ON ARRANGEMENTS OF PLANE REAL QUINTICS WITH RESPECT TO A PAIR OF LINES

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ABSTRACT. The arrangements of an M -curve of degree 5 with respect to a pair of lines in the projective plane are studied under the natural conditions of general position and maximality.

§1. PROBLEM SETTING AND FORMULATION OF THE RESULT

In this paper we continue the classification started in [1]. As a model of the projective plane $\mathbb{R}P^2$, we use the Poincaré disk. A real projective plane curve C_n of degree n is a real homogeneous polynomial in three variables x, y, z defined up to a nonzero constant factor. The set of zeros of C_n in the real (complex) projective plane $\mathbb{R}P^2$ (respectively, $\mathbb{C}P^2$) is called the set of real (respectively, complex) points of C_n and is denoted by $\mathbb{R}C_n$ (respectively, $\mathbb{C}C_n$).

Let C_1 and C'_1 be two distinct lines. The set $\mathbb{R}P^2 \setminus (\mathbb{R}C_1 \cup \mathbb{R}C'_1)$ consists of two connected components, each homeomorphic to an open disk. We denote by D_1 and D_2 the closures of these components. Our aim is to find a topological classification of the quadruples

$$(1.1) \quad (\mathbb{R}P^2, \mathbb{R}C_5 \cup \mathbb{R}C_1 \cup \mathbb{R}C'_1, \mathbb{R}C_5 \cup \mathbb{R}C_1, \mathbb{R}C_1)$$

under the following conditions of maximality and general position.

(i) The quintic curve C_5 is an M -curve, which means that the set $\mathbb{R}C_5$ of its real points consists of the odd branch J and six ovals that do not surround one another.

(ii) The curves $\mathbb{R}C_5, \mathbb{R}C_1, \mathbb{R}C'_1$ are in general position, i.e., they have pairwise transversal intersection, and the curve $\mathbb{R}C_5$ does not pass through the point of intersection of the lines $\mathbb{R}C_1$ and $\mathbb{R}C'_1$.

(iii) Each of the lines $\mathbb{R}C_1$ and $\mathbb{R}C'_1$ intersects the odd branch J of the curve $\mathbb{R}C_5$ at five points.

(iv) Let a and b be the numbers of the arcs of J that lie in the disks D_1 and D_2 (respectively) and have one endpoint lying on the line $\mathbb{R}C_1$ and the other on $\mathbb{R}C'_1$. In this paper we consider the case where $(a, b) = (1, 3)$.

Remark. By the equivalence of the disks D_1 and D_2 , we may assume that $(a, b) \in \{(1, 1), (1, 3), (1, 5), (3, 3), (3, 5)\}$. Classification for $(a, b) = (1, 1)$ was found in [1]. For $(1, 5)$ and $(3, 5)$ classification follows easily from that of the arrangements of a quintic curve with respect to a line; see [2]. For $(3, 3)$, classification was obtained recently by S. Yu. Orevkov in [3].

The following theorem gives the answer for $(a, b) = (1, 3)$.

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Theorem 1. *The union of a quintic curve and a pair of lines is realized by exactly 51 distinct topological models of quadruples (1.1) satisfying condition (i)–(iv); the numbers of the codes of the realizable models are shown in bold in Table 1 below.*

Remark. The proof of this theorem was completed by S. Yu. Orevkov in [3]; see §5 below for the details. Pictures of realizable models are listed in Subsection 0.7 of [3]; we do not reproduce them in our paper.

To prove Theorem 1, we use the same method that was applied to prove a similar theorem in [1]: first, we list all topological models of the quadruple (1.1) satisfying conditions (i)–(iv) that do not contradict the Bézout theorem and complex orientation formulas; second, to prove the nonrealizability of some models by algebraic curves, we use the Orevkov method based on the link theory (see [4]). After that, we construct algebraic curves by the Viro patchworking method (see [5, 6, 7]). As in [1], the enumeration of the admissible arrangements and the nonrealizability of models was proved by the second author, and the construction of algebraic models was proved by the first author.

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§2. ENUMERATION OF ADMISSIBLE TOPOLOGICAL MODEL

For uniformity of our pictures, we place the point $\mathbb{R}C_1 \cap \mathbb{R}C'_1$ at the ends of the horizontal diameter of the Poincaré disk, we select the disk D_2 so that it intersects the boundary of the Poincaré disk only at the point $\mathbb{R}C_1 \cap \mathbb{R}C'_1$, and we assume that the upper line in the pictures is $\mathbb{R}C_1$. The quadruples (1.1) will be coded as follows.

1) The points of intersection of the odd branch J with the line $\mathbb{R}C_1$ (respectively, $\mathbb{R}C'_1$) are enumerated from left to right along this line by the symbols 0, 1, 2, 3, 4 (respectively, from right to left by the symbols 5, 6, 7, 8, 9).

2) The location of J relative to $\mathbb{R}C_1 \cup \mathbb{R}C'_1$ is coded by a permutation $(i_1, i_2, \dots, i_{10})$ of the symbols 0, 1, ..., 9. This permutation shows the order in which the odd branch J passes through the points 0, 1, ..., 9. Here we assume that only the arc (i_1, i_{10}) of J that contains no points of the lines $\mathbb{R}C_1$ and $\mathbb{R}C'_1$ different from i_1 and i_{10} intersects the boundary of the Poincaré disk.

3) Permutations are equipped with additional symbols in angle brackets or in braces: a fragment $i_s \langle m \rangle i_{s+1}$ (respectively, $i_s \{m\} i_{s+1}$) means that if one moves along the arc (i_s, i_{s+1}) from i_s to i_{s+1} , then the region on the right (respectively, on the left) contains m ovals of the curve C_5 .

Remark. Clearly, for every model of the curve $\mathbb{R}C_5 \cup \mathbb{R}C_1 \cup \mathbb{R}C'_1$ in $\mathbb{R}P^2$, its code defined by the rules 1)–3) is not unique. We shall always choose a permutation minimal in the lexicographical order and write the additional symbols showing the location of ovals arbitrarily in a certain possible way.

First, under the conditions (ii)–(iv), we list all possible locations of the odd branch J with respect to $\mathbb{R}C_1 \cup \mathbb{R}C'_1$. By simple exhaustive search, we find that, up to symmetry, three arcs of J in the disk D_2 can connect a triple of points on $\mathbb{R}C_1$ with a triple of points on $\mathbb{R}C'_1$ only in six different ways depicted in Figure 1. After that, in each of these cases it is necessary to complete the odd branch J in all possible ways so that condition (iv) will be fulfilled. For example, for the first case in Figure 1 it is easy to get the two arrangements shown in Figure 2.

There are exactly 20 pairwise different arrangements for $\mathbb{R}C_1 \cup \mathbb{R}C'_1 \cup J$ that can be obtained from the six cases depicted in Figure 1. For each of these 20 arrangements the set $\mathbb{R}P^2 \setminus (\mathbb{R}C_1 \cup \mathbb{R}C'_1 \cup J)$ consists of 12 connected components (homeomorphic to an open disk). The next step is to find all admissible arrangements of the six ovals of a quintic curve in these 12 regions.

FIGURE 1

FIGURE 2

It turns out that, as in [1], a substantial part of the admissible arrangements of ovals is ruled out by the Bézout theorem, which we apply in two ways. First, if for a given model $\mathbb{R}C_1 \cup \mathbb{R}C'_1 \cup J$ one can produce a pencil of lines that covers some region and consists of lines, each having 7 points of intersection with $\mathbb{R}C_1 \cup \mathbb{R}C'_1 \cup J$, then this region can contain no ovals of our quintic curve. Second, if for a conjectural arrangement of ovals one smooths in some way the double points of $\mathbb{R}C_1 \cup \mathbb{R}C'_1 \cup J$, obtaining a number of nests with total number of weights exceeding 1, then such an arrangement of ovals is impossible. In particular, for the model with the code (0789614325), the second method proves that no oval can be put in any region. Thus, only 19 cases out of the 20 mentioned in the preceding paragraph remain.

After an application of the Bézout theorem (in the same manner as in [1]), some of the remaining cases contradict the classification of the unions of a line and a quintic curve as presented in [2]. Namely, for the model $\mathbb{R}C_1 \cup \mathbb{R}C'_1 \cup J$ depicted in Figure 3, the Bézout theorem allows the location of ovals only in regions marked by Latin letters. However, in [2] it was proved that any topological pair $(\mathbb{R}P^2, \mathbb{R}C_1 \cup \mathbb{R}C_5)$ must be homeomorphic to one of the two topological pairs depicted in the right upper corner of Figure 3. If we denote the number of ovals in a given region by the same Latin letter, then we obtain the following disjunction of linear systems:

$$(2.1) \quad \begin{cases} E = 2, \\ B + D = 4 \end{cases} \vee E = 6 \vee \begin{cases} A + C + F = 2, \\ B + D = 4 \end{cases} \vee A + C + F = 6.$$

In [2] it was also proved that a topological pair $(\mathbb{R}P^2, \mathbb{R}C'_1 \cup \mathbb{R}C_5)$ must be homeomorphic to one of the two topological pairs depicted in the right lower corner of Figure 3, whence it follows that

$$(2.2) \quad \begin{aligned} & \left\{ \begin{array}{l} D + E = 1, \\ B + C = 5 \end{array} \right. \vee \left\{ \begin{array}{l} D + E = 5, \\ B + C = 1 \end{array} \right. \vee \left\{ \begin{array}{l} F = 1, \\ B + C = 5 \end{array} \right. \vee \left\{ \begin{array}{l} F = 5, \\ B + C = 1 \end{array} \right. \\ & \vee \left\{ \begin{array}{l} A = 1, \\ B + C = 5 \end{array} \right. \vee \left\{ \begin{array}{l} A = 5, \\ B + C = 1. \end{array} \right. \end{aligned}$$

For this example, the nonnegative integer solutions of (2.1) and (2.2) provide the set of

FIGURE 3

admissible models, which are subject to subsequent steps of the study (see rows 6–13 in Table 1).

We complete the enumeration of the admissible topological models in precisely the same way as in [1]. We fix the orientation of the lines $\mathbb{R}C_1$, $\mathbb{R}C'_1$ and of the odd branch J ; we smooth all double points of $\mathbb{R}C_5 \cup \mathbb{R}C_1 \cup \mathbb{R}C'_1$ in accordance with the fixed orientations and get a nonsingular curve of degree 7. Applying the complex orientation formulas to $\mathbb{R}C_5$ and to the curve of degree 7 obtained, we see that the nine models with the codes $(012\{5\}98\langle 1 \rangle 34765)$, $(012\{a\}9876\langle 6-a \rangle 345)$, $(0\langle a \rangle 12\{6-a\}9876345)$, $(0\langle a \rangle 98\{6-a\}7612345)$, where $a \in \{1, 5\}$, and $(09\langle 5 \rangle 87\{1\}612345)$ contradict these formulas.

The list of codes of the remaining topological models (1.1), which are subject to further examination, is shown in the third column of Table 1.

§3. RESTRICTION BY LINK THEORY METHODS

The Orevkov method of finding restrictions on isotopy types of an algebraic curve is presented in the literature in detail. Besides the fundamental paper [4], see also [1, 8, 9]. Since an application of this method in our paper is absolutely the same as in [1], we confine ourselves to a short description and an example.

Suppose that a curve C_7 has only double points and all of them are generic. Suppose that a pencil L_P with center at $P \in \mathbb{R}P^2 \setminus \mathbb{R}C_7$ is in maximal general position relative to $\mathbb{R}C_7$, which means that (i) every line $l \in L_P$ intersects the curve $\mathbb{R}C_7$ in at least five distinct points, (ii) there is a line in L_P that intersects $\mathbb{R}C_7$ at seven points, and (iii) every line $l \in L_P$ contains at most one double point of the curve $\mathbb{R}C_7$ or at most one point of tangency with it. Consider the set $M = \mathbb{C}C_7 \cap \mathbb{C}L_P \subset \mathbb{C}P^2$, where $\mathbb{C}L_P$ is the pencil of complex lines with center at P . The set M is homeomorphic to a collection of circles glued pairwise at the double points of the curve $\mathbb{R}C_7$ and at the points of tangency of the

TABLE 1

No.	$\mathbb{R}C_1 \cup \mathbb{R}C'_1 \cup J$ and choice of pencil center	$\mathbb{R}C_1 \cup \mathbb{R}C'_1 \cup \mathbb{R}C_5$ (location of ovals)	h	Answer
1	(0127{P}634589)	(0/3)1{2}276345{1}89)	0	8.1 $\uplus_1$ 9.9*
2	— —	(01{5}2{1}7634589)	0	8.1 $\uplus_1$ 9.9
3	— —	(01{1}2{5}7634589)	>0	
4	— —	(0/3)127(2)6345{1}89)	>0	
5	— —	(0/4)1{1}2{1}7634589)	>0	
6	(0127(P)896345)	(0/2)1{3}2{1}7896345)	0	8.2 $\uplus_1$ 9.3*
7	— —	(0/5)1(1)27896345)	0	13 $\uplus_3$ 15.2
8	— —	(01{1}2{3}7{2}896345)	0	8.2 $\uplus_1$ 9.3
9	— —	(0/1)1(1)2789(4)6345)	>0	
10	— —	(0/1)127896345{5})	>0	
11	— —	(0/1)1(5)27896345)	>0	
12	— —	(0/5)127896345{1})	>0	
13	— —	(0/1)1{4}27896345{1})	>0	
14	(012967(P)8345)	(012{1}9(3)678(2)345)	0	8.2 $\uplus_1$ 9.4*
15	— —	(0129{2}678{3}3{1}45)	0	8.2 $\uplus_1$ 9.4
16	— —	(0129678(5)3(1)45)	0	restricted in [3, 4.1] PH
17	— —	(0129678(1)3(5)45)	>0	
18	— —	(0129(4)8(1)3(1)4567)	>0	
19	(012983456{P}7)	(0/2)12{1}9(3)834567)	0	8.1 $\uplus_1$ 9.6*
20	— —	(0/5)1298345{1}67)	0	8.1 $\uplus_1$ 9.6
21	— —	(0129(4)8(1)3(1)4567)	0	constructed in [3, 2.3]
22	— —	(01298(1)3(5)4567)	0	constructed in [3, 2.3]
23	— —	(01298(5)3(1)4567)	>0	
24	— —	(0/1)129(4)8345{1}67)	>0	
25	— —	(0/1)1298345{5}67)	>0	
26	— —	(012{1}9(3)8(2)34567)	>0	
27	(012983(P,*)4765)	(01{1}2983(5)4765)	0	constructed in [3, 2.3]
28	— —	(01{5}2983(1)4765)	0	constructed in [3, 2.3]
29	(012{P,*}9876345)	(0/1)1{4}2{1}9876345)	0	8.1 $\uplus_1$ 9.3*
30	— —	(01{1}298{4}7{1}6345)	0	8.1 $\uplus_1$ 9.3
31	— —	(01{1}29{1}876{4}345)	0	8.1 $\uplus_1$ 9.4
32	— —	(01{4}2{1}9876(1)345)	0	8.1 $\uplus_1$ 9.4*
33	— —	(01298{1}76(5)345)	>0	
34	— —	(012(5)9876{1}345)	>0	
35	— —	(01298{5}76(1)345)	>0	
36	— —	(012(1)9876{5}345)	>0	
37	(01(P,*)29876345)	(01(1)2(4)987(1)6345)	0	13 $\uplus_3$ 15.1
38	— —	(01(5)2987(1)345)	>0	
39	— —	(01(1)298763{5}45)	>0	
40	(01298763(P,*)45)	(01{1}298763(5)45)	>0	
41	— —	(01{1}29876(4)3(1)45)	0	restricted in [3, 0.1] PH
42	— —	(01{5}298763(1)45)	>0	
43	(0129{P,*}876345)	(01{1}29{5}876345)	>0	
44	(0129{P,*}876345)	(0129{1}8763{5}45)	>0	
45	(012987{P,*}6345)	(0129(1)87{5}6345)	>0	
46	— —	(0129(5)87{1}6345)	>0	
47	(0(P,*)129876345)	(0/4)129(1)876345{1})	0	8.1 $\uplus_1$ 9.1
48	(0129876345) (no necessary pencil)	(012987(1)6345{5})	—	restricted by reduction to $C_2 \cup C_5$ in [3]
49	— —	(01{5}29876345{1})	—	— —

No.	$\mathbb{R}C_1 \cup \mathbb{R}C'_1 \cup J$ and choice of pencil center	$\mathbb{R}C'_1 \cup \mathbb{R}C'_5 \cup \mathbb{R}C_5$ (location of ovals)	h	Answer
50	$(014563\{P\}2987)$	$(0145\{4\}63298\{2\}7)$	0	$8.2 \uplus_1 8.9$
51	— —	$(0145\{2\}632(2)9(2)87)$	0	$8.2 \uplus_1 9.9^*$
52	— —	$(01\langle 2 \rangle 45632(2)9\langle 2 \rangle 87)$	>0	
53	— —	$(0\langle 4 \rangle 145632\{2\}987)$	>0	
54	— —	$(014563298\{6\}7)$	>0	
55	$(01456783\{P\}29)$	$(0145678\{5\}329\{1\})$	0	$13 \uplus_3 15.3$
56	— —	$(0\langle 4 \rangle 145678\{1\}329\{1\})$	0	constructed in [3, 2.3]
57	— —	$(0\langle 3 \rangle 1\langle 2 \rangle 45678329\{1\})$	0	restricted in [3, 4.2] PH
58	— —	$(0\langle 3 \rangle 1\{1\}45678329\{2\})$	>0	
59	— —	$(0145678\{1\}329\{5\})$	>0	
60	$(014587\{P\}6329)$	$(0145876\{6\}329)$	0	$13 \uplus_3 15.4$
61	— —	$(0\langle 4 \rangle 145876\{2\}329)$	0	constructed in [3, 2.3]
62	— —	$(0145876\{2\}329\{4\})$	>0	
63	$(014783\{P\}2965)$	$(01\langle 2 \rangle 47832965\langle 4 \rangle)$	0	constructed in [3, 2.3]
64	— —	$(01\langle 6 \rangle 47832965)$	0	constructed in [3, 2.3]
65	$(07\langle P \rangle 89612345)$	$(0\{3\}7\{2\}8961\{1\}2345)$	0	$4.2 \uplus_1 9.5$
66	— —	$(0\{1\}78\langle 2 \rangle 961\{3\}2345)$	0	$4.2 \uplus_1 9.5^*$
67	— —	$(0\langle 5 \rangle 789612345\{1\})$	0	restricted in [3, 4.3] PH
68	— —	$(0\langle 1 \rangle 789612345\{5\})$	>0	
69	— —	$(0\langle 1 \rangle 789\langle 4 \rangle 612345\{1\})$	>0	
70	$(07\langle P \rangle 89632145)$	$(0\{4\}7\{2\}89632145)$	0	$8.2 \uplus_1 9.8$
71	— —	$(0\langle 2 \rangle 7896\{2\}321\{2\}45)$	0	$8.2 \uplus_1 9.8^*$
72	— —	$(07\{2\}89\langle 4 \rangle 632145)$	>0	
73	— —	$(0\{6\}789632145)$	>0	
74	$(0967\langle P \rangle 812345)$	$(0\{3\}9\{2\}6781\{1\}2345)$	0	$8.2 \uplus_1 9.2$
75	— —	$(096\langle 2 \rangle 78\{1\}1\{3\}2345)$	0	$8.2 \uplus_1 9.2^*$
76	— —	$(096\langle 5 \rangle 781\langle 1 \rangle 2345)$	0	$8.1 \uplus_1 9.8$
77	— —	$(096\langle 2 \rangle 781\{3\}234\{1\}5)$	0	$8.1 \uplus_1 9.8^*$
78	— —	$(096\langle 1 \rangle 781\langle 5 \rangle 2345)$	>0	
79	— —	$(096\langle 1 \rangle 78123\langle 5 \rangle 45)$	>0	
80	— —	$(09\langle 4 \rangle 6\langle 1 \rangle 781\langle 1 \rangle 2345)$	>0	
81	— —	$(09\langle 3 \rangle 6\langle 2 \rangle 781\{1\}2345)$	>0	
82	— —	$(09\langle 4 \rangle 6\langle 1 \rangle 78123\langle 1 \rangle 45)$	>0	
83	— —	$(096\langle 5 \rangle 78123\langle 1 \rangle 45)$	>0	
84	$(096783\{P\}2145)$	$(0\{2\}96\langle 2 \rangle 78321\{2\}45)$	0	$8.2 \uplus_2 11.2 \uplus_2 12.1$
85	— —	$(096783\langle 2 \rangle 21\{4\}45)$	>0	
86	— —	$(096\langle 4 \rangle 783\langle 2 \rangle 2145)$	>0	
87	— —	$(096783\langle 6 \rangle 2145)$	>0	
88	$(098123456\{P\}7)$	$(0\{1\}9\langle 3 \rangle 8\langle 2 \rangle 1234567)$	0	$8.1 \uplus_1 9.7^*$
89	— —	$(0\langle 1 \rangle 9812345\{5\}67)$	0	$13 \uplus_3 15.1$
90	— —	$(09\langle 3 \rangle 8\langle 2 \rangle 1234\{1\}567)$	0	$8.2 \uplus_1 9.1^*$
91	— —	$(09\langle 1 \rangle 81234\{3\}5\{2\}67)$	0	$8.2 \uplus_1 9.1$
92	— —	$(098\langle 5 \rangle 123\langle 1 \rangle 4567)$	0	$8.1 \uplus_1 9.7$
93	— —	$(0\langle 1 \rangle 9\langle 4 \rangle 812345\{1\}67)$	0	constructed in [3, 2.3]
94	— —	$(09\langle 4 \rangle 8\langle 1 \rangle 1\langle 1 \rangle 234567)$	>0	
95	— —	$(09\langle 4 \rangle 8\langle 1 \rangle 123\langle 1 \rangle 4567)$	>0	
96	— —	$(098\langle 1 \rangle 1\langle 5 \rangle 234567)$	>0	
97	— —	$(098\langle 5 \rangle 1\langle 1 \rangle 234567)$	>0	
98	— —	$(098\langle 1 \rangle 123\langle 5 \rangle 4567)$	>0	
99	— —	$(0\langle 2 \rangle 9\langle 3 \rangle 81234\{1\}567)$	>0	
100	— —	$(0\langle 1 \rangle 981234\{4\}5\{1\}67)$	>0	
101	— —	$(0\langle 2 \rangle 9\langle 3 \rangle 8\{1\}1234567)$	>0	
102	— —	$(0\langle 5 \rangle 9812345\{1\}67)$	>0	

No.	$\mathbb{R}C_1 \cup \mathbb{R}C'_1 \cup J$ and choice of pencil center	$\mathbb{R}C_1 \cup \mathbb{R}C'_1 \cup \mathbb{R}C_5$ (location of ovals)	h	Answer
103	$(098143\{P\}2567)$	$(098\langle 2 \rangle 1\{2\}4\{2\}32567)$	0	$8.2 \uplus_2 11.4 \uplus_2 12.1$
104	— —	$(098\langle 4 \rangle 1\{2\}432567)$	>0	
105	— —	$(0981\{4\}43\langle 2 \rangle 2567)$	>0	
106	— —	$(098143\langle 6 \rangle 2567)$	>0	
107	— —	$(09814325\{6\}67)$	>0	
108	— —	$(0981\{4\}4325\{2\}67)$	>0	
109	— —	$(09814\{4\}325\{2\}67)$	>0	
110	$(0983\{P\}214567)$	$(098\langle 4 \rangle 3\langle 2 \rangle 214567)$	0	$8.2 \uplus_1 9.7$
111	— —	$(0\{2\}98\langle 2 \rangle 321\{2\}4567)$	0	$8.1 \uplus_1 9.7^*$
112	— —	$(0983\langle 2 \rangle 21\{4\}4567)$	>0	
113	— —	$(0983\langle 6 \rangle 214567)$	>0	
114	— —	$(0\langle 2 \rangle 98\{2\}321\langle 2 \rangle 4567)$	>0	
115	$(0\{P, *\}987612345)$	$(0\{4\}9\{1\}8761\{1\}2345)$	0	$8.1 \uplus_1 9.2$
116	— —	$(0\{4\}987\{1\}61\{1\}2345)$	0	$8.1 \uplus_1 9.5$
117	— —	$(0\{1\}9876\langle 1 \rangle 1\{4\}2345)$	0	$8.1 \uplus_1 9.2^*$
118	— —	$(0\langle 1 \rangle 98\{1\}761\{4\}2345)$	0	$8.1 \uplus_1 9.5^*$
119	— —	$(09876\langle 1 \rangle 1234\{5\}5)$	>0	
120	— —	$(0\langle 1 \rangle 98\{5\}7612345)$	>0	
121	— —	$(0\{5\}9876\langle 1 \rangle 12345)$	>0	
122	$(098761\langle P, * \rangle 2345)$	$(09\langle 1 \rangle 8761\langle 5 \rangle 2345)$	>0	
123	— —	$(09\langle 5 \rangle 8761\langle 1 \rangle 2345)$	>0	
124	$(09876123\langle P, * \rangle 45)$	$(09\langle 1 \rangle 876123\langle 5 \rangle 45)$	>0	
125	— —	$(09\langle 5 \rangle 876123\langle 1 \rangle 45)$	>0	
126	$(0\{P, *\}987612345)$	$(0\langle 4 \rangle 98761\{1\}2345\{1\})$	0	restricted in $[3, 0.1]$ PH
127	(0987612345) (no necessary pencil)	$(09\langle 5 \rangle 87612345\{1\})$	—	restricted by reduction to $C_2 \cup C_5$ in $[3]$
128	— —	$(09\langle 1 \rangle 87612345\{5\})$	—	— —
129	$(2\langle P \rangle 109834567)$	$(2\{2\}10\langle 2 \rangle 9\langle 2 \rangle 834567)$	0	$8.2 \uplus_1 9.6^*$
130	— —	$(21\langle 2 \rangle 09834\{4\}567)$	0	$8.2 \uplus_1 9.6$
131	— —	$(2\{2\}109\langle 2 \rangle 8\langle 2 \rangle 34567)$	>0	
132	— —	$(21\langle 2 \rangle 09\langle 4 \rangle 834567)$	>0	
133	— —	$(21\langle 6 \rangle 09834567)$	>0	

lines in L_P with that curve. If we unglue all circles via some standard perturbation, we obtain a link $K(C_7, P)$. Let $b(C_7, P)$ be a braid whose closure coincides with $K(C_7, P)$. For what follows, it is essential that the braid $b(C_7, P)$ is uniquely determined (up to conjugation in the group B_7 of braids of seven strings) by the mutual arrangement of the model of C_7 and the pencil L_P in $\mathbb{R}P^2$.

It is known that the above braid must be quasipositive, which means that it can be written in the form $\prod_{j=1}^k \omega_j \sigma_{i_j} \omega_j^{-1}$, where σ_s , $s \in \{1, 2, \dots, 6\}$, are the standard generators of group B_7 , and ω_j , $j \in \{1, 2, \dots, k\}$, are some words written in the alphabet $\{\sigma_1, \dots, \sigma_6, \sigma_1^{-1}, \dots, \sigma_6^{-1}\}$. Now, suppose we are given a topological model of a conjectural curve $\mathbb{R}C_7$, together with any possible mutual location of the pencil L_P and this model. If we find that the braid $b(C_7, P)$ is not quasipositive, then this model is not realizable by any algebraic curve of degree 7. As a necessary condition of quasipositivity, we employ the Murasugi–Tristram inequality, which can be written in the form $h \leq 0$, where h is a linear combination of the signature and defect of the link $K(C_7, P)$ and the algebraic length of the braid $b(C_7, p)$.

Almost all steps of the application of the Orevkov method (finding a word that determines the braid $b(C_7, P)$, computation of the invariant h , etc.) were realized as a computer algorithm in the paper [4] and were improved in [1, 8, 9]. However, the choice of an appropriate point P for the center of the pencil is done “manually”. This choice is

indicated in the second column of Table 1, where the meaning of the angle brackets or braces is the same as in rule 3) for codes in §2 above, and the star * means that we pick the point P inside an oval in the respective region. After the position of P is chosen, it is necessary to examine all possible locations of models with respect to the pencil L_P . As an example, we examine the arrangement coded by (0981432567).

In Figure 4.1, the dotted lines of the pencil L_P are tangent lines that touch the odd branch of our quintic curve. The connected components of the complement to the model under study are denoted by Latin letters. These components must contain six ovals of the quintic curve, which lie outside one another. The admissible arrangements of ovals in the connected components have already been listed; for the example in question, see rows 103–109 of Table 1. In accordance with the Bézout theorem, the ovals can be located only in hatched areas of these components.

The mutual location of the odd branch J and the lines $\mathbb{R}C_1$ and $\mathbb{R}C'_1$ shown in Figure 4.2 is the same as in Figure 4.1. To turn Figure 4.1 into Figure 4.2, one should merely choose the line that intersects the model curve at the points 1–7 for the role of the boundary of the Poincaré disk. Finally, Figure 5 shows the affine part of the Poincaré disk of Figure 4.2 (the boundary of the Poincaré disk becomes the line at infinity, P becomes the point at infinity of the ordinate axis, the pencil L_P turns into the pencil of vertical parallel lines, and the hatched areas turn into hatched areas that must contain the six ovals).

FIGURE 4

FIGURE 5

Next, using Figure 5, we compose a code, the parameters of which are the numbers of ovals in the horizontal strips of the hatched areas shown in Figure 5. With the help of a computer, we run through all collections of parameters corresponding to arrangements of ovals lying in the components A–E and compute the invariant h for each collection. If

for every collection of parameters we obtain $h > 0$ (we mark this in the fourth column of Table 1), then no location with the given arrangement of ovals in the components A–E is realizable by an algebraic curve of degree 7.

§4. CONSTRUCTION BY THE VIRO METHOD

In this section we construct curves of degree 5 having some set of singular points and then smooth them by the Viro method.

4.1. Curves with singularities of types D_4 and A_4 .

Lemma 1. *There exists a curve of degree 5 with singularity D_4 at the point $(0 : 0 : 1)$ and A_4 at $(1 : 0 : 0)$ shown in Figure 6.1. This curve has the Newton polygon with vertices $(0, 3)$, $(0, 5)$, $(3, 0)$ and the chart shown in Figure 6.2.*

FIGURE 6

Proof. Applying the hyperbolism $(x : y : z) \mapsto (xy : yz : z^2)$ to the curve of degree 3 shown in Figure 7, we get the required curve. $\square$

FIGURE 7

The following lemma provides smoothing of the singularities of type D_4 . It was proved in [7] (for the details, see also Lemmas 1.1–1.4 in [8]).

Lemma 2. *For any numbers $0 < a_1 < a_2 < a_3$, there exist two curves of degree 3 passing through the points $(1 : a_i : 0)$, $i \in \{1, 2, 3\}$ and having the charts shown in Figure 8.*

The following lemma provides smoothing of the singularities of type A_4 .

Lemma 3. *There exist nine curves of degree 5 having the Newton polygon with vertices $(0, 5)$, $(5, 0)$, $(3, 0)$ and the charts shown in Figure 9.*

FIGURE 8

FIGURE 9

Proof. Three curves of degree 5 in Figure 10 were constructed in [9] (their codes in [9] are f_5^{100} , f_5^{110} , f_5^{010}). Each of them has two ovals, an odd branch, and a singularity of type E_8 at the point $(0 : 0 : 1)$. If we choose coordinate systems as shown in Figure 10, i.e., the axes $x = 0$ and $z = 0$ are the same and the axis $y = 0$ takes positions 1–9 consecutively, then we obtain curves with the charts depicted in Figures 9.1–9.9, respectively. $\square$

Theorem 4.1. *The 35 topological models that have symbol $\uplus_1$ in the fifth column of Table 1 are realizable by unions of a quintic curve and a pair of lines.*

Proof. Consider the triples of curves provided by Lemmas 1–3. The first element of every triple is a curve of degree 5 with the chart shown in Figure 6.2. The second element is a curve with one of the two charts shown in Figure 8. The third element is either a curve with one of the nine charts shown in Figure 9 or a curve with one of the nine new charts obtained from the charts shown in Figure 9 by the transformation $(x, y) \mapsto (-x, -y)$. Every triple satisfies the Viro patchworking theorem proved in [5]. As a result of every such patchworking, we obtain an arrangement of $\mathbb{R}C_5 \cup \mathbb{R}C_1 \cup \mathbb{R}C'_1$ that realizes the quadruple (1.1) and satisfies conditions (i)–(iv), where the axes $y = 0$ and $z = 0$ play the roles of the lines $\mathbb{R}C_1$ and $\mathbb{R}C'_1$. For example, the chart of a curve that realizes model 2 of Table 1 can be obtained by patchworking the charts shown in Figures 6.2, 8.1, and 9.9. In the last column of Table 1, we indicate the number of charts in Figures 8 and 9 that are involved in patchworking. The star means that a chart in Figure 9.9 was transformed by the rule $(x, y) \mapsto (-x, -y)$. $\square$

FIGURE 10

Remark. The number of distinct triples of charts considered in the proof equals $1 \times 2 \times (9 + 9) = 36$, but the patchworkings $8.1 \uplus_1 9.1^*$ and $8.1 \uplus_1 9.3^*$ give the same model, so that we get 35 instead of 36 topological models (1.1).

4.2. Curves with singularities of types D_4 and A_3 .

Lemma 4. *There exist two curves of degree 5, shown in Figures 11.1 and 11.3, having the Newton polygon with the vertices $(0, 3)$, $(4, 1)$, $(5, 0)$, $(3, 0)$, and having the charts shown in Figures 11.2 and 11.4, respectively.*

FIGURE 11

Proof. Patchworking of these curves is shown in Figures 11.2 and 11.4. □

Lemma 5. *There exist two curves of degree 5 having the Newton polygon with vertices $(0, 3)$, $(0, 5)$, $(4, 1)$ and having the charts shown in Figures 12.1 and 12.2, respectively.*

Proof. These curves are also obtained by patchworking; see Figure 12. $\square$

FIGURE 12

Theorem 4.2. *The two topological models shown in rows 84 and 103 are realizable by unions of a quintic curve and a pair of lines.*

Proof. We consider triples provided by Lemmas 2, 4, 5 and patchwork them as is shown in the fifth column of rows 84 and 103 of Table 1. The axes $y = 0$ and $z = 0$ play the role of the lines $\mathbb{R}C_1$ and $\mathbb{R}C'_1$. $\square$

4.3. Curves with singularity of type A_3 .

Lemma 6. *There exists a curve of degree 5 having the Newton polygon with the vertices $(0, 2)$, $(0, 5)$, $(5, 0)$ and the chart shown in Figure 13.*

FIGURE 13

Proof. Consider the arrangement of a conic K and two lines L and $y = 0$ shown in Figure 14.1 (the axis $y = 0$ is tangent to K at the origin O). Let K and L have equations $C_2 = 0$ and $L = 0$, respectively.

Let four points $a_1, a_2, a_3, a_4 \in K$ be close to O . Let $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ be equations of the lines a_1O, a_2O, a_3a_4 , respectively. Consider the curve of degree 3 with the equation $C_3 \equiv C_2L + tL_1L_2L_3 = 0$. If t is a small parameter ($|t| \ll 1$) with a

FIGURE 14

suitable sign, then this curve looks like that shown in Figure 14.1. Let C_5 denote the result of smoothing the union of the conic K and the cubic curve C_3 such that the double points at a_1, a_2, a_3, a_4 are smoothed as shown in Figure 14.2, but the A_3 -singularity at O remains intact. Choose a projective coordinate system with axes $x = 0, y = 0, x = 0$, as shown in Figure 14.2. In this coordinate system, the chart of the curve C_5 is as shown in Figure 13. $\square$

Lemma 7. *There exist four curves of degree 4 having the Newton polygon with vertices $(0, 0), (0, 2)$, and $(4, 0)$ and having the charts shown in Figure 15.*

FIGURE 15

Proof. Such curves are well known. In particular, in [9], one can find patchworking for the charts of Figures 15.1 and 15.2 (in [9] they have codes f^{000} and f^{010} , respectively). The charts shown in Figures 15.3 and 15.4 can easily be obtained from the charts of Figures 15.1 and 15.2: the axis $x = 0$ can be rotated suitably around the point $(0 : 1 : 0)$. $\square$

Theorem 4.3. *The five topological models marked with $\oplus_1$ in the fifth column of Table 1 (see rows 7, 37, 55, 60, 89) are realizable by unions of a quintic curve and a pair of lines.*

Proof. Since the curves provided by Lemmas 2 and 7 satisfy the Viro patchworking theorem [5], we can patchwork the chart of Figure 8 in turn with the charts of Figures 15.1–15.4. As a result, we get charts of curves that realize the required models. For the models of rows 55, 60, and 89, the axes $x = 0$ and $y = 0$ play the role of the lines

$\mathbb{R}C_1$ and $\mathbb{R}C'_1$, and for rows 7 and 37, the same role is played by the axes $y = 0$ and $z = 0$. $\square$

5. Completion of the classification by S. Yu. Orevkov in [3]. We studied 114 models of Table 1 after which the realizability question remained open for 18 models. We communicated this information to S. Yu. Orevkov, who studied a similar problem, and he completed the classification. In the paper [3], he proved that the 9 models of rows 21, 22, 27, 28, 56, 61, 63, 64, 93 of Table 1 are realized by algebraic curves of degree 7, and the 9 models of rows 16, 41, 48, 49, 57, 67, 126, 127, 128 are nonrealizable. Note that for the models of rows 41 and 126 the answer was announced in [3] without proof.

Remark. Also in [3], Orevkov enlarged the list of examples (which he obtained in earlier papers) of topological models that are nonrealizable by algebraic curves of the degree in question, but are realizable by pseudoholomorphic curves. In particular, in [3], it was proved that models 16, 41, 57, 67, 126 of Table 1 are realizable by pseudoholomorphic curves (we mark them by the symbol “PH” in the fifth column).

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