ON THE SINGULAR BRAID MONOID

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ABSTRACT. Garside's results and the existence of the greedy normal form for braids are shown to be true for the singular braid monoid. An analog of the presentation of J. S. Birman, K. H. Ko, and S. J. Lee for the classical braid group is also obtained for this monoid.

§1. INTRODUCTION

Various questions concerning braid groups and their generalizations have attracted attention during the last decades. Presentations of braid groups and algorithmic problems for the singular braid monoid are among these questions.

The canonical presentation of the braid group on n strings Br_n was given by Artin [5] and is well known. It has the generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ and the relations

$$(1) \quad \begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i - j| > 1, \ i, j = 1, \dots, n - 1; \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, & i = 1, \dots, n - 2. \end{cases}$$

Of course, there exist other presentations of the braid group. Birman, Ko and Lee [8] introduced a presentation with generators a_{ts} with $1 \leq s < t \leq n$, and the relations

$$(2) \quad \begin{cases} a_{ts} a_{rq} = a_{rq} a_{ts} & \text{for } (t - r)(t - q)(s - r)(s - q) > 0, \\ a_{ts} a_{sr} = a_{tr} a_{ts} = a_{sr} a_{tr} & \text{for } 1 \leq r < s < t \leq n. \end{cases}$$

The generators a_{ts} are expressed in terms of the canonical generators σ_i as follows:

$$a_{ts} = (\sigma_{t-1} \sigma_{t-2} \cdots \sigma_{s+1}) \sigma_s (\sigma_{s+1}^{-1} \cdots \sigma_{t-2}^{-1} \sigma_{t-1}^{-1}) \quad \text{for } 1 \leq s < t \leq n.$$

The *singular braid monoid* SB_n or *Baez–Birman monoid* [6, 7] is defined as a monoid with generators $\sigma_i, \sigma_i^{-1}, x_i, i = 1, \dots, n - 1$, and the relations

$$(3) \quad \begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i - j| > 1, \\ x_i x_j = x_j x_i & \text{if } |i - j| > 1, \\ x_i \sigma_j = \sigma_j x_i & \text{if } |i - j| \neq 1, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_i \sigma_{i+1} x_i = x_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_{i+1} \sigma_i x_{i+1} = x_i \sigma_{i+1} \sigma_i, \\ \sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1. \end{cases}$$

In pictures, σ_i corresponds to the canonical generator of the braid group and x_i represents the intersection of the i th and $(i+1)$ st strands as in Figure 1. The basic motivation for the introduction of this object was the Vassiliev–Goussarov theory of finite type invariants.

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FIGURE 1

FIGURE 2

The singular braid monoid on two strings is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}^+$, so that the word problem in this case is trivial. For the monoid with three strings this problem was solved by Jrai [16] and Dasbach and Gemein [11]. In the general case, this was done by Corran in the complicated and technical paper [10]. Here we hope to give a simple solution using almost nothing but F. A. Garside's arguments. The initial idea is simple and geometric: the behavior of the Garside *fundamental element* determined by the word Δ (for convenience, the definition is given below in formula (4)) with respect to the singular generators x_i is similar to that with respect to the braid generators σ_i :

$$x_i \Delta \doteq \Delta x_{n-i},$$

as shown in Figure 2.

2. THE WORD PROBLEM FOR THE SINGULAR BRAID MONOID

Following Garside's ideas, we consider the *positive singular braid monoid* SB_n^+ . It is defined as a monoid with the generators σ_i, x_i , $i = 1, \dots, n - 1$, and relations (3) except the last one, which concerns the invertibility of σ_i . We denote by $\mathcal{A}$ the alphabet $\{\sigma_i, x_i, (i = 1, \dots, n - 1)\}$. We say that two positive words A and B in the alphabet $\mathcal{A}$ are

positively equal if they are equal as elements of SB_n^+ . In this case we write $A \doteq B$. The identity of words is denoted by the symbol $\equiv$. We denote by $L(W)$ the *word-length* of the word W . As usual, we say that an element a in a monoid is a (left) divisor of an element b if there exists an element c such that $ac = b$ in the monoid; in this case we say that b is a (right) multiple of a . The relation of divisibility is a partial order in SB_n^+ , and the notions of the greatest common divisor and the least common multiple make sense.

The proofs of the statements below are the same as in Garside's paper [15] with some exceptions as, for example, the proof of Proposition 6, where we use the Mal'tsev rule. Proposition 6 was proved by Corran [10]; Theorem 3 was proved by Fenn, Rolfsen and Zhu [14] by different methods.

Proposition 1. *The following claims are true in SB_n^+ for $i, k = 1, \dots, n-1$. Given $\sigma_i A \doteq \sigma_k B$, where A and B are (positive) words in the alphabet $\mathcal{A}$, we have:*

$$\begin{aligned} \text{if } k = i, & \quad \text{then } A \doteq B, \\ \text{if } |k - i| = 1, & \quad \text{then } A \doteq \sigma_k \sigma_i Z, \quad B \doteq \sigma_i \sigma_k Z \quad \text{for some word } Z, \\ \text{if } |k - i| \geq 2, & \quad \text{then } A \doteq \sigma_k Z, \quad B \doteq \sigma_i Z \quad \text{for some word } Z; \end{aligned}$$

given $\sigma_i A \doteq x_k B$, we have:

$$\begin{aligned} \text{if } |k - i| = 1, & \quad \text{then } A \doteq \sigma_k x_i Z, \quad B \doteq \sigma_i \sigma_k Z \quad \text{for some } Z, \\ \text{if } |k - i| \neq 1, & \quad \text{then } A \doteq x_k Z, \quad B \doteq \sigma_i Z \quad \text{for some } Z; \end{aligned}$$

and given $x_i A \doteq x_k B$, we have:

$$\begin{aligned} \text{if } k = i, & \quad \text{then } A \doteq B, \\ \text{if } |k - i| \geq 2, & \quad \text{then } A \doteq x_k Z, \quad B \doteq x_i Z \quad \text{for some } Z, \\ \text{the case where } |k - i| = 1 & \quad \text{is impossible.} \end{aligned}$$

The same is true for the right multiples of σ_i or x_k .

Proof. Garside's proof works here. We apply induction on the length s of the word A and the length of the chain of transformations from $a_i A$ to $a_k B$, where a_i may be σ_i or x_i , and a_k may be σ_k or x_k . The cases of $s = 0, 1$ are obvious; so, suppose that the statement is true for $s \leq r$ and that for $s = r + 1$ it is true if the chain-length does not exceed t . As an example, we prove the last statement, which is formally not contained in Garside's considerations. So, let A, B be of word-length $r + 1$ and let $x_i A \doteq x_k B$, $|i - k| = 1$, through a transformation of chain-length $t + 1$. We may assume that $k = i + 1$ and let the successive words of the transformations be

$$W_1 \equiv x_i A, \dots, W_{t+2} \equiv x_{i+1} B.$$

We choose an arbitrary intermediate word W_g , say, somewhere in the middle of the chain. We have $W_g \equiv aV$, where a is a generator of SB_n^+ . First, suppose that a commutes with x_i, x_{i+1} ; then $x_i A \doteq aV \doteq x_{i+1} B$ and by induction we obtain

$$A \doteq aP, \quad V \doteq x_i P, \quad V \doteq x_{i+1} Q, \quad B \doteq aQ.$$

Thus, $x_i P \doteq x_{i+1} Q$, which is impossible by induction.

The cases where $a = x_{i-1}, x_i, x_{i+1}, x_{i+2}$ are also impossible by induction. The cases that need consideration are $a = \sigma_{i-1}, \sigma_i, \sigma_{i+1}, \sigma_{i+2}$. So, let $a = \sigma_{i-1}$; then $W_g \doteq \sigma_{i-1} V$. By induction we get

$$A \doteq \sigma_{i-1} \sigma_i P, \quad V \doteq \sigma_i x_{i-1} P, \quad V \doteq x_{i+1} Q, \quad B \doteq \sigma_{i-1} Q.$$

Hence, $\sigma_i x_{i-1} P \doteq x_{i+1} Q$, whence $x_{i-1} P \doteq \sigma_{i+1} x_i R$, $Q \doteq \sigma_i \sigma_{i+1} R$. Again by induction we get $P \doteq \sigma_{i+1} S$, $x_i R \doteq x_{i-1} S$, which is impossible. The remaining cases may be analyzed in the same way. $\square$

Corollary 1. *If $A \doteq P$, $B \doteq Q$, $AXB \doteq PYQ$ ($L(A) \geq 0$, $L(B) \geq 0$), then $X \doteq Y$. That is, the monoid SB_n^+ is left and right cancellative.*

Proposition 2. *For any word W in the alphabet $\mathcal{A} = \{\sigma_i, x_i \ (i = 1, \dots, n-1)\}$, let S be a word in the alphabet $\mathcal{S} = \{\sigma_i \ (i = 1, \dots, n-1)\}$ of maximal length such that $W \doteq ST$ for some word T . Suppose also that $W \doteq AV$ for some word A in the alphabet $\{\sigma_i\}$. Then S is divisible by A . The same is true for right division.*

Proof. We use induction on the length of S . If S has length 1, then A also has length 1 and the assertion follows from Proposition 1. Suppose the claim is true if the length is less than or equal to k , and let the length of S be equal to $k+1$. First, we consider the case where the length of A is equal to 1. This means that $A \equiv \sigma_j$. Let the first letter of S be σ_i : $S \equiv \sigma_i S'$, so that we have the situation $\sigma_i S' T \doteq \sigma_j V$. If $i = j$, then we are done. If $|i - j| \geq 2$, then $S' T \doteq \sigma_j X$, and induction yields $S' \doteq \sigma_j R'$. If $|i - j| = 1$, then $S' T \doteq \sigma_j \sigma_i Y$, and again by induction we obtain $S' \doteq \sigma_j \sigma_i Q'$. Suppose now that the length of A is greater than 1; then $A \equiv \sigma_j A'$ for some σ_j . Using the preceding case, we have $S \doteq \sigma_j S'$ and $S' T \doteq A' V$, and induction shows that $S' \doteq A' S''$. Hence, S is divisible by A . $\square$

Corollary 2. *For any element w of SB_n^+ there exists a unique greatest left (right) divisor that belongs to Br_n^+ .*

Garside's *fundamental element* in the braid group Br_n is defined by the following word:

$$(4) \quad \Delta \equiv \sigma_1 \dots \sigma_{n-1} \sigma_1 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1.$$

If we use Garside's notation $\Pi_t \equiv \sigma_1 \dots \sigma_t$, then $\Delta \equiv \Pi_{n-1} \dots \Pi_1$. We keep the same notation for the corresponding images in SB_n . Garside's transformation of words $\mathcal{R}$ is defined by the formula

$$\mathcal{R}(\sigma_i) \equiv \sigma_{n-i}.$$

This gives an automorphism of Br_n and the positive braid monoid Br_n^+ . We extend it to letters x_i and, thus, to SB_n^+ and to SB_n by

$$\mathcal{R}(x_i) \equiv x_{n-i}.$$

Proposition 3. *In SB_n^+ we have*

$$\begin{aligned} \sigma_i \Delta &\doteq \Delta \mathcal{R}(\sigma_i), \\ x_i \Delta &\doteq \Delta \mathcal{R}(x_i). \end{aligned}$$

Proof. For $i = 1$, we have

$$\begin{aligned} x_1 \Delta &\equiv x_1 \sigma_1 \dots \sigma_{n-1} \sigma_1 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \doteq \sigma_1 x_1 \sigma_2 \dots \sigma_{n-1} \sigma_1 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \\ &\doteq \sigma_1 x_1 \sigma_2 \sigma_1 \dots \sigma_{n-1} \sigma_2 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \\ &\doteq \sigma_1 \sigma_2 \sigma_1 x_2 \sigma_3 \dots \sigma_{n-1} \sigma_2 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \\ &\doteq \sigma_1 \sigma_2 \sigma_1 \sigma_3 \dots x_{n-2} \sigma_{n-1} \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \\ &\doteq \sigma_1 \sigma_2 \sigma_1 \sigma_3 \dots \sigma_{n-2} \sigma_{n-1} x_{n-1} \dots \sigma_1 \sigma_2 \sigma_1 \\ &\doteq \sigma_1 \sigma_2 \sigma_3 \dots \sigma_{n-2} \sigma_{n-1} \sigma_1 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 x_{n-1} \equiv \Delta x_{n-1}. \end{aligned}$$

For $i \geq 2$, we have

$$\begin{aligned}
x_i \Delta &\equiv x_i \sigma_1 \dots \sigma_{n-1} \sigma_1 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \\
&\doteq \sigma_1 \dots x_i \sigma_{i-1} \sigma_i \dots \sigma_{n-1} \sigma_1 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \\
&\doteq \sigma_1 \dots \sigma_{i-1} \sigma_i x_{i-1} \dots \sigma_{n-1} \sigma_1 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \doteq \Pi_{n-1} x_{i-1} \Pi_{n-2} \dots \Pi_1 \\
&\doteq \Pi_{n-1} \dots \Pi_{n-i+1} x_1 \Pi_{n-i} \Pi_{n-i-1} \dots \Pi_1 \\
&\doteq \Pi_{n-1} \dots \Pi_{n-i+1} \Pi_{n-i} \Pi_{n-i-1} x_{n-i} \Pi_{n-i-2} \dots \Pi_1 \\
&\doteq \Pi_{n-1} \dots \Pi_{n-i+1} \Pi_{n-i} \dots \Pi_1 x_{n-i} \doteq \Delta x_{n-i}.
\end{aligned}$$

□

The fundamental element has remarkable divisibility properties.

Proposition 4 ([15], Lemma 4). *In Br_n , there exist positive words W_r and V_r such that*

$$\Delta \doteq \sigma_r W_r \doteq V_r \sigma_r, \quad r = 1, \dots, n-1.$$

Proposition 5. *If W is an arbitrary positive word in the alphabet $\mathcal{A}$ such that either*

$$W \doteq \sigma_1 A_1 \doteq \sigma_2 A_2 \doteq \dots \doteq \sigma_{n-1} A_{n-1},$$

or

$$W \doteq B_1 \sigma_1 \doteq B_2 \sigma_2 \doteq \dots \doteq B_{n-1} \sigma_{n-1}$$

in SB_n^+ , then $W \doteq \Delta Z$ for some Z .

In the proof of the next proposition, we use the Mal'tsev rule [2]–[4]. Originally, it was formulated for groups. Here we need a version for monoids; the proof is the same. We consider monoids some generators of which are invertible. All relations except invertibility (such as the last relation in (3)) will be called *principal*. In the Mal'tsev rule the following situation is considered: two words A and C , where C contain no negative degrees of letters of the alphabet, represent one and the same element of the monoid. Then there exists a sequence of words

$$(5) \quad A \equiv A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_j \rightarrow \dots \rightarrow A_k \equiv C,$$

where each arrow means an elementary operation, which may be an application of one principal defining relation or insertion or deletion of an expression cc^{-1} or $c^{-1}c$, where c is a letter of our alphabet. In the first case, insertion is said to be *left* and in the second it is *right*. A right insertion

$$A_j \equiv Y_j Z_j \rightarrow Y_j c^{-1} c Z_j$$

is *correct* if the fragment Y_j is not changed in the sequence of transformations (5) till the elimination of c^{-1} . For left insertion the same condition should be fulfilled for the fragment Z_j . We use the Mal'tsev rule in the following form.

If all principal defining relations of a monoid contain only positive degrees of letters of the alphabet, then the transformation of a word A into a positive word C can be done by a sequence of operations whose insertions are all correct.

Proposition 6. *The canonical homomorphism*

$$SB_n^+ \rightarrow SB_n$$

is a monomorphism.

Proof. We need to prove that if two elements of SB_n^+ (expressed by positive words A and C) are equal in SB_n , then they are positively equal. If the elements determined by the words A and C are equal in SB_n , then there exists a sequence of words (5) where in any insertion or deletion of the expression cc^{-1} or $c^{-1}c$ the generator c is one of the σ_i 's.

Consider a subsequence of (5) that starts with the last (e.g., right) insertion and finishes with the deletion of the corresponding c^{-1} :

$$A_j \equiv Y_j Z_j \rightarrow Y_j c^{-1} c Z_j \rightarrow Y_j c^{-1} V_{j+1} \rightarrow \cdots \rightarrow Y_j c^{-1} c Z_{j+r} \rightarrow Y_j Z_{j+r}.$$

The fragment Y_j does not change in this subsequence because the insertion is correct. For any c there exists a positive word D_c such that $D_c c \doteq \Delta$. Define the sequence

$$\begin{aligned} \Delta A_j &\equiv \Delta Y_j Z_j \mapsto \cdots \mapsto \mathcal{R}(Y_j) \Delta Z_j \mapsto \cdots \mapsto \mathcal{R}(Y_j) D_c c Z_j \hookrightarrow \mathcal{R}(Y_j) D_c V_{j+1} \\ &\hookrightarrow \cdots \hookrightarrow \mathcal{R}(Y_j) D_c c Z_{j+r} \mapsto \cdots \mapsto \mathcal{R}(Y_j) \Delta Z_{j+r} \mapsto \cdots \mapsto \Delta Y_j Z_{j+r}, \end{aligned}$$

where $\mapsto$ denotes a positive operation and $\hookrightarrow$ denotes a positive operation or deletion. Thus, multiplying by Δ we eliminate one insertion. By induction, we construct a sequence of positive operations between $\Delta^m A \Delta^l$ and $\Delta^m C \Delta^l$ for some m and l . After cancellation, this means that A and C are positive equivalent. As was noted by V. V. Chaynikov, one can get a proof without invoking the Mal'tsev rule but using the fact that relations (3) are invariant with respect to the operation $\mathcal{R}$. $\square$

We introduce the lexicographical ordering among all positive words in the alphabet $\mathcal{A} = \{\sigma_1, \dots, \sigma_{n-1}, x_1, \dots, x_{n-1}\}$, assuming that $\sigma_1 < \sigma_2 < \cdots < \sigma_{n-1} < x_1 < x_2 < \cdots < x_{n-1}$. The *base* of a positive word W is the smallest positive word with respect to this ordering that is positively equal to W . The base is uniquely determined. If for a positive word A the element in SB_n^+ that represents A is not a multiple of Δ , then we write $\bar{A}$ for the base of A .

Theorem 1. *Every element w in SB_n can be expressed uniquely in the form $\Delta^m \bar{A}$, where m is an integer.*

Proof. First, suppose that an element w is represented by a positive word P . Among all positive words positively equivalent to P , we choose a word in the form $\Delta^t A$ with t maximal. Then A is not a multiple of Δ , and we have

$$P \doteq \Delta^t \bar{A}.$$

Now, let W be an arbitrary word in SB_n . Then we can put

$$W \equiv W_1(c_1)^{-1} W_2(c_2)^{-1} \cdots (c_k)^{-1} W_{k+1},$$

where each W_j is a positive word of nonnegative length and the c_l are generators σ_i , the only possible invertible generators. As has already been mentioned, for each c_l there exists a positive word D_l such that $c_l D_l \doteq \Delta$, so that $(c_l)^{-1} = D_l \Delta^{-1}$, whence

$$W = W_1 D_1 \Delta^{-1} W_2 D_2 \Delta^{-1} \cdots W_k D_k \Delta^{-1} W_{k+1}.$$

Hence, moving the factors Δ^{-1} to the left, we obtain $W = \Delta^k P$, where P is positive, and expressing this in the form $\Delta^t \bar{A}$, finally we get

$$(6) \quad W = \Delta^m \bar{A}.$$

It remains to show that the form (6) is unique. Suppose

$$(7) \quad \Delta^m \bar{A} = \Delta^p \bar{C}.$$

Let $p < m$, and let $m - p = t > 0$. Then (7) gives $\Delta^t \bar{A} = \bar{C}$, which is impossible. Consequently, $p = m$, whence $\bar{A} = \bar{C}$. Thus, by Proposition 6, we obtain $\bar{A} \doteq \bar{C}$, but the base is unique, so that $\bar{A} \equiv \bar{C}$ and the uniqueness of the form (6) is established. $\square$

The form of an element w established in this theorem is called the *Garside left normal form*, and the index m is called the *power* of w . The *Garside right normal form* is defined similarly, and the corresponding version of Theorem 1 is true. The Garside normal form also gives a solution to the word problem in the singular braid monoid.

Corollary 3. *Two words in the alphabet $\mathcal{A}$ represent one and the same element in SB_n if and only if their Garside normal forms (left or right) are identical.*

The Garside normal form for the braid groups was refined in the subsequent papers [1, 13, 12]. Namely, the *left-greedy form* (in the terminology of W. Thurston [13])

$$\Delta^t A_1 \dots A_k$$

was introduced, where the A_i are the successive longest possible *fragments of the word* Δ (in the terminology of Adyan [1]), or *positive permutation braids* (in the terminology of El-Rifai and Morton [12]). Certainly, the *right-greedy form* is defined similarly.

We consider these forms for the singular braid monoid. First, for any word W we move the greatest power of Δ to the left. To the right we have a positive word W' not divisible by Δ . Consider the decomposition $W' \doteq S_1 T$ as in Proposition 2. Then we take the fragments of the left-greedy form for S_1 : $S_{1,1} \dots S_{1,t}$. Among all the x_i -divisors of T , we choose the smallest in the lexicographical order: $T \doteq x_{i_1} T_1$. Consider the decomposition as in Proposition 2 for T_1 and continue this process. We get the form

$$W \doteq \Delta^t S_1 X_1 \dots S_k X_k,$$

where each S_i consists of fragments of the left-greedy form for the braid group and each X_i is a lexicographically ordered product of x_j . We call this form the *left-greedy form* for the singular braid monoid. The *right-greedy form* for the singular braid monoid is defined in the same way.

Theorem 2. *Each element of the singular braid monoid can be written uniquely in a left-greedy (right-greedy) form.*

Theorem 3. *For $n = 2$ the singular braid monoid SB_n is commutative and isomorphic to $\mathbb{Z} \oplus \mathbb{Z}^+$. For $n \geq 3$, the center of SB_n is the same as the center of Br_n and, thus, it is generated by Δ^2 .*

§3. CONJUGACY PROBLEM FOR THE SINGULAR BRAID MONOID

Let M be a monoid with unit group G . We say that two elements $u, v \in M$ are *conjugate* if $v = g^{-1}ug$ for some $g \in G$. This means that conjugate elements u and v in SB_n belong to one and the same orbit of the canonical action of the braid group Br_n on SB_n , and the *conjugacy problem* for the singular braid monoid will be understood precisely in this sense. Such an approach to conjugacy appears among others in monoid theory; see, e.g., [17].

Since the relations (3) are homogeneous with respect to both types of generators σ_i and x_i , the following three homomorphisms, which express the degree of an element with respect to σ_i , x_i and the total degree, are well defined: $\deg_\sigma : SB_n \rightarrow \mathbb{Z}$, $\deg_x : SB_n \rightarrow \mathbb{Z}^+$, and $\deg : SB_n \rightarrow \mathbb{Z}$.

Proposition 7. *The group of units of the monoid SB_n is equal to the image of the braid group Br_n .*

Proof. Since invertible elements must have invertible degree $\deg_x$, we have $\deg_x = 0$. $\square$

Garside's solution of the conjugacy problem for braid groups works in the case of the singular braid monoid. In the proof of the following theorem concerning the structure of the Cayley diagram $D(W)$ of a positive word W in the alphabet $\mathcal{A} = \{\sigma_i, x_k\}$ containing Δ , some additional considerations are necessary.

Theorem 4. *If $W \doteq \Delta V$ is any positive word (in the alphabet $\mathcal{A}$) containing Δ , then each node of $D(W)$ is incident with each edge σ_i , $i = 1, \dots, n - 1$.*

FIGURE 3

FIGURE 4

FIGURE 5

Proof. We follow Garside's proof, which employs induction on the order of a node equal to the word-length of an element. Proposition 4 gives the beginning of this induction: a statement for the identity element.

Assume that the statement of the theorem is true for all nodes of degree not exceeding m . Let C be an arbitrary node of order m , and let an edge σ_i start at C and end at a node D of order $m+1$, the case analyzed by Garside. By the inductive hypothesis, the node C is also incident to all σ_k , $k \neq i$. Then the braid relations and Proposition 1 show that the node D is also incident to all σ_k , $k \neq i$.

Now we consider the case of an edge x_i that starts at some node C of order m and ends at a node D of order $m+1$. By the inductive hypothesis, the node C is also incident to all σ_k , $k \neq i$. First, we treat the case of the generators σ_k with $|k-i| \neq 1$. If the edge σ_k ends at C , then $\sigma_k x_i = x_i \sigma_k$, which means that the edge σ_k also ends at D . If σ_k starts at C , then we can use Proposition 1 to obtain the fragment of the Cayley graph depicted in Figure 3 a). Let $|k-i| = 1$. If the edge σ_k starts at C , then Proposition 1 yields the fragment of the Cayley graph depicted in Figure 3 b). Now, consider the case where the edge σ_k ends at C (and starts at some node B). The edge σ_i must be incident to C , so that it either ends or starts at C . In the first case, Proposition 1 gives us the fragment of the Cayley graph depicted in Figure 3 c). We complete the proof in this case by using the relation $\sigma_i \sigma_k x_i = x_k \sigma_i \sigma_k$. In the last case, the edge σ_i starts at C . Consider the node B where σ_k starts. If the edge σ_i ends at B , then we use the relation $\sigma_i \sigma_k x_i = x_k \sigma_i \sigma_k$ and we are done. So, suppose that the edge σ_i also starts at B . Using Proposition 1, we obtain the fragment of the Cayley graph depicted in Figure 4 a). Using Proposition 1 twice, we obtain the fragment of the Cayley graph depicted in Figure 4 b). Finally, using Proposition 1 once again, we come to the fragment of the Cayley graph depicted

FIGURE 6

in Figure 5 c). To complete the proof in this case we use the relation $\sigma_i \sigma_k x_i = x_k \sigma_i \sigma_k$, arriving at Figure 5 d). $\square$

The definition of the *summit set* for the singular braid monoid is the same as Garside's.

Theorem 5. *In SB_n , two elements are conjugate if and only if their summit sets are identical.*

Since for any element of SB_n the summit set is finite and is obtained algorithmically, this theorem gives a solution of the conjugacy problem.

§4. BIRMAN-KO-LEE PRESENTATION FOR THE SINGULAR BRAID MONOID

For the singular braid monoid, we prove the existence of an analog of the presentation (2) of J. S. Birman, K. H. Ko, and S. J. Lee. For $1 \leq s < t \leq n$ and $1 \leq p < q \leq n$, consider the elements of SB_n defined by

$$(8) \quad \begin{cases} a_{ts} = (\sigma_{t-1}\sigma_{t-2}\cdots\sigma_{s+1})\sigma_s(\sigma_{s+1}^{-1}\cdots\sigma_{t-2}^{-1}\sigma_{t-1}^{-1}) & \text{for } 1 \leq s < t \leq n, \\ a_{ts}^{-1} = (\sigma_{t-1}\sigma_{t-2}\cdots\sigma_{s+1})\sigma_s^{-1}(\sigma_{s+1}^{-1}\cdots\sigma_{t-2}^{-1}\sigma_{t-1}^{-1}) & \text{for } 1 \leq s < t \leq n, \\ b_{qp} = (\sigma_{q-1}\sigma_{q-2}\cdots\sigma_{p+1})x_p(\sigma_{p+1}^{-1}\cdots\sigma_{q-2}^{-1}\sigma_{q-1}^{-1}) & \text{for } 1 \leq p < q \leq n. \end{cases}$$

Geometrically, the generators $a_{s,t}$ and $b_{s,t}$ are depicted in Figure 6.

Theorem 6. *The singular braid monoid SB_n has a presentation with generators a_{ts} , a_{ts}^{-1} for $1 \leq s < t \leq n$ and b_{qp} for $1 \leq p < q \leq n$ and with the relations*

$$(9) \quad \begin{cases} a_{ts}a_{rq} = a_{rq}a_{ts} & \text{for } (t-r)(t-q)(s-r)(s-q) > 0, \\ a_{ts}a_{sr} = a_{tr}a_{ts} = a_{sr}a_{tr} & \text{for } 1 \leq r < s < t \leq n, \\ a_{ts}a_{ts}^{-1} = a_{ts}^{-1}a_{ts} = 1 & \text{for } 1 \leq s < t \leq n, \\ a_{ts}b_{rq} = b_{rq}a_{ts} & \text{for } (t-r)(t-q)(s-r)(s-q) > 0, \\ a_{ts}b_{ts} = b_{ts}a_{ts} & \text{for } 1 \leq s < t \leq n, \\ a_{ts}b_{sr} = b_{tr}a_{ts} & \text{for } 1 \leq r < s < t \leq n, \\ a_{sr}b_{tr} = b_{ts}a_{sr} & \text{for } 1 \leq r < s < t \leq n, \\ a_{tr}b_{ts} = b_{sr}a_{tr} & \text{for } 1 \leq r < s < t \leq n, \\ b_{ts}b_{rq} = b_{rq}b_{ts} & \text{for } (t-r)(t-q)(s-r)(s-q) > 0. \end{cases}$$

Proof. We follow the proof of Birman, Ko, and Lee [8]; we begin with the presentation of SB_n with the help of the generators $\sigma_i, \sigma_i^{-1}, x_i$, $i = 1, \dots, n-1$, and the system of relations (3). We add the new generators a_{ts}, a_{ts}^{-1} for $1 \leq s < t \leq n$ and b_{qp} for $1 \leq p < q \leq n$ defined by (8). Since relations (9) are described by isotopies of singular braids, they must be consequences of (3), and we may add them too.

In the special case where $t = s+1$, relations (8) imply that $a_{(s+1)s} = \sigma_s$, $a_{(s+1)s}^{-1} = \sigma_s^{-1}$, $b_{(s+1)s} = x_s$; therefore, we may omit the generators $\sigma_1, \dots, \sigma_{n-1}$, $\sigma_1^{-1}, \dots, \sigma_{n-1}^{-1}$ to obtain a presentation with the generators a_{ts} , a_{ts}^{-1} , b_{pq} . Now, the defining relations are (9) and

$$(10) \quad a_{(i+1)i}a_{(j+1)j} = a_{(j+1)j}a_{(i+1)i} \quad \text{if } |i-j| > 1,$$

$$(11) \quad a_{(i+1)i}a_{(i+2)(i+1)}a_{(i+1)i} = a_{(i+2)(i+1)}a_{(i+1)i}a_{(i+2)(i+1)},$$

$$(12) \quad a_{ts} = (a_{t(t-1)}a_{(t-1)(t-2)} \cdots a_{(s+2)(s+1)})a_{(s+1)s}(a_{(s+2)(s+1)}^{-1} \cdots a_{(t-1)(t-2)}^{-1}a_{t(t-1)}^{-1}),$$

$$(13) \quad a_{ts}^{-1} = (a_{t(t-1)}a_{(t-1)(t-2)} \cdots a_{(s+2)(s+1)})a_{(s+1)s}^{-1}(a_{(s+2)(s+1)}^{-1} \cdots a_{(t-1)(t-2)}^{-1}a_{t(t-1)}^{-1}),$$

$$(14) \quad b_{(i+1)i}b_{(j+1)j} = b_{(j+1)j}b_{(i+1)i} \quad \text{if } |i-j| > 1,$$

$$(15) \quad b_{(i+1)i}a_{(j+1)j} = a_{(j+1)j}b_{(i+1)i} \quad \text{if } |i-j| \neq 1,$$

$$(16) \quad a_{(i+1)i}a_{(i+2)(i+1)}b_{(i+1)i} = b_{(i+2)(i+1)}a_{(i+1)i}a_{(i+2)(i+1)},$$

$$(17) \quad a_{(i+2)(i+1)}a_{(i+1)i}b_{(i+2)(i+1)} = b_{(i+1)i}a_{(i+2)(i+1)}a_{(i+1)i},$$

$$(18) \quad b_{qp} = (a_{q(q-1)}a_{(q-1)(q-2)} \cdots a_{(p+2)(p+1)})b_{(p+1)p}(a_{(p+2)(p+1)}^{-1} \cdots a_{(q-1)(q-2)}^{-1}a_{q(q-1)}^{-1})$$

for $1 \leq p < q \leq n$.

Now we prove that relations (10)–(18) are consequences of (9). Birman, Ko, and Lee proved that relations (10)–(12) are consequences of the first two relations in (9). The proof for (13) is the same as for (12). Relations (14) are special cases of the last relation in (9). Relations (15) are special cases of the fourth and the fifth relations in (9). To deduce (16), we use the sixth relation in (9):

$$a_{(i+1)i}a_{(i+2)(i+1)}b_{(i+1)i} = a_{(i+1)i}b_{(i+2)i}a_{(i+2)(i+1)},$$

and then the seventh relation in (9):

$$a_{(i+1)i}b_{(i+2)i}a_{(i+2)(i+1)} = b_{(i+2)(i+1)}a_{(i+1)i}a_{(i+2)(i+1)}.$$

To deduce (17), we use the second, eighth, fifth, and again second relations in (9):

$$\begin{aligned} a_{(i+2)(i+1)}a_{(i+1)i}b_{(i+2)(i+1)} &= a_{(i+1)i}a_{(i+2)i}b_{(i+2)(i+1)} = a_{(i+1)i}b_{(i+1)i}a_{(i+2)i} \\ &= b_{(i+1)i}a_{(i+1)i}a_{(i+2)i} = b_{(i+1)i}a_{(i+2)(i+1)}a_{(i+1)i}. \end{aligned}$$

To eliminate (18), we apply the second relation in (9) to change the middle pair $a_{(p+2)(p+1)}b_{(p+1)p}$ in (18) to $b_{(p+2)p}a_{(p+1)p}$. Then we apply this process to the pair $a_{(p+3)(p+2)}b_{(p+2)p}$. Ultimately, this process will move the original central letter $b_{(p+1)p}$ to the leftmost position, where it becomes b_{qp} . Free cancellation eliminates everything to the right of it, and we are done. $\square$

Now we consider the *positive singular braid monoid* with respect to generators a_{ts} and $b_{t,s}$ for $1 \leq s < t \leq n$. Its relations are (9) except the one concerning the invertibility of a_{ts} . We say that positive words A and B in the alphabets a_{ts} and $b_{t,s}$ are *positively equivalent* if they are equal as elements of this monoid. In this case, as in the preceding section, we write $A \doteq B$.

The *fundamental element* of Birman, Ko, and Lee is given by the *fundamental word* δ by the formula

$$\delta \equiv a_{n(n-1)}a_{(n-1)(n-2)} \cdots a_{21} \equiv \sigma_{n-1}\sigma_{n-2} \cdots \sigma_2\sigma_1.$$

For us it is convenient to express its divisibility by any generator a_{ts} (which was proved in [8]) in the following form.

Proposition 8. *The fundamental word δ is positively equivalent to a word that begins or ends with any given generator a_{ts} . The explicit expression for left divisibility is*

$$\delta \doteq a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \cdots a_{(t+1)s}a_{t(t-1)} \cdots a_{(s+2)(s+1)}a_{s(s-1)} \cdots a_{21}.$$

FIGURE 7

Proposition 9. *For the fundamental word δ , we have the following commutation formulas:*

$$\begin{cases} a_{ts}\delta \doteq \delta a_{(t+1)(s+1)} & \text{for } 1 \leq r < s < t < n, \\ a_{ns}\delta \doteq \delta a_{(s+1)1}, \\ b_{ts}\delta \doteq \delta b_{(t+1)(s+1)} & \text{for } 1 \leq r < s < t < n, \\ b_{ns}\delta \doteq \delta b_{(s+1)1}. \end{cases}$$

Proof. For the generators a_{ts} , this was proved in [8]. Suppose that $1 \leq r < s < t < n$. Then, using relations (9), we obtain

$$\begin{aligned} b_{ts}\delta &\doteq b_{ts}a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \cdots a_{(t+1)s}a_{t(t-1)} \cdots a_{(s+2)(s+1)}a_{s(s-1)} \cdots a_{21} \\ &\doteq a_{ts}b_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \cdots a_{(t+1)s}a_{t(t-1)} \cdots a_{(s+2)(s+1)}a_{s(s-1)} \cdots a_{21} \\ &\doteq a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \cdots b_{ts}a_{(t+1)s}a_{t(t-1)} \cdots a_{(s+2)(s+1)}a_{s(s-1)} \cdots a_{21} \\ &\doteq a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \cdots a_{(t+1)s}b_{(t+1)t}a_{t(t-1)} \cdots a_{(s+2)(s+1)}a_{s(s-1)} \cdots a_{21} \\ &\doteq a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \cdots a_{(t+1)s}a_{t(t-1)}b_{(t+1)(t-1)} \cdots a_{(s+2)(s+1)}a_{s(s-1)} \cdots a_{21} \\ &\doteq a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \cdots a_{(t+1)s}a_{t(t-1)} \cdots b_{(t+1)(s+2)}a_{(s+2)(s+1)}a_{s(s-1)} \cdots a_{21} \\ &\doteq a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \cdots a_{(t+1)s}a_{t(t-1)} \cdots a_{(s+2)(s+1)}b_{(t+1)(s+1)}a_{s(s-1)} \cdots a_{21} \\ &\doteq a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \cdots a_{(t+1)s}a_{t(t-1)} \cdots a_{(s+2)(s+1)}a_{s(s-1)} \cdots a_{21}b_{(t+1)(s+1)} \\ &\doteq \delta b_{(t+1)(s+1)}. \end{aligned}$$

For $t = n$, we have

$$\begin{aligned} b_{ns}\delta &\doteq b_{ns}a_{n(n-1)}a_{(n-1)(n-2)} \cdots a_{21} \doteq a_{n(n-1)}b_{(n-1)s}a_{(n-1)(n-2)} \cdots a_{21} \\ &\doteq \cdots \\ &\doteq a_{(n-1)n}a_{(n-1)(n-2)} \cdots a_{(s+2)(s+1)}b_{(s+1)s}a_{(s+1)s} \cdots a_{21} \\ &\doteq a_{(n-1)n}a_{(n-1)(n-2)} \cdots a_{(s+1)s}b_{(s+1)s}a_{s(s-1)} \cdots a_{21} \\ &\doteq a_{(n-1)n}a_{(n-1)(n-2)} \cdots a_{s(s-1)}b_{(s+1)(s-1)}a_{(s-1)(s-2)} \cdots a_{21} \\ &\doteq a_{(n-1)n}a_{(n-1)(n-2)} \cdots a_{s(s-1)}a_{(s-1)(s-2)} \cdots a_{21}b_{(s+1)1}. \end{aligned}$$

□

Geometrically, this commutation is shown in Figure 7.

Analogs of other results proved by Birman, Ko, and Lee (which are also analogs of the statements of §§2 and 3 of the present paper) remain valid for the singular braid monoid. They were proved by Chaynikov in [9].

REFERENCES

- [1] S. I. Adyan, *Fragments of the word Δ in the braid group*, Mat. Zametki **36** (1984), no. 1, 25–34; English transl., Math. Notes **36** (1984), no. 1–2, 505–510. MR0757642 (86b:20048)
- [2] A. I. Mal'tsev, *On including of associative systems in groups. I*, Mat. Sb. **6** (1939), no. 2, 331–336. (Russian) MR0002152 (2:7d)
- [3] ———, *On including of associative systems in groups. II*, Mat. Sb. **8** (1940), no. 2, 251–264. (Russian) MR0002895 (2:128b)
- [4] P. S. Novikov, *On the algorithmic unsolvability of the word problem in group theory*, Trudy Mat. Inst. Steklov. **44** (1955), 3–143. (Russian) MR0075197 (17:706b)
- [5] E. Artin, *Theorie der Zöpfe*, Abh. Math. Semin. Univ. Hamburg **4** (1925), 47–72.
- [6] J. C. Baez, *Link invariants of finite type and perturbation theory*, Lett. Math. Phys. **26** (1992), no. 1, 43–51. MR1193625 (93k:57006)
- [7] J. S. Birman, *New points of view in knot theory*, Bull. Amer. Math. Soc. (N.S.) **28** (1993), no. 2, 253–287. MR1191478 (94b:57007)
- [8] J. S. Birman, K. H. Ko, and S. J. Lee, *A new approach to the word and conjugacy problems in the braid groups*, Adv. Math. **139** (1998), no. 2, 322–353. MR1654165 (99m:20082)
- [9] V. V. Chaynikov, *Garside structure for singular braid monoid in Birman, Ko, Lee generators*, Comm. Algebra **34** (2006), no. 6, 1981–1995. MR2235075 (2007c:20092)
- [10] R. Corran, *A normal form for a class of monoids including the singular braid monoids*, J. Algebra **223** (2000), no. 1, 256–282. MR1738262 (2001f:20123)
- [11] O. T. Dasbach and B. Gemein, *The word problem for the singular braid monoid*, Arxiv.math.GT/9809070.
- [12] E. El-Rifai and H. R. Morton, *Algorithms for positive braids*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 479–497. MR1315459 (96b:20052)
- [13] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston, *Word processing in groups*, Jones and Bartlett Publishers, Boston, MA, 1992. MR1161694 (93i:20036)
- [14] R. Fenn, D. Rolfsen, and J. Zhu, *Centralisers in the braid group and singular braid monoid*, Enseign. Math. (2) **42** (1996), no. 1–2, 75–96. MR1395042 (97h:20047)
- [15] F. A. Garside, *The braid group and other groups*, Quart. J. Math. Oxford Ser. (2) **20** (1969), 235–254. MR0248801 (40:2051)
- [16] A. Jrai, Jr., *On the monoid of singular braids*, Topology Appl. **96** (1999), no. 2, 109–119. MR1702305 (2000f:57013)
- [17] M. S. Putcha, *Conjugacy classes and nilpotent variety of a reductive monoid*, Canad. J. Math. **50** (1998), no. 4, 829–844. MR1638627 (99f:20095)

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