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FIVE-VERTEX MODEL WITH FIXED BOUNDARY CONDITIONS

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Abstract. The exactly solvable five-vertex model on a square lattice with fixed boundary conditions is considered. Application of the algebraic Bethe ansatz makes it possible to express the partition function and the boundary correlation functions of the nonhomogeneous model in the determinantal form. The relationship established between the homogeneous model and plane partitions helps to calculate its partition function.

§1. Introduction

The study of exactly solvable vertex models of classical statistical physics has been actual for many years [1, 2]. One of the basic vertex models, the so-called six-vertex model, has been investigated intensively both for periodic and fixed boundary conditions; see [1]–[8].

Quite recently, it was realized that the methods used for the investigation of integrable models can be applied efficiently to the solution of certain problems of enumerative combinatorics [9, 10]. For example, the six-vertex model with domain wall boundary conditions is related to enumeration of the domino tilings of Aztec diamonds and to enumeration of alternating sign matrices [11]–[13], while the four-vertex and the phase models are related to enumeration of plane partitions (3-dimensional Young diagrams); see [14]–[19].

The existence of the determinantal representation of the partition functions and boundary correlation functions is substantial in this direction.

The five-vertex model is a special case of the six-vertex model with one vertex frozen out. For periodic boundary conditions, this model was used, in particular, in the study of interacting domain walls [20, 21] and directed percolation [22].

The five-vertex model on a square lattice is determined by five different configurations of arrows pointed both in and out of each lattice site. A statistical weight w_k ($k = 2, 3, 4, 5, 6$) is ascribed to each admissible type of vertices (Figure 1). Representing the arrows pointing up or to the right by solid lines, one can get an alternative description of the vertices in terms of lines floating through the lattice sites. Since the bonds of a lattice may be only in two states — either with a line or without it, there is a one-to-one correspondence between the admissible configurations of arrows on a lattice and the networks of lines — the nests of lattice paths.

For the general case of a nonhomogeneous model, the statistical weights w_k depend on the coordinates (i, j) of lattice sites. The partition function of the model is

$$Z = \sum_{\text{(config)}} \prod_{\text{(vertices)}} w_2^{l^2}(i,j) w_3^{l^3}(i,j) w_4^{l^4}(i,j) w_5^{l^5}(i,j) w_6^{l^6}(i,j), \tag{1}$$

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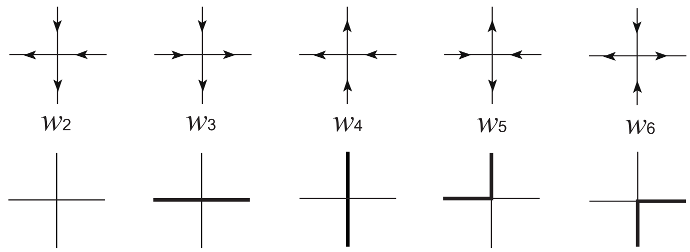

FIGURE 1. Five admissible types of vertices in terms of arrows and lines.

where the product is taken over all lattice sites and summation is over all admissible configurations of arrows. Each of the allowed configurations gives rise to a set of numbers $l^k = 0, 1$ with $\sum_{k=2}^{6} l^k = 1$ at each lattice site. For the homogeneous model, the weights w_k do not depend on the position of a lattice site.

In this paper we shall consider the model on a $2N \times (M+1)$ square lattice with the following boundary conditions: all arrows on the left and right boundaries are pointing to the left, while the arrows on the top and bottom of the first N columns (counting from the left) are pointing inwards and the arrows on the top and bottom of the last N columns are pointing outwards. This condition will be called the fixed boundary condition. We shall always assume that $M + 1 \geq 2N$.

A typical configuration of arrows and the corresponding nest of lattice paths on a lattice with fixed boundary conditions is represented in Figure 2.

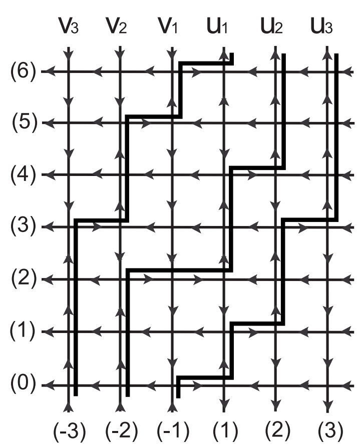

FIGURE 2. A typical configuration of arrows and the nest of lattice paths.

The nonhomogeneous model introduced above describes a propagation of domain walls with fixed endpoints in an anisotropic medium. These walls satisfy the reflection condition, because, due to the definition of the vertices (Figure 1), the paths do not osculate.

We shall apply the quantum inverse scattering method (QISM) [23]–[27] to the solution of the nonhomogeneous model with fixed boundary conditions. This approach allows us to represent the partition function of the model and its boundary correlation functions in the determinantal form and to obtain explicit answers for the limiting values of vertex weights.

The homogeneous model cannot be treated by the QISM approach. However, it will be demonstrated that the partition function of the model and its correlation functions are closely related to the theory of plane partitions, or 3-dimensional Young diagrams, placed into a finite box.

The paper is organized as follows. In §2 we discuss the QISM approach to the solution of the model with fixed boundary conditions. In §3 the boundary correlation functions are expressed in the determinantal form. In §4 the relationship between the homogeneous model and the row-strict plane partitions is established. The partition function and the boundary correlation functions of the model are calculated. In the Appendix, the L-operator and the R-matrix of the model are derived from the L-operator and the R-matrix of the six-vertex model.

§2. The solution of the nonhomogeneous model

To apply the QISM to the investigation of the nonhomogeneous model we use the spin description of the model. The spin up state on the vertical bond corresponds to the line pointing up, while the spin down state corresponds to the line pointing down. The spin up state on the ith horizontal bond $\binom{1}{0}_i \equiv |\leftarrow\rangle_i$ corresponds to the horizontal line pointing to the left, and the spin down state $\binom{0}{1}_i \equiv |\rightarrow\rangle_i$ to the line pointing to the right. With each vertical bond and with each horizontal bond of the grid we associate the space $\mathbb{C}^2$. The spin up states and the spin down states form a natural basis in this space. The space associated with all columns of the lattice $\mathcal{V} = (\mathbb{C}^2)^{\otimes 2N}$ is called the auxiliary space, while the space associated with all rows of the lattice $\mathcal{H} = (\mathbb{C}^2)^{\otimes(M+1)}$ is called the quantum space. In each lattice site in the space $\mathcal{V}\otimes\mathcal{H}$, an operator acts. This operator acts nontrivially only in a single auxiliary space $\mathbb{C}^2$ and in a single quantum space $\mathbb{C}^2$ and is called the L-operator. In all other spaces it acts as a unit operator.

The L-operator of the five-vertex model can be expressed in the form

$$L(n|u) = \begin{pmatrix} ue_n & \sigma_n^- \\ \sigma_n^+ & uI - u^{-1}e_n \end{pmatrix} = ee_n + (I-e)(uI - u^{-1}e_n) + \sigma^-\sigma_n^+ + \sigma^+\sigma_n^- \tag{2}$$

(see [28, 29]), where the parameter u is in $\mathbb{C}$, the $\sigma^{z,\pm}$ are the Pauli matrices, and $e = \frac{1}{2}(\sigma^z + 1)$ is the projection to a spin up state. The matrix with subindex n acts nontrivially only in the nth space: $s_n = I\otimes\cdots\otimes I\otimes s\otimes I\otimes\cdots\otimes I$.

The operator-valued matrix (2) satisfies the intertwining relation

$$R(u,v)\,(L(n|u)\otimes L(n|v)) = (L(n|v)\otimes L(n|u))\,R(u,v), \tag{3}$$

where $R(u,v)$ is a (4×4)-matrix equal to

$$R(u,v) = \begin{pmatrix} f(v,u) & 0 & 0 & 0 \\ 0 & g(v,u) & 1 & 0 \\ 0 & 0 & g(v,u) & 0 \\ 0 & 0 & 0 & f(v,u) \end{pmatrix}, \tag{4}$$

where

$$f(v,u) = \frac{u^2}{u^2 - v^2}, \quad g(v,u) = \frac{uv}{u^2 - v^2}. \tag{5}$$

The vertical monodromy matrix is the product of L-operators:

$$T(u) = L(M|u)L(M-1|u)\cdots L(0|u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{6}$$

The commutation relations of the entries of the monodromy matrix are given by the same R-matrix (4),

$$R(u,v)\,(T(u)\otimes T(v)) = (T(v)\otimes T(u))\,R(u,v). \tag{7}$$

The most important relations are:

$$
\begin{aligned}
C(u)B(v) &= g(u,v)\left\{A(u)D(v) - A(v)D(u)\right\}, \\
A(u)B(v) &= f(u,v)B(v)A(u) + g(v,u)B(u)A(v), \\
D(u)B(v) &= f(v,u)B(v)D(u) + g(u,v)B(u)D(v), \\
[B(u), B(v)] &= [C(u), C(v)] = 0.
\end{aligned} \tag{8}
$$

The L-operator (2) satisfies the equation

$$
e^{\zeta\sigma_n^z} L(n|u) e^{-\zeta\sigma_n^z} = e^{-\zeta\sigma^z} L(n|u) e^{\zeta\sigma^z}, \tag{9}
$$

where ζ is an arbitrary parameter. This identity and the definition of the monodromy matrix (6) show that

$$
e^{\zeta S^z} T(u) e^{-\zeta S^z} = e^{-\frac{\zeta}{2}\sigma^z} T(u) e^{\frac{\zeta}{2}\sigma^z}, \tag{10}
$$

where $S^z = \frac{1}{2}\sum_{i=0}^{M} \sigma_i^z$ is an operator of the z-component of the total spin. The formulas

$$
\begin{aligned}
S^z B(u) &= B(u)\left(S^z - 1\right), \\
S^z C(u) &= C(u)\left(S^z + 1\right)
\end{aligned} \tag{11}
$$

follow from (10) and mean that the operator $B(u)$ reduces the total spin of the system, while $C(u)$ increases it.

The generating vector of the quantum space $\mathcal{H}$ is a state with all spins up,

$$
|\Leftarrow\rangle = \bigotimes_{i=0}^{M} |\leftarrow\rangle_i = \bigotimes_{i=0}^{M} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i . \tag{12}
$$

This vector is annihilated by the operator $C(u)$,

$$
C(u)|\Leftarrow\rangle = 0, \tag{13}
$$

and is an eigenvector of the operators $A(u)$ and $D(u)$:

$$
A(u)|\Leftarrow\rangle = \alpha_{M+1}(u)|\Leftarrow\rangle, \quad D(u)|\Leftarrow\rangle = \delta_{M+1}(u)|\Leftarrow\rangle \tag{14}
$$

with the eigenvalues

$$
\alpha_{M+1}(u) = u^{M+1}, \quad \delta_{M+1}(u) = (u - u^{-1})^{M+1}. \tag{15}
$$

The total spin of the generating vector is equal to $\frac{1}{2}(M+1)$:

$$
S^z|\Leftarrow\rangle = \frac{1}{2}(M+1)|\Leftarrow\rangle. \tag{16}
$$

The representation

$$
\begin{aligned}
u^M B(u) &= u^{2M} \sum_{k=0}^{M} e_M \cdots e_{k+1}\sigma_k^- + \cdots + (-1)^M \sigma_M^- e_{M-1} \cdots e_0, \\
u^M C(u) &= u^{2M} \sum_{k=0}^{M} \sigma_k^+ e_{k-1} \cdots e_0 + \cdots + (-1)^M e_M \cdots e_2 \sigma_0^+
\end{aligned} \tag{17}
$$

follows from the definitions (2) and (6).

A principal object in the further considerations will be the vector generated by the multiple action of the operators $B(u)$ on the state $|\Leftarrow\rangle$:

$$
|\Psi_N(u_1, u_2, \ldots, u_N)\rangle = \prod_{i=1}^{N} B(u_i)|\Leftarrow\rangle. \tag{18}
$$

From (11) it follows that

$$S^z \prod_{i=1}^{N} B(u_i)|\Leftarrow\rangle = \frac{1}{2}(M+1-2N)\prod_{i=1}^{N} B(u_i)|\Leftarrow\rangle, \tag{19}$$

which means that the total spin of this vector is equal to $\frac{1}{2}(M+1-2N)$.

The vector conjugate to (18) is defined as

$$\langle\Psi_N(u_1,u_2,\dots,u_N)| = \langle\Leftarrow|\prod_{i=1}^{N} C(u_i). \tag{20}$$

It is easy to verify that $\langle\Leftarrow|B(u)=0$.

The matrix entries of the L-operator (2) can be represented as dots with attached arrows (Figure 3). The entry $L_{11}(n|u)$ corresponds to the vertex (1) (Figure 3), where

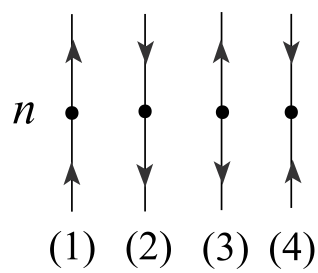

FIGURE 3. The vertex representation of the matrix entries of the L-operator.

a dot stands for the operator ue_n acting on a local quantum space. The only nonzero matrix entry ${}_n\langle\leftarrow|ue_n|\leftarrow\rangle_n$ of this operator determines the vertex (4) (Figure 1) with the weight $w_4=u$. The entry $L_{22}(n|u)$ corresponds to the vertex (2) (Figure 3) with a dot standing for the operator $uI-u^{-1}e_n$, which has nonzero entries ${}_n\langle\leftarrow|uI-u^{-1}e_n|\leftarrow\rangle_n$ and ${}_n\langle\rightarrow|uI-u^{-1}e_n|\rightarrow\rangle_n$, giving rise to the vertices (2) and (3) (Figure 1) with the weights equal to $w_2=u-u^{-1}$ and $w_3=u$, respectively. The entries $L_{12}(n|u)=\sigma_n^-$ and $L_{21}(n|u)=\sigma_n^+$ correspond to the vertices (3) and (4) (Figure 3). The nonzero entries ${}_n\langle\rightarrow|\sigma_n^-|\leftarrow\rangle_n$ and ${}_n\langle\leftarrow|\sigma_n^+|\rightarrow\rangle_n$ of these operators determine the vertices (5) and (6) (Figure 1) with the weights $w_5=w_6=1$.

The entries of the monodromy matrix (6) are expressed as sums over all possible configurations of arrows with different boundary conditions on a 1-dimensional lattice with $M+1$ sites (Figure 4). For instance, the operator $B(u)$, which, by the definition (6), is equal to

$$B(u) = \sum_{k_M,\dots,k_1=1}^{2} L_{1k_M}(M|u)L_{k_M k_{M-1}}(M-1|u)\cdots L_{k_1 2}(0|u),$$

corresponds to the boundary conditions where the top and the bottom arrows are pointed outwards (configuration (B)). The operator $C(u)$ corresponds to the boundary conditions where the arrows on the top and bottom of the lattice are pointed inwards (configuration (C)). The operators $A(u)$ and $D(u)$ correspond to the boundary conditions where the arrows on the top and bottom of the lattice are pointed in one direction — up and down, respectively (configurations (A) and (D)).

Consider the scalar product of the state vectors (18) and (20):

$$W_M(u_1,\dots,u_N;v_1,\dots,v_N) = \langle\Leftarrow|C(v_1)\cdots C(v_N)B(u_1)\cdots B(u_N)|\Leftarrow\rangle, \tag{21}$$

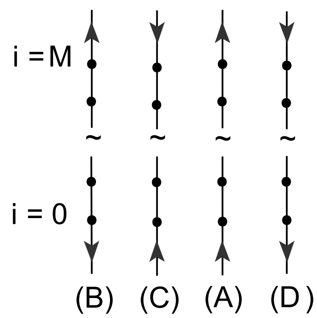

FIGURE 4. Graphic representation of the entries of the monodromy matrix.

where $\{u\}$ and $\{v\}$ are sets of independent parameters. The matrix entries of (21) can be represented as a two-dimensional $2N \times (M+1)$ square lattice. The first N vertical rows of the lattice (numbered by $(-N, \dots, -1)$) are associated with the operators $C(v_j)$, and the last N rows (numbered by $(1, \dots, N)$) with the operators $B(u_j)$. The horizontal rows of the lattice are associated with the local quantum spaces; moreover, the row with number i ($0 \le i \le M$) corresponds to the ith quantum space. The graphic representation of the operators $B(u)$ and $C(u)$ shows that the matrix entry (21) is equal to the sum over all admissible configurations of arrows on a square lattice with fixed boundary conditions (Figure 2). This means that the scalar product (21) is equal to the partition function of the anisotropic five-vertex model with the vertex weights given by the matrix entries of the L-operator (2) and depending on the number of the lattice column:

$$Z_M(u_1, \dots, u_N; v_1, \dots, v_N) = W_M(u_1, \dots, u_N; v_1, \dots, v_N)$$

$$= \sum_{(\text{config})} \prod_{k=-1}^{-N} (v_{-k} - v_{-k}^{-1})^{l_k^2} (v_{-k})^{l_k^3 + l_k^4} \prod_{j=1}^{N} (u_j - u_j^{-1})^{l_j^2} (u_j)^{l_j^3 + l_j^4}, \tag{22}$$

where the summation is taken over all admissible configurations of arrows on the lattice.

For arbitrary values of the parameters u_j, v_j, the scalar product (21) is evaluated with the help of the commutation relations (8) [27, 14]:

$$W_M(u_1, \dots, u_N; v_1, \dots, v_N)$$

$$= \left\{ \prod_{N \ge j > k \ge 1} g(v_j, v_k) \prod_{N \ge l > n \ge 1} g(u_m, u_l) \right\} \det Q_M. \tag{23}$$

The entries of the $(N \times N)$-matrix Q_M are equal to

$$(Q_M(v_j, u_k))_{jk}$$

$$= \frac{\alpha_{M+1}(v_j)\delta_{M+1}(u_k) \left(\frac{u_k}{v_j}\right)^{N-1} - \alpha_{M+1}(u_k)\delta_{M+1}(v_j) \left(\frac{u_k}{v_j}\right)^{-N+1}}{\frac{u_k}{v_j} - \left(\frac{u_k}{v_j}\right)^{-1}}$$

$$= \frac{\left[v_j(u_k - u_k^{-1})\right]^{M+1} \left(\frac{u_k}{v_j}\right)^{N-1} - \left[u_k(v_j - v_j^{-1})\right]^{M+1} \left(\frac{u_k}{v_j}\right)^{-N+1}}{\frac{u_k}{v_j} - \left(\frac{u_k}{v_j}\right)^{-1}}, \tag{24}$$

where the $\alpha_{M+1}(u)$ and $\delta_{M+1}(u)$ are as defined in (15).

Thus, we have obtained a representation of the partition function of the inhomogeneous five-vertex model in a determinantal form.

Now, consider the special case where the main contribution to the partition function is given by the vertices (2), (3), and (4) (Figure 1). In this limit the weights w_2, w_3, w_4 tend to infinity [30]. If $v_j \to \infty$ for $1 \le j \le N$, the partition function is equal to

$$Z_M^{(\infty,u)} = \lim_{\{v\}\to\infty} \prod_{i=1}^{N} v_i^{-M} W(u_1,\dots,u_N;v_1,\dots,v_N). \tag{25}$$

To calculate this limit, first we fix $v_2,\dots,v_N$ and let $v_1 \to \infty$. Then we let $v_2 \to \infty$, with $v_3,\dots,v_N$ fixed. Repeating this procedure, we obtain

$$Z_M^{(\infty,u)} = \prod_{k=1}^{N} u_k^{M+1} \prod_{N\ge l>n\ge 1} \frac{1}{u_l^2 - u_n^2} \det V, \tag{26}$$

where V is the $(N\times N)$-matrix with the entries

$$\begin{aligned} V_{jk} &= \sum_{n=0}^{j-1} (-1)^n \binom{M+1}{n} u_k^{2(j-1-n)}, \qquad 1 \le j \le N-1; \\ V_{Nk} &= -\sum_{n=N}^{M+1} (-1)^n \binom{M+1}{n} u_k^{-2(n-N+1)}. \end{aligned} \tag{27}$$

Since all weights u_j, v_j tend to ∞, the determinant (23) can be calculated, and the partition function (22) becomes equal to the combinatorial coefficient

$$\begin{aligned} Z_M^{(\infty)} &= \lim_{\{u,v\}\to\infty} \prod_{k=1}^{N} (u_k v_k)^{-M} W(u_1,\dots,u_N;v_1,\dots,v_N) \\ &= \det U = \frac{(M+1)!}{N!(M+1-N)!}, \end{aligned} \tag{28}$$

where the nonzero entries of the $(N\times N)$-matrix U are $U_{jj+1} = 1$, $1 \le j \le N-1$, and $U_{N1} = (-1)^{N+1}\binom{M+1}{N}$.

§3. Boundary correlation functions

Consider the correlation functions describing the probabilities of the local states on the boundary. The probability that on the bottom row of a lattice the arrow between the $(k-1)$st and the kth columns is pointed to the right is determined by the one-point correlation function:

$$P_k = \frac{1}{Z_M} \langle \Leftarrow | C(v_N)\cdots C(v_1) B(u_1)\cdots B(u_{k-1}) g_0 B(u_k)\cdots B(u_N) | \Leftarrow \rangle, \tag{29}$$

where $g_0 = 1 - e_0$ is a projection on the spin down state, and Z_M is the partition function (22).

The probability that in $K+1$ bottom rows of a lattice all arrows between the -1st and 1st columns are pointed to the left is determined by the correlation function

$$G^{(K)} = \frac{1}{Z_M} \langle \Leftarrow | C(v_N)\cdots C(v_1) e_0 \cdots e_K B(u_1)\cdots B(u_N) | \Leftarrow \rangle. \tag{30}$$

To calculate the above correlation functions, we represent the monodromy matrix (6) in the form

$$T(u) = \begin{pmatrix} A_M(u) & B_M(u) \\ C_M(u) & D_M(u) \end{pmatrix} \begin{pmatrix} ue_0 & \sigma_0^- \\ \sigma_0^+ & uI - u^{-1}e_0 \end{pmatrix}. \tag{31}$$

This shows that the entries of the monodromy matrix defined on $M+1$ lattice sites are related to the entries of the monodromy matrix defined on M sites:

$$
\begin{aligned}
A(u) &= uA_M(u)e_0 + B_M(u)\sigma_0^+,\\
B(u) &= A_M(u)\sigma_0^- + uB_M(u) - u^{-1}B_M(u)e_0,\\
C(u) &= uC_M(u)e_0 + D_M(u)\sigma_0^+,\\
D(u) &= C_M(u)\sigma_0^- + uD_M(u) - u^{-1}D_M(u)e_0.
\end{aligned}
$$

As a consequence of these identities, we can obtain the following commutation relations:

$$
\begin{aligned}
e_0B(u) &= (u-u^{-1})B_M(u)e_0,\\
C(u)e_0 &= ue_0C_M(u),
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
g_0B(u) &= A_M(u)\sigma_0^- + uB_M(u)g_0,\\
B(u)\sigma_0^- &= u\sigma_0^- B_M(u),\\
C(u)\sigma_0^- &= e_0D_M(u).
\end{aligned}
\tag{33}
$$

We recall that, by the definition (2), the operators with different indices commute.

Now we turn to the calculation of the function (30) at $K=1$. Since the projection $e_0 = e_0^2$ satisfies $e_0|\Leftarrow\rangle = |\Leftarrow\rangle, \langle\Leftarrow|e_0 = \langle\Leftarrow|$, we obtain

$$
\begin{aligned}
G^{(1)} &= \frac{1}{Z_M}\langle\Leftarrow|C(v_N)\cdots C(v_1)e_0B(u_1)\cdots B(u_N)|\Leftarrow\rangle\\
&= \frac{1}{Z_M}\langle\Leftarrow|C(v_N)\cdots C(v_1)e_0^2B(u_1)\cdots B(u_N)|\Leftarrow\rangle\\
&= \frac{1}{Z_M}\prod_{j=1}^{N} v_j(u_j-u_j^{-1})\\
&\qquad\times\langle\Leftarrow|e_0C_M(v_N)\cdots C_M(v_1)B_M(u_1)\cdots B_M(u_N)e_0|\Leftarrow\rangle\\
&= \frac{1}{Z_M}\prod_{j=1}^{N} v_j(u_j-u_j^{-1})\\
&\qquad\times\langle\Leftarrow|C_M(v_N)\cdots C_M(v_1)B_M(u_1)\cdots B_M(u_N)|\Leftarrow\rangle\\
&= \prod_{j=1}^{N} v_j(u_j-u_j^{-1})\frac{Z_{M-1}}{Z_M}.
\end{aligned}
\tag{34}
$$

The partition function Z_{M-1} of the model on a square $2N\times M$ lattice is given by the expression (23), with the matrix Q_M replaced by the matrix Q_{M-1} (24).

This result easily extends to the function (30) with an arbitrary K:

$$
G^{(K)} = \prod_{j=1}^{N}\left[v_j(u_j-u_j^{-1})\right]^K\frac{Z_{M-K}}{Z_M},
$$

where Z_{M-K} is the partition function of the model on a $2N\times(M-K)$ square lattice.

The correlation function P_1 (see (29)) is related to the correlation function $G^{(1)}$ (see (30)):

$$
P_1 = 1 - G^{(1)}. \tag{35}
$$

To calculate the correlation function P_N (see (29)), we shall use relations (32), (33), property (14), and the condition $g_0| \Leftarrow\rangle = 0$:

$$
\begin{aligned}
P_N &= \frac{1}{Z_M} \langle \Leftarrow | C(v_N) \cdots C(v_1) B(u_1) \cdots B(u_{N-1}) g_0 B(u_N) | \Leftarrow \rangle \\
&= \frac{1}{Z_M} \langle \Leftarrow | C(v_N) \cdots C(v_1) B(u_1) \cdots B(u_{N-1}) \sigma_0^- A_M(u_N) | \Leftarrow \rangle \\
&= \frac{1}{Z_M} \alpha_M(u_N) \langle \Leftarrow | C(v_N) \cdots C(v_1) B(u_1) \cdots B(u_{N-1}) \sigma_0^- | \Leftarrow \rangle \\
&= \frac{1}{Z_M} \alpha_M(u_N) \prod_{j=1}^{N-1} u_j \\
&\quad \times \langle \Leftarrow | C(v_N) \cdots C(v_1) \sigma_0^- B_M(u_1) \cdots B_M(u_{N-1}) | \Leftarrow \rangle \\
&= \frac{1}{Z_M} \alpha_M(u_N) \prod_{j=1}^{N-1} u_j \\
&\quad \times \langle \Leftarrow | C(v_N) \cdots C(v_2) e_0 D_M(v_1) B_M(u_1) \cdots B_M(u_{N-1}) | \Leftarrow \rangle \\
&= \frac{1}{Z_M} \alpha_M(u_N) \prod_{j=1}^{N-1} u_j \prod_{k=2}^{N} v_k \\
&\quad \times \langle \Leftarrow | C_M(v_N) \cdots C_M(v_2) D_M(v_1) B_M(u_1) \cdots B_M(u_{N-1}) | \Leftarrow \rangle
\end{aligned} \tag{36}
$$

We substitute one of the main formulas of the algebraic Bethe ansatz [23, 24, 26], derived by successive application of the commutation relations (8), namely,

$$
\begin{aligned}
D(v) \prod_{j=1}^{N} B(u_j) | \Leftarrow \rangle &= \delta_{M+1}(v) \prod_{s=1}^{N} f(u_s, v) \prod_{j=1}^{N} B(u_j) | \Leftarrow \rangle \\
&+ \sum_{n=1}^{N} \delta_{M+1}(u_n) g(v, u_n) \prod_{s=1, s\neq n}^{N} f(u_s, u_n) B(v) \prod_{j=1, j\neq n}^{N} B(u_j) | \Leftarrow \rangle
\end{aligned} \tag{37}
$$

in a form-factor of the operator $D_M(v_1)$,

$$
\mathcal{D}_M(v_1) \equiv \langle \Leftarrow | C_M(v_N) \cdots C_M(v_2) D_M(v_1) B_M(u_1) \cdots B_M(u_{N-1}) | \Leftarrow \rangle, \tag{38}
$$

and use formula (23), obtaining:

$$
\begin{aligned}
\mathcal{D}_M(v_1) &= \delta_M(v_1) \prod_{s=1}^{N-1} f(u_s, v_1) \\
&\times \left[\prod_{N \geq j > k \geq 2} g(v_j, v_k) \prod_{N-1 \geq l > m \geq 1} g(u_m, u_l) \right] \\
&\times \left\{ \det Q_{M-1} + \sum_{n=1}^{N} \det Q_{M-1}^{(n)} \right\},
\end{aligned} \tag{39}
$$

where Q_{M-1} is the matrix defined in (24), and the matrix $Q_{M-1}^{(n)}$ is equal to

$$
\begin{aligned}
(Q_{M-1})_{jk}^{(n)} &= Q_{M-1}(v_j, u_k) \quad \text{for} \quad k \neq n, \\
(Q_{M-1})_{jn}^{(n)} &= -\frac{\delta_M(u_n)}{\delta_M(v_1)} \left(\frac{u_n}{v_1} \right)^N Q_{M-1}(v_j, v_1).
\end{aligned}
$$

Identity (39) can be represented in the form

$$\mathcal{D}_M(v_1) = \delta_M(v_1) \prod_{s=1}^{N-1} f(u_s, v_1) \times \left[\prod_{N \geq j > k \geq 2} g(v_j, v_k) \prod_{N-1 \geq l > m \geq 1} g(u_m, u_l) \right] \det \{Q_{M-1} + H\}, \tag{40}$$

where H is an $(N \times N)$-matrix of rank one, equal to

$$H_{jk} = -\frac{\delta_M(u_n)}{\delta_M(v_1)} \left(\frac{u_n}{v_1}\right)^N Q_{M-1}(v_j, v_1).$$

The final expression for the correlation function (36) takes the form

$$P_N = \frac{1}{Z_M} \alpha_M(u_N) \prod_{j=1}^{N-1} u_j \prod_{k=2}^{N} v_k \mathcal{D}_M(v_1). \tag{41}$$

The determinantal representations for the correlation functions (36) with $1 < k < N$ are derived with the help of the commutation relations (8) and formula (23) and are more cumbersome than the answers obtained above.

From (28) it follows that in the limit as all weights u_j, v_j tend to ∞, the correlation function looks like this:

$$G^{(K)} = \frac{Z_{M-K}^{(\infty)}}{Z_M^{(\infty)}} = \frac{(M-K+1)!(M+1-N)!}{(M-K+1-N)!(M+1)!}.$$

The representation (17) implies that, in the same limit,

$$\lim_{u_1,\ldots,u_k \to \infty} (u_1, \ldots, u_k)^{-M} [g_0, B(u_1) \cdots B(u_k)] = 0, \quad k > 1,$$

whence $P_N = P_1$. Then, by (35), we have

$$P_N = P_1 = 1 - G^{(1)} = \frac{N}{M+1}.$$

§4. The homogeneous model and plane partitions

Now, consider the homogeneous model with all weights equal: $w_2 = w_3 = w_4 = w_5 = w_6 \equiv w$. It should be mentioned that this case cannot be treated by the QISM approach, as described in the preceding sections. The partition function of this model is equal to

$$Z = w^{2N(M+1)} \sum_{\text{(config)}} 1, \tag{42}$$

where, as in (1), the summation is taken over all admissible configurations of arrows on a lattice.

To enumerate all admissible configurations of arrows on a $2N \times (M+1)$ square lattice with fixed boundary conditions, it is more convenient to use the description of the model in terms of lines flowing through the vertices of the lattice and to represent these configurations as nests of lattice paths. A path starts at one of the bottom N left vertices and terminates at one of the top N right vertices, going only to the east and to the north along the arrows pointed to the right and upwards. The paths do not osculate and several steps are allowed both in the horizontal and in the vertical direction. If the first N columns of the lattice have the numbers $(-N, \ldots, -1)$ and the last N columns have the numbers $(1, \ldots, N)$, the bottom row and the top row having the number 0 and M, respectively, then the path with the number m goes from the vertex $(-N+m-1;0)$ to the vertex $(m;M)$, $1 \le m \le N$. A generic nest of paths is represented in Figure 5.

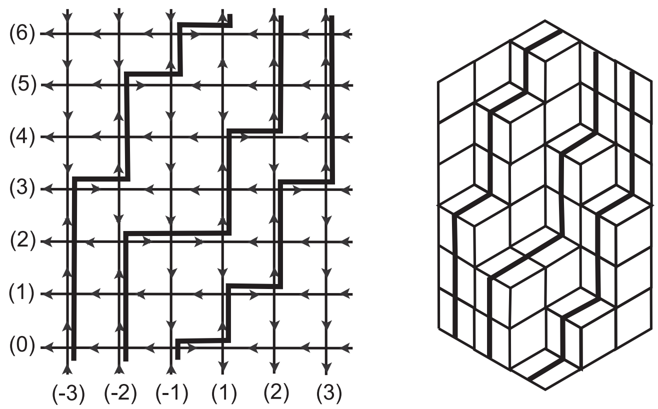

FIGURE 5. The generic configuration of the admissible lattice paths on a lattice with fixed boundary conditions, and the corresponding 3-dimensional Young diagram.

Each set of admissible lattice paths can be expressed as an $(N \times N)$-matrix π_{ij}. The mth path corresponds to the mth column of this matrix, and the entry π_{jm} is equal to the number of cells in the jth column of the lattice under the mth path. The numbering of the columns under the mth path starts with the last right column (the end of the path) and finishes at the last left one (the beginning of the path). The matrix

$$\pi = \begin{pmatrix} 6 & 4 & 3 \\ 5 & 2 & 1 \\ 3 & 2 & 0 \end{pmatrix} \tag{43}$$

corresponds to the set of paths in Figure 5.

An array of nonnegative integers $(\pi_{i,j})$ that are monotone nonincreasing functions of both i and j $(i,j=1,2,\ldots)$ is called a plane partition π [9]. The integers $\pi_{i,j}$ are the parts of that plane partition. Each plane partition has a 3-dimensional diagram, which can be interpreted as a stack of unit cubes — a 3-dimensional Young diagram. The number $|\pi| = \sum \pi_{i,j}$ is the volume (or the weight) of π. If $i \le r$, $j \le s$ and $\pi_{ij} \le t$ for all parts of the plane partition, we say that π is contained in a box with side lengths r, s, t. If, moreover, $\pi_{ij} > \pi_{ij+1}$, i.e., if the parts of π decay along each row, then π is called a row-strict plane partition. The 3-dimensional Young diagram that corresponds to the plane partition (43) is represented in Figure 2.

The partition function of the model (42) under consideration is proportional to the number of admissible lattice paths, and hence, to the number of all row-strict (column-strict) plane partitions in an $N \times N \times M$ box. Applying the MacMahon formula

$$S(L,N,M) = \prod_{j=1}^{N} \prod_{k=1}^{L} \frac{M+1+j-k}{j+k-1} \tag{44}$$

for the row-strict plane partitions in a $L \times N \times M$ box [9, 10], we see that the partition function of the homogeneous model (42) looks like this:

$$Z = w^{2N(M+1)} \prod_{j=1}^{N} \prod_{k=1}^{N} \frac{M+1+j-k}{j+k-1}. \tag{45}$$

The analysis of the admissible lattice paths shows that the probability that in the bottom $K+1$ rows of a lattice all arrows between -1 and 1 columns are pointed to the left (i.e., are not segments of lattice paths) is equal to the ratio of the number of the row-strict plane partitions in the $N \times N \times (M-K)$ box to the number of the row-strict plane partitions in the $N \times N \times M$ box:

$$G^{(K)} = \prod_{j=1}^{N} \prod_{k=1}^{N} \frac{M-K+1+j-k}{M+1+j-k}. \tag{46}$$

The probability that in the bottom row of a lattice the arrow between the $N-1$ and N columns is pointed to the right (i.e., is a segment of a lattice path) is the ratio between the number of row-strict plane partitions in the $(N-1) \times N \times M$ box and the number of row-strict plane partitions in the $N \times N \times M$ box:

$$P_N = \prod_{j=1}^{N} \frac{N-1+j}{M-N+1+j} = \frac{(2N-1)!(M-N+1)!}{(N-1)!(M+1)!}. \tag{47}$$

§5. Conclusion

To conclude, we mention that, although the five-vertex model is a special case of the six-vertex model, no determinantal representations for the partition function and for the boundary correlation functions of the nonhomogeneous model on a lattice with fixed boundary conditions can be obtained by a simple limit because no similar representations for the six-vertex model are known.

The quantum Hamiltonian that commutes with the transfer matrix of the five-vertex model is a non-Hermitian Hamiltonian that is used for the description of the totally asymmetric simple exclusion process [31, 32, 33] of the nonequilibrium statistical physics. The approach developed in this paper makes it possible to obtain determinantal representations for the correlation functions of the corresponding quantum model.

§6. Appendix

We demonstrate that the L-operator (2) and the R-matrix (4) of the five-vertex model are special limits of the L-operator and R-matrix of the six-vertex model, respectively.

The L-operator of the six-vertex model

$$L_{6v}(n|u) = \begin{pmatrix} ue^{\gamma\sigma_n^z} - u^{-1}e^{-\gamma\sigma_n^z} & \sigma_n^-\left(e^{2\gamma} - e^{-2\gamma}\right) \\ \sigma_n^+\left(e^{2\gamma} - e^{-2\gamma}\right) & ue^{-\gamma\sigma_n^z} - u^{-1}e^{\gamma\sigma_n^z} \end{pmatrix} \tag{48}$$

satisfies the intertwining relation (3) with the R-matrix

$$\widetilde{R}(u,v) = \begin{pmatrix} \widetilde{f}(v,u) & 0 & 0 & 0 \\ 0 & \widetilde{g}(v,u) & 1 & 0 \\ 0 & 1 & \widetilde{g}(v,u) & 0 \\ 0 & 0 & 0 & \widetilde{f}(v,u) \end{pmatrix}, \tag{49}$$

where

$$\widetilde{f}(v,u) = \frac{u^2e^{2\gamma} - v^2e^{-2\gamma}}{u^2 - v^2}, \quad \widetilde{g}(v,u) = \frac{uv}{u^2 - v^2}\left(e^{2\gamma} - e^{-2\gamma}\right). \tag{50}$$

Consider the following transformation of the L-operator (48):

$$\breve{L}(n|u) = e^{x\sigma_n^z} L_{6v}(n|u) e^{-\omega\sigma^z}. \tag{51}$$

This L-operator is intertwined by the transformed R-matrix:

$$\check{R}(u,v) = \left(1\otimes e^{-x\sigma^z}\right)\widetilde{R}(u,v)\left(1\otimes e^{x\sigma^z}\right)$$

(52)
$$= \begin{pmatrix} \widetilde{f}(v,u) & 0 & 0 & 0 \\ 0 & \widetilde{g}(v,u) & e^{2x} & 0 \\ 0 & e^{-2x} & \widetilde{g}(v,u) & 0 \\ 0 & 0 & 0 & \widetilde{f}(v,u) \end{pmatrix}.$$

To prove this, we note that the L-operator satisfies (9), namely,

$$e^{x\sigma_n^z}\check{L}(n|u)e^{-x\sigma_n^z} = e^{-x\sigma^z}\check{L}(n|u)e^{x\sigma^z},$$

while the R-matrix satisfies the commutation relations

$$\left(e^{\omega\sigma^z}\otimes e^{\omega\sigma^z}\right)\widetilde{R}(u,v) = \widetilde{R}(u,v)\left(e^{\omega\sigma^z}\otimes e^{\omega\sigma^z}\right),$$
$$e^{-x\sigma_n^z}\widetilde{R}(u,v) = \widetilde{R}(u,v)e^{-x\sigma_n^z}.$$

Our claim is a consequence of the following chain of identities:

$$\widetilde{R}(u,v)L_{6v}(n|u)\otimes L_{6v}(n|v) = L_{6v}(n|v)\otimes L_{6v}(n|u)\widetilde{R}(u,v),$$
$$\widetilde{R}(u,v)e^{-x\sigma_n^z}\check{L}(n|u)e^{\omega\sigma^z}\otimes e^{-x\sigma_n^z}\check{L}(n|v)e^{\omega\sigma^z}$$
$$= e^{-x\sigma_n^z}\check{L}(n|v)e^{\omega\sigma^z}\otimes e^{-x\sigma_n^z}\check{L}(n|u)e^{\omega\sigma^z}\widetilde{R}(u,v),$$
$$\widetilde{R}(u,v)\check{L}(n|u)\otimes e^{x\sigma^z}\check{L}(n|v)e^{-x\sigma^z} = \check{L}(n|v)\otimes e^{x\sigma^z}\check{L}(n|u)e^{-x\sigma^z}\widetilde{R}(u,v),$$
$$\widetilde{R}(u,v)\big(I\otimes e^{x\sigma^z}\big)\check{L}(n|u)\otimes\check{L}(n|v)\big(I\otimes e^{-x\sigma^z}\big)$$
$$= \big(I\otimes e^{x\sigma^z}\big)\check{L}(n|v)\otimes\check{L}(n|u)\big(I\otimes e^{-x\sigma^z}\big)\widetilde{R}(u,v).$$

We rewrite the L-operator (51) in a matrix form:

$$L(n|u) = \begin{pmatrix} ue^{-\omega+(x+\gamma)\sigma_n^z} - u^{-1}e^{-\omega+(x-\gamma)\sigma_n^z} & e^{(\omega+x\sigma_n^z)}\sigma_n^-\left(e^{2\gamma}-e^{-2\gamma}\right) \\ e^{(-\omega+x\sigma_n^z)}\sigma_n^+\left(e^{2\gamma}-e^{-2\gamma}\right) & ue^{\omega+(x-\gamma)\sigma_n^z} - u^{-1}e^{\omega+(x+\gamma)\sigma_n^z} \end{pmatrix}$$
$$\equiv \begin{pmatrix} L_{11}(n|u) & L_{12}(n|u) \\ L_{21}(n|u) & L_{22}(n|u) \end{pmatrix},$$

where

$$L_{11}(n|u) = \begin{pmatrix} ue^{-\omega+x+\gamma} - u^{-1}e^{-\omega+x-\gamma} & 0 \\ 0 & ue^{-\omega-x-\gamma} - u^{-1}e^{-\omega-x+\gamma} \end{pmatrix}_n,$$
$$L_{22}(n|u) = \begin{pmatrix} ue^{\omega+x-\gamma} - u^{-1}e^{\omega+x+\gamma} & 0 \\ 0 & ue^{\omega-x+\gamma} - u^{-1}e^{\omega-x-\gamma} \end{pmatrix}_n,$$
$$L_{12}(n|u) = \begin{pmatrix} 0 & 0 \\ e^{\omega-x}\left(e^{2\gamma}-e^{-2\gamma}\right) & 0 \end{pmatrix}_n,$$
$$L_{21}(n|u) = \begin{pmatrix} 0 & e^{-\omega+x}\left(e^{2\gamma}-e^{-2\gamma}\right) \\ 0 & 0 \end{pmatrix}_n.$$

The matrix with the subscript n is on the nth place in the direct product: $s_n = I\otimes\cdots\otimes I\otimes s\otimes I\otimes\cdots\otimes I$.

We put $x=\omega=\gamma$ in (51) and use the fact that relation (3) is invariant with respect to the transformation $u\to xu$. Then the L-operator of the five-vertex model is defined as the limit

$$L(n|u) = \lim e^{-2\gamma}\check{L}(n|e^{\gamma}u).$$

The corresponding limit of the R-matrix (52) is (4).

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