j \leq 4; \end{aligned}$$

— of degree 1:

$$(1.4) \quad p_1u_2, p_1u_1 + p_3u_2, p_3u_1 + p_2u_2,$$

$$(1.5) \quad p_3u_3, p_4u_3, p_4u_1 + p_2u_3, p_2u_2 + p_1^{k-1}u_3;$$

— of degree 2:

$$(1.6) \quad p_3v_1, p_4v_1, p_1^{k-1}v_2, p_3v_2, p_4v_2, p_1v_1 + p_2v_2,$$

$$(1.7) \quad p_2v_1 + p_4u_1^2,$$

$$(1.8) \quad p_1v_3, p_2v_3, p_4v_3,$$

$$(1.9) \quad u_1u_2, u_2u_3, u_3^2, p_2v_2 + p_3v_3, p_2v_2 + p_4u_2^2;$$

— of degree 3:

$$(1.10) \quad p_3w_2, p_4w_2,$$

$$(1.11) \quad u_2^3, u_2v_2, p_1w_1 + u_1v_2, p_3w_1 + u_2v_1, p_4w_1 + p_2w_2,$$

$$(1.12) \quad u_3v_3, p_3w_1 + u_1v_3, u_1v_3 + p_1^{k-2}u_3v_2, p_1w_1 + u_2v_3,$$

$$(1.13) \quad p_4w_1 + u_3v_1, p_1w_2 + u_3v_2, u_1^2u_3 + u_1v_1 + p_2w_1;$$

— of degree 4:

$$(1.14) \quad v_1^2, v_1v_2, v_2^2 + p_1^2t,$$

$$(1.15) \quad v_1v_3, v_2v_3, v_3^2, \\ u_2w_1, u_2w_2, u_3w_2, u_1w_2 + u_3w_1;$$

— of degree 5:

$$(1.16) \quad v_2w_1 + p_3u_2t,$$

$$(1.17) \quad v_3w_2, v_2w_2 + p_1u_3t, v_1w_2 + p_4u_1t, v_3w_1 + p_3u_1t,$$

$$(1.18) \quad u_1u_3w_1 + v_1w_1 + p_2u_1t + \delta_{k,2}p_4u_2t;$$

— of degree 6:

$$(1.19) \quad w_2^2, w_1^2 + u_1^2t,$$

$$(1.20) \quad w_1w_2 + u_1u_3t;$$

here, $\delta_{i,j}$ is the Kronecker delta. Moreover, we induce a grading on the algebra $\mathcal{A}_1$ from that on $K[\mathcal{X}_1]$.

Next, we consider the set

$$(1.21) \quad \mathcal{X}_2 = \left(\mathcal{X}_1 \setminus \{u_3, v_3\} \right) \cup \{u'_3\}$$

with $\mathcal{X}_1$ as in (1.1). A grading on the algebra $\mathcal{K}[\mathcal{X}_2]$ is introduced by the requirement that the relations (1.2) be true for the elements of the set $\mathcal{X}_1 \setminus \{u_3, v_3\}$, and $\deg u'_3 = 1$. Then we introduce the algebra $\mathcal{A}_2 = K[\mathcal{X}_2]/I_2$, where the ideal I_2 is generated by the generators of the ideal I_1 in (1.3), (1.4), (1.6), (1.7), (1.10), (1.11), (1.14), (1.16), (1.19),

and by the elements

$$(1.22) \quad p_2 u_2 + p_1^{k-1} u'_3, \quad p_4 u_2 + p_3 u'_3, \quad p_2 u'_3, \quad p_4 u'_3;$$

$$(1.23) \quad p_2 v_2 + p_3 u_2 u'_3, \quad u_1 u_2 + p_1^{k-2} v_2, \quad (u'_3)^2,$$

$$(1.24) \quad p_1 w_1 + u_2^2 u'_3, \quad p_3 w_1 + p_1^{k-1} w_2, \quad p_4 w_1 + u'_3 v_1, \\ p_1 w_2 + u'_3 v_2, \quad p_2 w_1 + u_1^2 u'_3;$$

$$(1.25) \quad u_2 w_1 + p_1^{k-1} t, \quad u_2 w_2 + p_3 t, \quad u'_3 w_1 + p_2 t, \quad u'_3 w_2 + p_4 t,$$

$$(1.26) \quad v_1 w_2, \quad v_2 w_2 + p_1 u'_3 t, \quad u_1^2 w_2 + v_1 w_1,$$

$$(1.27) \quad w_1 w_2 + v_1 t.$$

Since the ideal I_2 is homogeneous, the grading on the algebra $K[\mathcal{X}_2]$ induces a grading on $\mathcal{A}_2$.

Theorem 1.1. *Let $R = R_k$ with $k \in \mathbb{N} \setminus \{1\}$.*

1) *If k is even, then the Hochschild cohomology algebra $\mathrm{HH}^*(R)$ is isomorphic, as a graded algebra, to $\mathcal{A}_1$.*

2) *If k is odd, then $\mathrm{HH}^*(R) \simeq \mathcal{A}_2$ as graded K -algebras.*

This description of the Hochschild cohomology algebra immediately implies the following statement.

Corollary 1.2. *Let G be the semidihedral group of order 2^n ($n \geq 4$), and let K be an algebraically closed field of characteristic 2. Then, with $k = 2^{n-2}$, we have $\mathrm{HH}^*(K[G]) \simeq \mathcal{A}_1$.*

In the proof of Theorem 1.1 we compute also the dimensions of the groups $\mathrm{HH}^n(R)$. Since these results are of independent interest, we collect them in the following separate statement.

Proposition 1.3. *Let $R = R_k$. Then we have*

$$\dim_K \mathrm{HH}^n(R) = \begin{cases} k + 3 + 8 \left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \equiv 0 \pmod{4}, \\ k + 6 + 8 \left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \equiv 1 \pmod{4}, \\ k + 7 + 8 \left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \equiv 2 \pmod{4}, \\ k + 8 + 8 \left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Remark 1.4. Proposition 1.3 extends the corresponding result on the group algebras of the semidihedral groups, obtained in [17], to the entire family of algebras R_k .

§2. RESOLUTION

Let $R = R_k$ be the algebra defined in §1. The algebra R admits the following set as a K -basis:

$$\mathcal{B}_R = \{(xy)^i \mid 0 \leq i \leq k\} \cup \{(yx)^i \mid 1 \leq i \leq k-1\} \\ \cup \{x(yx)^i \mid 0 \leq i \leq k-1\} \cup \{y(xy)^i \mid 0 \leq i \leq k-1\},$$

which consists of all nonzero paths of the quiver of R (this quiver contains one vertex and two loops x and y). The set $\mathcal{B}_R$ is called the *standard* basis of R . In its turn, the enveloping algebra Λ of R has a K -basis consisting of elements of the form

$$(2.1) \quad u \otimes v \quad \text{with } u, v \in \mathcal{B}_R.$$

This basis of Λ will be denoted by $\mathcal{B}_\Lambda$; we also call it the *standard* basis.

We introduce the following grading on Λ : for an element of the form (2.1), we define its *degree* as the sum of the lengths of the paths u and v , and then we put $\Lambda = \sum_{n \geq 0} \Lambda_n$, where Λ_n is the K -vector space spanned by the elements (2.1) with degree n .

Multiplication from the right by an element $\lambda \in \Lambda$ induces an endomorphism λ^* of the left Λ -module Λ . In the sequel, this homomorphism will often be denoted simply by λ .

We consider the following elements of the algebra $\Lambda = R^e$:

$$(2.2) \quad Y^- = y \otimes 1 - 1 \otimes y, \quad Y^+ = y \otimes 1 + 1 \otimes y,$$

$$(2.3) \quad X^- = x \otimes 1 - 1 \otimes x,$$

$$(2.4) \quad \varphi_1 = \sum_{i=0}^{k-1} (yx)^i \otimes (xy)^{k-1-i},$$

$$(2.5) \quad \varphi_2 = -x \otimes 1 - 1 \otimes x + \sum_{i=0}^{k-2} y(xy)^i \otimes y(xy)^{k-2-i},$$

$$(2.6) \quad \rho^- = x(yx)^{k-1} \otimes 1 - 1 \otimes x(yx)^{k-1},$$

$$(2.7) \quad \rho^+ = x(yx)^{k-1} \otimes 1 + 1 \otimes x(yx)^{k-1},$$

$$(2.8) \quad \psi = (yx)^{k-1} \otimes (xy)^{k-1},$$

$$(2.9) \quad \tau = Y^+ \cdot X^-, \quad \sigma = X^+ \cdot \varphi_1.$$

Furthermore, we define the homomorphisms

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} : \Lambda \rightarrow \Lambda^2, \quad \Psi = \begin{pmatrix} \rho^+ & \psi \end{pmatrix} : \Lambda^2 \rightarrow \Lambda,$$

$$\Upsilon = \begin{pmatrix} Y^+ \\ 0 \end{pmatrix} : \Lambda \rightarrow \Lambda^2.$$

In the category of (left) Λ -modules we construct the following bicomplex $\mathcal{B}_{\bullet\bullet}$ lying in the first quadrant of the plane (i.e., its rows and columns are numbered by $0, 1, 2, \dots$):

$$(2.10) \quad \begin{array}{ccccccc} & & & \dots & & \dots & \\ & & & \sigma \downarrow & & \Psi \downarrow & \\ & & \Lambda & \xleftarrow{-\tau} & \Lambda & \xleftarrow{-Y^-} & \Lambda & \xleftarrow{-Y^+} & \dots \\ & & \Phi \downarrow & & \rho^- \downarrow & & \rho^- \downarrow & & \\ & \Lambda & \xleftarrow{(Y^-, X^-)} & \Lambda^2 & \xleftarrow{\Upsilon} & \Lambda & \xleftarrow{Y^-} & \Lambda & \xleftarrow{-Y^+} & \dots \\ & \sigma \downarrow & & \Psi \downarrow & & \rho^+ \downarrow & & \rho^+ \downarrow & \\ & \Lambda & \xleftarrow{-\tau} & \Lambda & \xleftarrow{-Y^-} & \Lambda & \xleftarrow{-Y^+} & \Lambda & \xleftarrow{-Y^-} & \Lambda & \xleftarrow{-Y^+} & \dots \\ & \downarrow \Phi & & \rho^- \downarrow & & \rho^- \downarrow & & \rho^- \downarrow & & \rho^- \downarrow & \\ \Lambda & \xleftarrow{(Y^-, X^-)} & \Lambda^2 & \xleftarrow{\Upsilon} & \Lambda & \xleftarrow{Y^-} & \Lambda & \xleftarrow{Y^+} & \Lambda & \xleftarrow{Y^-} & \Lambda & \xleftarrow{Y^+} & \dots \end{array}$$

Theorem 2.1. *The total complex $Q_\bullet = \text{Tot}(\mathcal{B}_{\bullet\bullet})$ of the bicomplex (2.10) is a minimal projective resolution of the Λ -module R .*

Remark 2.2. Direct inspection shows that, indeed, the diagram (2.10) represents a bicomplex; i.e., its rows and columns are differential sequences and all squares are anti-commutative.

Remark 2.3. For the role of an augmentation map $\mu: Q_0 = \Lambda \rightarrow R$, we take the canonical map induced by multiplication in R : $\mu(r \otimes s) = rs$. Then the exactness of the corresponding sequence at the member Q_0 , that is, the relation $\text{Im } d_0^Q = \text{Ker } \mu$, is a well-known fact.

We shall use several auxiliary statements in the proof of Theorem 2.1.

Lemma 2.4. *Let $Y^\pm: \Lambda \rightarrow \Lambda$ be a homomorphism of multiplication (from the right) by the corresponding element (see (2.2)). Then*

$$\text{Ker } Y^- = \text{Im } Y^+, \quad \text{Ker } Y^+ = \text{Im } Y^-.$$

Proof. We only prove the first relation. The second can be proved similarly. It is clear that $\text{Im } Y^+ \subset \text{Ker } Y^-$.

Let $\lambda \in \text{Ker } Y^-$, and let

$$\lambda = \sum_{u,v \in \mathcal{B}_R} \lambda(u,v) u \otimes v$$

be the expansion of this element with respect to the standard basis of the algebra Λ . Subtracting the element

$$\left(\sum_{v \in \mathcal{B}_R} \left(\sum_{i=0}^{k-1} \lambda(y(xy)^i, v) (yx)^i \otimes v + \sum_{i=1}^k \lambda((xy)^i, v) x(yx)^{i-1} \otimes v \right) \right) \cdot Y^+ \in \text{Ker } Y^-,$$

from λ , we obtain an element whose expansion does not involve summands that are multiples of a basic element $u \otimes v$ such that u is a path with the first letter y . Hence, we may assume that the initial element λ has such property. Then the relation $\lambda \cdot y \otimes 1 = \lambda \cdot 1 \otimes y$ implies that

$$\begin{aligned} (2.11) \quad & \sum_{v \in \mathcal{B}_R} \left(\lambda(1, v) y \otimes v + \sum_{i=0}^{k-1} \lambda(x(yx)^i, v) (xy)^{i+1} \otimes v + \sum_{i=1}^{k-1} \lambda((yx)^i, v) y(xy)^i \otimes v \right) \\ &= \sum_{v \in \mathcal{B}_R} \left(\lambda(1, v) 1 \otimes yv + \sum_{i=0}^{k-1} \lambda(x(yx)^i, v) x(yx)^i \otimes yv + \sum_{i=1}^{k-1} \lambda((yx)^i, v) (xy)^i \otimes yv \right). \end{aligned}$$

Here the left-hand side is equal to zero, because the right-hand side in (2.11) involves no basis vectors in $\mathcal{B}_\Lambda$ that are included in the linear combination presented on the left-hand side of (2.11). Consequently, all coefficients $\lambda(u, v)$ on the left in (2.11) are zero, whence $\lambda = 0$. $\square$

The following lemma implies the exactness of the above sequence at the member Q_1 .

Lemma 2.5. $\text{Ker } d_0^Q \subset \text{Im } d_1^Q$.

Proof. The proof repeats word for word the proof of Lemma 2.4 in [5]. Indeed, we note that the matrix of the differential d_1 is congruent to the diagonal matrix $\text{diag}(Y^+, X^+)$ modulo $(\text{Rad } \Lambda)^2$, but the matrix of the differential d_1 in [5, (2.3)] is congruent to the same diagonal matrix (again modulo $(\text{Rad } \Lambda)^2$). $\square$

Lemma 2.6. *If $(\lambda_1, \lambda_2) \in \text{Ker } d_1^Q$, then $\lambda_2 \in \text{Im } \tau^*$.*

Proof. The relation $d_1^Q(\lambda_1, \lambda_2) = 0$ is equivalent to the system

$$(2.12) \quad \begin{aligned} \lambda_1 Y^+ + \lambda_2 \varphi_1 &= 0, \\ \lambda_2 \varphi_2 &= 0. \end{aligned}$$

By Lemma 2.4, the first equation in (2.12) is satisfied by an element λ exactly when we have $\lambda_2 \cdot \Xi = 0$ with

$$\Xi := \varphi_1 \cdot Y^- = \sum_{j=0}^{k-1} (yx)^j y \otimes (xy)^{k-1-j} - \sum_{j=0}^{k-1} (yx)^j \otimes y(xy)^{k-1-j}.$$

Hence, we need to check that if $\lambda \in \Lambda$ satisfies the relations

$$(2.13) \quad \begin{aligned} \lambda \cdot \Xi &= 0, \\ \lambda \cdot \varphi_2 &= 0, \end{aligned}$$

then $\lambda \in \text{Im } \tau^*$. It is useful to observe that the reverse implication is also valid (this is verified by direct calculation): if $\lambda \in \text{Im } \tau^*$, then λ satisfies (2.13).

Assume that λ satisfies (2.13). Let $\lambda^{(t)}$ be the homogeneous component in λ of degree t . A direct calculation gives $\lambda^{(0)} = 0 = \lambda^{(1)}$.

We fix $t \geq 2$ such that $\lambda^{(i)} = 0$ for all $i < t$. We shall modify the element λ successively by adding elements in $\text{Im } \tau^*$ so as to get an element in which all homogeneous components of degrees $i \leq t$ are zero. After finitely many steps, we replace the initial λ by the zero element, whence the desired inclusion will follow.

Step 1. First we consider the case where $t = 2\ell$ with $1 \leq \ell \leq k-1$. We represent the homogeneous component $\lambda^{(t)}$ as a linear combination of the standard basis elements in Λ (with coefficients $\alpha_i, \dots, f_i$ in K):

$$(2.14) \quad \begin{aligned} \lambda^{(t)} &= \sum_{i=0}^{\ell} \alpha_i (xy)^i \otimes (xy)^{\ell-i} + \sum_{i=0}^{\ell-1} \beta_i (xy)^i \otimes (yx)^{\ell-i} \\ &\quad + \sum_{i=1}^{\ell} \gamma_i (yx)^i \otimes (xy)^{\ell-i} + \sum_{i=1}^{\ell-1} \varepsilon_i (yx)^i \otimes (yx)^{\ell-i} \\ &\quad + \sum_{i=0}^{\ell-1} a_i x(yx)^i \otimes x(yx)^{\ell-1-i} + \sum_{i=0}^{\ell-1} b_i x(yx)^i \otimes y(xy)^{\ell-1-i} \\ &\quad + \sum_{i=0}^{\ell-1} c_i y(xy)^i \otimes x(yx)^{\ell-1-i} + \sum_{i=0}^{\ell-1} f_i y(xy)^i \otimes y(xy)^{\ell-1-i}. \end{aligned}$$

Substituting (2.14) in the first relation in (2.13), we obtain the following equations for the scalar coefficients in (2.14):

$$(2.15) \quad \alpha_0 + \sum_{i=1}^{\ell} \gamma_i = 0 = \alpha_0 + \sum_{i=1}^{\ell-1} \gamma_i - c_{\ell-1},$$

$$(2.16) \quad b_0 - \sum_{i=1}^{\ell} \gamma_i = 0,$$

$$(2.17) \quad \sum_{i=0}^{\ell-1} a_i = 0,$$

$$(2.18) \quad \alpha_0 = \beta_0 = \alpha_{\ell} = 0.$$

Similarly, the second relation in (2.13) is equivalent to the following:

$$(2.19) \quad \alpha_i + b_i = 0 \quad \text{for } 0 \leq i \leq \ell - 1,$$

$$(2.20) \quad c_{i-1} + \varepsilon_i = 0 \quad \text{for } 1 \leq i \leq \ell - 1,$$

$$(2.21) \quad \beta_i = f_i = 0 \quad \text{for } 0 \leq i \leq \ell - 1,$$

$$(2.22) \quad \alpha_\ell = \gamma_\ell = c_{\ell-1} = 0.$$

Since $\alpha_0 = 0$, we also have $b_0 = 0$ (see (2.18) and (2.19)). Replacing λ by the element

$$\lambda + \left(\sum_{i=1}^{\ell-1} \alpha_i (xy)^i \otimes (xy)^{\ell-1-i} \right) \cdot \tau,$$

we obtain an element such that in the canonical expansion of it the coefficients of $(xy)^i \otimes (xy)^{\ell-i}$ ($1 \leq i \leq \ell-1$) are zero. Hence, we may assume that the initial element λ has the property that $\alpha_i = 0$ for $1 \leq i \leq \ell-1$. By (2.19), we also have $b_i = 0$ for $1 \leq i \leq \ell-1$.

Similarly, replacing λ by the element

$$\lambda - \left(\sum_{i=1}^{\ell-1} \varepsilon_i y(xy)^{i-1} \otimes x(yx)^{\ell-1-i} \right) \cdot \tau,$$

we may assume that $\varepsilon_i = 0$ for all $1 \leq i \leq \ell-1$. Then, by (2.20), also $c_i = 0$ for $0 \leq i \leq \ell-2$. Next, if there exists an index $i \leq \ell-1$ with $\gamma_i \neq 0$, then we pick the smallest j such that $\gamma_j \neq 0$. Adding the element

$$\gamma_j (yx)^j \otimes (xy)^{\ell-1-j} \cdot \tau$$

to λ , we obtain an element for which the coefficients of $(yx)^i \otimes (xy)^{\ell-i}$ with $i \leq j$ are zero (observe that $j < \ell-1$). Repeating this procedure, we obtain an element for which the coefficients of $(yx)^i \otimes (xy)^{\ell-i}$ with $i \leq \ell-1$ are zero. Hence, we may assume that in the initial λ we have $\gamma_i = 0$ for $1 \leq i \leq \ell-1$. If there exist nonzero coefficients a_i ($0 \leq i \leq \ell-1$), then a similar procedure (invoking elements of the form $x(yx)^{j-1} \otimes x(yx)^{\ell-1-j} \cdot \tau$) results in an element for which the coefficients of $x(yx)^i \otimes x(yx)^{\ell-1-i}$ for $0 \leq i \leq \ell-2$ are zero. Hence, we may assume that, in the initial element λ , we have $a_i = 0$ for $0 \leq i \leq \ell-2$, and, by (2.17), $a_{\ell-1} = 0$. Consequently, we have $\lambda^{(t)} = 0$ in the resulting element λ .

Step 2. Now we put $t = 2\ell + 1$ with $1 \leq \ell < k-1$ and write

$$(2.23) \quad \begin{aligned} \lambda^{(t)} = & \sum_{i=0}^{\ell} \alpha_i (xy)^i \otimes x(xy)^{\ell-i} + \sum_{i=0}^{\ell} \beta_i (xy)^i \otimes y(xy)^{\ell-i} \\ & + \sum_{i=1}^{\ell} \gamma_i (yx)^i \otimes x(yx)^{\ell-i} + \sum_{i=1}^{\ell} \varepsilon_i (yx)^i \otimes y(yx)^{\ell-i} \\ & + \sum_{i=0}^{\ell} a_i x(yx)^i \otimes (xy)^{\ell-i} + \sum_{i=0}^{\ell-1} b_i x(yx)^i \otimes (yx)^{\ell-i} \\ & + \sum_{i=0}^{\ell} c_i y(xy)^i \otimes (xy)^{\ell-i} + \sum_{i=0}^{\ell-1} f_i y(xy)^i \otimes (yx)^{\ell-i}, \end{aligned}$$

where the coefficients $\alpha_i, \dots, f_i$ lie in K . Substitution of (2.23) in the two relations in (2.13) gives the following equations:

$$(2.24) \quad \alpha_0 + \sum_{i=1}^{\ell} \gamma_i = 0,$$

$$(2.25) \quad b_0 - \sum_{i=1}^{\ell} \gamma_i = \sum_{i=0}^{\ell} a_i = 0,$$

$$(2.26) \quad \alpha_i + b_i = 0 \quad \text{for } 0 \leq i \leq \ell - 1,$$

$$(2.27) \quad c_{i-1} + \varepsilon_i = 0 \quad \text{for } 1 \leq i \leq \ell,$$

$$(2.28) \quad \beta_i = 0 \quad \text{for } 0 \leq i \leq \ell,$$

$$(2.29) \quad f_i = 0 \quad \text{for } 0 \leq i \leq \ell - 1,$$

$$(2.30) \quad \alpha_{\ell} + a_{\ell} = c_{\ell} = \alpha_{\ell} = \gamma_{\ell} = a_0 = b_0 = 0.$$

Adding the element

$$\left(\sum_{i=0}^{\ell-1} \alpha_i (xy)^i \otimes x(yx)^{\ell-1-i} \right) \cdot \tau$$

to λ , we obtain an element for which the coefficients of $(xy)^i \otimes x(yx)^{\ell-i}$ with $0 \leq i \leq \ell - 1$ are zero. Hence, we may assume that, in the initial λ , we have $\alpha_i = 0$ for all $0 \leq i \leq \ell - 1$, and then also $b_i = 0$ for $0 \leq i \leq \ell - 1$. Similarly, using elements of the form

$$y(xy)^{i-1} \otimes (xy)^{\ell-i} \cdot \tau, \quad 1 \leq i \leq \ell,$$

we may assume that $\varepsilon_i = 0$ for $1 \leq i \leq \ell$, and then $c_i = 0$ for $0 \leq i \leq \ell - 1$. If there exists an index $i \leq \ell - 1$ with $\gamma_i \neq 0$, we pick the smallest index j such that $\gamma_j \neq 0$ (note that $j < \ell - 1$). Adding the element

$$\gamma_j (yx)^j \otimes x(yx)^{\ell-1-j} \cdot \tau$$

to λ , we obtain an element for which the coefficients of $(yx)^i \otimes x(yx)^{\ell-i}$ with $i \leq j$ are zero. Repeating this process, we obtain an element for which the coefficients of $(yx)^i \otimes x(yx)^{\ell-i}$ with $i \leq \ell - 1$ are zero. Hence, we may assume that in the initial element λ we have $\gamma_i = 0$ for $1 \leq i \leq \ell - 1$. Similarly, using elements of the form

$$x(yx)^j \otimes (xy)^{\ell-1-j} \cdot \tau,$$

we may assume that in the initial λ we have $a_i = 0$ for $0 \leq i \leq \ell$, and then we deduce that $\lambda^{(t)} = 0$.

Step 3. Next, we consider the case where $t = 2k - 1$. Again, the homogeneous component $\lambda^{(2k-1)}$ can be presented in the form (2.23) (by putting $\ell = k - 1$). As at Step 2, we obtain the same relations (2.24)–(2.30) except the first, which must be replaced by the following relation:

$$\alpha_0 + \sum_{i=1}^{\ell} \gamma_i - c_{k-1} = 0.$$

But the second equation in (2.13) shows that $c_{k-1} = 0$, so that we can apply the arguments of Step 2, obtaining the desired statement for the case under consideration.

Step 4. Now we put $t = 2k$. We present the homogeneous component $\lambda^{(2k)}$ in the form

$$\begin{aligned}
 \lambda^{(2k)} = & \sum_{i=0}^k \alpha_i (xy)^i \otimes (xy)^{k-i} + \sum_{i=1}^{k-1} \beta_i (xy)^i \otimes (yx)^{k-i} \\
 & + \sum_{i=1}^{k-1} \gamma_i (yx)^i \otimes (xy)^{k-i} + \sum_{i=1}^{k-1} \varepsilon_i (yx)^i \otimes (yx)^{k-i} \\
 & + \sum_{i=0}^{k-1} a_i x(yx)^i \otimes x(yx)^{k-1-i} + \sum_{i=0}^{k-1} b_i x(yx)^i \otimes y(xy)^{k-1-i} \\
 & + \sum_{i=0}^{k-1} c_i y(xy)^i \otimes x(yx)^{k-1-i} + \sum_{i=0}^{k-1} f_i y(xy)^i \otimes y(xy)^{k-1-i}.
 \end{aligned} \tag{2.31}$$

Substituting (2.31) in (2.13) leads to the relations

$$\begin{aligned}
 \alpha_0 + \sum_{i=1}^{k-1} \gamma_i - c_{k-1} &= 0, \quad b_0 - \sum_{i=1}^{k-1} \gamma_i - \alpha_k = 0, \quad \sum_{i=0}^{k-1} a_i = 0, \\
 \alpha_i + b_i &= 0 \quad \text{for } 0 \leq i \leq k-1, \\
 c_{i-1} + \varepsilon_i &= 0 \quad \text{for } 1 \leq i \leq k-1, \\
 \beta_i &= 0 \quad \text{for } 1 \leq i \leq k-1, \\
 f_i &= 0 \quad \text{for } 0 \leq i \leq k-1, \\
 \alpha_k + c_{k-1} &= 0.
 \end{aligned}$$

Applying reductions similar to that at Step 1, we put successively the following restrictions on λ :

- a) $\alpha_i = 0$ for $0 \leq i \leq k-1$ (we get this by using the elements $(xy)^i \otimes (xy)^{k-1-i} \cdot \tau$); then we also have $b_i = 0$ for $0 \leq i \leq k-1$;
 - b) $\alpha_k = 0 = c_{k-1}$ (here, we use the element $(yx)^{k-1} \otimes 1 \cdot \tau$);
 - c) $\varepsilon_i = 0$ for $1 \leq i \leq k-1$ (here, we use the elements $y(xy)^{i-1} \otimes x(yx)^{k-1-i} \cdot \tau$); then we also have $c_i = 0$ for $0 \leq i \leq k-2$;
 - d) $\gamma_i = 0$ for $1 \leq i \leq k-1$ (we use the elements $(yx)^i \otimes (xy)^{k-1-i} \cdot \tau$);
 - e) $a_i = 0$ for $0 \leq i \leq k-1$ (we use the elements $x(yx)^i \otimes x(yx)^{k-2-i} \cdot \tau$).
- Clearly, we obtain $\lambda^{(2k)} = 0$.

Step 5. Put $t = 2\ell + 1$ with $k \leq \ell \leq 2k-1$. Denote $\bar{\ell} = \ell - k$. We write the homogeneous component $\lambda^{(t)}$ in the form

$$\begin{aligned}
 \lambda^{(t)} = & \sum_{i=\bar{\ell}+1}^k \alpha_i (xy)^i \otimes x(yx)^{\ell-i} + \sum_{i=\bar{\ell}+1}^k \beta_i (xy)^i \otimes y(xy)^{\ell-i} \\
 & + \sum_{i=\bar{\ell}+1}^{k-1} \gamma_i (yx)^i \otimes x(yx)^{\ell-i} + \sum_{i=\bar{\ell}+1}^{k-1} \varepsilon_i (yx)^i \otimes y(xy)^{\ell-i} \\
 & + \sum_{i=\bar{\ell}}^{k-1} a_i x(yx)^i \otimes (xy)^{\ell-i} + \sum_{i=\bar{\ell}+1}^{k-1} b_i x(yx)^i \otimes (yx)^{\ell-i} \\
 & + \sum_{i=\bar{\ell}}^{k-1} c_i y(xy)^i \otimes (xy)^{\ell-i} + \sum_{i=\bar{\ell}+1}^{k-1} f_i y(xy)^i \otimes (yx)^{\ell-i}.
 \end{aligned} \tag{2.32}$$

If $\ell > k$, then $\lambda^{(t)} \cdot \Xi = 0$; hence, we only need to analyze the second equation in (2.13). Substitution of (2.32) in (2.13) gives the following relations:

$$(2.33) \quad \alpha_i + b_i = 0 \quad \text{for } \bar{\ell} + 1 \leq i \leq k-1,$$

$$(2.34) \quad c_{i-1} + \varepsilon_i = 0 \quad \text{for } \bar{\ell} + 1 \leq i \leq k-1,$$

$$(2.35) \quad \beta_i = 0 \quad \text{for } \bar{\ell} + 1 \leq i \leq k-1,$$

$$(2.36) \quad f_i = 0 \quad \text{for } \bar{\ell} + 1 \leq i \leq k-1,$$

$$(2.37) \quad \beta_k + c_{k-1} = 0.$$

(Observe that for $\ell = 2k-1$ this list of relations reduces to relation (2.37) only.) If $\ell = k$, then we join one additional relation to the above relations, namely

$$\alpha_k + \sum_{i=1}^{k-1} \gamma_i + \sum_{i=0}^{k-1} a_i = 0.$$

Now we can argue as before to prove the desired statement.

Step 6. Put $t = 2\ell$ with $k < \ell \leq 2k$ and denote $\bar{\ell} = \ell - k$. Then the homogeneous component $\lambda^{(t)}$ takes the form

$$(2.38) \quad \begin{aligned} \lambda^{(t)} = & \sum_{i=\bar{\ell}}^k \alpha_i (xy)^i \otimes (xy)^{\ell-i} + \sum_{i=\bar{\ell}+1}^k \beta_i (xy)^i \otimes (yx)^{\ell-i} \\ & + \sum_{i=\bar{\ell}}^{k-1} \gamma_i (yx)^i \otimes (xy)^{\ell-i} + \sum_{i=\bar{\ell}+1}^{k-1} \varphi_i (yx)^i \otimes (yx)^{\ell-i} \\ & + \sum_{i=\bar{\ell}}^{k-1} a_i x(yx)^i \otimes x(yx)^{\ell-1-i} + \sum_{i=\bar{\ell}}^{k-1} b_i x(yx)^i \otimes y(xy)^{\ell-1-i} \\ & + \sum_{i=\bar{\ell}}^{k-1} c_i y(xy)^i \otimes x(yx)^{\ell-1-i} + \sum_{i=\bar{\ell}}^{k-1} f_i y(xy)^i \otimes y(xy)^{\ell-1-i}. \end{aligned}$$

Again we have $\lambda^{(t)} \cdot \Xi = 0$; hence, we only need to analyze the second equation in (2.13). Substitution of (2.38) in (2.13) gives the following relations:

$$\begin{aligned} \alpha_i + b_i &= 0 & \text{for } \bar{\ell} \leq i \leq k-1, \\ c_{i-1} + \varepsilon_i &= 0 & \text{for } \bar{\ell} + 1 \leq i \leq k-1, \\ \beta_i &= 0 & \text{for } \bar{\ell} + 1 \leq i \leq k-1, \\ f_i &= 0 & \text{for } \bar{\ell} \leq i \leq k-1, \\ \beta_k + c_{k-1} &= 0. \end{aligned}$$

(If $\ell = 2k$, we have no relations.) Now, arguing as above, we obtain the desired statement with the help of suitable reductions of the element λ . $\square$

Lemma 2.7. $\text{Ker}(\sigma Y^+)^* \subset \text{Im } d_0^Q$.

Proof. Put

$$\begin{aligned} \Theta := \sigma Y^+ = & \sum_{j=0}^{k-1} (xy)^{j+1} \otimes (xy)^{k-1-j} + \sum_{j=0}^{k-1} y(xy)^j \otimes x(yx)^{k-1-j} \\ & + \sum_{j=0}^{k-1} x(yx)^j \otimes y(xy)^{k-1-j} + \sum_{j=0}^{k-1} (yx)^j \otimes (yx)^{k-j}. \end{aligned}$$

Observe that $\text{Im } d_0^Q \subset \text{Ker } \Theta^*$.

Assume that the element $\lambda \in \Lambda$ satisfies

$$(2.39) \quad \lambda \cdot \Theta = 0.$$

As in the proof of Lemma 2.6, we consider a nonzero component $\lambda^{(t)}$ of the element λ with the smallest possible value of t . Clearly, $\lambda^{(0)} = 0$. We shall change the element λ by subsequently adding elements of $\text{Im } d_0^Q = \text{Im}((Y^-)^*, (X^-)^*)$ to λ , in order to obtain an element in which all homogeneous components of degree $i \leq t$ are zero.

Step 1. First, we consider the case where $t = 2\ell + 1$ with $0 \leq \ell \leq k - 1$. We write the homogeneous component $\lambda^{(t)}$ in the form (2.23) (with coefficients $\alpha_i, \dots, f_i$ in K). Substitution of (2.23) in (2.39) gives the following relations:

$$\begin{aligned} \sum_{i=0}^{\ell} \alpha_i + \sum_{i=0}^{\ell-1} b_i + a_{\ell} &= 0, \\ \sum_{i=1}^{\ell} \varepsilon_i + \sum_{i=0}^{\ell} c_i + \beta_0 &= 0. \end{aligned}$$

Next, we put successively additional restrictions on λ .

a) We may assume that $\beta_i = 0$ for $1 \leq i \leq \ell$. For this, we can replace λ by an element of the form

$$\lambda - \left(\sum_{i=1}^{\ell} \beta_i x(yx)^{i-1} \otimes y(xy)^{\ell-i} \right) \cdot Y^-$$

in which the coefficients of $(xy)^i \otimes y(xy)^{\ell-i}$, $i = 1, \dots, \ell$, are zero.

b) We may assume that $\gamma_i = 0$ for $1 \leq i \leq \ell$. For this, we can replace λ by an element of the form

$$\lambda + \left(\sum_{i=1}^{\ell} \gamma_i (yx)^i \otimes (yx)^{\ell-i} \right) \cdot X^-.$$

c) We may assume that $f_i = 0$ for $0 \leq i \leq \ell - 1$. For this, we can replace λ by an element of the form

$$\lambda - \left(\sum_{i=0}^{\ell-1} f_i (yx)^i \otimes (yx)^{\ell-i} \right) \cdot Y^-.$$

d) We may assume that $a_i = 0$ for $0 \leq i \leq \ell - 1$. For this, we can replace λ by an element of the form

$$\lambda + \left(\sum_{i=0}^{\ell-1} a_i x(yx)^i \otimes y(xy)^{\ell-1-i} \right) \cdot X^-.$$

Note that, unlike the previous cases, some components $\lambda^{(j)}$ with $j > t$ can be changed under such a replacement.

e) We may assume that $\alpha_i = 0$ for $0 \leq i \leq \ell$, $b_i = 0$ for $0 \leq i \leq \ell - 1$, and $a_{\ell} = 0$. Indeed, we have the following easily verified relation, in which the right-hand

side contains two sums of the corresponding elements in the expansion (2.23):

$$\begin{aligned}\Omega &:= \sum_{j=0}^{\ell-1} \left(\sum_{i=\ell-j}^{\ell} \alpha_i + \sum_{i=\ell-j}^{\ell-1} b_i \right) x(yx)^{\ell-1-j} \otimes x(yx)^j \cdot Y^- \\ &\quad + \sum_{j=0}^{\ell-1} \left(\sum_{i=\ell-j}^{\ell} \alpha_i + \sum_{i=\ell-1-j}^{\ell-1} b_i \right) (xy)^{\ell-1-j} \otimes (yx)^{j+1} \cdot X^- \\ &= \sum_{j=0}^{\ell} \alpha_j (xy)^j \otimes x(yx)^{\ell-j} + \sum_{j=0}^{\ell-1} b_j x(yx)^j \otimes (yx)^{\ell-j} + a_\ell \cdot 1 \otimes x(yx)^\ell.\end{aligned}$$

Replacing λ by $\lambda - \Omega$, we obtain an element for which the expansion in the standard basis contains the following group of two elements:

$$\omega := a_\ell (x(yx)^\ell \otimes 1 - 1 \otimes x(yx)^\ell).$$

Since $\mu(\omega) = 0$, we have $\omega \in \text{Im } d_0^Q$. Then we subtract ω from λ , obtaining an element for which the coefficient of the basis element $x(yx)^\ell \otimes 1$ is equal to zero.

f) We may assume that $\varepsilon_i = 0$ for $0 \leq i \leq \ell - 1$, $c_i = 0$ for $1 \leq i \leq \ell$, and $\beta_0 = 0$. We consider the following identity, in which the right-hand side is the sum of the corresponding elements in the expansion (2.23):

$$\begin{aligned}\Omega_1 &:= \sum_{j=0}^{\ell} \left(\sum_{i=\ell-j}^{\ell} c_i + \sum_{i=\ell+1-j}^{\ell} \varepsilon_i \right) (yx)^{\ell-j} \otimes (xy)^j \cdot Y^- \\ &\quad + \sum_{j=0}^{\ell-1} \left(\sum_{i=\ell-j}^{\ell} c_i + \sum_{i=\ell-j}^{\ell} \varepsilon_i \right) y(xy)^{\ell-1-j} \otimes y(xy)^j \cdot X^- \\ &= \sum_{j=0}^{\ell} c_j y(xy)^j \otimes (xy)^{\ell-j} + \sum_{j=1}^{\ell} \varepsilon_j (yx)^j \otimes y(xy)^{\ell-j} + \beta_0 \cdot 1 \otimes y(xy)^\ell.\end{aligned}$$

It remains to replace λ by $\lambda - \Omega_1$.

Clearly, if λ satisfies the assumptions of cases a)–f), then $\lambda^{(t)} = 0$.

Step 2. Next, we consider the case where $t = 2\ell$ with $1 \leq \ell \leq k - 1$. We write the homogeneous component $\lambda^{(t)}$ in the form (2.14). Substituting this in (2.39), we obtain the following equations for the coefficients in (2.14):

$$\begin{aligned}\sum_{i=0}^{\ell} \alpha_i + \sum_{i=0}^{\ell-1} b_i &= 0, \\ \sum_{i=1}^{\ell-1} \varepsilon_i + \sum_{i=0}^{\ell-1} c_i + \beta_0 + \gamma_\ell &= 0.\end{aligned}$$

As at Step 1, we successively put additional restrictions on λ .

a) We may assume that $\beta_i = 0$ for $1 \leq i \leq \ell - 1$. For this, we can replace λ by an element of the form

$$\lambda - \left(\sum_{i=1}^{\ell-1} \beta_i (xy)^{i-1} \otimes (yx)^{\ell-i} \right) \cdot Y^-.$$

b) We may assume that $\gamma_i = 0$ for $1 \leq i \leq \ell - 1$. For this, we can replace λ by an element of the form

$$\lambda - \left(\sum_{i=1}^{\ell-1} \gamma_i y(xy)^{i-1} \otimes (xy)^{\ell-i} \right) \cdot X^-.$$

c) We may assume that $f_i = 0$ for $0 \leq i \leq \ell - 1$. For this, we can replace λ by an element of the form

$$\lambda - \left(\sum_{i=0}^{\ell-1} f_i (yx)^i \otimes y(xy)^{\ell-1-i} \right) \cdot Y^-.$$

d) We may assume that $a_i = 0$ for $0 \leq i \leq \ell - 1$. For this, we can replace λ by an element of the form

$$\lambda - \left(\sum_{i=0}^{\ell-1} a_i (xy)^i \otimes x(yx)^{\ell-1-i} \right) \cdot X^-.$$

e) We may assume that $\alpha_i = 0$ for $0 \leq i \leq \ell$ and that $b_i = 0$ for $0 \leq i \leq \ell - 1$. As at Step 1, for this we use the following identity, in which the right-hand side is the sum of the corresponding elements in (2.14):

$$\begin{aligned} \Omega' &:= \sum_{j=0}^{\ell} \left(\sum_{i=\ell-j+1}^{\ell} \alpha_i + \sum_{i=\ell-j+1}^{\ell-1} b_i \right) x(yx)^{\ell-j} \otimes (xy)^{j-1} \cdot Y^- \\ &\quad + \sum_{j=0}^{\ell} \left(\sum_{i=\ell-j+1}^{\ell} \alpha_i + \sum_{i=\ell-j}^{\ell-1} b_i \right) (xy)^{\ell-j} \otimes y(xy)^{j-1} \cdot X^- \\ &= \sum_{j=0}^{\ell} \alpha_j (xy)^j \otimes (xy)^{\ell-j} + \sum_{j=0}^{\ell-1} b_j x(yx)^j \otimes y(xy)^{\ell-j-1}. \end{aligned} \quad (2.40)$$

It remains to replace λ by $\lambda - \Omega'$.

f) Finally, we may assume that $\varepsilon_i = 0$ for $1 \leq i \leq \ell - 1$, $c_i = 0$ for $0 \leq i \leq \ell - 1$, and $\beta_0 = \gamma_\ell = 0$. Again, we use an identity similar to (2.40). To present this identity, it is convenient to introduce the following auxiliary notation:

$$\varepsilon_0 := \beta_0, \quad \varepsilon_\ell := \gamma_\ell.$$

Then we have

$$\begin{aligned} \Omega'_1 &:= \sum_{j=0}^{\ell} \left(\sum_{i=\ell-j}^{\ell-1} c_i + \sum_{i=\ell-j}^{\ell} \varepsilon_i \right) y(xy)^{\ell-1-j} \otimes (yx)^j \cdot X^- \\ &\quad + \sum_{j=0}^{\ell-1} \left(\sum_{i=\ell-1-j}^{\ell-1} c_i + \sum_{i=\ell-j}^{\ell} \varepsilon_i \right) (yx)^{\ell-1-j} \otimes x(yx)^j \cdot Y^- \\ &= \sum_{j=0}^{\ell-1} c_j y(xy)^j \otimes x(yx)^{\ell-1-j} + \sum_{j=0}^{\ell} \varepsilon_j (yx)^j \otimes (yx)^{\ell-j}, \end{aligned} \quad (2.41)$$

and we replace λ by $\lambda - \Omega'_1$.

Step 3. We consider the case where $t = 2k$. We write the homogeneous component $\lambda^{(2k)}$ in the form (2.31). Substituting this in (2.39), we obtain the equation

$$\sum_{i=0}^k \alpha_i + \sum_{i=0}^{k-1} b_i + \sum_{i=1}^{k-1} \varepsilon_i + \sum_{i=0}^{k-1} c_i = 0. \quad (2.42)$$

We repeat the arguments of cases a)–d) at Step 2; this allows us to assume that in the initial λ the coefficients β_i, γ_i, f_i , and a_i are zero for all corresponding values of the

indices. Finally, we consider the following analogs of (2.40) and (2.41):

$$\begin{aligned}
\Omega'' &:= \sum_{j=0}^k \left(\sum_{i=k-j+1}^k \alpha_i + \sum_{i=k-j+1}^{k-1} b_i \right) x(yx)^{k-j} \otimes (xy)^{j-1} \cdot Y^- \\
&\quad + \sum_{j=1}^k \left(\sum_{i=k-j+1}^k \alpha_i + \sum_{i=k-j}^{k-1} b_i \right) (xy)^{k-j} \otimes y(xy)^{j-1} \cdot X^- \\
&= \sum_{j=0}^k \alpha_j (xy)^j \otimes (xy)^{k-j} + \sum_{j=0}^{k-1} b_j x(yx)^j \otimes y(xy)^{k-j-1} \\
&\quad - \left(\sum_{i=0}^k \alpha_i + \sum_{i=0}^{k-1} b_i \right) \cdot 1 \otimes (xy)^k, \\
\Omega_1'' &:= \sum_{j=1}^{k-1} \left(\sum_{i=k-j}^{k-1} c_i + \sum_{i=k-j}^{k-1} \varepsilon_i \right) y(xy)^{k-1-j} \otimes (yx)^j \cdot X^- \\
&\quad + \sum_{j=0}^{k-1} \left(\sum_{i=k-1-j}^{k-1} c_i + \sum_{i=k-j}^{k-1} \varepsilon_i \right) (yx)^{k-1-j} \otimes x(yx)^j \cdot Y^- \\
&= \sum_{j=0}^{k-1} c_j y(xy)^j \otimes x(yx)^{k-1-j} + \sum_{j=1}^{k-1} \varepsilon_j (yx)^j \otimes (yx)^{k-j} \\
&\quad - \left(\sum_{i=0}^{k-1} c_i + \sum_{i=1}^k \varepsilon_i \right) \cdot 1 \otimes (xy)^k.
\end{aligned}$$

Replacing λ by $\lambda - \Omega'' - \Omega_1''$ and using (2.42), we may assume that all coefficients $\alpha_i, b_i, \varepsilon_i, c_i$ are zero for the corresponding values of the indices.

Step 4. Assume that $t > 2k$. Since Θ is a homogeneous element of degree $2k$, we have $\lambda^{(t)} \cdot \Theta = 0$. On the other hand, for any homogeneous element $\zeta \in \Lambda$ of degree greater than $2k$, we have $\mu(\zeta) = 0$, whence, $\lambda^{(t)} \in \text{Im } d_0^Q$. $\square$

Lemma 2.8. $\text{Ker } \tau^* = \text{Im } \sigma^* + \text{Im}(Y^-)^*$.

Proof. It remains to prove the inclusion

$$\text{Ker } \tau^* \subset \text{Im } \sigma^* + \text{Im}(Y^-)^*.$$

Assume that $\lambda \in \text{Ker } \tau^*$. We consider a nonzero homogeneous component $\lambda^{(t)}$ of the element λ with the smallest possible t . Clearly, $\lambda^{(0)} = 0$.

Step 1. Consider the case where $t = 2\ell + 1$ with $0 \leq \ell < k-1$. We write the homogeneous component $\lambda^{(t)}$ in the form (2.23) (with $\alpha_i, \dots, f_i \in K$). We present detailed arguments under the additional assumption that $\ell > 0$. The proof in the case where $\ell = 0$ is obtained by minor modifications.

Substitution of (2.23) in the equation

$$(2.43) \quad \lambda \cdot \tau = 0$$

gives the following relations:

$$(2.44) \quad \alpha_i + b_{i-1} = 0 \quad \text{for } 1 \leq i \leq \ell,$$

$$(2.45) \quad \varepsilon_i + c_i = 0 \quad \text{for } 1 \leq i \leq \ell,$$

$$(2.46) \quad \gamma_{i-1} - \gamma_i = 0 \quad \text{for } 2 \leq i \leq \ell,$$

$$(2.47) \quad a_{i-1} - a_i = 0 \quad \text{for } 1 \leq i \leq \ell,$$

$$(2.48) \quad \alpha_0 - \gamma_1 = \beta_0 + c_0 = 0,$$

$$(2.49) \quad \alpha_0 = a_0 = a_\ell = \gamma_\ell = 0.$$

Relations (2.46) (and (2.49)) imply that

$$\gamma_\ell = \gamma_{\ell-1} = \cdots = \gamma_2 = \gamma_1 = 0.$$

Furthermore, relations (2.47) yield

$$a_\ell = a_{\ell-1} = \cdots = a_1 = a_0 = 0.$$

Now we successively put additional restrictions on λ .

a) We may assume that $\beta_0 = c_0 = 0$. For this, we can replace λ by $\lambda + \beta_0 1 \otimes (xy)^\ell \cdot Y^-$.

b) We may assume that $\alpha_i = 0$ for $1 \leq i \leq \ell$, and $b_i = 0$ for $0 \leq i \leq \ell - 1$. For this, we can replace λ by

$$\lambda - \left(\sum_{i=1}^{\ell} \alpha_i x(yx)^{i-1} \otimes x(yx)^{\ell-i} \right) \cdot Y^-.$$

c) We may assume that $\varepsilon_i = c_i = 0$ for $1 \leq i \leq \ell$. For this, we can replace λ by

$$\lambda + \left(\sum_{i=1}^{\ell} \varepsilon_i (yx)^i \otimes (xy)^{\ell-i} \right) \cdot Y^-.$$

d) We may assume that $\beta_i = 0$ for $1 \leq i \leq \ell$. For this, we can replace λ by

$$\lambda + \left(\sum_{i=1}^{\ell} \beta_i (xy)^i \otimes (xy)^{\ell-i} \right) \cdot Y^-.$$

e) Finally, we may assume that $f_i = 0$ for $0 \leq i \leq \ell - 1$. For this, we can replace λ by

$$\lambda - \left(\sum_{i=0}^{\ell-1} f_i (yx)^i \otimes (yx)^{\ell-i} \right) \cdot Y^-.$$

It is clear that if the initial λ satisfies the additional assumptions of cases a)–e), then $\lambda^{(t)} = 0$.

Step 2. Now we consider the case where $t = 2\ell$ with $1 \leq \ell \leq k - 1$. We write the homogeneous component $\lambda^{(t)}$ in the form (2.14). Then equation (2.43) implies the following relations:

$$\begin{aligned} \alpha_i + b_{i-1} &= 0 & \text{for } 1 \leq i \leq \ell, \\ \varepsilon_i + c_i &= 0 & \text{for } 1 \leq i \leq \ell - 1, \\ \gamma_{i-1} - \gamma_i &= 0 & \text{for } 2 \leq i \leq \ell, \\ a_{i-1} - a_i &= 0 & \text{for } 1 \leq i \leq \ell - 1, \end{aligned}$$

$$\beta_0 + c_0 = 0, \quad \alpha_0 = \gamma_1 = \gamma_\ell = a_0 = a_{\ell-1} = 0.$$

Slightly modifying the arguments used at Step 1, we can replace the initial λ by an element for which all homogeneous components of degree not greater than t are zero. The details of these arguments are left to the reader.

It should be noted that, in the corresponding replacements of elements at Steps 1 and 2, we used only elements in $\text{Im } Y^-$.

Step 3. Put $t = 2k - 1$. We consider the representation of $\lambda^{(2k-1)}$ in the form (2.23) with $\ell = k - 1$. For the coefficients in (2.23), we obtain relations (2.44)–(2.48) (with $\ell := k - 1$), and also the relations

$$\alpha_0 - a_0 = \gamma_{k-1} - a_{k-1} = a_0 - \gamma_{k-1} = 0.$$

Applying the arguments of Step 1, we can additionally assume that in (2.23) the following coefficients are zero: β_0, c_0, α_i for $1 \leq i \leq k - 1, b_i$ for $0 \leq i \leq k - 2, \varepsilon_i$ and c_i for $1 \leq i \leq k - 1$, and β_i for $1 \leq i \leq k - 1$. Furthermore, we have

$$a := \alpha_0 = a_0 = a_1 = \cdots = a_{k-1} = \gamma_1 = \gamma_2 = \cdots = \gamma_{k-1}.$$

It follows that $\lambda^{(2k-1)} = a \cdot \sigma$.

Step 4. Put $t = 2k$. We write $\lambda^{(2k)}$ in the form (2.31). Then equation (2.43) implies that

$$(2.50) \quad \alpha_i + b_{i-1} = 0 \quad \text{for } 1 \leq i \leq k - 1,$$

$$(2.51) \quad \varepsilon_i + c_i = 0 \quad \text{for } 1 \leq i \leq k - 1,$$

$$(2.52) \quad \gamma_{i-1} - \gamma_i = 0 \quad \text{for } 2 \leq i \leq k - 1,$$

$$(2.53) \quad a_{i-1} - a_i = 0 \quad \text{for } 1 \leq i \leq k - 1,$$

$$(2.54) \quad \alpha_0 - \gamma_1 + c_0 = 0, \quad \gamma_{k-1} - \alpha_k - b_{k-1} = 0.$$

As at Step 2, we may additionally assume that the following coefficients are zero: α_i for $1 \leq i \leq k - 1, b_i$ for $0 \leq i \leq k - 2, \varepsilon_i$ and c_i for $1 \leq i \leq k - 1, \beta_i$ for $1 \leq i \leq k - 1$, and f_i for $0 \leq i \leq k - 1$ (the details are left to the reader).

Moreover, we may assume that $\alpha_0 = b_{k-1} = 0$. Indeed, for this, we can replace λ by an element of the form

$$\lambda + \left(\alpha_0 1 \otimes x(yx)^{k-1} + b_{k-1} x(yx)^{k-1} \otimes 1 \right) \cdot Y^-.$$

Finally, relations (2.52), (2.53), and (2.54) imply that

$$\gamma := \gamma_1 = \gamma_2 = \cdots = \gamma_{k-1} = \alpha_k = c_0,$$

$$a := a_0 = a_1 = \cdots = a_{k-1}.$$

It remains to replace λ by the element

$$\lambda - (\gamma y \otimes 1 + ax \otimes 1) \cdot \sigma.$$

We notice that some homogeneous component of λ of degree greater than $2k$ is changed under this replacement.

Step 5. Now we consider the case where $t = 2\ell + 1$ with $k \leq \ell \leq 2k - 1$. Put $\bar{\ell} := \ell - k$. We write $\lambda^{(t)}$ in the form (2.32). We give detailed arguments for $\bar{\ell} < k - 1$. If $\bar{\ell} = k - 1$ (i.e., $\ell = 2k - 1$), then only minor modifications are needed. Substituting (2.32) in (2.43), we obtain

$$(2.55) \quad \alpha_i + b_{i-1} = 0 \quad \text{for } \bar{\ell} + 2 \leq i \leq k - 1,$$

$$(2.56) \quad \varepsilon_i + c_i = 0 \quad \text{for } \bar{\ell} + 1 \leq i \leq k - 1,$$

$$(2.57) \quad \gamma_{i-1} - \gamma_i = 0 \quad \text{for } \bar{\ell} + 2 \leq i \leq k - 1,$$

$$(2.58) \quad a_{i-1} - a_i = 0 \quad \text{for } \bar{\ell} + 2 \leq i \leq k - 1,$$

$$(2.59) \quad \alpha_{\bar{\ell}} + a_{\bar{\ell}} - a_{\bar{\ell}+1} = 0,$$

$$(2.60) \quad \gamma_{k-1} - b_{k-1} - \alpha_k = 0.$$

Again, we shall put additional restrictions on λ .

a) We may assume that $\alpha_i = 0$ for $\bar{\ell} + 2 \leq i \leq k - 1$ and $b_i = 0$ for $\bar{\ell} + 1 \leq i \leq k - 2$. For this, we can replace λ by

$$\lambda - \left(\sum_{i=\bar{\ell}+2}^{k-1} \alpha_i x(yx)^{i-1} \otimes x(yx)^{\ell-i} \right) \cdot Y^-.$$

b) We may assume that $\varepsilon_i = c_i = 0$ for $\bar{\ell} + 1 \leq i \leq k - 1$. For this, we can replace λ by

$$\lambda + \left(\sum_{i=\bar{\ell}+1}^{k-1} \varepsilon_i (yx)^i \otimes (xy)^{\ell-i} \right) \cdot Y^-.$$

c) We may assume that $\beta_i = 0$ for $\bar{\ell} + 1 \leq i \leq k$. For this, we can replace λ by

$$\lambda + \left(\sum_{i=\bar{\ell}+1}^k \beta_i (xy)^i \otimes (xy)^{\ell-i} \right) \cdot Y^-.$$

d) We may assume that $f_i = 0$ for $\bar{\ell} + 1 \leq i \leq k - 1$. For this, we can replace λ by

$$\lambda - \left(\sum_{i=\bar{\ell}+1}^{k-1} f_i (yx)^i \otimes (yx)^{\ell-i} \right) \cdot Y^-.$$

e) Finally, we may assume that $\alpha_{\bar{\ell}+1} = c_{\bar{\ell}} = b_{k-1} = 0$. For this, we can replace λ by

$$\lambda - \left(\alpha_{\bar{\ell}+1} x(yx)^{\bar{\ell}} \otimes x(yx)^{k-1} + c_{\bar{\ell}} (yx)^{\bar{\ell}} \otimes (xy)^k - b_{k-1} x(yx)^{k-1} \otimes x(yx)^{\bar{\ell}} \right) \cdot Y^-.$$

Relations (2.57) and (2.60) imply that

$$\gamma := \gamma_{\bar{\ell}+1} = \gamma_{\bar{\ell}+2} = \cdots = \gamma_{k-2} = \gamma_{k-1} = \alpha_k.$$

Relations (2.58) and (2.59) imply that

$$a := a_{\bar{\ell}} = a_{\bar{\ell}+1} = \cdots = a_{k-2} = a_{k-1}.$$

It remains to replace λ by

$$\lambda - \left(\gamma \cdot (yx)^{\bar{\ell}+1} \otimes 1 + a \cdot 1 \otimes (xy)^{\bar{\ell}+1} \right) \cdot \sigma. \quad \square$$

Step 6. Finally, we consider the case where $t = 2\ell$ with $k < \ell \leq 2k$.

Put $\bar{\ell} := \ell - k$. We write $\lambda^{(t)}$ in the form (2.38). For $\bar{\ell} < k - 1$, equation (2.43) implies the following relations:

$$(2.61) \quad \alpha_i + b_{i-1} = 0 \quad \text{for } \bar{\ell} + 1 \leq i \leq k - 1,$$

$$(2.62) \quad \varepsilon_i + c_i = 0 \quad \text{for } \bar{\ell} + 1 \leq i \leq k - 1,$$

$$(2.63) \quad \gamma_{i-1} - \gamma_i = 0 \quad \text{for } \bar{\ell} + 2 \leq i \leq k - 1,$$

$$(2.64) \quad a_{i-1} - a_i = 0 \quad \text{for } \bar{\ell} + 1 \leq i \leq k - 1,$$

$$(2.65) \quad c_{\ell} + \gamma_{\ell} - \gamma_{\ell+1} = 0,$$

$$(2.66) \quad \gamma_{k-1} - b_{k-1} - \alpha_k = 0.$$

Then, as in the preceding cases, we modify the initial element λ subsequently to obtain a new element in which all homogeneous components with degree not greater than t are zero.

Next, for $\bar{\ell} = k - 1$, equation (2.43) implies the sole relation

$$\alpha_k - \gamma_{k-1} + b_{k-1} - c_{k-1} = 0.$$

As above, we can put successively additional restrictions on λ , which allows us to assume that $\alpha_{k-1} = 0$, $f_{k-1} = 0$, $\beta_k = 0$. Furthermore, replacing λ by

$$\lambda - \left(c_{k-1}(yx)^{k-1} \otimes x(yx)^{k-1} - b_{k-1}x(yx)^{k-1} \otimes (xy)^{k-2} \right) \cdot Y^-,$$

we may also assume that $c_{k-1} = b_{k-1} = 0$. Then $\alpha_k = \gamma_{k-1}$. It remains to replace λ by

$$\lambda - \left(a_{k-1}x(yx)^{k-1} \otimes 1 + \gamma_{k-1}(yx)^{k-1} \otimes y \right) \cdot \sigma.$$

Finally, for $\bar{\ell} = k$ we have $\lambda^{(4k)} \cdot \tau = 0$. It is clear that $\lambda^{(4k)} \in \text{Im } \sigma$.

Completion of the proof of Theorem 2.1. We consider the spectral sequence of the bi-complex $\mathcal{B}_{\bullet\bullet}$ in (2.10),

$$(2.67) \quad H_p^v H_q^h(\mathcal{B}_{\bullet\bullet}) \Rightarrow H_{p+q}(\text{Tot}(\mathcal{B}_{\bullet\bullet})),$$

and prove that the second page of this spectral sequence degenerates.

By Lemma 2.4, every column with number $i > 0$ on the first page of this spectral sequence contains two nonzero terms, namely,

$$\text{Coker } \tau \xrightarrow{\bar{\Phi}} \text{Ker}(Y^-, X^-) / \text{Im } \Upsilon$$

if i is odd, and

$$\text{Coker}(Y^-, X^-) \xrightarrow{\bar{\sigma}} \text{Ker } \tau / \text{Im } Y^-$$

if i is even. Here, $\bar{\Phi}$ and $\bar{\sigma}$ denote the homomorphisms induced by the homomorphisms Φ and σ , respectively. We shall prove that these homomorphisms are in fact isomorphisms.

a) $\bar{\Phi}$ is a monomorphism. Indeed, if $\Phi(\lambda) \in \text{Im } \Upsilon$ with $\lambda \in \Lambda$, then

$$\begin{aligned} \lambda \cdot \varphi_1 &= \lambda' \cdot Y^+, \\ \lambda \cdot \varphi_2 &= 0; \end{aligned}$$

i.e., $(-\lambda', \lambda) \in \text{Ker } d_1^Q$. By Lemma 2.6, we have $\lambda \in \text{Im } \tau$, which proves the claim.

b) $\bar{\Phi}$ is an epimorphism. Let $(\lambda_1, \lambda_2) \in \text{Ker}(Y^-, X^-)$. By Lemma 2.5, there exist $q_1, q_2 \in \Lambda$ such that $(\lambda_1, \lambda_2) = d_1^Q(q_1, q_2)$; i.e., we have

$$\begin{aligned} \lambda_1 &= q_1 Y^+ + q_2 \varphi_1, \\ \lambda_2 &= q_2 \varphi_2. \end{aligned}$$

Consequently, $(\lambda_1, \lambda_2) - \Phi(q_2) \in \text{Im } \Upsilon$.

c) $\bar{\sigma}$ is a monomorphism. Assume that $\sigma^*(\lambda) \in \text{Im}(Y^-)^* = \text{Ker}(Y^+)^*$; i.e., $\lambda \in \text{Ker}(\sigma Y^+)^*$. By Lemma 2.7, $\lambda \in \text{Im } d_0^Q = \text{Im}(Y^-, X^-)$.

d) $\bar{\sigma}$ is an epimorphism. Let $\lambda \in \text{Ker } \tau$. By Lemma 2.8, there exist $q_1, q_2 \in \Lambda$ such that $\lambda = q_1 \sigma + q_2 Y^-$. Consequently, $\bar{\sigma}(q_1 + \text{Im}(Y^-, X^-)) = \lambda + \text{Im } Y^-$.

Finally, in the 0th column of the first page of the spectral sequence, there is a unique nonzero term $\text{Coker}(Y^-, X^-) \simeq R$. Consequently, we have proved that $\mu: Q_\bullet \xrightarrow{\mu} R$ is a Λ -projective resolution of the module R . Its minimality follows immediately from the construction. $\square$

Corollary 2.9. *Let l_n denote the rank of the free Λ -module Q_n . Then*

$$(2.68) \quad l_n = \begin{cases} 1 + 2 \left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \equiv 0 \pmod{4}, \\ 2 + 2 \left\lfloor \frac{n}{4} \right\rfloor & \text{in other cases.} \end{cases}$$

Remark 2.10. Let $Q_\bullet \xrightarrow{\mu} R$ be the minimal Λ -projective resolution of the algebra R , as described in Theorem 2.1. In the sequel, in the decomposition of the m th member $Q_m = \bigoplus_{i+j=m} B_{ij}$ of this resolution, we always arrange the direct summands with respect to the increase of the second index j . The corresponding decomposition of the module Q_m is said to be *standard*.

We consider the bicomplex $\mathcal{A}_{\bullet\bullet}$ that consists of the first two rows of the bicomplex $\mathcal{B}_{\bullet\bullet}$ with numbers 0 and 1 (the other rows in $\mathcal{A}_{\bullet\bullet}$ are zero). Set $X_\bullet = \text{Tot}(\mathcal{A}_{\bullet\bullet})$.

Proposition 2.11. *There is a short exact sequence of complexes*

$$0 \rightarrow X_\bullet \xrightarrow{i} Q_\bullet \xrightarrow{\pi} Q_\bullet[-4] \rightarrow 0$$

that splits in each degree.

Proof. Let $i_n: X_n \rightarrow Q_n$ be the inclusion of the direct summand, and let $\pi_n: Q_n \rightarrow Q_{n-4}$ be the projection onto the corresponding direct summand ($n \geq 4$). Then the statement follows immediately from the construction of the bicomplex $\mathcal{B}_{\bullet\bullet}$. $\square$

§3. ADDITIVE STRUCTURE OF THE COHOMOLOGY ALGEBRA

In the sequel, we assume that $\text{char } K = 2$. Put $R = R_k$ ($k \geq 2$). Let $Q_\bullet \xrightarrow{\mu} R$ be the minimal Λ -projective resolution of the algebra R , described in §2. To calculate the cohomology $\text{HH}^n(R)$, we use the complex

$$(3.1) \quad \left(\text{Hom}_\Lambda(Q_n, R), \delta^n = \text{Hom}_\Lambda(d_n^Q, R) \right)_{n \geq 0}.$$

For any $n \geq 0$, the module Q_n is a free Λ -module of rank l_n (see Corollary 2.9); hence, any element f in $\text{Hom}_\Lambda(Q_n, R)$ is identified with the collection of its values $f(e^{(i)}) \in R$ at the elements of the canonical basis of the module Q_n (that is, the i th coordinate of $e^{(i)}$ is equal to $1 \otimes 1$, and the others are zero).

We observe that if $f = w^*: \Lambda \rightarrow \Lambda$ is a homomorphism of multiplication on the right by $w \in \Lambda$, then, in accordance with the identification mentioned above, the induced homomorphism of Abelian groups

$$\text{Hom}_\Lambda(f, R): \text{Hom}(\Lambda, R) \simeq R \rightarrow \text{Hom}(\Lambda, R) \simeq R$$

acts as follows: $r \in R$ is taken to $w * r$ (here, $*$ corresponds to the Λ -module structure on R).

We simplify the notation in (2.2), (2.3), (2.6), (2.7), denoting $Y^\pm$, X^- , and $\rho^\pm$ by Y , X , and ρ , respectively. Then the differential

$$\delta^0: \text{Hom}_\Lambda(Q_0, R) \simeq R \rightarrow \text{Hom}_\Lambda(Q_1, R) \simeq R^2$$

can be described by the following formula:

$$\delta^0(r) = (Y * r, X * r) = (yr - ry, xr - rx)$$

for any $r \in R$.

Proposition 3.1. 1) *The following set is a K -basis of the vector space $\text{HH}^0(R)$:*

$$(3.2) \quad \{(xy)^i + (yx)^i \mid 1 \leq i \leq k-1\} \cup \{1, x(yx)^{k-1}, y(xy)^{k-1}, (xy)^k\}.$$

2) *The set formed by the elements*

$$(3.3) \quad ((xy)^i + (yx)^i, 0) \quad \text{for } 1 \leq i \leq k-1,$$

$$(3.4) \quad (0, (xy)^i + (yx)^i) \quad \text{for } 1 \leq i \leq k-1, \quad (y(xy)^i, x(yx)^i) \quad \text{for } 1 \leq i \leq k-1$$

is a K -basis for the vector space $\text{Im } \delta^0$.

Proof. The first statement follows from [4, III.14]. To prove the second statement, it suffices to calculate $\delta^0(r)$ for r running through the standard basis of the algebra R and then to pick a basis in the resulting collection of values of the differential δ^0 . $\square$

Corollary 3.2. $\dim_K \text{HH}^0(R) = k + 3$, and $\dim_K B^1(R) = 3k - 3$.

Remark 3.3. In the sequel, we use a brief notation for several elements in (3.2):

$$(3.5) \quad p_1 := xy + yx, \quad p_2 := x(yx)^{k-1}, \quad p_3 := y(xy)^{k-1}, \quad p_4 := (xy)^k.$$

Observe that $p_1^i = (xy)^i + (yx)^i$ ($1 \leq i \leq k-1$), $p_1^k = 2(xy)^k = 0$.

The differential

$$\delta^1: \text{Hom}_\Lambda(Q_1, R) \rightarrow \text{Hom}_\Lambda(Q_2, R)$$

can be described as follows: for any $r_1, r_2 \in R$, we have

$$\delta^1(r_1, r_2) = (Y * r_1, \varphi_1 * r_1 + \varphi_2 * r_2).$$

Now, assume that $(r_1, r_2) \in \text{Ker } \delta^1$, i.e.,

$$(3.6) \quad \begin{aligned} Y * r_1 &= yr_1 + r_1y = 0, \\ \varphi_1 * r_1 + \varphi_2 * r_2 &= \sum_{i=0}^{k-1} (yx)^i \cdot r_1 \cdot (xy)^{k-1-i} \\ &\quad + xr_2 + r_2x + \sum_{i=0}^{k-2} (yx)^i y \cdot r_2 \cdot y(xy)^{k-2-i} = 0. \end{aligned}$$

We write the components of this 1-cocycle in the form

$$(3.7) \quad r_1 = \sum_{w \in \mathcal{B}_R} \lambda_w w, \quad r_2 = \sum_{w \in \mathcal{B}_R} \mu_w w$$

($\lambda_w, \mu_w \in K$). The first equation in (3.6) yields the following system of equations for the coordinates of the expansions in (3.7):

$$(3.8) \quad \begin{aligned} \lambda_{(xy)^i} + \lambda_{(yx)^i} &= 0 & \text{for } 1 \leq i \leq k-1, \\ \lambda_{x(yx)^i} &= 0 & \text{for } 0 \leq i \leq k-2. \end{aligned}$$

Next, the second equation in (3.6) is equivalent to the system

$$(3.9) \quad \begin{aligned} \mu_{(xy)^i} + \mu_{(yx)^i} &= 0 & \text{for } 1 \leq i \leq k-1, \\ \mu_{y(xy)^i} &= 0 & \text{for } 0 \leq i \leq k-3, \\ \lambda_1 + \mu_{y(xy)^{k-2}} &= 0, \\ k \cdot \lambda_y + (k+1) \cdot \mu_x &= 0. \end{aligned}$$

Now, analyzing relations (3.8) and (3.9), we obtain the following statement.

Proposition 3.4. a) Assume that k is even. Then the elements

$$(3.10) \quad \left((xy)^i + (yx)^i, 0 \right) \quad \text{for } 1 \leq i \leq k-1,$$

$$(3.11) \quad \left(0, (xy)^i + (yx)^i \right) \quad \text{for } 1 \leq i \leq k-1,$$

$$(3.12) \quad \left(y(xy)^i, 0 \right) \quad \text{for } 0 \leq i \leq k-1,$$

$$(3.13) \quad \left(0, x(yx)^i \right) \quad \text{for } 1 \leq i \leq k-1,$$

$$(3.14) \quad \left(1, y(xy)^{k-2} \right), \left(x(yx)^{k-1}, 0 \right), \left((xy)^k, 0 \right),$$

$$(3.15) \quad \left(0, 1 \right), \left(0, y(xy)^{k-1} \right), \left(0, (xy)^k \right)$$

form a K -basis of the vector space $\text{Ker } \delta^1$.

b) Assume that k is odd. To obtain a basis of the vector space $\text{Ker } \delta^1$, it suffices to replace the element $(y, 0)$ in the set described above by $(0, x)$.

c) For any $k \geq 2$, the elements

$$(3.16) \quad \left((xy)^i + (yx)^i, 0 \right) \quad \text{for } 1 \leq i \leq k-1,$$

$$(3.17) \quad \left(y(xy)^i, 0 \right) \quad \text{for } 2 \leq i \leq k-1,$$

$$(3.18) \quad \left(0, (xy)^i + (yx)^i \right) \quad \text{for } 1 \leq i \leq k-1,$$

$$(3.19) \quad \left(0, x(yx)^i \right) \quad \text{for } 1 \leq i \leq k-1,$$

$$(3.20) \quad \left(yxy, (xy)^k \right), \left(0, y(xy)^{k-1} \right)$$

form a K -basis of the vector space $\text{Im } \delta^1$.

Corollary 3.5. $\dim_K \text{Ker } \delta^1 = 4k + 3$, and $\dim_K \text{Im } \delta^1 = 4k - 3$.

Proposition 3.6. a) Assume that k is even. Then the cohomology classes of the elements listed in (3.12), (3.14), and (3.15) form a K -basis of the vector space $\text{HH}^1(R)$.

b) Assume that k is odd. To obtain a basis of the vector space $\text{HH}^1(R)$, it suffices to replace the cohomology class of the element $(y, 0)$ in the set indicated in a) by the cohomology class of the element $(0, x)$.

Proof. This follows from Proposition 3.4 and Proposition 3.1, part (2). $\square$

Corollary 3.7. $\dim_K \text{HH}^1(R) = k + 6$.

Under the above assumptions, the differential δ^2 is described by the following rule: for any $r_1, r_2 \in R$, we have

$$\delta^2(r_1, r_2) = (Y * r_1, \rho * r_1 + \tau * r_2).$$

Assume that $(r_1, r_2) \in \text{Ker } \delta^2$, i.e.,

$$(3.21) \quad Y * r_1 = 0, \quad \rho * r_1 + \tau * r_2 = 0.$$

We represent the components of this 2-cocycle in the form (3.7). The first equation in (3.21) is equivalent to (3.8), and the second is equivalent to the relation

$$\mu_{(xy)^i} + \mu_{(yx)^i} = 0 \quad \text{for } 1 \leq i \leq k-2$$

(it should be noted that $\rho * r = 0$ for any $r \in R$).

Arguing as above, it is easy to prove the following statements.

Proposition 3.8. 1) *The elements*

$$(3.22) \quad \left((xy)^i + (yx)^i, 0 \right) \quad \text{for } 1 \leq i \leq k-1,$$

$$(3.23) \quad \left(0, (xy)^i + (yx)^i \right) \quad \text{for } 1 \leq i \leq k-2,$$

$$(3.24) \quad \left(y(xy)^i, 0 \right) \quad \text{for } 0 \leq i \leq k-1,$$

$$(3.25) \quad \left(0, x(yx)^i \right) \quad \text{for } 0 \leq i \leq k-1,$$

$$(3.26) \quad \left(0, y(xy)^i \right) \quad \text{for } 0 \leq i \leq k-1,$$

$$(3.27) \quad \left(1, 0 \right), \left(x(yx)^{k-1}, 0 \right), \left((xy)^k, 0 \right),$$

$$(3.28) \quad \left(0, 1 \right), \left(0, (xy)^{k-1} \right),$$

$$(3.29) \quad \left(0, (yx)^{k-1} \right), \left(0, (xy)^k \right)$$

form a K -basis of the vector space $\text{Ker } \delta^2$.

2) *The elements*

$$(3.30) \quad \left((xy)^i + (yx)^i, 0 \right) \quad \text{for } 1 \leq i \leq k-1,$$

$$(3.31) \quad \left(0, (xy)^i + (yx)^i \right) \quad \text{for } 2 \leq i \leq k-1,$$

$$(3.32) \quad \left(y(xy)^i, 0 \right) \quad \text{for } 1 \leq i \leq k-1$$

form a K -basis of the vector space $\text{Im } \delta^2$.

Corollary 3.9. $\dim_K \text{Ker } \delta^2 = 5k + 4$, and $\dim_K \text{Im } \delta^2 = 3k - 4$.

Proposition 3.10. A K -basis of the vector space $\text{HH}^2(R)$ can be formed by the cohomology classes of the elements listed in (3.27) and (3.28) and the cohomology classes of the elements

$$(3.33) \quad \left(0, y(xy)^i \right) \quad \text{for } 0 \leq i \leq k-2,$$

$$(3.34) \quad \left(y, 0 \right), \left(0, x \right), \left(yxy, 0 \right).$$

Corollary 3.11. $\dim_K \text{HH}^2(R) = k + 7$.

The differential

$$\delta^3: \text{Hom}_\Lambda(Q_3, R) \rightarrow \text{Hom}_\Lambda(Q_4, R)$$

can be described as follows: for any $r_1, r_2 \in R$,

$$(3.35) \quad \delta^3(r_0, r_1) = (Y * r_1, \rho * r_1 + Y * r_2, \sigma * r_2).$$

The condition $(r_1, r_2) \in \text{Ker } \delta^3$ is equivalent to the relations

$$Y * r_1 = 0, \quad Y * r_2 = 0.$$

Here, we use the fact that $\rho * r = 0 = \sigma * r$ for any $r \in R$. Representing the components r_1, r_2 of this cocycle in the form (3.7), we obtain relations connecting the coefficients λ_w, μ_w ($w \in \mathcal{B}_R$) in the corresponding decompositions. Now, we easily establish the following statements.

Proposition 3.12. 1) *The elements*

$$(3.36) \quad \left((xy)^i + (yx)^i, 0 \right) \quad \text{for } 1 \leq i \leq k-1,$$

$$(3.37) \quad \left(0, (xy)^i + (yx)^i \right) \quad \text{for } 1 \leq i \leq k-1,$$

$$(3.38) \quad \left(y(xy)^i, 0 \right) \quad \text{for } 0 \leq i \leq k-1,$$

$$(3.39) \quad \left(0, y(xy)^i \right) \quad \text{for } 0 \leq i \leq k-1,$$

$$(3.40) \quad \left(1, 0 \right), \left(x(yx)^{k-1}, 0 \right), \left((xy)^k, 0 \right),$$

$$(3.41) \quad \left(0, 1 \right), \left(0, x(yx)^{k-1} \right), \left(0, (xy)^k \right)$$

form a K -basis of the vector space $\text{Ker } \delta^3$.

2) *The elements*

$$(3.42) \quad \left((xy)^i + (yx)^i, 0, 0 \right) \quad \text{for } 1 \leq i \leq k-1,$$

$$(3.43) \quad \left(0, (xy)^i + (yx)^i, 0 \right) \quad \text{for } 1 \leq i \leq k-1,$$

$$(3.44) \quad \left(y(xy)^i, 0, 0 \right) \quad \text{for } 1 \leq i \leq k-1,$$

$$(3.45) \quad \left(0, y(xy)^i, 0 \right) \quad \text{for } 1 \leq i \leq k-1$$

form a K -basis of the vector space $\text{Im } \delta^3$.

Corollary 3.13. $\dim_K \text{Ker } \delta^3 = 4k + 4$, and $\dim_K \text{Im } \delta^3 = 4k - 4$.

Proposition 3.14. *The classes of the elements listed in (3.39), (3.40), and (3.41), and the cohomology classes of the elements*

$$(3.46) \quad \left(y, 0 \right), \left(0, xy + yx \right)$$

form a K -basis of the vector space $\text{HH}^3(R)$.

Corollary 3.15. $\dim_K \text{HH}^3(R) = k + 8$.

Proposition 3.16. *For any $n \geq 4$, we have*

$$(3.47) \quad \dim_K \text{HH}^n(R) = \dim_K \text{HH}^{n-4}(R) + 8.$$

Proof. Proposition 2.11 implies that we have the following exact sequence of complexes:

$$0 \rightarrow \text{Hom}_\Lambda(Q_\bullet[-4], R) \xrightarrow{\pi^*} \text{Hom}_\Lambda(Q_\bullet, R) \xrightarrow{i^*} \mathcal{X}^\bullet \rightarrow 0,$$

where $\mathcal{X}^\bullet = \text{Hom}_\Lambda(X_\bullet, R)$.

In its turn, this sequence leads to the long exact cohomology sequence

$$(3.48) \quad \dots \xrightarrow{\Delta^{n-1}} \text{HH}^{n-4}(R) \xrightarrow{\pi^*} \text{HH}^n(R) \xrightarrow{i^*} H^n(\mathcal{X}^\bullet) \xrightarrow{\Delta^n} \text{HH}^{n-3}(R) \xrightarrow{\pi^*} \dots$$

Now, we need to compute the cohomology of the auxiliary complex $\mathcal{X}^\bullet$.

Lemma 3.17. *For any $n \geq 4$, we have*

$$(3.49) \quad \dim_K H^n(\mathcal{X}^\bullet) = 8.$$

Proof. As usual, we identify $\mathcal{X}^n = \text{Hom}_\Lambda(X_n, R)$, where $n \geq 1$, and R^2 (but $\mathcal{X}^0$ is identified with R). Let $f_y: R \rightarrow R$ denote the homomorphism (of Abelian groups) such that $f_y(r) = Y * r = yr + ry$. For $n \geq 3$, the differential $d_{\mathcal{X}^\bullet}^n$ is described by the formula

$d_{\mathcal{X}^\bullet}^n(r_1, r_2) = (f_y(r_1), f_y(r_2))$. Therefore, for $n \geq 4$ we have $H^n(\mathcal{X}^\bullet) \simeq \left(\text{Ker } f_y / \text{Im } f_y \right)^2$. Arguing as in the proof of Proposition 3.4, we see that $\text{Ker } f_y / \text{Im } f_y$ is generated by the classes of the elements $1, y, x(yx)^{k-1}, (xy)^k$, and (3.49) follows. $\square$

We continue the proof of Proposition 3.16. Lemma 3.17 shows that, to establish (3.47), it suffices to prove that the connecting homomorphisms Δ^n in the sequence (3.48) are zero if $n \geq 3$.

We consider the following diagram, which is used in the construction of the connecting homomorphism:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_\Lambda(Q_{n-4}, R) & \xrightarrow{\pi^*} & \text{Hom}_\Lambda(Q_n, R) & \xrightarrow{i^*} & \mathcal{X}^n \longrightarrow 0 \\ & & \delta^{n-4} \downarrow & & \delta^n \downarrow & & d_{\mathcal{X}}^n \downarrow \\ 0 & \longrightarrow & \text{Hom}_\Lambda(Q_{n-3}, R) & \xrightarrow{\pi^*} & \text{Hom}_\Lambda(Q_{n+1}, R) & \xrightarrow{i^*} & \mathcal{X}^{n+1} \longrightarrow 0. \end{array}$$

Let $f = (r_1, r_2) \in \mathcal{X}^n$ be an n -cocycle, $n \geq 3$. We denote by $\nu: Q_n = X_n \oplus Q_{n-4} \rightarrow X_n$ the projection onto the direct summand. Then we define $\tilde{f} = f \circ \nu: Q_n \rightarrow R$; in other words, the map $\tilde{f}$ is obtained from f via extension by zero to the direct summand complementary to X_n in Q_n , i.e., $\tilde{f} = (r_1, r_2, 0)$. It is clear that $i^*(\tilde{f}) = f$. If $n \geq 4$, then the matrix of the differential d_n^Q has the following block structure:

$$d_n^Q = \left(\begin{array}{cc|cc} Y & \rho & 0 & O_{1, l'-3} \\ 0 & Y & \rho & O_{1, l'-3} \\ \hline O_{l-2, 2} & & & d_{n-4}^Q \end{array} \right)$$

with $l = l_n, l' = l_{n+1}$ (see formula (2.68)). In this case, we see that

$$\delta^n(\tilde{f}) = (Y * r_1, Y * r_2, O_{l'-2}) = 0,$$

whence $\Delta^n(\text{cl} f) = 0$. If $n = 3$, then (3.35) shows directly that $\delta^3(\tilde{f}) = 0$; consequently, $\Delta^3(\text{cl} f) = 0$. $\square$

Corollary 3.18. *Put $R = R_k$. Then*

$$\dim_K \text{HH}^n(R) = \begin{cases} k + 3 + 8 \left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \equiv 0 \pmod{4}, \\ k + 6 + 8 \left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \equiv 1 \pmod{4}, \\ k + 7 + 8 \left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \equiv 2 \pmod{4}, \\ k + 8 + 8 \left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

In the course of the proof of Proposition 3.16, for $n \geq 4$, we established isomorphism

$$\text{HH}^n(R) \simeq \text{HH}^{n-4}(R) \oplus H^n(\mathcal{X}^\bullet).$$

This isomorphism and the proof of Lemma 3.17 lead to the following two statements.

Proposition 3.19. *The vector space $\text{HH}^4(R)$ has a K -basis formed by the cohomology classes of the following elements:*

$$\begin{aligned} & (1, 0, 0), \quad (y, 0, 0), \quad (x(yx)^{k-1}, 0, 0), \quad ((xy)^k, 0, 0), \\ & (0, 1, 0), \quad (0, y, 0), \quad (0, x(yx)^{k-1}, 0), \quad (0, (xy)^k, 0), \\ & (0, 0, (xy)^i + (yx)^i) \quad \text{for } 1 \leq i \leq k-1, \\ & (0, 0, 1), \quad (0, 0, x(yx)^{k-1}), \quad (0, 0, y(xy)^{k-1}), \quad (0, 0, (xy)^k). \end{aligned}$$

Proposition 3.20. a) If k is even, then the vector space $\mathrm{HH}^5(R)$ has a K -basis formed by the cohomology classes of the following elements:

$$\begin{aligned} & \left(1, \mathrm{O}_3\right), \left(y, \mathrm{O}_3\right), \left(x(yx)^{k-1}, \mathrm{O}_3\right), \left((xy)^k, \mathrm{O}_3\right), \\ & \left(0, 1, \mathrm{O}_2\right), \left(0, y, \mathrm{O}_2\right), \left(0, x(yx)^{k-1}, \mathrm{O}_2\right), \left(0, (xy)^k, \mathrm{O}_2\right), \\ & \left(\mathrm{O}_2, y(xy)^i\right) \quad \text{for } 0 \leq i \leq k-1, \\ & \left(\mathrm{O}_2, 1, y(xy)^{k-2}\right), \left(\mathrm{O}_2, x(yx)^{k-1}, 0\right), \left(\mathrm{O}_2, (xy)^k, 0\right), \\ & \left(\mathrm{O}_3, 1\right), \left(\mathrm{O}_3, y(xy)^{k-1}\right), \left(\mathrm{O}_3, (xy)^k\right). \end{aligned}$$

b) If k is odd, then, in order to get a basis of the vector space $\mathrm{HH}^5(R)$, it suffices to replace the cohomology class of the element $\left(\mathrm{O}_2, y, 0\right)$ in the set described above in part a) by the cohomology class of the element $\left(\mathrm{O}_3, x\right)$.

§4. GENERATORS AND RELATIONS

We recall briefly an interpretation of the Yoneda product in the algebra

$$\mathrm{HH}^*(R) = \bigoplus_{m \geq 0} \mathrm{Ext}_{\Lambda}^m(R, R),$$

used in [3]. Let

$$\mu: Q_{\bullet} \rightarrow R$$

be the minimal Λ -projective resolution described in §2. Let

$$\mathrm{Hom}_{\Lambda}(Q_{\bullet}, R) = \left(\mathrm{Hom}_{\Lambda}(Q_n, R), \delta^n\right)$$

be the complex (3.1). Then for cocycles $f \in \mathrm{Ker} \delta^n$ and $g \in \mathrm{Ker} \delta^t$ we have

$$\mathrm{cl}g \cdot \mathrm{cl}f = \mathrm{cl}(\mu T^0(g)T^t(f)),$$

where $T^i(h)$ denotes the i th translate of the cocycle h . In the sequel, we shall describe the translates $T^i(h)$ ($i \geq 0$) with the help of matrices that correspond to the standard decompositions of the modules Q_i (see Remark 2.10).

Case 1. First, assume that k is even. We consider the following homogeneous elements of $\mathrm{HH}^*(R)$:

$$(4.0) \quad \text{— of degree 0 :} \quad p_1, p_2, p_3, p_4 \text{ in (3.5);}$$

$$(4.1) \quad \text{— of degree 1 :} \quad u_1 := (1, y(xy)^{k-2}), u_2 := (0, 1), u_3 := (y, 0);$$

$$(4.2) \quad \text{— of degree 2 :} \quad v_1 := (y, 0), v_2 := (0, y), v_3 := (0, x);$$

$$(4.3) \quad \text{— of degree 3 :} \quad w_1 := (0, 1), w_2 := (0, y);$$

$$(4.4) \quad \text{— of degree 4 :} \quad t := (0, 0, 1).$$

Proposition 4.1. Assume that k is even. In the algebra $\mathrm{HH}^*(R)$, the elements of the set

$$(4.5) \quad \mathcal{V}_1 = \{p_1, p_2, p_3, p_4, u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, t\}$$

satisfy the following relations:

$$\begin{aligned}
(4.6) \quad & p_1^k = p_2^2 = p_3^2 = p_4^2 = 0, \\
& p_i p_j = 0 \text{ for } 1 \leq i < j \leq 4; \\
(4.7) \quad & p_1 u_1 = p_3 u_2, \ p_3 u_1 = p_2 u_2, \ p_1 u_2 = 0, \\
(4.8) \quad & p_3 u_3 = p_4 u_3 = 0, \ p_4 u_1 = p_2 u_3, \ p_2 u_2 = p_1^{k-1} u_3; \\
(4.9) \quad & p_3 v_1 = p_4 v_1 = p_1^{k-1} v_2 = p_3 v_2 = p_4 v_2 = 0, \ p_1 v_1 = p_2 v_2, \\
(4.10) \quad & p_2 v_1 = p_4 u_1^2, \\
(4.11) \quad & p_1 v_3 = p_2 v_3 = p_4 v_3 = 0, \\
(4.12) \quad & u_1 u_2 = u_2 u_3 = u_3^2 = 0, \ p_2 v_2 = p_3 v_3 = p_4 u_2^2; \\
(4.13) \quad & p_3 w_2 = p_4 w_2 = 0, \\
(4.14) \quad & u_2^3 = u_2 v_2 = 0, \ p_1 w_1 = u_1 v_2, \ p_3 w_1 = u_2 v_1, \ p_4 w_1 = p_2 w_2, \\
(4.15) \quad & u_3 v_3 = 0, \ p_3 w_1 = u_1 v_3 = p_1^{k-2} u_3 v_2, \ p_1 w_1 = u_2 v_3, \\
(4.16) \quad & p_4 w_1 = u_3 v_1, \ p_1 w_2 = u_3 v_2, \ u_1^2 u_3 = u_1 v_1 + p_2 w_1; \\
(4.17) \quad & v_1^2 = v_1 v_2 = 0, \ v_2^2 = p_1^2 t, \\
(4.18) \quad & v_1 v_3 = v_2 v_3 = v_3^2 = 0, \\
& u_2 w_1 = u_2 w_2 = u_3 w_2 = 0, \ u_1 w_2 = u_3 w_1; \\
(4.19) \quad & v_2 w_1 = p_3 u_2 t, \\
(4.20) \quad & v_3 w_2 = 0, \ v_1 w_2 = p_4 u_1 t, \ v_2 w_2 = p_1 u_3 t, \ v_3 w_1 = p_3 u_1 t, \\
(4.21) \quad & u_1 u_3 w_1 = \begin{cases} v_1 w_1 + p_2 u_1 t & \text{if } k > 2, \\ v_1 w_1 + (p_2 u_1 + p_4 u_2) t & \text{if } k = 2; \end{cases} \\
(4.22) \quad & w_2^2 = 0, \ w_1^2 = u_1^2 t, \\
(4.23) \quad & w_1 w_2 = u_1 u_3 t.
\end{aligned}$$

Proof. Relations (4.6), (4.7), (4.8), (4.9), (4.11), and (4.13) are verified directly. To prove the remaining relations, we need to compute the translates of suitable order for the elements in $\mathcal{Y}_1$ that have positive degree.

Proposition 2.11 implies the following statement.

Lemma 4.2. *For any $i \geq 0$, the projection onto the direct summand $\pi_{i+4}: Q_{i+4} = X_{i+4} \oplus Q_i \rightarrow Q_i$ is the i th translate $T^i(t)$ of the cocycle t .*

Suitable translates of the other elements in $\mathcal{Y}_1$ are presented in the following lemma.

Lemma 4.3. *For the role of translates of the elements in $\mathcal{Y}_1 \setminus \{t\}$ that have a positive degree, we can take the homomorphisms determined by the following matrices:*

$$\begin{aligned}
T^0(u_1) &= \begin{pmatrix} 1 \otimes 1, y(xy)^{k-2} \otimes 1 \end{pmatrix}, \\
T^1(u_1) &= \begin{pmatrix} 1 \otimes 1 & \sum_{i=0}^{k-2} x(yx)^i \otimes (xy)^{k-2-i} \\ 0 & \sum_{i=0}^{k-2} y(xy)^i \otimes (xy)^{k-2-i} + \sum_{i=0}^{k-2} (xy)^i \otimes y(xy)^{k-2-i} \end{pmatrix}, \\
T^2(u_1) &= \begin{pmatrix} 1 \otimes 1 & (yx)^{3k-5} \otimes y \\ 0 & x \otimes 1 + 1 \otimes x + y \otimes y(xy)^{k-2} \end{pmatrix}, \\
T^3(u_1) &= \begin{pmatrix} 1 \otimes 1 & 0 & \star \\ 0 & 1 \otimes 1 & \star \end{pmatrix}
\end{aligned}$$

with

$$(T^3(u_1))_{13} = \begin{cases} 0 & \text{if } k > 2, \\ xyx \otimes xy + (xy)^2 \otimes x & \text{if } k = 2, \end{cases}$$

$$(T^3(u_1))_{23} = \sum_{i=0}^{k-2} x(yx)^i \otimes x(yx)^{k-2-i} + (yx)^{k-1} \otimes y(xy)^{k-2} + y(xy)^{k-2} \otimes (xy)^{k-1};$$

$$T^0(u_2) = (0, 1 \otimes 1), \quad T^1(u_2) = \begin{pmatrix} 0 & \sum_{i=0}^{k-2} (yx)^i \otimes y(xy)^{k-2-i} \\ 0 & 1 \otimes 1 \end{pmatrix},$$

$$T^2(u_2) = \begin{pmatrix} 0 & 1 \otimes (xy)^{k-1} + 1 \otimes (yx)^{k-1} \\ 0 & y \otimes 1 + 1 \otimes y \end{pmatrix};$$

$$T^0(u_3) = (y \otimes 1, 0), \quad T^1(u_3) = \begin{pmatrix} y \otimes 1 & \sum_{i=1}^{k-1} i(yx)^i \otimes (xy)^{k-1-i} \\ 0 & \sum_{i=0}^{k-2} (i+1)y(xy)^i \otimes y(xy)^{k-2-i} \end{pmatrix};$$

$$T^0(v_1) = (y \otimes 1, 0), \quad T^1(v_1) = \begin{pmatrix} y \otimes 1 & \sum_{i=1}^{k-1} (yx)^i \otimes x(yx)^{k-1-i} \\ 0 & \sum_{i=0}^{k-1} y(xy)^i \otimes (yx)^{k-1-i} \end{pmatrix},$$

$$T^2(v_1) = \begin{pmatrix} y \otimes 1 & \rho & \star \\ 0 & 1 \otimes xy + x \otimes y & \star \end{pmatrix}$$

with

$$(T^2(v_1))_{13} = \begin{cases} 0 & \text{if } k > 2, \\ xyx \otimes yxy + yx \otimes (xy)^2 & \text{if } k = 2, \end{cases}$$

$$(T^2(v_1))_{23} = y(xy)^{k-1} \otimes (yx)^{k-1} + y(xy)^{k-1} \otimes (xy)^{k-1};$$

$$T^0(v_2) = (0, y \otimes 1), \quad T^1(v_2) = \begin{pmatrix} 0 & yx \otimes 1 + x \otimes y \\ 0 & y \otimes y \end{pmatrix},$$

$$T^2(v_2) = \begin{pmatrix} 0 & 0 & y(xy)^{k-1} \otimes (xy)^{k-1} \\ 0 & 0 & X \cdot (x \otimes y + y \otimes x) \end{pmatrix};$$

$$T^0(v_3) = (0, 1 \otimes x), \quad T^1(v_3) = \begin{pmatrix} 0 & 1 \otimes x^2 \\ 0 & yx \otimes 1 + 1 \otimes xy \end{pmatrix},$$

$$T^2(v_3) = \begin{pmatrix} 0 & 1 \otimes x^2 & (yx)^{k-1} \otimes x(yx)^{k-1} \\ 0 & y \otimes y & x \otimes x^2 + x^2 \otimes x \end{pmatrix};$$

$$T^0(w_1) = (0, 1 \otimes 1), \quad T^1(w_1) = \begin{pmatrix} 0 & 1 \otimes 1 & \sum_{i=0}^{k-2} x(yx)^i \otimes x(yx)^{k-2-i} \\ 0 & 0 & \star \end{pmatrix}$$

with

$$(T^1(w_1))_{23} = \sum_{i=0}^{k-1} (xy)^i \otimes (yx)^{k-1-i} + \sum_{i=0}^{k-2} (xy)^i \otimes (xy)^{k-1-i} + \sum_{i=1}^{k-1} (yx)^i \otimes (yx)^{k-1-i};$$

$$T^2(w_1) = \begin{pmatrix} 0 & 1 \otimes 1 & 0 & \star \\ 0 & 0 & \varphi_2 & \star \end{pmatrix}$$

with

$$\begin{aligned} (T^2(w_1))_{14} &= (yx)^{k-1} \otimes x(yx)^{k-2} + x(yx)^{k-2} \otimes (xy)^{k-1} + x(yx)^{k-1} \otimes (xy)^{k-2}, \\ (T^2(w_1))_{24} &= \sum_{j=0}^{k-1} (xy)^j \otimes (yx)^{k-1-j} + x \otimes y(xy)^{k-2} + (yx)^{k-1} \otimes 1; \\ T^3(w_1) &= \begin{pmatrix} 0 & 1 \otimes 1 & 0 & \star \\ 0 & 0 & 1 \otimes 1 & \sum_{i=0}^{k-2} x(yx)^i \otimes (xy)^{k-2-i} \end{pmatrix} \end{aligned}$$

with

$$\begin{aligned} (T^3(w_1))_{14} &= \begin{cases} 0 & \text{if } k > 2, \\ yx \otimes xy + (xy)^2 \otimes 1 & \text{if } k = 2; \end{cases} \\ T^0(w_2) &= (0, y \otimes 1), \quad T^1(w_2) = \begin{pmatrix} 0 & 1 \otimes 1 & \star \\ 0 & 0 & \star \end{pmatrix} \end{aligned}$$

with

$$\begin{aligned} (T^1(w_2))_{13} &= \sum_{i=1}^{k-1} ix(yx)^i \otimes (xy)^{k-1-i} + \sum_{i=1}^{k-1} i(yx)^i \otimes x(yx)^{k-1-i}, \\ (T^1(w_2))_{23} &= \sum_{i=0}^{k-2} (i+1)y(xy)^i \otimes (yx)^{k-1-i} + \sum_{i=1}^{k-1} i(xy)^i \otimes y(xy)^{k-1-i}; \\ T^2(w_2) &= \begin{pmatrix} 0 & y \otimes 1 & x(yx)^{k-1} \otimes 1 & (yx)^{k-1} \otimes (xy)^{k-1} \\ 0 & 0 & y \otimes x + yx \otimes 1 & 0 \end{pmatrix}, \\ T^3(w_2) &= \begin{pmatrix} 0 & y \otimes 1 & 1 \otimes x(yx)^{k-1} & 0 \\ 0 & 0 & 1 \otimes y & \sum_{i=1}^{k-1} i(yx)^i \otimes (xy)^{k-1-i} \end{pmatrix}. \end{aligned}$$

The *proof* of the lemma is a direct verification of the relations $\mu T^0(b) = b$, $d_{i-1} T^i(b) = T^{i-1}(b) d_{i+\deg b}$ ($i > 0$), where $b \in \mathcal{Y}_1 \setminus \{t\}$ with $\deg b > 0$.

Remark 4.4. For what follows, it should be noted that the formulas for the translates of the elements u_1, u_2, v_1, v_2, w_1 , and t remain valid for odd values of k .

Now the proof of Proposition 4.1 is completed using direct calculations with the matrices described in Lemma 4.3, and we leave this to the reader. $\square$

Proposition 4.5. *Assume that k is even. The set $\mathcal{Y}_1$ (see (4.5)) generates $\mathrm{HH}^*(R)$ as a K -algebra.*

Proof. Let $\mathcal{H}$ denote the K -subalgebra of $\mathrm{HH}^*(R)$ generated by the set $\mathcal{Y}_1 \cup \{1\}$ (here 1 denotes the unit element of the algebra $\mathrm{HH}^*(R)$). We introduce also the intermediate subalgebra $\mathcal{H}'$ generated by the set $\bigcup_{i=0}^5 \mathrm{HH}^i(R)$. First, we show that $\mathrm{HH}^*(R) = \mathcal{H}'$, and after that we check that $\mathcal{H}' = \mathcal{H}$.

The following auxiliary statement is proved by a direct calculation (cf. the proof of Lemma 4.3).

Lemma 4.6. Put $v := (1, 0) \in \text{HH}^2(R)$. The translates $T^i(v)$, $i \leq 5$, can be presented by the maps

$$\begin{aligned}
 T^0(v) &= (1 \otimes 1 \quad 0), \\
 T^1(v) &= \begin{pmatrix} 1 \otimes 1 & \sum_{i=0}^{k-2} x(yx)^i \otimes x(yx)^{k-2-i} \\ 0 & \sum_{i=0}^{k-1} (xy)^i \otimes (yx)^{k-1-i} \end{pmatrix}, \\
 T^2(v) &= \begin{pmatrix} 1 \otimes 1 & 0 & 0 \\ 0 & x \otimes 1 + 1 \otimes x & (xy)^{k-1} \otimes (yx)^{k-1} \end{pmatrix}, \\
 T^3(v) &= \begin{pmatrix} 1 \otimes 1 & 0 & 0 & 0 \\ 0 & 1 \otimes 1 & \sum_{i=0}^{k-2} x(yx)^i \otimes x(yx)^{k-2-i} & 0 \end{pmatrix}, \\
 (4.24) \quad T^4(v) &= \begin{pmatrix} 1 \otimes 1 & 0 & 0 & 0 \\ 0 & 1 \otimes 1 & 0 & 0 \\ 0 & 0 & 1 \otimes 1 & 0 \end{pmatrix}, \\
 (4.25) \quad T^5(v) &= \begin{pmatrix} 1 \otimes 1 & 0 & 0 & 0 \\ 0 & 1 \otimes 1 & 0 & 0 \\ 0 & 0 & 1 \otimes 1 & \sum_{i=0}^{k-2} x(yx)^i \otimes x(yx)^{k-2-i} \\ 0 & 0 & 0 & \sum_{i=0}^{k-1} (xy)^i \otimes (yx)^{k-1-i} \end{pmatrix}.
 \end{aligned}$$

Lemma 4.7. Put $z := u_1^4 \in \text{HH}^4(R)$. The translates $T^i(z)$, $i \leq 3$, can be presented by the maps

$$\begin{aligned}
 T^0(z) &= (1 \otimes 1 \quad 0 \quad 0), \\
 T^1(z) &= \begin{pmatrix} 1 \otimes 1 & \sum_{i=0}^{k-2} x(yx)^i \otimes x(yx)^{k-2-i} & 0 & 0 \\ 0 & \sum_{i=0}^{k-1} (xy)^i \otimes (yx)^{k-1-i} & 0 & 0 \end{pmatrix}, \\
 T^2(z) &= \begin{pmatrix} 1 \otimes 1 & 0 & 0 & 0 \\ 0 & x \otimes 1 + 1 \otimes x & (xy)^{k-1} \otimes (yx)^{k-1} & 0 \end{pmatrix}, \\
 T^3(z) &= \begin{pmatrix} 1 \otimes 1 & 0 & 0 & 0 \\ 0 & 1 \otimes 1 & \sum_{i=0}^{k-2} x(yx)^i \otimes x(yx)^{k-2-i} & 0 \end{pmatrix}.
 \end{aligned}$$

Moreover, for $i \geq 4$, we have

$$(4.26) \quad T^i(z) = \left(\begin{array}{c|c} \text{id}_{\Lambda^2} & 0 \\ \hline 0 & T^{i-4}(z) \end{array} \right).$$

Proof. Using the translates $T^j(u_1)$ of the element u_1 described in Lemma 4.3, we easily show that z can be presented by the cocycle $(1, 0, 0)$. Then the formulas for $T^i(z)$ with $i \leq 4$ are verified directly. Finally, formula (4.26) is established by induction. $\square$

Remark 4.8. We observe that the formulas for translates of the elements v and z presented in Lemmas 4.6 and 4.7 remain valid for any odd k .

We continue the proof of Proposition 4.5. Let $f = (r_1, r_2, f') \in \text{Ker } \delta^n \subset \text{Hom}_\Lambda(Q_n, R)$, where $n > 5$ and $r_1, r_2 \in R$, $f' \in \text{Hom}_\Lambda(Q_{n-4}, R)$ (here we use the identification $\mathcal{X}^n = \text{Hom}_\Lambda(X_n, R)$ and R^2). Since $\delta^n(f) = (Y * r_1, Y * r_2, \delta^{n-4}(f'))$, we have $f' \in \text{Ker } \delta^{n-4}$.

a) Assume additionally that $n = 6$. Let $g \in \text{HH}^4(R)$ be an element represented by the cocycle $(r_1, r_2, 0)$. Using (4.24), we obtain

$$T^0(g) \cdot T^4(v) = (r_1 \otimes 1, r_2 \otimes 1, 0) T^4(v) = (r_1 \otimes 1, r_2 \otimes 1, 0, 0).$$

Consequently, we have $g \cdot v = (r_1, r_2, 0, 0)$ and $f - gv = (0, 0, f')$. With the help of Lemma 4.2, we show that $f' \cdot t = (0, 0, f')$. Therefore, $f = gv + f't \in \mathcal{H}'$.

b) If $n = 7$, then we consider the element $g := (r_1, r_1, 0, 0) \in \text{HH}^5(R)$. Arguing as above (and using (4.25)), we see that $f = gv + f't \in \mathcal{H}'$.

c) Finally, assume that $n \geq 8$. We consider the element $g \in \text{HH}^{n-4}(R)$ represented by the cocycle $(r_1, r_2, 0)$. Using (4.26), we obtain

$$\begin{aligned} T^0(g) \cdot T^{n-4}(z) &= (r_1 \otimes 1, r_2 \otimes 1, 0) \left(\begin{array}{c|c} \text{id}_{\Lambda^2} & 0 \\ \hline 0 & T^{n-8}(z) \end{array} \right) \\ &= (r_1 \otimes 1, r_2 \otimes 1, 0). \end{aligned}$$

Hence, $g \cdot z = (r_1, r_2, 0) \in \text{HH}^n(R)$. As in part a) of the proof, we obtain $f = gz + f't$. Applying the inductive hypothesis to g and f' , we conclude that $f \in \mathcal{H}'$.

Now we prove the inclusion $\mathcal{H}' \subset \mathcal{H}$. We remark that, as in the proof of Proposition 4.1, when calculating products of elements of the set $\mathcal{Y}_1$, we use the translates listed in Lemma 4.3. By Remark 3.3, we have $\text{HH}^0(R) \subset \mathcal{H}$.

Next, the basis elements of the $\text{HH}^1(R)$ indicated in Proposition 3.6 satisfy the following relations:

$$\begin{aligned} (4.27) \quad & \left(y(xy)^i, 0 \right) = p_1^i u_3 \quad (0 \leq i \leq k-1), \\ & \left(x(yx)^{k-1}, 0 \right) = \begin{cases} p_2 u_1 & \text{if } k > 2, \\ p_2 u_1 + p_4 u_2 & \text{if } k = 2, \end{cases} \\ & \left((xy)^k, 0 \right) = p_4 u_1, \quad \left(0, y(xy)^{k-1} \right) = p_3 u_2, \quad \left(0, (xy)^k \right) = p_4 u_2. \end{aligned}$$

Thus, $\text{HH}^1(R) \subset \mathcal{H}$.

A direct calculation with the help of Lemma 4.3 shows that

$$v = \begin{cases} u_1^2 & \text{if } k > 2, \\ u_1^2 + v_3 & \text{if } k = 2. \end{cases}$$

Moreover, we obtain the following relations for the basis elements of $\text{HH}^2(R)$ indicated in Proposition 3.10:

$$(4.28) \quad \left(yxy, 0 \right) = p_1 v_1, \quad \left(x(yx)^{k-1}, 0 \right) = p_2 u_1^2, \quad \left((xy)^k, 0 \right) = p_4 u_1^2,$$

$$(4.29) \quad \begin{aligned} \left(0, 1 \right) &= u_2^2, \quad \left(0, y(xy)^i \right) = p_1^i v_2 \quad (0 \leq i \leq k-1), \\ \left(0, (xy)^{k-1} \right) &= u_1 u_3 + v_1. \end{aligned}$$

This implies that $\text{HH}^2(R) \subset \mathcal{H}$.

As above, we obtain the following relations for the basis elements of $\text{HH}^3(R)$ indicated in Proposition 3.14:

$$\begin{aligned} \left(1, 0 \right) &= u_1^3, \quad \left(y, 0 \right) = u_1 v_1, \quad \left(x(yx)^{k-1}, 0 \right) = p_2 u_1^3, \quad \left((xy)^k, 0 \right) = p_4 u_1^3, \\ \left(0, y(xy)^i, 0 \right) &= p_1^i w_2 \quad (0 \leq i \leq k-1), \\ \left(0, xy + yx \right) &= u_1 v_2, \quad \left(0, x(yx)^{k-1} \right) = p_2 w_1, \quad \left(0, (xy)^k \right) = p_4 w_1, \end{aligned}$$

and this shows that $\text{HH}^3(R) \subset \mathcal{H}$.

Next, the inclusion $\mathrm{HH}^4(R) \subset \mathcal{H}$ is deduced from the following relations for basis elements of $\mathrm{HH}^4(R)$ indicated in Proposition 3.19:

$$\begin{aligned}
 (4.30) \quad & \begin{aligned} & \left(1, 0, 0\right) = u_1^4, \quad \left(y, 0, 0\right) = u_1^2 v_1, \\ & \left(x(yx)^{k-1}, 0, 0\right) = p_2 u_1^4, \quad \left((xy)^k, 0, 0\right) = p_4 u_1^4, \\ & \left(0, 1, 0\right) = \begin{cases} u_1 w_1 & \text{if } k > 2, \\ u_1 w_1 + p_3 t & \text{if } k = 2, \end{cases} \\ & \left(0, x(yx)^{k-1}, 0\right) = p_2 u_1 w_1, \quad \left(0, (xy)^k, 0\right) = p_4 u_1 w_1, \\ & \left(0, y, 0\right) = u_3 w_1 + p_2 t, \end{aligned}
 \end{aligned}$$

and also from the fact that the remaining elements of the mentioned basis have the form $q \cdot t$, where q runs through the basis of the vector space $\mathrm{HH}^0(R)$ indicated in (3.2).

Finally, the inclusion $\mathrm{HH}^5(R) \subset \mathcal{H}$ is deduced from the following relations for the basis elements of $\mathrm{HH}^5(R)$ indicated in Proposition 3.20:

$$\begin{aligned}
 & \left(1, \mathrm{O}_3\right) = u_1^5, \quad \left(x(yx)^{k-1}, \mathrm{O}_3\right) = p_2 u_1^5, \quad \left((xy)^k, \mathrm{O}_3\right) = p_4 u_1^5, \\
 & \left(y, \mathrm{O}_3\right) = u_1^3 v_1, \quad \left(0, 1, \mathrm{O}_2\right) = u_1^2 w_1, \quad \left(0, x(yx)^{k-1}, \mathrm{O}_2\right) = p_2 u_1^2 w_1, \\
 & \left(0, (xy)^k, \mathrm{O}_2\right) = p_4 u_1^2 w_1,
 \end{aligned}$$

and also from the fact that the remaining elements of the mentioned basis have the form $q \cdot t$, where q runs through the basis of the vector space $\mathrm{HH}^1(R)$ indicated in Proposition 3.6. This completes the proof of the inclusion $\mathcal{H}' \subset \mathcal{H}$. $\square$

Let $\mathcal{A}_1 = K[\mathcal{X}_1]/I_1$ be the graded K -algebra defined in §1, where $\mathcal{X}_1$ is as in (1.1) and I_1 is the corresponding ideal of relations (see (1.3)–(1.20)). The (nonzero) images of the monomials in $K[\mathcal{X}_1]$ under the canonical epimorphism $K[\mathcal{X}_1] \rightarrow \mathcal{A}_1$ are also called monomials. Any element $a \in \mathcal{A}_1$ is represented as a linear combination of monomials (with coefficients in K). Propositions 4.1 and 4.5 imply that there exists a surjective homomorphism $\varphi: \mathcal{A}_1 \rightarrow \mathrm{HH}^*(R)$ of graded K -algebras that takes the generators in the set $\mathcal{X}_1$ to the corresponding generators in $\mathcal{Y}_1$ (see (4.5)). Note that no ambiguity is caused by employing the same letter to denote elements of the two sets that correspond to each other. Let $\mathcal{A}_1 = \bigoplus_{m \geq 0} \mathcal{A}_1^m$ be the direct decomposition of the algebra $\mathcal{A}_1$ into homogeneous direct summands. Now, part 1) of Theorem 1.1 is a consequence of the following statement.

Proposition 4.9. *For any $m \geq 0$,*

$$\dim_K \mathcal{A}_1^m = \dim_K \mathrm{HH}^m(R).$$

Before proving this proposition, we state the following auxiliary fact.

Lemma 4.10. *In the algebra $\mathcal{A}_1$, the following relations are true:*

$$\begin{aligned}
 & p_1^2 u_1 = p_1 u_1^2 = p_3 u_1^2 = p_1 u_1 u_3 = 0, \\
 & p_1 u_1 v_2 = u_1^2 v_2 = u_1 u_3 v_2 = p_1 u_3 w_1 = u_2^2 v_1 = u_3 v_2 w_1 = 0.
 \end{aligned}$$

The *proof* of the lemma is a direct verification of the relations mentioned above, with the use of the defining relations of the algebra $\mathcal{A}_1$, and we leave this to the reader. $\square$

Proof of Proposition 4.9. On the polynomial ring $K[\mathcal{X}_1]$, we introduce a lexicographic order such that

$$w_2 > u_3 > w_1 > v_3 > v_2 > v_1 > u_2 > u_1 > t > p_4 > p_3 > p_2 > p_1.$$

By definition, the *reduction* of an element f in $\mathcal{A}_1$ is the process of replacement of some submonomials in f by other elements of $\mathcal{A}_1$ in accordance with the following rules ($a \mapsto b$ means the replacement of every occurrence of the monomial a by the element b):

$$(4.31) \quad \begin{array}{ll} w_1^2 \mapsto u_1^2 t, & w_1 w_2 \mapsto u_1 u_3 t, \\ v_2 w_2 \mapsto p_1 u_3 t, & v_2 w_1 \mapsto p_3 u_2 t, \\ v_1 w_2 \mapsto p_4 u_1 t, & v_3 w_1 \mapsto p_3 u_1 t, \\ v_2^2 \mapsto p_1^2 t, & u_1 w_2 \mapsto u_3 w_1, \\ p_1 w_2 \mapsto u_3 v_2, & p_2 w_2 \mapsto u_3 v_1 \mapsto p_4 w_1, \\ p_1 w_1 \mapsto u_2 v_3 \mapsto u_1 v_2, & u_1^2 u_3 \mapsto u_1 v_1 + p_2 w_1, \\ p_1^{k-2} u_3 v_2 \mapsto p_3 w_1 \mapsto u_1 v_3 \mapsto u_2 v_1, & p_2 v_1 \mapsto p_4 u_1^2, \\ p_3 v_3 \mapsto p_2 v_2 \mapsto p_1 v_1 \mapsto p_4 u_2^2, & p_2 u_3 \mapsto p_4 u_1, \\ p_1^{k-1} u_3 \mapsto p_2 u_2 \mapsto p_3 u_1, & p_3 u_2 \mapsto p_1 u_1, \\ u_1 u_3 w_1 \mapsto \begin{cases} v_1 w_1 + p_2 u_1 t & \text{if } k > 2, \\ v_1 w_1 + (p_2 u_1 + p_4 u_2) t & \text{if } k = 2. \end{cases} \end{array}$$

Any replacement on the above list is called an *elementary step of reduction*. Under such a step, any nonzero monomial turns into a strictly smaller element with respect to the lexicographic order. Hence, after finitely many steps, we obtain monomials to which we cannot apply any elementary step of reduction. We say that a presentation of an element $a \in \mathcal{A}_1$ as a linear combination of monomials is in *normal form* if reduction cannot be applied to any of these monomials. Since these elementary steps correspond to some relations satisfied in the algebra $\mathcal{A}_1$ (see (1.3)–(1.20) and Lemma 4.10), any element $a \in \mathcal{A}_1$ admits at least one presentation in the normal form.

Put $q_i = \dim_K \mathcal{A}_1^i$. We denote the number of monomials in $\mathcal{A}_1^i$ presented in the normal form by $\tilde{q}_i$. It is clear that $\tilde{q}_i \geq q_i$. Since there is an epimorphism $\mathcal{A}_1^i \rightarrow \mathrm{HH}^i(R)$, we see that $q_i \geq \dim_K \mathrm{HH}^i(R)$. Consequently, it suffices to show that

$$(4.32) \quad \tilde{q}_i = \dim_K \mathrm{HH}^i(R).$$

We shall prove that all (nonzero) monomials having normal form are on the following list.

Monomials of degree $4n$:

$$(4.33) \quad \left\{ w_1 u_1^{4(n-j)-3} t^j \right\}_{j=0}^{n-1}, \left\{ p_2 w_1 u_1^{4(n-j)-3} t^j \right\}_{j=0}^{n-1}, \left\{ p_4 w_1 u_1^{4(n-j)-3} t^j \right\}_{j=0}^{n-1},$$

$$(4.34) \quad \left\{ w_1 v_1 u_1^{4(n-j-1)-1} t^j \right\}_{j=0}^{n-2}, \left\{ v_1 u_1^{4(n-j)-2} t^j \right\}_{j=0}^{n-1}, \left\{ u_1^{4(n-j)} t^j \right\}_{j=0}^n,$$

$$(4.35) \quad \left\{ p_2 u_1^{4(n-j)} t^j \right\}_{j=0}^n, \left\{ p_4 u_1^{4(n-j)} t^j \right\}_{j=0}^n, \left\{ p_1^i t^n \right\}_{i=1}^{k-1}, p_3 t^n,$$

$$(4.36) \quad u_3 w_1 t^{n-1};$$

monomials of degree $4n + 1$:

$$(4.37) \quad \left\{ w_1 u_1^{4(n-j)-2} t^j \right\}_{j=0}^{n-1}, \left\{ p_2 w_1 u_1^{4(n-j)-2} t^j \right\}_{j=0}^{n-1}, \left\{ p_4 w_1 u_1^{4(n-j)-2} t^j \right\}_{j=0}^{n-1},$$

$$(4.38) \quad \left\{ w_1 v_1 u_1^{4(n-j)-1} t^j \right\}_{j=0}^{n-1}, \left\{ v_1 u_1^{4(n-j)-1} t^j \right\}_{j=0}^{n-1}, \left\{ u_1^{4(n-j)+1} t^j \right\}_{j=0}^n,$$

$$(4.39) \quad \left\{ p_2 u_1^{4(n-j)+1} t^j \right\}_{j=0}^n, \left\{ p_4 u_1^{4(n-j)+1} t^j \right\}_{j=0}^n, u_2 t^n, p_4 u_2 t^n, p_1 u_1 t^n, p_3 u_1 t^n,$$

$$(4.40) \quad \left\{ u_3 t^n p_1^i \right\}_{i=0}^{k-2};$$

monomials of degree $4n + 2$:

$$(4.41) \quad \left\{ w_1 v_1 u_1^{4(n-j)-3} t^j \right\}_{j=0}^{n-1}, \left\{ w_1 u_1^{4(n-j)-1} t^j \right\}_{j=0}^{n-1}, \left\{ p_2 w_1 u_1^{4(n-j)-1} t^j \right\}_{j=0}^{n-1},$$

$$(4.42) \quad \left\{ p_4 w_1 u_1^{4(n-j)-1} t^j \right\}_{j=0}^{n-1}, \left\{ v_1 u_1^{4(n-j)} t^j \right\}_{j=0}^n, \left\{ u_1^{4(n-j)+2} t^j \right\}_{j=0}^n,$$

$$(4.43) \quad \left\{ p_2 u_1^{4(n-j)+2} t^j \right\}_{j=0}^n, \left\{ p_4 u_1^{4(n-j)+2} t^j \right\}_{j=0}^n, p_1 u_1 t^n, u_2^2 t^n, p_4 u_2^2 t^n,$$

$$(4.44) \quad \left\{ v_2 t^n p_1^i \right\}_{i=0}^{k-2}, u_3 u_1 t^n, v_3 t^n;$$

monomials of degree $4n + 3$:

$$(4.45) \quad \left\{ w_1 v_1 u_1^{4(n-j)-3} t^j \right\}_{j=0}^{n-1}, \left\{ w_1 u_1^{4(n-j)} t^j \right\}_{j=0}^n, \left\{ p_2 w_1 u_1^{4(n-j)} t^j \right\}_{j=0}^n,$$

$$(4.46) \quad \left\{ p_4 w_1 u_1^{4(n-j)} t^j \right\}_{j=0}^n, \left\{ v_1 u_1^{4(n-j)+1} t^j \right\}_{j=0}^n, \left\{ u_1^{4(n-j)+3} t^j \right\}_{j=0}^n,$$

$$(4.47) \quad \left\{ p_2 u_1^{4(n-j)+3} t^j \right\}_{j=0}^n, \left\{ p_4 u_1^{4(n-j)+3} t^j \right\}_{j=0}^n, w_2 t^n, v_1 u_2 t^n, v_2 u_1 t^n,$$

$$(4.48) \quad \left\{ u_3 v_2 t^n p_1^i \right\}_{i=0}^{k-3};$$

(here, $n \in \mathbb{N} \cup \{0\}$). It is easily seen that all monomials on this list have normal form.

Let

$$(4.49) \quad f = (w_2)^{\alpha_2} u_3^{\beta_3} w_1^{\alpha_1} v_3^{\gamma_3} v_2^{\gamma_2} v_1^{\gamma_1} u_2^{\beta_2} u_1^i t^j p_4^{\varepsilon_4} p_3^{\varepsilon_3} p_2^{\varepsilon_2} p_1^{\ell}$$

be a nonzero monomial represented in the normal form. Using relations satisfied in the algebra $\mathcal{A}_1$, we see that in (4.49) we have

$$\alpha_1, \alpha_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{0, 1\}, \quad \beta_2 \in \{0, 1, 2\},$$

$$i, j, \ell \in \mathbb{N} \cup \{0\}, \quad \text{and} \quad \ell \leq k - 1.$$

Next, we consider several cases successively.

1) Assume that the monomial f involves the factor w_2 . Then, by some steps of reduction and monomial relations satisfied in the algebra $\mathcal{A}_1$ (see (1.3)–(1.20) and Lemma 4.10), we conclude that f does not contain $u_3, w_1, v_3, v_2, v_1, u_2, u_1, p_4, p_3, p_2, p_1$. Consequently,

$$f = w_2 t^j.$$

2) Assume that f does not involve w_2 , but involves u_3 . As above, we conclude that f does not contain $v_3, v_1, u_2, p_2, p_3, p_4$. Hence,

$$f = u_3 w_1^{\alpha_1} v_2^{\gamma_2} v_1^{\gamma_1} u_1^i t^j p_1^{\ell}.$$

Here $i \leq 1$, because of the reduction $u_1^2 u_3 \mapsto u_1 v_1 + p_2 w_1$; furthermore, $\ell \leq k - 2$ (the reduction $p_1^{k-1} u_3 \mapsto p_3 u_1$), and for $\gamma_2 = 1$ we have $\ell \leq k - 3$ (the reduction $p_1^{k-2} u_3 v_2 \mapsto u_2 v_1$). Also, $\alpha_1 + \gamma_2 + i \leq 1$ (because $u_1 u_3 v_2 = u_3 v_2 w_1 = 0$ and the

monomial $u_1 u_3 w_1$ can be reduced). Finally, if $\alpha_1 = 1$, then $\ell = 0$ (because $p_1 u_3 w_1 = 0$). Consequently, f coincides with one of the following monomials:

$$u_3 t^j p_1^\ell \ (0 \leq \ell \leq k-2), \ u_3 v_2 t^j p_1^\ell \ (0 \leq \ell \leq k-3), \ u_3 w_1 t^j, \ u_3 u_1 t^j.$$

3) Assume that f involves neither w_2 nor u_3 , but involves w_1 . Arguing as above, we conclude that f does not contain v_3, v_2, u_2, p_3, p_1 . Moreover, in (4.49) we have $\gamma_1 + \varepsilon_4 + \varepsilon_2 \leq 1$ (because $p_4 v_1 = p_2 p_4 = 0$ and we have the reduction $p_2 v_1 \mapsto p_4 u_1^2$). Consequently, f coincides with one of the following monomials:

$$w_1 u_1^i t^j, \ p_2 w_1 u_1^i t^j, \ p_4 w_1 u_1^i t^j, \ w_1 v_1 u_1^i t^j$$

with $i, j \in \mathbb{N} \cup \{0\}$.

4) Assume that w_2, u_3, w_1 do not occur and v_3 occurs in f . Then it is easily seen that f does not involve $v_2, v_1, u_2, u_1, p_4, p_3, p_2, p_1$. Consequently, f is a monomial of the form $f = v_3 t^j$.

5) Assume that w_2, u_3, w_1, v_3 do not occur and v_2 occurs in f . Then f does not involve v_1, u_2, p_4, p_3, p_2 either. Hence, $f = v_2 u_1^i t^j p_1^\ell$, where $\ell \leq k-2$ (because $p_1^{k-1} v_2 = 0$) and $i \leq 1$ (because $u_1^2 v_2 = 0$); moreover, if $i = 1$, then $\ell = 0$ (because $p_1 u_1 v_2 = 0$). Consequently, f coincides with one of the following monomials:

$$v_2 t^j p_1^\ell \ (0 \leq \ell \leq k-2), \ v_2 u_1 t^j.$$

6) Assume that w_2, u_3, w_1, v_3, v_2 do not occur and v_1 occurs in f . Then f does not involve p_4, p_3, p_2, p_1 . In (4.49) we have $\beta_2 \leq 1$ because $u_2^2 v_1 = 0$; moreover, if $i \geq 1$, then $\beta_2 = 0$ (because $u_1 u_2 = 0$). Consequently, f coincides with one of the following monomials:

$$v_1 u_1^i t^j, \ v_1 u_2 t^j \ (i, j \in \mathbb{N} \cup \{0\}).$$

7) Assume that $w_2, u_3, w_1, v_3, v_2, v_1$ do not occur and u_2 occurs in f . Then f does not involve u_1, p_3, p_2, p_1 . Consequently, f is one of the following monomials:

$$u_2 t^j, \ p_4 u_2 t^j, \ u_2^2 t^j, \ p_4 u_2^2 t^j.$$

8) Assume that $w_2, u_3, w_1, v_3, v_2, v_1, u_2$ do not occur and u_1 occurs in f . Then in (4.49) we have $\ell \leq 1$ (because $p_1^2 u_1 = 0$) and $\ell + \sum_{r=2}^4 \varepsilon_r \leq 1$ (because $p_r p_s = 0$ for $r \neq s$); moreover, if $i \geq 2$, then $\varepsilon_3 = \ell = 0$ (because $p_1 u_1^2 = p_3 u_1^2 = 0$). Consequently, f is one of the following monomials:

$$u_1^i t^j, \ p_2 u_1^i t^j, \ p_4 u_1^i t^j, \ p_1 u_1 t^j, \ p_3 u_1 t^j$$

with $i \in \mathbb{N}, j \in \mathbb{N} \cup \{0\}$.

9) Finally, assume that $w_2, u_3, w_1, v_3, v_2, v_1, u_2, u_1$ do not occur in f . In this case, clearly, f coincides with one of the monomials:

$$t^j p_1^\ell \ (0 \leq \ell \leq k-1), \ t^j p_2, \ t^j p_3, \ t^j p_4.$$

Therefore, we have shown that any monomial having normal form is on the list above. It is easily seen that

$$\begin{aligned} \tilde{q}_{4n} &= 8n + k + 3, & \tilde{q}_{4n+1} &= 8n + k + 6, \\ \tilde{q}_{4n+2} &= 8n + k + 7, & \tilde{q}_{4n+3} &= 8n + k + 8. \end{aligned}$$

Now (4.32) follows from Corollary 3.18. $\square$

Case 2. Now we assume that k is odd. We consider the following homogeneous elements of $\mathrm{HH}^*(R)$:

- of degree 0 : p_1, p_2, p_3, p_4 in (3.5);
- of degree 1 : u_1, u_2 in (4.1) and $u'_3 := (0, x)$;
- of degree 2 : v_1, v_2 in (4.2);
- of degree 3 : w_1, w_2 in (4.3);
- of degree 4 : t in (4.4).

Proposition 4.11. *Assume that k is odd. In the algebra $\mathrm{HH}^*(R)$, the elements of the set*

$$(4.50) \quad \mathcal{Y}_2 = \{p_1, p_2, p_3, p_4, u_1, u_2, u'_3, v_1, v_2, w_1, w_2, t\}$$

satisfy the following relations: (4.6), (4.7), (4.9), (4.10), (4.13), (4.14), (4.17), (4.19), (4.22), and the relations

$$(4.51) \quad p_2 u_2 = p_1^{k-1} u'_3, \quad p_4 u_2 = p_3 u'_3, \quad p_2 u'_3 = p_4 u'_3 = 0;$$

$$(4.52) \quad p_2 v_2 = p_3 u_2 u'_3, \quad u_1 u_2 = p_1^{k-2} v_2, \quad (u'_3)^2 = 0;$$

$$(4.53) \quad p_1 w_1 = u_2^2 u'_3, \quad p_3 w_1 = p_1^{k-1} w_2, \quad p_4 w_1 = u'_3 v_1, \quad p_1 w_2 = u'_3 v_2, \quad p_2 w_1 = u_1^2 u'_3;$$

$$(4.54) \quad u_2 w_1 = p_1^{k-1} t, \quad u_2 w_2 = p_3 t, \quad u'_3 w_1 = p_2 t, \quad u'_3 w_2 = p_4 t;$$

$$(4.55) \quad v_1 w_2 = 0, \quad v_2 w_2 = p_1 u'_3 t, \quad u_1^2 w_2 = v_1 w_1;$$

$$(4.56) \quad w_1 w_2 = v_1 t.$$

As in the proof of Proposition 4.1, we need to compute the translates for the elements in $\mathcal{Y}_2$ for which they were not described earlier (see Remark 4.4).

Lemma 4.12. *For the role of the translates of the elements u'_3 and w_2 , we can take the homomorphisms determined by the following matrices:*

$$\begin{aligned} T^0(u'_3) &= \begin{pmatrix} 0 & x \otimes 1 \end{pmatrix}, \quad T^1(u'_3) = \begin{pmatrix} 0 & \sum_{i=1}^{k-2} i(yx)^i \otimes (xy)^{k-1-i} \\ 0 & x \otimes 1 + \sum_{i=1}^{k-2} iy(xy)^i \otimes y(xy)^{k-2-i} \end{pmatrix}, \\ T^0(w_2) &= \begin{pmatrix} 0 & y \otimes 1 \end{pmatrix}, \quad T^1(w_2) = \begin{pmatrix} 0 & 1 \otimes 1 & \star \\ 0 & 0 & \star \end{pmatrix} \end{aligned}$$

with

$$\begin{aligned} (T^1(w_2))_{13} &= \sum_{i=0}^{k-1} (i+1)x(yx)^i \otimes (xy)^{k-1-i} + \sum_{i=0}^{k-1} (i+1)(yx)^i \otimes x(xy)^{k-1-i}, \\ (T^1(w_1))_{23} &= \sum_{i=0}^{k-1} (i+1)(xy)^i \otimes y(xy)^{k-1-i} + \sum_{i=1}^{k-2} iy(xy)^i \otimes (yx)^{k-1-i}, \\ T^2(w_2) &= \begin{pmatrix} 0 & y \otimes 1 & \rho & 0 \\ 0 & 0 & y \otimes x + yx \otimes 1 & 1 \otimes y(xy)^{k-1} \end{pmatrix}, \\ T^3(w_2) &= \begin{pmatrix} 0 & y \otimes 1 & 0 & 0 \\ 0 & 0 & 1 \otimes y & \star \end{pmatrix} \end{aligned}$$

with

$$(T^3(w_2))_{24} = \sum_{i=0}^{k-1} (i+1)(yx)^i \otimes (xy)^{k-1-i}.$$

The proof of this lemma and the derivation of Proposition 4.11 from it are left to the reader.

Proposition 4.13. *Assume that k is odd. The set $\mathcal{Y}_2$ (see (4.50)) generates $\mathrm{HH}^*(R)$ as a K -algebra.*

Proof. Let $\mathcal{H}$ denote a K -subalgebra of $\mathrm{HH}^*(R)$ generated by the set $\mathcal{Y}_2 \cup \{1\}$. As in the proof of Proposition 4.5, we introduce also the subalgebra $\mathcal{H}'$ generated by the set $\bigcup_{i=0}^5 \mathrm{HH}^i(R)$. Then the proof of the fact that $\mathrm{HH}^*(R) = \mathcal{H}'$ repeats word for word the corresponding part of the proof of Proposition 4.5.

Now, we establish the inclusion $\mathcal{H}' \subset \mathcal{H}$. The inclusion $\mathrm{HH}^0(R) \subset \mathcal{H}$ follows from Remark 3.3. The inclusion $\mathrm{HH}^1(R) \subset \mathcal{H}$ is derived from (4.27) and the relations

$$(y(xy)^i, 0) = p_1^i u'_3 \quad (0 \leq i \leq k-1), \quad (x(yx)^{k-1}, 0) = p_2 u_1.$$

Next, in the group $\mathrm{HH}^2(R)$, relations (4.28), (4.29), and

$$(0, x) = u_2 u'_3, \quad (0, (xy)^{k-1}) = u_1 u'_3$$

are satisfied. This proves the inclusion $\mathrm{HH}^2(R) \subset \mathcal{H}$. Finally, the proof of the fact that $\mathrm{HH}^i(R) \subset \mathcal{H}$ for $i = 3, 4, 5$ repeats word for word the corresponding part of the proof of Proposition 4.5, except that, in the group $\mathrm{HH}^4(R)$, instead of (4.30) we use the relation $(0, y, 0) = u_1 w_2$. $\square$

Let $\mathcal{A}_2 = K[\mathcal{X}_2]/I_2$ be the graded K -algebra defined in §1, where I_2 is the corresponding ideal of relations. Propositions 4.11 and 4.13 imply that there exists a surjective homomorphism $\varphi: \mathcal{A}_2 \rightarrow \mathrm{HH}^*(R)$ of graded K -algebras that takes the generators in the set $\mathcal{X}_2$ to the corresponding generators in $\mathcal{Y}_2$. Let $\mathcal{A}_2 = \bigoplus_{m \geq 0} \mathcal{A}_2^m$ be the direct decomposition of the algebra $\mathcal{A}_2$ into homogeneous direct summands. Now, part 2) of Theorem 1.1 is a consequence of the following statement.

Proposition 4.14. *For any $m \geq 0$ we have*

$$(4.57) \quad \dim_K \mathcal{A}_2^m = \dim_K \mathrm{HH}^m(R).$$

Proof. The proof of (4.57) is similar to that of Proposition 4.9. First we need the following auxiliary fact, which is established by direct calculations.

Lemma 4.15. *In the algebra $\mathcal{A}_2$, the following relations are satisfied:*

$$\begin{aligned} p_1^2 u_1 &= p_1 u_1^2 = p_3 u_1^2 = p_4 u_1 u_2 = p_1 u_1 v_2 = 0, \\ u_1^2 u_2 &= u_1 u_2^2 = u_1 u_2 v_1 = u_1 u'_3 v_2 = u_2^2 v_1 = 0, \\ p_1 v_1 &= p_4 u_2^2, \quad u_1 u_2 u'_3 = u_2 v_1. \end{aligned}$$

Next, on the polynomial ring $K[\mathcal{X}_2]$ we introduce a lexicographic order such that

$$w_2 > u'_3 > w_1 > v_2 > v_1 > u_2 > u_1 > t > p_4 > p_3 > p_2 > p_1.$$

Then we consider the following list of elementary steps of reduction:

$$(4.58) \quad \begin{array}{ll} w_1^2 \mapsto u_1^2 t, & w_1 w_2 \mapsto v_1 t, \\ u_1^2 w_2 \mapsto v_1 w_1, & v_2 w_2 \mapsto p_1 u'_3 t, \\ v_2 w_1 \mapsto p_3 u_2 t, & u'_3 w_2 \mapsto p_4 t, \\ u'_3 w_1 \mapsto p_2 t, & u_2 w_1 \mapsto p_1^{k-1} t, \\ u_2 w_2 \mapsto p_3 t, & v_2^2 \mapsto p_1^2 t, \\ p_1 w_2 \mapsto u'_3 v_2, & p_2 w_2 \mapsto u'_3 v_1 \mapsto p_4 w_1, \\ u_2^2 u'_3 \mapsto p_1 w_1 \mapsto u_1 v_2, & p_1^{k-1} w_2 \mapsto p_3 w_1 \mapsto u_2 v_1, \\ u_1^2 u'_3 \mapsto p_2 w_1, & u_1 u_2 u'_3 \mapsto u_2 v_1, \\ p_3 u_2 u'_3 \mapsto p_2 v_2 \mapsto p_1 v_1 \mapsto p_4 u_2^2, & p_2 v_1 \mapsto p_4 u_1^2, \\ p_1^{k-2} v_2 \mapsto u_1 u_2, & p_3 u'_3 \mapsto p_2 u_2, \\ p_1^{k-1} u'_3 \mapsto p_2 u_2 \mapsto p_3 u_1, & p_3 u_2 \mapsto p_1 u_1. \end{array}$$

As in the proof of Proposition 4.9, we define a normal form of an element $f \in \mathcal{A}_2$ (with respect to the steps of reduction listed in (4.58)) and then we check that any element $f \in \mathcal{A}_2$ admits at least one normal form. Finally, analyzing successively several cases we prove that all (nonzero) monomials presented in the normal form belong to the following list.

Monomials of degree $4n$:

a) the monomials in (4.33)–(4.35), b) $w_2 u_1 t^{n-1}$;

monomials of degree $4n + 1$:

a) the monomials in (4.37)–(4.39), b) $\left\{ u'_3 t^n p_1^i \right\}_{i=0}^{k-2}$;

monomials of degree $4n + 2$:

a) the monomials in (4.41)–(4.43), b) $\left\{ v_2 t^n p_1^i \right\}_{i=0}^{k-3}$, $u'_3 u_1 t^n$, $u'_3 u_2 t^n$, $u_2 u_1 t^n$;

monomials of degree $4n + 3$:

a) the monomials in (4.45)–(4.47), b) $\left\{ u'_3 v_2 t^n p_1^i \right\}_{i=0}^{k-3}$

(here, $n \in \mathbb{N} \cup \{0\}$). Now, as in the proof of Proposition 4.9, we can obtain (4.57). The details of the corresponding arguments are left to the reader. $\square$

REFERENCES

- [1] S. F. Siegel and S. J. Witherspoon, *The Hochschild cohomology ring of a group algebra*, Proc. London Math. Soc. (3) **79** (1999), 131–157. MR1687539 (2000b:16016)
- [2] K. Erdmann and Th. Holm, *Twisted bimodules and Hochschild cohomology for self-injective algebras of class A_n* , Forum Math. **11** (1999), 177–201. MR1680594 (2001c:16018)
- [3] A. I. Generalov, *Hochschild cohomology of algebras of dihedral type*, I: *The family $D(3K)$ in characteristic 2*, Algebra i Analiz **16** (2004), no. 6, 53–122; English transl., St. Petersburg Math. J. **16** (2005), no. 6, 961–1012. MR2117449 (2005i:16013)
- [4] K. Erdmann, *Blocks of tame representation type and related algebras*, Lecture Notes in Math., vol. 1428, Springer-Verlag, Berlin, 1990. MR1064107 (91c:20016)
- [5] A. I. Generalov, *Hochschild cohomology of algebras of quaternion type*, I: *Generalized quaternion groups*, Algebra i Analiz **18** (2006), no. 1, 55–107; English transl., St. Petersburg Math. J. **18** (2007), no. 1, 37–76. MR2225213 (2007d:16021)
- [6] A. I. Generalov, A. A. Ivanov, and S. O. Ivanov, *Hochschild cohomology of algebras of quaternion type*. II. *The family $Q(2\mathcal{B})_1$ in characteristic 2*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **349** (2007), 53–134; English transl., J. Math. Sci. (New York) **151** (2008), no. 3, 2961–3009.
- [7] A. I. Generalov, *Hochschild cohomology of algebras of quaternion type*. III. *Algebras with a small parameter*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **356** (2008), 46–84; English transl. in J. Math. Sci. (New York).
- [8] K. Erdmann and A. Skowroński, *The stable Calabi–Yau dimension of tame symmetric algebras*, J. Math. Soc. Japan **58** (2006), no. 1, 97–128. MR2204567 (2007b:16037)
- [9] N. Yu. Kosovskaya, *Bimodule resolution for Liu–Schulz algebras*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **321** (2005), 213–223; English transl., J. Math. Sci. (New York) **136** (2006), no. 3, 3951–3956. MR2138419 (2006e:16012)
- [10] A. I. Generalov and N. Yu. Kosovskaya, *Hochschild cohomology of the Liu–Schulz algebras*, Algebra i Analiz **18** (2006), no. 4, 39–82; English transl., St. Petersburg Math. J. **18** (2007), no. 4, 539–572. MR2262583 (2007e:16013)
- [11] A. I. Generalov, *The Hochschild cohomology of the integer group ring of a dihedral group*. I. *The even case*, Algebra i Analiz **19** (2007), no. 5, 70–123; English transl., St. Petersburg Math. J. **19** (2008), no. 5, 723–763. MR2381942 (2009e:16015)
- [12] T. Hayami, *Hochschild cohomology ring of the integral group ring of dihedral groups*, Tsukuba J. Math. **31** (2007), 99–127. MR2337122 (2008d:16016)
- [13] K. Erdmann, Th. Holm, and N. Snashall, *Twisted bimodules and Hochschild cohomology for self-injective algebras of class A_n* . II, Algebr. Represent. Theory **5** (2002), 457–482. MR1935856 (2004a:16013)

- [14] A. I. Generalov and M. A. Kachalova, *Bimodule resolution of the Möbius algebra*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **321** (2005), 36–66; English transl., J. Math. Sci. (New York) **136** (2006), no. 3, 3850–3866. MR2138411 (2006a:16014)
- [15] M. A. Kachalova, *The Hochschild cohomology for the Möbius algebra*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **330** (2006), 173–200; English transl., J. Math. Sci. (New York) **140** (2007), no. 5, 699–715. MR2253573 (2007f:16016)
- [16] Yu. V. Volkov and A. I. Generalov, *Hochschild cohomology for self-injective algebras of tree class D_n . I*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **343** (2007), 121–182; English transl., J. Math. Sci. (New York) **147** (2007), no. 5, 7042–7073. MR2469415
- [17] Th. Holm, *Hochschild cohomology of tame blocks*, J. Algebra **271** (2004), 798–826. MR2025551 (2005c:20019)
- [18] S. Eilenberg and S. MacLane, *Cohomology theory in abstract groups. I*, Ann. of Math. (2) **48** (1947), 51–78. MR0019092 (8:367f)
- [19] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, NJ, 1956. MR0077480 (17:1040e)
- [20] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2) **78** (1963), 267–288. MR0161898 (28:5102)
- [21] M. Gerstenhaber and S. D. Schack, *Algebraic cohomology and deformation theory*, Deformation Theory of Algebras and Structures, and Applications (Il Ciocco, 1986) (M. Hazewinkel, M. Gerstenhaber, eds.), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 247, Kluwer Acad. Publ., Dordrecht, 1988, pp. 11–264. MR0981619 (90c:16016)

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