SPECTRAL SYNTHESIS IN THE KERNEL OF A CONVOLUTION OPERATOR ON WEIGHTED SPACES

R. S. YULMUKHAMETOV

ABSTRACT. Weighted spaces of analytic functions on a bounded convex domain $D \subset \mathbb{C}^p$ are treated. Let $U = \{u_n\}_{n=1}^\infty$ be a monotone decreasing sequence of convex functions on D such that $u_n(z) \rightarrow \infty$ as $\text{dist}(z, \partial D) \rightarrow 0$. The symbol $H(D, U)$ stands for the space of all $f \in H(D)$ satisfying $|f(z)| \exp(-u_n(z)) \rightarrow 0$ as $\text{dist}(z, \partial D) \rightarrow 0$, for all $n \in \mathbb{N}$. This space is endowed with a locally convex topology with the aid of the seminorms $p_n(f) = \sup_{z \in D} |f(z)| \exp(-u_n(z))$, $n = 1, 2, \dots$. Clearly, every functional $S \in H^*(D)$ is a continuous linear functional on $H(D, U)$, and the corresponding convolution operator $M_S : f \rightarrow S_w(f(z+w))$ acts on $H(D, U)$. All elementary solutions of the equation

$$(*) \quad M_S[f] = 0,$$

i.e., all solutions of the form $z^\alpha e^{\langle a, z \rangle}$, $\alpha \in \mathbb{Z}_+^p$, $a \in \mathbb{C}^p$, belong to $H(D, U)$. It is shown that the system $E(S)$ of elementary solutions is dense in the space of solutions of equation $(*)$ that belong to $H(D, U)$.

INTRODUCTION

For a domain D in $\mathbb{C}^p$, let $H(D)$ denote the space of functions holomorphic in D , and let $H^*(D)$ be the strong dual of $H(D)$. By the Hahn–Banach theorem, every $S \in H^*(D)$ can be extended up to a functional on the space $C(D)$ of continuous functions and can be represented in the form

$$(1) \quad S(f) = \int_D f d\mu, \quad f \in H(D),$$

where μ is a measure supported on a compact subset of D . Since the support of μ is compact, we can introduce the following convolution operator generated by S : $M_S[f](\zeta) = S_z(f(z+\zeta))$, $f \in H(D)$. The values of M_S are functions defined and holomorphic near $\zeta = 0$. Let $E(S)$ denote the set of “elementary” solutions of the homogeneous convolution equation

$$(2) \quad M_S[f](\zeta) \equiv 0, \quad f \in H(D);$$

i.e., $E(S)$ consists of solutions of the form $f(z) = z^\alpha \exp\langle z, \xi \rangle$, where $\alpha = (\alpha_1, \dots, \alpha_p)$, $\alpha_i \geq 0$, are nonnegative integers, and $\langle z, \xi \rangle = \sum_{i=1}^p z_i \xi_i$, $z^\alpha = z_1^{\alpha_1} \cdots z_p^{\alpha_p}$, $|\alpha|$ being smaller than the multiplicity of the root ξ . We remind the reader that if $\widehat{S}(\lambda) = S_z(\exp\langle \lambda, z \rangle)$ is the Laplace transform of S , and the function $f(z) = z^\alpha \exp\langle z, \xi \rangle$ is an elementary solution of (2), then $\widehat{S}(\xi) = 0$. Denoting by $W(D, S)$ the set of all solutions of equation (2), we ask whether $E(S)$ is dense in the space $W(D, S)$ of all solutions. If the answer is in the positive, the space $W(D, S)$ is said to *admit spectral synthesis*.

2000 *Mathematics Subject Classification.* Primary 32A50, 45E10, 46E10.

Key words and phrases. Weighted spaces of analytic functions, convolution operator, spectral synthesis.

Supported by RFBR (grant 06-01-00516-a.)

In dimension 1, a positive answer was obtained in [2] for convex domains. For some specific types of domains $D \subset \mathbb{C}^p$, the completeness of the system of elementary solutions was established in [3, 4, 5]. In [6], the presence of spectral synthesis was announced in the case of arbitrary multidimensional convex domains. A new version of that proof provides a solution of the spectral synthesis problem in weighted spaces of holomorphic functions. We begin with the definition of these spaces.

Let D be a bounded convex domain in $\mathbb{C}^p$, and let $U = \{u_n\}_{n=1}^\infty$ be a monotone decreasing sequence of convex functions on D such that $u_n(z) \rightarrow \infty$ as $\text{dist}(z, \partial D) \rightarrow 0$. We denote by $H(D, U)$ the space of functions $f \in H(D)$ satisfying $|f(z)| \exp(-u_n(z)) \rightarrow 0$ as $\text{dist}(z, \partial D) \rightarrow 0$, for all $n \in \mathbb{N}$. A locally convex topology is introduced in this space with the help of the seminorms $p_n(f) = \sup_{z \in D} |f(z)| \exp(-u_n(z))$, $n = 1, 2, \dots$. Clearly, an arbitrary $S \in H^*(D)$ is a continuous linear functional on $H(D, U)$, and the corresponding convolution operator M_S acts on $H(D, U)$. All elementary solutions of equation (2) belong to $H(D, U)$. It is natural to ask whether the system $E(S)$ is dense in the space of all solutions of (2) that belong to $H(D, U)$. In this paper, a positive answer is given under certain conditions on the weights. The proof follows the usual pattern involving the Laplace transformation. For a continuous linear functional $S \in H^*(D, U)$, its Laplace transform is defined by $\hat{S}(\lambda) = S_z(\exp\langle \lambda, z \rangle)$, $\lambda \in \mathbb{C}^p$. The following theorem was proved in [8].

Theorem A. *Let $U = \{u_n, n \in \mathbb{N}\}$ be a monotone decreasing sequence of convex functions on a bounded convex domain D , and let $v_n(\lambda) = \sup\{\text{Re}\langle \lambda, \bar{z} \rangle - u_n(z) : z \in D\}$, $\lambda \in \mathbb{C}^p$, $n \in \mathbb{N}$, be the Young conjugate functions for $u_n(z)$. Suppose that the following conditions are satisfied:*

- 1) *for every n there exists a number c_n with*
- (3)
$$v_{n+1}(\lambda) \geq v_n(\lambda) + \ln(1 + |\lambda|^2) + c_n, \quad \lambda \in \mathbb{C}^p;$$
- 2) *for every n we have $v_n \in C^2(\mathbb{C}^p)$, and there exists $b_n > 0$ such that*

$$\sum_{k,m=1}^{2p} \frac{\partial^2 v_n(\lambda)}{\partial x_k \partial x_m} y_k y_m \geq \frac{12p + b_n}{1 + |\lambda|^2} |y|^2, \quad \lambda = (x_1 + ix_2, \dots, x_{2p-1} + ix_{2p}) \in \mathbb{C}^p$$

for all $y = (y_1, \dots, y_{2p}) \in \mathbb{R}^{2p}$.

Let $P(v_n)$ denote the space of entire functions F with the norm

$$\|F\|_n = \sup\{|F(\lambda)| \exp(-v_n(\lambda)) : \lambda \in \mathbb{C}^p\},$$

and let $P(D, U)$ be the union of these spaces with the topology of inductive limit. Under the above assumptions, the Laplace transformation $L : S \mapsto \hat{S}$ is a topological isomorphism of the strong dual space $H^*(D, U)$ onto $P(D, U)$.

In [7], this theorem was announced without restrictions on the quadratic forms corresponding to v_n .

For every natural number m , put

$$(4) \quad D'_m = \left\{ z \in D : \text{dist}(z, \partial D) > \frac{1}{m} \right\}.$$

Let $D_m, D'_m \subset D_m \subset D'_{m+1}$, be a sequence of convex domains with C^2 -boundary. As in [15] we agree that the functions

$$u_m(z) = \begin{cases} 0 & \text{if } z \in \overline{D}_m, \\ +\infty & \text{if } z \in D \setminus \overline{D}_m \end{cases}$$

are convex. If $U = (u_m)_{m=1}^\infty$, then $H(D, U) = H(D)$. It is well known (see [1]) that the Laplace transformation establishes a topological isomorphism between $H^*(D)$ and

the space $P(D)$ of the entire functions F that obey the following condition: there is a compact subset K of D and a constant C such that $|F(\lambda)| \leq C \exp(h_K(\lambda))$, $\lambda \in \mathbb{C}^p$, where $h_K(\lambda) = \sup\{\operatorname{Re}\langle \lambda, \bar{z} \rangle : z \in K\}$ is the support function for K . The space $P(D)$ admits the following description. Let h_m denote the support function for $\overline{D}_m$, and let $P(h_m)$ be the Banach space of entire functions with the norm $\|F\|_m = \sup\{|F(\lambda)| \exp(-h_m(\lambda)) : \lambda \in \mathbb{C}^p\}$. Then $P(D)$ is the union of all $P(h_m)$ with the topology of inductive limit. Moreover, the Young conjugates to u_m are h_m , and $P(D) = P(D, U)$. So, the Laplace transformation is a topological isomorphism between $H^*(D, U)$ and $P(D, U)$; see [1].

Theorem 1. *Let $U = \{u_n, n \in \mathbb{N}\}$ be a monotone decreasing sequence of convex functions on a bounded domain D . Suppose that the Laplace transformation is a topological isomorphism between $H^*(D, U)$ and $P(D, U)$. If the Young conjugates v_n for u_n satisfy (3), then, for any $S \in H^*(D)$, an arbitrary solution of (2) belonging to $H(D, U)$ can be approximated by linear combinations of elementary solutions in the topology of $H(D, U)$.*

The paper is organized as follows.

In §1, by using methods described in preceding work of various authors, we reduce the spectral synthesis problem in question to the problem of completeness of the polynomials in certain weighted spaces of entire functions.

In §2, we study this completeness problem itself independently.

Finally, in §3, we deduce Theorem 1 from the results of §§1, 2.

§1. SPECTRAL SYNTHESIS AND COMPLETENESS OF POLYNOMIALS

Let $U = \{u_n, n \in \mathbb{N}\}$ be a monotone decreasing sequence of convex functions defined on a bounded convex domain D and satisfying the assumptions of Theorem 1. As usual, $X^\perp$ stands for the annihilator of a subset X of a linear topological space.

For $S \in H^*(D)$, we denote by $W(D, U, S)$ the set of solutions of equation (2) in the space $H(D, U)$. By the Banach theorem, spectral synthesis for $W(D, U, S)$ is equivalent to the identification of the annihilators $E^\perp(S)$ and $W^\perp(D, U, S)$. We need the following properties of the space $H(D, U)$.

Lemma 1. *Under the assumptions of Theorem 1, we have the following statements.*

1. $H(D, U)$ is a complete locally convex reflexive space, i.e., $(H^*(D, U))^* = H(D, U)$.
2. Differentiation operators act continuously on $H(D, U)$.
3. Putting $J(D, U, S) = \{f \in P(D, U) : f/\widehat{S} \in H(\mathbb{C}^p)\}$, we have $L(E^\perp(S)) = J(D, U, S)$.
4. Denoting by $I(D, U, S)$ the closure of $I'(S) = \{f \in P(D, U) : f/\widehat{S} \text{ is a polynomial}\}$ in $P(D, U)$, we have $L(W^\perp(D, U, S)) = I(S)$.
5. The elementary solutions are complete in the space of solutions of equation (2) if and only if $I(D, U, S) = J(D, U, S)$.

Proof. The first statement can be proved by standard methods on the basis of results of [9].

By (3), the operator of multiplication by λ_1 (say) acts continuously on $P(D, U)$. Under the Laplace transformation isomorphism, multiplication by λ_1 turns into a continuous operator A on $H^*(D, U)$, specifically, the operator of “convolution multiplication” by the functional δ' , $\delta'(f) = \frac{\partial f}{\partial z_1}(0)$, $f \in H(D, U)$. By statement 1, A^* acts continuously on $H(D, U)$, and it can easily be shown (see [14]) that A^* coincides with differentiation by the variable λ_1 .

Statements 3 and 4 are proved much as the corresponding statements for $H(D)$; see [10]. We have already established the prerequisites, i.e., the reflexivity of $H(D, U)$ and its

invariance under differentiation (the latter is a consequence of the Banach completeness theorem).

Restated in more detail, the last statement of the lemma means that spectral synthesis occurs in the kernel of a convolution operator on $H(D, U)$ if and only if every function of the form $f\hat{S}$ and belonging to $P(D, U)$ can be approximated by functions of the form $p\hat{S}$, where p is a polynomial. Taking the topology of $P(D, U)$ into account, we can formulate this condition as follows: for every $f\hat{S} \in P(D, U)$, there exists $m \in \mathbb{N}$ and a sequence p_n of polynomials such that

$$|f(\lambda) - p_n(\lambda)| \leq \varepsilon_n \frac{e^{v_m(\lambda)}}{|\hat{S}(\lambda)|}, \quad \lambda \in \mathbb{C}^p,$$

where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Thus, we arrive at the problem of completeness of the polynomials in a certain weighted space of entire functions. $\square$

§2. COMPLETENESS OF THE POLYNOMIALS IN WEIGHTED SPACES OF ENTIRE FUNCTIONS

For a plurisubharmonic (p.s.h.) function v on $\mathbb{C}^p$, we denote by $P(v)$ the space of entire functions with the norm $\|F\|_{P(v)} := \sup_{\lambda \in \mathbb{C}^p} (|F(\lambda)| \exp(-v(\lambda)))$, and by $P_2(v)$ the Hilbert space of entire functions with the norm $\|F\|_{P_2(v)}^2 := \int_{\mathbb{C}^p} |F(\lambda)|^2 \exp(-2v(\lambda)) dm(\lambda)$. Let $D(z, r)$ be the ball in $\mathbb{C}^p$ centered at z and of radius r . In what follows, we often impose the following condition on a p.s.h. function:

$$(R) \quad |\varphi(\lambda) - \varphi(\zeta)| \leq B, \quad \zeta \in D(\lambda, (1 + |\lambda|^2)^{-s}), \quad \lambda \in \mathbb{C}^p.$$

(Then we say that φ satisfies condition (R) with constants $B, s > 0$.) The following theorem is the main result of this section.

Theorem 2. *Let $v(\lambda)$ be a p.s.h. function on $\mathbb{C}^p$ satisfying (R). Suppose also that there exist constants $\delta, \Delta > 0$ such that*

$$\delta(1 + |\lambda|) \leq v(\lambda) \leq \Delta(1 + |\lambda|), \quad \lambda \in \mathbb{C}^p.$$

Then every function in $P(v)$ can be approximated by polynomials in the norm of $P(v + (ps + p + 1) \ln(1 + |\lambda|^2))$.

In essence, the proof is based on a theorem in [11], Lemma 2.2 in [10], and the Hörmander theorem on L_2 -estimates. We start with formulating these statements in a form suitable for the present setting.

Theorem B (see [11]). *Put $\varphi(\lambda) = \sup^* \varphi_\alpha(\lambda)$, $\lambda \in \mathbb{C}^p$, where φ_α is a family of p.s.h. functions on $\mathbb{C}^p$ that filters to the right. Then the functions in $P_2(\varphi)$ can be approximated by functions in $\bigcup_\alpha P_2(\varphi_\alpha)$ in the norm of $P_2(\varphi + \ln(1 + |\lambda|^2))$. In particular, if the φ_α are of logarithmic growth, i.e., $\varphi_\alpha(\lambda) = O(\ln |\lambda|)$ as $\lambda \rightarrow \infty$, we deal with approximation by polynomials.*

Lemma C (see [10, Lemma 12.2]). *Suppose φ is a p.s.h. function on $\mathbb{C}^p$, and for some $\varepsilon > 0$ a function $f \in P(\varphi + \varepsilon|\lambda|)$ is the product of some functions $f_k \in P(\frac{1}{N}(\varphi + \varepsilon|\lambda|))$, $k = 1, 2, \dots, N$. If $\frac{1}{N}(\varphi(\lambda) + \varepsilon|\lambda|) \leq C + \varepsilon|\lambda|$, $\lambda \in \mathbb{C}^p$, then f can be approximated by polynomials in the norm of the space $P(\varphi + 4\varepsilon|\lambda|)$.*

It should be noted that the spectral synthesis problem under study will be reduced to the problem of weighted uniform approximation by polynomials, whereas we are going to employ Theorem B and Hörmander's L_2 -estimates, where weighted integral norms are involved. For this reason, we explain how to pass to weighted integral norms from weighted uniform norms and *vice versa*.

Lemma 2. 1. If φ is a p.s.h. function on $\mathbb{C}^p$ and $F \in P(\varphi)$, then

$$\|F\|_{P_2(\varphi+p \ln(1+|\lambda|^2))} \leq \sqrt{2S_p} \|F\|_{P(\varphi)}.$$

2. If a p.s.h. function φ on $\mathbb{C}^p$ satisfies (R), and $F \in P_2(\varphi)$, then

$$\|F\|_{P(\varphi+ps \ln(1+|\lambda|^2))} \leq \frac{e^B}{\sqrt{V_p}} \|F\|_{P_2(\varphi)}.$$

Here S_p and V_p are the area of the unit sphere and the volume of the unit ball in $\mathbb{C}^p$.

Proof. Statement 1 is trivial. The other estimate is also simple: it can be deduced from the fact that $|F|^2$ is subharmonic. $\square$

Lemma 3. Let φ be a p.s.h. function on $\mathbb{C}^p$ satisfying (R). Then for every $a \in \mathbb{C}^p$ there exist an entire function F_a such that $F_a(a) = \exp \varphi(a)$ and

$$\int_{\mathbb{C}^p} |F_a(\lambda)|^2 e^{-2\varphi(\lambda) - (2sp+3p+1) \ln(1+|\lambda|^2)} dm(\lambda) \leq C_p,$$

where $C_p = \frac{3^{2s+2}}{1+4s^2} (6e^{2B} \pi(1+4s^2))^p$.

Proof. We proceed by induction on the dimension of the space. For $p = 1$, we put

$$\alpha(t) = \begin{cases} 0 & \text{if } |t| \geq 1, \\ 1 & \text{if } |t| \leq \frac{1}{2}, \\ 2(1 - |t|) & \text{if } \frac{1}{2} \leq |t| \leq 1. \end{cases}$$

Then $|\partial\alpha(t)/\partial\bar{t}| \leq 1$. Let $a \in \mathbb{C}$ be arbitrary. We define

$$g(z) = \frac{e^{\varphi(a)}}{(z-a)} \frac{\partial}{\partial\bar{z}} \alpha((z-a)(1+|a|^2)^s), \quad z \in \mathbb{C},$$

and consider the equation

$$(2.1) \quad \frac{\partial}{\partial\bar{z}} u(z) = g(z).$$

By the properties of $\alpha(z)$ and the conditions imposed on $\varphi(z)$, we have

$$\int_{\mathbb{C}} |g(z)|^2 e^{-2\varphi(z)} dm(z) \leq 4\pi e^{2B} (1+|a|^2)^{2s}.$$

The elementary estimate

$$\sup_{|z-\lambda| \leq 1} \frac{1+|z|^2}{1+|\lambda|^2} \leq \frac{1+(1+|\lambda|)^2}{1+|\lambda|^2} \leq \frac{3+2|\lambda|^2}{1+|\lambda|^2} \leq 3$$

shows that

$$\int_{\mathbb{C}} |g(z)|^2 e^{-2\varphi(z) - (2s+1) \ln(1+|z|^2)} dm(z) \leq 4\pi 3^{2s+1} e^{2B} (1+|a|^2)^{-1}.$$

By the Hörmander theorem, equation (2.1) has a solution u satisfying

$$\int_{\mathbb{C}} |u(z)|^2 e^{-2\varphi(z) - (2s+3) \ln(1+|z|^2)} dm(z) \leq 2\pi 3^{2s+1} e^{2B} (1+|a|^2)^{-1}.$$

Since $|z-a|^2 \leq (1+|z|)^2(1+|a|)^2 \leq 4(1+|a|^2)(1+|z|^2)$, we see that

$$\int_{\mathbb{C}} |z-a|^2 |u(z)|^2 e^{-2\varphi(z) - (2s+4) \ln(1+|z|^2)} dm(z) \leq 8\pi 3^{2s+1} e^{2B}.$$

By the definition of u , the function $F_a(z) = \alpha((z-a)(1+|a|^2)^s) \exp \varphi(a) - (z-a)u(z)$, $z \in \mathbb{C}$, is entire and $F_a(a) = \exp \varphi(a)$. Moreover,

$$\int_{\mathbb{C}} |F_a(z)|^2 e^{-2\varphi(z) - (2s+4) \ln(1+|z|^2)} dm(z) \leq 50\pi 3^{2s} e^{2B}.$$

Suppose the claim is proved for $\mathbb{C}^{p-1}$. Take $a \in \mathbb{C}^p$. There is no loss of generality in assuming that a belongs to the $(p-1)$ -dimensional subspace $\{z_1 = 0\}$. By the inductive hypothesis, there exists an entire function f of $z' = (z_2, \dots, z_p) \in \mathbb{C}^{p-1}$ such that $f(a') = e^{\varphi(a')}$, $a' = (a_2, \dots, a_p)$, and

$$\int_{\mathbb{C}^{p-1}} |f(z')|^2 e^{-2\varphi(z') - (2s(p-1)+3(p-1)+1) \ln(1+|z'|^2)} dm(z') \leq C_{p-1}.$$

The expression $\varphi(a')$ is understood as the value of φ at $(0, a') \in \mathbb{C}^p$. Put $g(z) = f(z)z_1^{-1} \bar{\partial} \alpha(z_1(1+|z'|^2)^s)$, $z = (z_1, \dots, z_p) \in \mathbb{C}^p$, where f is viewed as a function on $\mathbb{C}^p$ depending only on $z' = (z_2, \dots, z_p)$, and $\alpha(w)$, $w \in \mathbb{C}$, has already been defined. Direct inspection shows that $|\bar{\partial} \alpha(z_1(1+|z'|^2)^s)|^2 \leq (1+4s^2)(1+|z'|^2)^{2s}$. The properties of $\alpha(w)$ imply $|g(z)|^2 \leq 4|f(z')|^2(1+4s^2)(1+|z'|^2)^{4s}$; moreover, $g(z) = 0$ if $|z_1| \geq (1+|z'|^2)^{-s}$. Thus,

$$\begin{aligned} & \int_{\mathbb{C}^p} |g(z)|^2 e^{-2\varphi(z) - (2sp+3p-2) \ln(1+|z|^2)} dm(z) \\ & \leq 4(1+4s^2) \int_{\mathbb{C}^{p-1}(z')} |f(z')|^2 (1+|z'|^2)^{4s} e^{-2\varphi(z') - (2sp+3p-2) \ln(1+|z'|^2)} \\ & \quad \times \int_{|z_1| \leq (1+|z'|^2)^{-s}} e^{2(\varphi(z') - \varphi(z)) + (2sp+3p-2) \ln \frac{1+|z'|^2}{1+|z|^2}} dm(z_1) dm(z'). \end{aligned}$$

Condition (R) for φ shows that

$$\int_{\mathbb{C}^p} |g(z)|^2 e^{-2\varphi(z) - (2sp+3p-2) \ln(1+|z|^2)} dm(z) \leq 4\pi(1+4s^2)e^{2B}C_{p-1}.$$

Now we consider the equation $\bar{\partial}u = g$. By the Hörmander theorem, this equation has a solution u with

$$\int_{\mathbb{C}^p} |u(z)|^2 e^{-2\varphi(z) - (2sp+3p) \ln(1+|z|^2)} dm(z) \leq 2\pi(1+4s^2)e^{2B}C_{p-1}.$$

Then

$$(2.2) \quad \int_{\mathbb{C}^p} |z_1 u(z)|^2 e^{-2\varphi(z) - (2sp+3p+1) \ln(1+|z|^2)} dm(z) \leq 2\pi(1+4s^2)e^{2B}C_{p-1}.$$

We put $F_a(z) = f(z')\alpha(z_1(1+|z'|^2)^s) - z_1 u(z)$, $z \in \mathbb{C}^p$. The definition of u implies that F_a is an entire function. By the inductive hypothesis, $F_a(a) = \exp \varphi(a)$. Next, from the properties of α we deduce that

$$\int_{\mathbb{C}^p} |f(z')|^2 \alpha(z_1(1+|z'|^2)^s) e^{-2\varphi(z) - (2sp+3p+1) \ln(1+|z|^2)} dm(z) \leq \pi e^{2B}C_{p-1}.$$

Together with (2.2) and the triangle inequality for norms, this estimate yields

$$\int_{\mathbb{C}^p} |F_a(z)|^2 e^{-2\varphi(z) - (2sp+3p+1) \ln(1+|z|^2)} dm(z) \leq e^{2B}6\pi(1+4s^2)C_{p-1}.$$

Therefore, we can take $C_1 = 50\pi e^{2B}3^{2s}$, $C_p = 6\pi e^{2B}(1+4s^2)C_{p-1}$, $p > 1$.

Lemma 3 is proved. $\square$

Corollary. *Let φ be a p.s.h. function on $\mathbb{C}^p$ satisfying (R). Then the function F_a in Lemma 3 obeys the uniform estimate*

$$|F_a(\lambda)|^2 \leq c_p e^{2\varphi(\lambda) + (3sp+3p+1)\ln(1+|\lambda|^2)}, \quad \lambda \in \mathbb{C}^p,$$

where $c_p = \frac{e^{2B} 3^{2s+2}}{V_p(1+4s^2)} (6\pi e^{2B}(1+4s^2))^p$.

Proof. This is an immediate consequence of statement 2 in Lemma 2. $\square$

Lemma 4. *Suppose that φ is a nonnegative p.s.h. function on $\mathbb{C}^p$ satisfying (R) and such that $\varphi(\lambda) \leq \Delta(1+|\lambda|)$, $\lambda \in \mathbb{C}^p$. Then for every $\sigma > 0$ the functions in the space $P(\varphi + \sigma|\lambda|)$ admit approximation by polynomials in the norm of $P(\varphi + 9\sigma|\lambda|)$.*

Proof. We introduce the following function of the variable λ :

$$w(\lambda) = \sup^* \{u(\lambda) : u \text{ is p.s.h., } u(z) = O(\ln|z|), u(z) \leq \varphi(z) + 8\sigma|z|, z \in \mathbb{C}^p\}.$$

Put $N = [\Delta/\sigma] + 3$, where $[x]$ denotes the integral part of x . Then $(\Delta + 2\sigma)/N \leq \sigma$. The function $u(\lambda) = (\varphi(\lambda) + \sigma|\lambda|)/N$ satisfies condition (R) with constants $(B + \sigma)/N$ and s . By the corollary to Lemma 3, for every $a \in \mathbb{C}^p$ there exists a function F_a that equals $\exp((\varphi(a) + \sigma|a|)/N)$ at a and satisfies

$$|F_a(\lambda)| \leq c_p e^{\frac{1}{N}(\varphi(\lambda) + \sigma|\lambda|) + (3sp+3p+1)\ln(1+|\lambda|^2)}, \quad \lambda \in \mathbb{C}^p,$$

where c_p is defined as in the corollary to Lemma 3 and depends only on the dimension p and the numbers B, s, Δ, σ . We write the upper estimate for F_a in the form $|F_a(\lambda)| \leq c'_p \exp(\frac{1}{N}(\varphi(\lambda) + 2\sigma|\lambda|))$, where c'_p depends only on p, B, Δ, s, σ , and N . The function $f_a = (F_a/c'_p)^N$ equals $(c'_p)^{-N} e^{\varphi(a) + \sigma|a|}$ at a and satisfies the inequality $|f_a(\lambda)| \leq \exp(\varphi(\lambda) + 2\sigma|\lambda|)$, $\lambda \in \mathbb{C}^p$. By the choice of N , the functions f_a satisfy the assumptions of Lemma C with 2σ in the role of ε . That lemma shows that each f_a can be approximated by polynomials in the norm of the space $P(\varphi + 8\sigma|\lambda|)$; i.e., for every $\varepsilon > 0$ there exists a polynomial $p_{a,\varepsilon}$ with $|f_a(\lambda) - p_{a,\varepsilon}(\lambda)| \leq \varepsilon \exp(\varphi(\lambda) + 8\sigma|\lambda|)$, $\lambda \in \mathbb{C}^p$. Then the polynomial $q_{a,\varepsilon} = \frac{1}{1+\varepsilon} p_{a,\varepsilon}$ obeys the inequalities $|q_{a,\varepsilon}(\lambda)| \leq \exp(\varphi(\lambda) + 8\sigma|\lambda|)$ for all λ , and $|q_{a,\varepsilon}(a)| \geq ((c'_p)^{-N} \exp(\varphi(a) + \sigma|a|) - \varepsilon \exp(\varphi(a) + 8\sigma|a|))/(1+\varepsilon)$. The upper estimate and the definition of w show that $w(\lambda) \geq \ln|q_{a,\varepsilon}(\lambda)|$, $\lambda \in \mathbb{C}^p$. At the point a we use the lower estimate for $q_{a,\varepsilon}(a)$ and let ε tend to zero, obtaining $w(a) \geq \varphi(a) + \sigma|a| - N \ln c'_p$, $a \in \mathbb{C}^p$. It follows that the space $P(\varphi + \sigma|\lambda|)$ is included in $P(w)$. By statement 1 of Lemma 2, we have $P(w) \subset P_2(w + p \ln(1 + |\lambda|^2))$. By the definition of w , the function $w + p \ln(1 + |\lambda|^2)$ is the upper envelope of a family of p.s.h. functions of logarithmic growth.

By Theorem B, the functions belonging to $P_2(w + p \ln(1 + |\lambda|^2))$ (and *a fortiori* those belonging to $P(\varphi + \sigma|\lambda|)$) can be approximated by polynomials in the norm of $P_2(w + (p+1) \ln(1 + |\lambda|^2))$, consequently, also in the norm of $P_2(\varphi + 8\sigma|\lambda|)$. Since we can replace φ by $\varphi + 8\sigma|\lambda|$ in statement 2 of Lemma 2, the latter function satisfies condition (R) with the constants $B + 8\sigma$ and s . Taking statement 2 of Lemma 2 into account, we see that the functions belonging to $P(\varphi + \sigma|\lambda|)$ can be approximated by polynomials in the norm of $P(\varphi + 8\sigma|\lambda| + ps \ln(1 + |\lambda|^2))$, consequently, in the norm of $P(\varphi + 9\sigma|\lambda|)$. This finishes the proof of Lemma 4. $\square$

Proof of Theorem 2. Suppose a function v satisfies the assumptions of Theorem 2. Then so does also the function $v_1(\lambda) = v(\lambda) + p \ln(1 + |\lambda|^2)$ with the constants $B + p \ln 3$ and s in condition (R) and with $\Delta + p$ in place of Δ . For $\alpha \in (0; 1)$, we denote the function $(1 - \alpha)v_1$ by φ_α , and apply Lemma 4 with $\sigma_\alpha = \frac{\alpha\delta}{10}$ to it. This shows that all functions belonging to $P(\varphi_\alpha + \sigma_\alpha|\lambda|)$ can be approximated by polynomials in the norm of

$P(\varphi_\alpha + 9\sigma_\alpha|\lambda|)$. Since $v_1(\lambda) \geq v(\lambda) \geq \delta(1+|\lambda|) \geq \delta|\lambda|$ and $\varphi_\alpha(\lambda) + 9\sigma_\alpha|\lambda| + p \ln(1+|\lambda|^2) \leq v_1(\lambda) - \frac{\alpha\delta}{10}|\lambda| + p \ln(1+|\lambda|^2) \leq v_1(\lambda) + \text{const}$ for all λ , statement 1 of Lemma 2 implies that for any $\alpha \in (0; 1)$ the functions in $P(\varphi_\alpha + \sigma_\alpha|\lambda|)$ can be approximated by polynomials in the norm of $P_2(v_1)$. The function $\varphi_\alpha + \sigma_\alpha|\lambda|$ satisfies condition (R) with the constants $(1 - \alpha)(B + p \ln 3)\sigma_\alpha$ and s . By Lemma 2,

$$\bigcup_{\alpha \in (0; 1)} P(\varphi_\alpha + \sigma_\alpha|\lambda|) = \bigcup_{\alpha \in (0; 1)} P_2(\varphi_\alpha + \sigma_\alpha|\lambda|).$$

Thus, the functions in these unions can be approximated by polynomials in the norm of $P_2(v_1)$. Since $\varphi_\alpha(\lambda) + \sigma_\alpha|\lambda| \leq v_1(\lambda)$ and $\lim_{\alpha \rightarrow 0}(\varphi_\alpha(\lambda) + \sigma_\alpha|\lambda|) = v_1(\lambda)$ for every $\lambda \in \mathbb{C}^p$, we see that $\sup^*(\varphi_\alpha(\lambda) + \sigma_\alpha|\lambda|) = v_1(\lambda)$. By Theorem B, the functions belonging to $P_2(v_1)$ can be approximated by functions belonging to $\bigcup_{\alpha \in (0; 1)} P_2(\varphi_\alpha + \sigma_\alpha|\lambda|)$ in the norm of $P_2(v_1 + \ln(1 + |\lambda|^2))$. Hence, the functions in $P_2(v_1)$ can be approximated by polynomials in the norm of $P_2(v_1(\lambda) + \ln(1 + |\lambda|^2))$. Since the function $v_1(\lambda) + \ln(1 + |\lambda|^2) = v(\lambda) + (p + 1) \ln(1 + |\lambda|^2)$ satisfied condition (R), statement 2 of Lemma 2 shows that the functions in $P(v)$ can be approximated by polynomials in the norm of $P(v + (ps + p + 1) \ln(1 + |\lambda|^2))$. The proof of Theorem 2 is complete. $\square$

§3. PROOF OF THEOREM 1

In §1, we reduced the spectral synthesis problem in the kernel of the convolution operator M_S , $S \in H^*(D)$, to the question of completeness of the polynomials in the inductive limit of the weighted spaces $P(v_m - \ln|\widehat{S}|)$. We start with a description of these spaces in terms of weights more suitable for an application of Theorem 2. We formulate two simple statements.

Lemma 5. *Suppose u is a lower semicontinuous function on $\overline{D}$ and denote by $v(\lambda) = \sup\{\text{Re}\langle \lambda, z \rangle - u(z) : z \in D\}$ the Young conjugate to u . Then v satisfies the Lipschitz condition $|v(\lambda_1) - v(\lambda_2)| \leq M(D)|\lambda_1 - \lambda_2|$, $\lambda_1, \lambda_2 \in \mathbb{C}^p$, where $M(D) = \max\{|z|, z \in \overline{D}\}$.*

This is a direct consequence of the definition of the Young conjugate function.

Lemma 6. *Let w and v be subharmonic functions on $\mathbb{R}^m$ satisfying $w(0) > -\infty$ and $v(x) + w(x) \leq C(1 + |x|)$, $w(x) \leq c(1 + |x|)$, $x \in \mathbb{R}^m$. Then*

$$v(x) \leq (4C + 2^{m+1}c)(1 + |x|) - 2^m w(0), \quad x \in \mathbb{R}^m.$$

Proof. In dimension 2 ($m = 2$), the claim follows from a lemma in [13]. For $m > 2$ the proof is similar.

For every $m \in \mathbb{N}$, we introduce the following families of functions:

$$K_m = \{t : t \text{ is p.s.h. in } \mathbb{C}^p, t(\lambda) \leq v_m(\lambda) - \ln|\widehat{S}(\lambda)|, \lambda \in \mathbb{C}^p\}.$$

Since the determining set of the functional S is compact, the family K_m is nonempty for m sufficiently large. Each of these families is upper bounded. Indeed, take an infinitely differentiable nonnegative function α depending only on $|z|$, equal to 0 for $|z| > 1$ and with integral 1. For every locally integrable function f , we introduce its regularization

$$f_\varepsilon(z) = \int_{\mathbb{C}^p} \frac{1}{\varepsilon^{2p}} \alpha\left(\frac{\zeta - z}{\varepsilon}\right) f(\zeta) dm(\zeta),$$

where $\varepsilon > 0$ and integration is with respect to Lebesgue measure in $\mathbb{C}^p$. It is known (see [12]) that the regularizations are infinitely differentiable and, if f is (pluri)subharmonic, then f_ε is also (pluri)subharmonic and $f_\varepsilon(z) \geq f(z)$ for every $\varepsilon > 0$. Take $\varepsilon = 1$. Since the regularization $(v_m - \ln|\widehat{S}|)_\varepsilon$ is continuous, it is bounded on each compact set. The properties of regularizations of p.s.h. functions show that for every $t \in K_m$ and all λ

we have $t(\lambda) \leq t_\varepsilon(\lambda) \leq (v_m - \ln |\widehat{S}|)_\varepsilon(\lambda)$. So, the upper envelope $V_m(\lambda) = \sup^* \{t(\lambda) : t \in K_m\}$ is a p.s.h. function. If F is entire and $\ln |F(\lambda)| \leq v_m(\lambda) - \ln |\widehat{S}(\lambda)|$ for all $\lambda \in \mathbb{C}^p$, then $\ln |F| \in K_m$. By the definition of V_m , we obtain $\ln |F(\lambda)| \leq V_m(\lambda)$ for all λ . Consequently,

$$(3.1) \quad \sup_{\mathbb{C}^p} |F(\lambda)| e^{-V_m(\lambda)} = \sup_{\mathbb{C}^p} |F(\lambda)| \frac{|\widehat{S}(\lambda)|}{e^{v_m(\lambda)}}, \quad \lambda \in \mathbb{C}^p. \quad \square$$

Lemma 7. *Suppose a functional S admits a representation in the form (1), and u is a lower semicontinuous function on D bounded from above on some set D_m (see (4)) containing the support of μ . Put $v(\lambda) = \sup \{\operatorname{Re} \langle \lambda, z \rangle - u(z) : z \in D\}$,*

$$K = \{t : t \text{ is p.s.h. in } \mathbb{C}^p, t(\lambda) \leq v(\lambda) - \ln |\widehat{S}(\lambda)|, \lambda \in \mathbb{C}^p\},$$

$$V(\lambda) = \sup^* \{t(\lambda) : t \in K\}, \quad w(z) = \int_{\mathbb{C}^p} \alpha(\zeta - z) V(\zeta) dm(\zeta).$$

Then the following statements hold true:

1. $V(\lambda) \leq w(\lambda) \leq V(\lambda) + M(D)$, $\lambda \in \mathbb{C}^p$, $M(D) = \sup \{|z| : z \in D\}$.
2. There exists a constant B depending only on D , S , and u such that $|\operatorname{grad} w(\lambda)| \leq B \sqrt{|\lambda|^2 + 1}$, $\lambda \in \mathbb{C}^p$.

Proof. The left inequality in item 1 is a direct consequence of the properties of regularization for p.s.h. functions. Next, the definition of V implies that

$$w(z) \leq \int_{\mathbb{C}^p} \alpha(\zeta - z) (v(\zeta) - \ln |\widehat{S}(\zeta)|) dm(\zeta).$$

Since the function $\ln |\widehat{S}(\zeta)|$ is subharmonic and the function v is Lipschitzian (Lemma 5), we have

$$w(z) \leq v(z) + M(D) - \ln |\widehat{S}(z)|.$$

This means that the function $w(z) - M(D)$ belongs to K , and the right inequality in item 1 follows.

Now, we estimate the gradient of w . Let $M = \sup |\operatorname{grad} \alpha(z)|$. Then

$$|\operatorname{grad} w(\lambda)| \leq M \int_{D(\lambda, 1)} |V(z)| dm(z), \quad \lambda \in \mathbb{C}^p,$$

and we must estimate the integral of $|V|$ over the ball $D(\lambda, 1)$ in $\mathbb{C}^p$. For this, we apply Lemma 6 to the functions V and $\ln |\widehat{S}|$. By the assumptions of that lemma, the functional S is representable in the form (1) and $\operatorname{supp}(\mu) \subset D_m$; therefore,

$$(3.2) \quad |\widehat{S}(\lambda)| \leq |\mu|(D) \exp \sup_{z \in D_m} \operatorname{Re} \langle \lambda, z \rangle = |\mu|(D) e^{h_m(\lambda)}, \quad \lambda \in \mathbb{C}^p.$$

Consequently,

$$(3.3) \quad \ln |\widehat{S}(\lambda)| \leq \ln |\mu|(D) + M(D)|\lambda| \leq (\ln^+ |\mu|(D) + M(D))(|\lambda| + 1), \quad \lambda \in \mathbb{C}^p.$$

If $\inf \{u(z) : z \in D\} = u_0$, the definition of V shows that

$$(3.4) \quad V(\lambda) + \ln |\widehat{S}(\lambda)| \leq v(\lambda) \leq \sup_{z \in D} \operatorname{Re} \langle \lambda, z \rangle - u_0 \leq (M(D) + u_0^-)(|\lambda| + 1)$$

for all $\lambda \in \mathbb{C}^p$. First, we assume that $\widehat{S}(0) \neq 0$. By (3.3) and (3.4), the assumptions of Lemma 6 are fulfilled for V and $\ln |\widehat{S}|$. Consequently, there is a constant A depending only on D , S , and u and such that $V(\lambda) \leq A(|\lambda| + 1)$, $\lambda \in \mathbb{C}^p$. But if $\widehat{S}(0) = 0$, we take λ_0 with $|\lambda_0| < 1$ and $\widehat{S}(\lambda_0) \neq 0$, and then estimate the functions $V(\lambda + \lambda_0) + \ln |\widehat{S}(\lambda + \lambda_0)|$ and $\ln |\widehat{S}(\lambda + \lambda_0)|$, obtaining an estimate for $V(\lambda + \lambda_0)$.

Now we estimate V from below. Let $\sup\{u(z) : z \in D_m\} = M'$. Then $v(\lambda) \geq \sup\{\operatorname{Re}\langle \lambda, z \rangle - u(z) : z \in D_m\} \geq h_m(\lambda) - M'$, $\lambda \in \mathbb{C}^p$. Taking (3.2) into account, we obtain $V(\lambda) \geq -M' - \ln |\mu|(D)$. Thus, there exists a constant B' such that $|V(\lambda)| \leq B'(|\lambda| + 1)$, $\lambda \in \mathbb{C}^p$.

Lemma 7 is proved. $\square$

Proof. Using Lemmas 5–7, we can prove Theorem 1. In §1, the problem of spectral synthesis in the kernel of M_S was reduced to the problem of completeness of the polynomials in the inductive limit of the spaces $P(v_k - \ln |\widehat{S}|)$. Next, by (3.1), it suffices to prove that the polynomials are complete in the inductive limit of the spaces $P(V_k)$. Put $w_k(\lambda) = \int_{\mathbb{C}^p} \alpha(\zeta - \lambda) V_k(\zeta) dm(\zeta)$. The first statement of Lemma 7 implies that the spaces $P(w_k)$ and $P(V_k)$ coincide, and it suffices to prove that the polynomials are complete in the inductive limit of the spaces $P(w_k)$.

We show that each w_k satisfies the assumptions of Theorem 2. By the second statement of Lemma 7, w_k satisfies (R) with some constant B and $s = \frac{1}{2}$. We estimate each w_k from below. By assumption, S is representable as in (1). Let m be the smallest natural number such that $\operatorname{supp} \mu \subset D_m$ (see (4)). Take an arbitrary index $k > m$ and put $r_k = \sup\{u_k(z) : z \in D_k\}$. The Young conjugates obey the estimate $v_k(\lambda) \geq \sup\{\operatorname{Re}\langle \lambda, z \rangle - r_k : z \in D_k\} = h_k(\lambda) - r_k$. Together with (3.4), this shows that

$$(3.5) \quad \frac{e^{v_k(\lambda)}}{|\widehat{S}(\lambda)|} \geq \frac{1}{|\mu|(D)e^{r_k}} e^{h_k(\lambda) - h_m(\lambda)}, \quad \lambda \in \mathbb{C}^p.$$

Since $D_s + D(0, \frac{1}{s(s+1)}) \subset D_{s+1}$ for any s , $h_{s+1}(\lambda) \geq h_s(\lambda) + \frac{1}{s(s+1)}|\lambda|$ for all $\lambda \in \mathbb{C}^p$. Thus, from (3.5) we deduce that $v_k(\lambda) - \ln |\widehat{S}(\lambda)| \geq -\ln |\mu|(D) - r_k + |\lambda|/(m(m+1))$. By statement 1 of Lemma 7, the same is true for the regularizations: $V_k(\lambda) \geq -\ln |\mu|(D) - r_k + |\lambda|/(m(m+1))$, $\lambda \in \mathbb{C}^p$. It remains to estimate the w_k from above. Let $b_k = \inf\{u_k(z) : z \in D\}$. Then $v_k(\lambda) \leq \sup\{\operatorname{Re}\langle \lambda, z \rangle - b_k : z \in D\} \leq M(D)|\lambda| - b_k \leq (M(D) + |b_k|)(|\lambda| + 1)$, whence $V_k(\lambda) + \ln |\widehat{S}(\lambda)| \leq v_k(\lambda) \leq (M(D) + |b_k|)(|\lambda| + 1)$. Assume first that $\widehat{S}(0) \neq 0$. We apply Lemma 6 to the functions V_k and $\ln |\widehat{S}|$. By (3.2) and the last estimate, it follows that $\exp V_k$ is of the first order and finite type. By Lemma 7, the regularization w_k admits the estimate $w_k(\lambda) \leq \Delta(|\lambda| + 1)$, $\lambda \in \mathbb{C}^p$. But if $\widehat{S}(0) = 0$, we take λ_0 , $|\lambda_0| < 1$, with $\widehat{S}(\lambda_0) \neq 0$ and apply Lemma 6 to the functions $V_k(\lambda + \lambda_0)$ and $\ln |\widehat{S}(\lambda + \lambda_0)|$. Thus, every w_k with $k > m$ satisfies the assumptions of Theorem 2 with $s = \frac{1}{2}$ and some B, Δ , and δ . By that theorem the functions in $P(w_k)$ can be approximated by polynomials in the norm of $P(w_k + (2p+1)\ln(1+|\lambda|^2))$. By condition (3), the functions in $P(w_k)$ can be approximated by polynomials in the norm of $P(w_{k+2p+1})$. Consequently, the polynomials form a dense subset of the inductive limit $P(D, U)$.

Theorem 1 is proved. $\square$

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INSTITUTE OF MATHEMATICS WITH COMPUTING CENTRE, 112 CHERNYSHEVSKY STREET, UFA 450077, RUSSIA

E-mail address: Yulmukhametov@mail.ru

Received 2/APR/2007

Translated by S. V. KISLYAKOV