DISPERSION ESTIMATES FOR DISCRETE SCHRÖDINGER AND KLEIN–GORDON EQUATIONS

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ABSTRACT. The long-time asymptotics is derived for solutions of the discrete 3-dimensional Schrödinger and Klein–Gordon equations.

§1. INTRODUCTION

We consider the 3-dimensional discrete version of the Schrödinger equation,

$$(1.1) \quad \begin{cases} i\dot{\psi}(x, t) = H\psi(x, t) := (-\Delta + V(x))\psi(x, t), \\ \psi|_{t=0} = \psi_0, \end{cases} \quad \left| \begin{array}{l} x \in \mathbb{Z}^3, \quad t \in \mathbb{R}, \end{array} \right.$$

where Δ stands for the difference Laplacian in $\mathbb{Z}^3$, defined by

$$\Delta\psi(x) = \sum_{|y-x|=1} \psi(y) - 6\psi(x), \quad x \in \mathbb{Z}^3, \quad \psi : \mathbb{Z}^3 \rightarrow \mathbb{C}.$$

We denote by $\mathcal{V}$ the set of real-valued finitely supported functions on the lattice $\mathbb{Z}^3$. Concerning the potential V , we assume that $V \in \mathcal{V}$. Then for any $\psi_0 \in l^2 = l^2(\mathbb{Z}^3)$ there exists a unique solution $\psi(x, t) \in C(\mathbb{R}, l^2)$ of the Cauchy problem (1.1), and the charge conservation law $\|\psi(\cdot, t)\|_{l^2} = \text{const}$ is fulfilled.

It is well known that, with the help of the Fourier–Laplace transformation relative to the variable t , the properties of the nonstationary equation can be deduced from the properties of the resolvent $R(\omega) = (H - \omega)^{-1}$ of the Schrödinger operator H .

We are going to use the weighted Hilbert spaces $l^2_\sigma = l^2_\sigma(\mathbb{Z}^3)$ with the norms

$$\|u\|_{l^2_\sigma} = \|(1 + x^2)^{\sigma/2} u\|_{l^2}, \quad \sigma \in \mathbb{R}.$$

Let

$$B(\sigma, \sigma') = \mathcal{L}(l^2_\sigma, l^2_{\sigma'})$$

denote the space of bounded linear operators from l^2_σ to $l^2_{\sigma'}$.

The spectrum of the operator H consists of the continuous spectrum and real eigenvalues μ_j , $j = 1, \dots, n$. Note that $n \leq N$, where N is the number of points in the support of V (see [4, Chapter I, Theorem 13^{bis}]).

The continuous spectrum of H coincides with the interval $[0, 12]$, which is the range of the symbol $\phi(\theta) = 4(\sin^2 \frac{\theta_1}{2} + \sin^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta_3}{2})$ of the difference Laplace operator $H_0 = -\Delta$.

We pay special attention to the points $\omega_k = 4k \in [0, 12]$, $k = 0, 1, 2, 3$, which are critical values of the symbol, i.e., the values of the symbol at the critical points.

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Our main results are as follows. For “a generic potential” $V \in \mathcal{V}$ (see Definition 3.3), we obtain:

- a) the existence of the limits $R(\omega \pm i0)$ (“limiting absorption principle”) on the continuous spectrum in the norm of $B(\sigma; -\sigma)$ with $\sigma > 3/2$;
- b) the Puiseux expansion for the resolvent at the singular spectral points ω_k :

$$(1.2) \quad R(\omega_k + \omega) = D_k + \mathcal{O}(\sqrt{\omega}), \quad \omega \rightarrow 0,$$

in the norm of $B(\sigma; -\sigma)$ with $\sigma > 7/2$.

Then, for initial data $\psi_0 \in l_\sigma^2$ with $\sigma > 11/2$ we deduce the following long-time asymptotics:

$$(1.3) \quad \left\| e^{-itH} - \sum_{j=1}^n e^{-it\mu_j} P_j \right\|_{B(\sigma, -\sigma)} = \mathcal{O}(t^{-3/2}), \quad t \rightarrow \infty.$$

Here the P_j are the orthogonal projections in l^2 onto the eigenspaces of H corresponding to the discrete eigenvalues μ_j .

We also obtain similar results for the discrete Klein-Gordon equation

$$(1.4) \quad \begin{cases} \ddot{\psi}(x, t) = (\Delta - m^2 - V(x)) \psi(x, t), \\ \psi|_{t=0} = \psi_0, \quad \dot{\psi}|_{t=0} = \pi_0, \end{cases} \quad \left| \begin{array}{l} x \in \mathbb{Z}^3, \quad t \in \mathbb{R}. \end{array} \right.$$

Some comments on previous results in this direction are in order. For the first time, the difference Schrödinger equation was considered by Èskina in [14]. She proved the limiting absorption principle for the matrix entries of the resolvent. The asymptotic expansion of the matrix entries of the resolvent $R(\omega)$ at the critical points ω_k was obtained by Islami and Vainberg [5] in the 2-dimensional case. They used this expansion to prove the long-time asymptotics for the solutions of the Cauchy problem for the difference wave equation. The present paper differs from [5] mainly by the fact that here all asymptotic expansions are valid in the weighted function spaces l_σ^2 rather than on compact sets. Such expansions may be useful for the study of nonlinear evolutionary equations.

The asymptotic expansion of the resolvent and the long-time asymptotics (1.3) for hyperbolic PDEs in $\mathbb{R}^n$ (continuous case) were obtained earlier in [10, 2, 3], and for the Schrödinger equation in [6, 7, 11]; see also [12] for an up-to-date review. We use the main ideas of those papers.

The results of the present paper extend the results of [8] and [9] from 1- and 2-dimensional difference equations to 3-dimensional ones. In the 3-dimensional case, we encounter more difficult analytic problems, because of the diversity of the types of critical points.

The paper is organized as follows. In §2 we prove the limiting absorption principle and derive the Puiseux asymptotic of the free resolvent. In §3 we extend these results to the case of a perturbed resolvent. In §4 we prove the long-time asymptotics (1.3). In §5 we consider the discrete Klein–Gordon equation. In Appendix C we apply our results to construct asymptotic scattering states.

§2. FREE RESOLVENT

We start with the investigation of the unperturbed problem for equation (1.1) with $V(x) = 0$. The discrete Fourier transform of $u(x) \in l^2(\mathbb{Z}^3)$ is defined by the formula

$$\hat{u}(\theta) = \sum_{x \in \mathbb{Z}^3} u(x) e^{i\theta x}, \quad \theta \in T^3 := \mathbb{R}^3 / 2\pi\mathbb{Z}^3.$$

After the Fourier transformation, the operator $H_0 = -\Delta$ becomes the operator of multiplication by $\phi(\theta) := 6 - 2 \sum_{j=1}^3 \cos \theta_j = 4 \sum_{j=1}^3 \sin^2 \frac{\theta_j}{2}$:

$$(2.1) \quad -\widehat{\Delta u}(\theta) = \phi(\theta) \widehat{u}(\theta), \quad \theta \in T^3.$$

Thus, the spectrum of H_0 coincides with the range of the function ϕ , i.e., $\text{Spec } H_0 = \Sigma := [0, 12]$. We denote by $R_0(\omega) = (H_0 - \omega)^{-1}$ the resolvent of the difference Laplacian. Then the kernel of the resolvent $R_0(\omega)$ looks like this:

$$(2.2) \quad R_0(\omega, x - y) = \frac{1}{8\pi^3} \int_{T^3} \frac{e^{-i\theta(x-y)}}{\phi(\theta) - \omega} d\theta, \quad \omega \in \mathbb{C} \setminus \Sigma.$$

Lemma 2.1. *The free resolvent $R_0(\omega)$ is an analytic function of $\omega \in \mathbb{C} \setminus \Sigma$ with values in $\mathcal{B}(\sigma, \sigma')$ for any $\sigma, \sigma' \in \mathbb{R}$.*

Proof. For a fixed $\omega \in \mathbb{C} \setminus \Sigma$, we have $\phi(\theta) - \omega \neq 0$ for $\theta \in T^3$. Therefore, $\phi(\theta + i\xi) - \omega \neq 0$ for $\theta \in T^3$ and $\xi \in \mathbb{R}^3$ provided $\xi \neq 0$ is sufficiently small. Hence, the function $1/(\phi(\theta) - \omega)$ admits analytic continuation to a complex neighborhood of the torus $\{\theta + i\xi : \theta \in T^3, \xi \in \mathbb{R}^3 : |\xi| < \delta(\omega)\}$ with $\delta(\omega) > 0$. Therefore, the Paley–Wiener arguments show that

$$|R_0(\omega, x - y)| \leq C(\delta) e^{-\delta|x-y|}$$

for any $\delta < \delta(\omega)$. Hence, $R_0(\omega) \in \mathcal{B}(\sigma, \sigma')$ by the Schur lemma. $\square$

2.1. Limiting absorption principle. Now we are interested in the traces of the analytic function $R_0(\omega)$ at the cut Σ . Consider

$$(2.3) \quad R_0(\omega \pm i\varepsilon, z) = \frac{1}{8\pi^3} \int_{T^3} \frac{e^{-i\theta z}}{\phi(\theta) - \omega \mp i\varepsilon} d\theta, \quad z \in \mathbb{Z}^3, \quad \omega \in \Sigma, \quad \varepsilon > 0.$$

Note that the limiting distribution $\frac{1}{\phi(\theta) - \omega \mp i0}$ is well defined if ω is not a critical value of the function $\phi(\theta)$, i.e., $\omega \neq 0, 4, 8, 12$. The following limiting absorption principle is valid.

Proposition 2.2. *For $\sigma > 3/2 + k$, the following limits exist as $\varepsilon \rightarrow 0+$:*

$$(2.4) \quad \partial_\omega^k R_0(\omega \pm i\varepsilon) \xrightarrow{\mathcal{B}(\sigma, -\sigma)} \partial_\omega^k R_0(\omega \pm i0), \quad \omega \in \Sigma \setminus \{0, 4, 8, 12\}.$$

Proof. i) Let $k = 0$. First, we follow [14] to prove (2.4) for any fixed z . Let $\chi_j(\theta)$, $j = 1, \dots, l$, be a sufficiently fine partition of unity on the torus T^3 , which will be specified below. Then

$$(2.5) \quad R_0(\omega \pm i\varepsilon, z) = \sum_{j=1}^l \frac{1}{8\pi^3} \int_{D_j} \frac{\chi_j(\theta) e^{-i\theta z}}{\phi(\theta) - \omega \mp i\varepsilon} d\theta = \sum_{j=1}^l P_j(\omega \pm i\varepsilon, z),$$

where D_j is the support of the function χ_j . If $\{\phi(\theta) = \omega\} \cap D_j = \emptyset$, then the function $P_j(\omega \pm i\varepsilon, z)$ is continuous for $\varepsilon \geq 0$ and

$$(2.6) \quad |P_j(\omega \pm i\varepsilon, z)| < C_j < \infty, \quad z \in \mathbb{Z}^3, \quad \varepsilon \geq 0.$$

Now, let $S_j = \{\phi(\theta) = \omega\} \cap D_j$. Then any $\theta \in D_j$ can be represented uniquely as $\theta = s + tn(s)$, where $s \in S_j$, and $n(s)$ is the outward unit normal to S_j at the point s . We introduce the new variables (s, t) . Then

$$(2.7) \quad P_j(\omega \pm i\varepsilon, z) = \frac{1}{8\pi^3} \int_{S_j} e^{-isz} ds \int_{-a(s)}^{b(s)} \frac{\chi_j(s + tn(s)) e^{-itn(s)z} J(s, t)}{t\psi(s + tn(s)) \mp i\varepsilon} dt,$$

where $J(s, t)$ is the Jacobian, ψ is a smooth function, and $a(s), b(s) > 0$. Observe that $J(s, t)|_{t=0} = 1$ and $\psi(s + tn(s))|_{t=0} = |\nabla \phi(s)| \neq 0$, because $\omega \in \Sigma \setminus \{0, 4, 8, 12\}$ is not a critical value of $\phi(\theta)$. We need the following statement.

Lemma 2.3. *Let $\varphi(t, z)$ be a smooth function satisfying*

$$(2.8) \quad |\varphi(t, z)| \leq C, \quad |\partial_t \varphi(t, z)| \leq C|z|, \quad t \in [-\delta, \delta], \quad z \in \mathbb{Z}^3,$$

and let $\psi(t)$ be a smooth function such that $\psi(t) \neq 0$ if $t \in [-\delta, \delta]$. Consider

$$F(\pm\varepsilon, z) = \int_{-\delta}^{\delta} \frac{\varphi(t, z)}{t\psi(t) \mp i\varepsilon} dt.$$

Then $F(\pm\varepsilon, z) \rightarrow F(\pm 0, z)$ as $\varepsilon \rightarrow 0+$ for all $z \in \mathbb{Z}^3$, and

$$\sup_{\varepsilon \in (0, 1]} |F(\pm\varepsilon, z)| \leq C \left(\log(1 + |z|) + 1 \right), \quad z \in \mathbb{Z}^3.$$

Proof. We rewrite $F(\pm\varepsilon, z)$ as

$$\begin{aligned} F(\pm\varepsilon, z) &= \varphi(0, z) \int_{-\delta}^{\delta} \frac{dt}{t\psi(0) \mp i\varepsilon} - \varphi(0, z) \int_{-\delta}^{\delta} \frac{(\psi(t) - \psi(0))tdt}{(t\psi(t) \mp i\varepsilon)(t\psi(0) \mp i\varepsilon)} \\ &\quad + \int_{-\delta}^{\delta} \frac{\varphi(t, z) - \varphi(0, z)}{t\psi(t) \mp i\varepsilon} dt. \end{aligned}$$

Then

$$(2.9) \quad \begin{aligned} F(\pm\varepsilon, z) \rightarrow F(\pm 0, z) &= \pm i\pi \frac{\varphi(0, z)}{\psi(0)} - \frac{\varphi(0, z)}{\psi(0)} \int_{-\delta}^{\delta} \frac{\psi(t) - \psi(0)}{t\psi(t)} dt \\ &\quad + \int_{-\delta}^{\delta} \frac{\varphi(t, z) - \varphi(0, z)}{t\psi(t)} dt \end{aligned}$$

as $\varepsilon \rightarrow 0+$. By (2.8), the first and second summands on the right in (2.9) can be estimated by a constant independent of $|z| \in \mathbb{Z}^3$. We estimate the third term on the right-hand side of (2.9). For $|z| < 1/\delta$, this term can also be estimated by a constant. For $|z| > 1/\delta$, we use (2.8) to show that

$$\begin{aligned} \int_{-\delta}^{\delta} \left| \frac{\varphi(t, z) - \varphi(0, z)}{t\psi(t)} \right| dt &= \int_{|t| < 1/|z|} \dots + \int_{1/|z| < |t| < \delta} \dots \\ &\leq \frac{1}{|z|} C|z| + C \log |z| \leq C \log |z|. \end{aligned}$$

The lemma is proved. $\square$

Lemma 2.3 implies that for all $z \in \mathbb{Z}^3$, $P_j(\omega \pm i\varepsilon, z) \rightarrow P_j(\omega \pm i0, z)$ as $\varepsilon \rightarrow 0+$, and

$$\sup_{\varepsilon \in (0, 1]} |P_j(\omega \pm i\varepsilon, z)| \leq C_j (\log(1 + |z|) + 1).$$

Obviously, the entire resolvent R_0 has similar properties. Hence, by the Lebesgue dominated convergence theorem,

$$\sum_{x, y \in \mathbb{Z}^3} (1 + |x|^2)^{-\sigma} |R_0(\omega \pm i\varepsilon, x - y) - R_0(\omega \pm i0, x - y)|^2 (1 + |y|^2)^{-\sigma} \rightarrow 0$$

as $\varepsilon \rightarrow 0+$, with $\sigma > 3/2$. Then the Hilbert–Schmidt norm of the difference $R_0(\omega \pm i\varepsilon) - R_0(\omega \pm i0)$ converges to zero. This proves Proposition 2.2 in the case where $k = 0$.

ii) For $k \neq 0$ we use integration by parts. For instance, let $k = 1$. Since $\nabla\phi(\theta) \neq 0$ for $\theta \in D_j$, there exists $i \in \{1, 2, 3\}$ such that $\partial_i\phi(\theta) \neq 0$ for $\theta \in D_j$. Consequently,

$$\begin{aligned} P'_j(\omega \pm i\varepsilon, z) &= \frac{1}{8\pi^3} \int_{D_j} \frac{\chi_j(\theta)e^{-i\theta z}}{(\phi(\theta) - \omega \mp i\varepsilon)^2} d\theta \\ &= -\frac{1}{8\pi^3} \int_{D_j} \partial_i \left(\frac{1}{\phi(\theta) - \omega \mp i\varepsilon} \right) \frac{\chi_j(\theta)e^{-i\theta z}}{\partial_i\phi(\theta)} d\theta \\ &= \frac{1}{8\pi^3} \int_{D_j} \frac{1}{\phi(\theta) - \omega \mp i\varepsilon} \partial_i \left(\frac{\chi_j(\theta)e^{-i\theta z}}{\partial_i\phi(\theta)} \right) d\theta. \end{aligned}$$

The rest of the proof is similar to the case of $k = 0$. Differentiation of the exponential yields an additional factor z_i , and then the value of σ increases by one. $\square$

2.2. Asymptotics near critical points. In the sequel, we shall need information on the behavior of the resolvent $R_0(\omega)$ near the critical points ω_k . We consider the “elliptic” points $\omega_1 = 0$, $\omega_4 = 12$ and the “hyperbolic” points $\omega_2 = 4$, $\omega_3 = 8$ separately.

2.2.1. Elliptic points. Here we construct the Puiseux expansion of the free resolvent $R_0(\omega)$ near the point $\omega_1 = 0$ (the expansion near the point $\omega_4 = 12$ can be constructed similarly).

Proposition 2.4. *Let $N = 0, 1, 2, \dots$, and let $\sigma > N + 3/2$. Then the following expansion is valid in $\mathcal{B}(\sigma, -\sigma)$:*

$$(2.10) \quad R_0(\omega) = \sum_{k=0}^N A_k \omega^{k/2} + \mathcal{O}(\omega^{(N+1)/2}), \quad |\omega| \rightarrow 0, \quad \arg \omega \in (0, 2\pi).$$

Here $A_k \in \mathcal{B}(\sigma, -\sigma)$ with $\sigma > k + 1/2$.

Proof. The resolvent $R_0(\omega \pm i0)$ is represented by the integral (2.2). Fix $0 < \delta < 1$ and consider $0 < |\omega| < \delta^2/2$. We identify T^3 with the cube $[-\pi, \pi]^3$ and represent $R_0(\omega, z)$, $z = x - y$, as the sum

$$R_0(\omega, z) = \frac{1}{8\pi^3} \int_{B_\delta} \frac{e^{-i\theta z}}{\phi(\theta) - \omega} d\theta + \frac{1}{8\pi^3} \int_{T^3 \setminus B_\delta} \frac{e^{-i\theta z}}{\phi(\theta) - \omega} d\theta = R_{01}(\omega, z) + R_{02}(\omega, z),$$

where B_δ is the ball of radius δ . Since $\phi(\theta) = |\theta|^2 + \mathcal{O}(|\theta|^4)$, it follows that $R_{02}(\omega, z)$ is an analytic function of ω in $|\omega| \leq \delta^2/2$, and

$$|\partial_\omega^j R_{02}(\omega, z)| \leq \frac{C_{j,N}}{(|z| + 1)}, \quad |\omega| \leq \delta^2/2, \quad z \in \mathbb{Z}^3.$$

Hence, it suffices to prove an asymptotics of type (2.10) for R_{01} . For simplicity, we suppose that $\phi(\theta) = |\theta|^2$. (If $\phi(\theta) = |\theta|^2 + \mathcal{O}(|\theta|^4)$, the proof is similar, differing only in some technical details.) We choose a system of coordinates in which the direction of the axis θ_3 coincides with the direction of the vector z , and we rewrite $R_{01}(\omega, z)$ as

$$R_{01}(\omega, z) = \frac{1}{8\pi^3} \int_{|n|=1} \left(\int_0^\delta \frac{e^{-ir|z|n_3 r^2} dr}{r^2 - \omega} \right) dS(n).$$

Here $r = |\theta|$, $\theta = rn$. Then

$$\begin{aligned} (2.11) \quad R_{01}(\omega, z) &= \frac{1}{8\pi^3} \int_{S_+} \left(\int_0^\delta \frac{e^{-ir|z|n_3 r^2} dr}{r^2 - \omega} + \int_0^\delta \frac{e^{ir|z|n_3 r^2} dr}{r^2 - \omega} \right) dS(n) \\ &= \frac{1}{8\pi^3} \int_{S_+} \left(\int_{-\delta}^\delta \frac{e^{-ir|z|n_3 r^2} dr}{r^2 - \omega} \right) dS(n), \end{aligned}$$

where $S_+ = \{|n| = 1, n_3 > 0\}$. The integrand on the right-hand side of (2.11) has one simple pole in the lower half-plane. The Cauchy residue theorem yields

$$(2.12) \quad \begin{aligned} R_{01}(\omega, z) &= -\frac{i\sqrt{\omega}}{32\pi^4} \int_{S_+} e^{i\sqrt{\omega}|z|n_3} dS(n) + \frac{1}{8\pi^3} \int_{S_+} \left(\int_{\Gamma_\delta} \frac{e^{-ir|z|n_3} r^2 dr}{r^2 - \omega} \right) dS(n) \\ &= R_{01}^1(\omega, z) + R_{01}^2(\omega, z). \end{aligned}$$

Here $\Gamma_\delta = \{|r| = \delta, \operatorname{Im} r < 0\}$. For the first summand, an asymptotics of the type (2.10) is evident. For the second summand we have

$$\begin{aligned} R_{01}^2(\omega, z) &= \frac{1}{8\pi^3} \int_0^{2\pi} d\beta \int_0^{\pi/2} d\alpha \int_{\Gamma_\delta} \frac{e^{-ir|z|\cos\alpha} r^2 \sin\alpha dr}{r^2 - \omega} \\ &= \frac{i}{4\pi^2|z|} \int_{\Gamma_\delta} \frac{r dr}{r^2 - \omega} \int_0^{\pi/2} de^{-ir|z|\cos\alpha} \\ &= \frac{i}{4\pi^2|z|} \int_{\Gamma_\delta} \frac{r(1 - e^{-ir|z|}) dr}{r^2 - \omega}, \quad |z| \neq 0, \end{aligned}$$

where α is the angle between z and θ . Since $r^2 = \delta^2$ on Γ_δ , the operator-valued function $R_{01}^2(\omega)$ is analytic for $|\omega| < \delta^2/2$. Moreover, the function $R_{01}^2(\omega)$ and all its derivatives with respect to ω are bounded in $\mathcal{B}(\sigma, -\sigma)$ for $\sigma > 1/2$. Hence, R_{01}^2 admits an expansion of type (2.10). $\square$

Remark 2.5. The expansion (2.10) can be differentiated $N + 1$ times in $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > N + 3/2$:

$$\partial_\omega^r R_0(\omega) = \partial_\omega^r \left(\sum_{k=0}^N A_k \omega^{k/2} \right) + \mathcal{O}(\omega^{(N+1)/2-r}), \quad 1 \leq r \leq N + 1.$$

Proof. For the proof, we note that each differentiation of the resolvent with respect to ω increases the order of the pole of the integrand on the right in (2.11) by one. Therefore, the calculation of the corresponding residue leads to differentiation of the exponential, which results in an extra factor of $|z|$. Hence, σ increases by one. $\square$

2.2.2. Hyperbolic points. Here we construct the Puiseux expansion of the free resolvent $R_0(\omega)$ near the “hyperbolic” point $\omega_2 = 4$ (the expansion near the point $\omega_3 = 8$ can be constructed similarly). The main contribution to (2.2) is given by the corresponding critical points $(0, 0, \pi)$, $(0, \pi, 0)$, and $(\pi, 0, 0)$ of hyperbolic type.

Proposition 2.6. *Let $N = -1, 0, 1, \dots$, and let $\sigma > 2N + 7/2$. Then in $\mathcal{B}(\sigma, -\sigma)$ we have*

$$(2.13) \quad R_0(4 + \omega) = \sum_{k=0}^N E_k \omega^k + \sqrt{\omega} \sum_{k=0}^N B_k \omega^k + \mathcal{O}(\omega^{N+1}), \quad |\omega| \rightarrow 0, \quad \operatorname{Im} \omega > 0.$$

Here the operators E_k, B_k belong to $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 2k + 3/2$. A similar expansion is valid for $\operatorname{Im} \omega < 0$.

Proof. For $\omega = \omega_2 = 4$, the denominator of the integral (2.2) vanishes along the curve $\phi(\theta) = 4$. We shall study the main contribution of the points $(0, 0, \pi)$, $(0, \pi, 0)$, and $(\pi, 0, 0)$ of the surface, which are critical points of $\phi(\theta)$. The contribution of the other points of the curve can be computed by methods of Subsection 2.1. For definiteness, consider the integral over a neighborhood of the point $(\pi, 0, 0)$. Let $\zeta(\theta)$ be a smooth

cutoff function equal to 1 in a neighborhood of the point $(\pi, 0, 0)$ (the other properties of $\zeta(\theta)$ will be specified below). For $\text{Im } \omega > 0$, denote

$$\begin{aligned} Q(\omega, z) &= \frac{1}{8\pi^3} \int \frac{e^{-iz\theta} \zeta(\theta) d\theta}{\phi(\theta) - 4 - \omega} \\ &= \frac{e^{-iz_3\pi}}{8\pi^3} \int \frac{e^{-iz\theta'} \zeta_1(\theta') d\theta'}{4\sin^2 \frac{\theta_1}{2} + 4\sin^2 \frac{\theta_2}{2} - 4\sin^2 \frac{\theta'_3}{2} - \omega}, \end{aligned}$$

where $\theta'_3 = \theta_3 - \pi$, $\theta' = (\theta_1, \theta_2, \theta'_3)$, and $\zeta_1(\theta') = \zeta(\theta)$. We assume that $\zeta_1(\theta')$ is symmetric in θ_1, θ_2 and θ'_3 . Then the exponential in the numerator can be replaced by its even part, so that we have

$$Q(\omega, z) = \frac{e^{-iz_3\pi}}{\pi^3} \int_0^\infty \int_0^\infty \int_0^\infty \frac{\cos(z_1\theta_1) \cos(z_2\theta_2) \cos(z_3\theta_3) \zeta_1(\theta) d\theta}{4\sin^2 \frac{\theta_1}{2} + 4\sin^2 \frac{\theta_2}{2} - 4\sin^2 \frac{\theta_3}{2} - \omega} = \frac{e^{-iz_3\pi}}{\pi^3} Q_1(\omega, z).$$

We want to obtain an expansion of the form (2.13) for Q_1 . We change the variables: $s_i = 2\sin \frac{\theta_i}{2}$, and choose a cutoff function ζ such that $\zeta_1(\theta) = \zeta_2(|s|^2)$, with a smooth function ζ_2 . Then

$$\begin{aligned} Q_1(\omega, z) &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{F(z, s_1^2, s_2^2, s_3^2) \zeta_2(|s|^2) ds}{s_1^2 + s_2^2 - s_3^2 - \omega}, \\ F(z, s_1^2, s_2^2, s_3^2) &= \prod_i \frac{2\cos(2z_i \arcsin s_i/2)}{\sqrt{4 - s_i^2}}. \end{aligned}$$

Now we use the cylindrical variables: $s_1 = \tau \cos \varphi$, $s_2 = \tau \sin \varphi$, $s_3 = s_3$. Then

$$\begin{aligned} (2.14) \quad Q_1(\omega, z) &= \int_0^\infty \int_0^\infty \frac{F_1(z, \tau^2, s_3^2) \zeta_2(\tau^2 + s_3^2) \tau d\tau ds_3}{\tau^2 - s_3^2 - \omega}, \\ F_1(z, \tau^2, s_3^2) &= \int_0^{\pi/2} F(z, \tau^2 \cos^2 \varphi, \tau^2 \sin^2 \varphi, s_3^2) d\varphi. \end{aligned}$$

We change the variables once again:

$$\rho_1 = \tau^2 - s_3^2 = R^2 \cos 2\psi, \quad \rho_2 = 2\tau s_3 = R^2 \sin 2\psi,$$

where R, ψ are the polar coordinates on the plane (τ, s_3) . Then $|\rho|^2 = \rho_1^2 + \rho_2^2 = R^4$, whence $|\rho| = R^2$, $\tau^2 = (|\rho| + \rho_1)/2$, $s_3^2 = (|\rho| - \rho_1)/2$, $d\rho_1 d\rho_2 = 4|\rho| d\tau ds_3$, and

$$(2.15) \quad Q_1(\omega, z) = \int_0^\infty \left(\int_{\mathbb{R}} \frac{h(|\rho|, \rho_1, z)}{(\rho_1 - \omega)|\rho|} \sqrt{|\rho| + \rho_1} d\rho_1 \right) d\rho_2,$$

where $h(|\rho|, \rho_1, z) = F_1(z, \frac{|\rho| + \rho_1}{2}, \frac{|\rho| - \rho_1}{2}) \zeta_2(|\rho|)/(4\sqrt{2})$. Now we can specify the final required property of our cutoff function:

$$\text{supp } \zeta_2(|\rho|) \cap \{\rho \in \mathbb{R}^2 : \rho_2 \geq 0\} \subset \Pi = \{(\rho_1, \rho_2) : -\delta \leq \rho_1 \leq \delta, 0 \leq \rho_2 \leq \delta\}$$

with some $0 < \delta < 1$.

Let $0 < |\omega| \leq \delta/2$, $\text{Im } \omega > 0$. Denote $r = |\rho|$. The function $h(r, \rho_1, z)$ can be expanded in the following finite Taylor series with respect to ρ_1 :

$$(2.16) \quad h(r, \rho_1, z) = h_0(r, z) + h_1(r, z)\rho_1 + \cdots + h_N(r, z)\rho_1^N + H_N(r, \rho_1, z)\rho_1^N,$$

where $h_k(r, z)$ is a polynomial in z of order $2k$, and

$$\begin{aligned} (2.17) \quad |H_N(r, \rho_1, z)| &\leq C|z|^{2N}, \\ |\partial_{\rho_1} H_N(r, \rho_1, z)| &\leq C|z|^{2N+2}, \quad (\rho_1, \rho_2) \in [-\delta, \delta] \times [0, \delta]. \end{aligned}$$

We substitute (2.16) in (2.15). Then

$$(2.18) \quad Q_1(\omega, z) = \sum_{k=0}^N J_k(\omega, z) + \tilde{J}_N(\omega, z),$$

where

$$J_k(\omega, z) = \int_{\Pi} \frac{h_k(r, z) \rho_1^k \sqrt{r + \rho_1}}{(\rho_1 - \omega)r} d\rho_1 d\rho_2,$$

$$\tilde{J}_N(\omega, z) = \int_{\Pi} \frac{H_N(r, \rho_1, z) \rho_1^N d\rho_1 d\rho_2}{(\rho_1 - \omega)r} \sqrt{r + \rho_1}.$$

Step i. First, we consider $J_k(\omega, z)$, $k = 0, 1, \dots, N$:

$$(2.19) \quad \begin{aligned} J_k(\omega, z) &= \int_{\Pi} \frac{h_k(r, z) \sqrt{r + \rho_1}}{r} \left(\rho_1^{k-1} + \omega \rho_1^{k-2} + \dots + \omega^{k-1} + \frac{\omega^k}{\rho_1 - \omega} \right) d\rho_1 d\rho_2 \\ &= \sum_{j=0}^{k-1} a_{k,j}(z) \omega^j + \omega^k \int_{\Pi} \frac{h_k(r, z) \sqrt{r + \rho_1}}{(\rho_1 - \omega)r} d\rho_1 d\rho_2 \\ &= \sum_{j=0}^{k-1} a_{k,j}(z) \omega^j + \omega^k \int_0^\delta \frac{h_k(r, z)}{\sqrt{r}} dr \int_0^\pi \frac{\sqrt{1 + \cos \psi} d\psi}{\cos \psi - \omega/r}, \end{aligned}$$

where the $a_{k,j}(z)$ are polynomials of order $2k$. We calculate the integral

$$(2.20) \quad \begin{aligned} \int_0^\pi \frac{\sqrt{1 + \cos \psi} d\psi}{\cos \psi - \omega/r} &= \int_0^\pi \frac{2\sqrt{2} d(\sin \frac{\psi}{2})}{1 - 2\sin^2 \frac{\psi}{2} - \frac{\omega}{r}} \\ &= \int_0^1 \frac{-\sqrt{2} dt}{t^2 - \frac{r-\omega}{2r}} = \frac{-\sqrt{r}}{\sqrt{r-\omega}} \log \frac{1 - \sqrt{\frac{r-\omega}{2r}}}{1 + \sqrt{\frac{r-\omega}{2r}}} + \frac{\pi i \sqrt{r}}{\sqrt{r-\omega}}. \end{aligned}$$

Here $\sqrt{r} \geq 0$, the function $z = \sqrt{r - \omega}$ is analytic for $\text{Im } \omega > 0$ with values in the domain $\text{Im } z < 0$, $\text{Re } z > 0$, and the function $\zeta = \log w$ is analytic for $|w| < 1$, $\text{Im } w > 0$, where $\log(-1) = \pi i$.

Substituting (2.20) in (2.19), we get

$$(2.21) \quad J_k(\omega, z) = \sum_{j=0}^{k-1} a_{k,j}(z) \omega^j + \omega^k \int_0^\delta \left(\pi i - \log \frac{1 - \sqrt{\frac{r-\omega}{2r}}}{1 + \sqrt{\frac{r-\omega}{2r}}} \right) \frac{h_k(r, z) dr}{\sqrt{r-\omega}}.$$

We expand $h_k(r, z)$ in the following finite Taylor series with respect to r :

$$(2.22) \quad h_k(r, z) = h_{k,0}(z) + h_{k,1}(z)r + \dots + h_{k,N-k}(z)r^{N-k} + H_{k,N-k}(r, z)r^{N-k},$$

where the $h_{k,j}(z)$ are polynomials of order $2(k+j)$, and $|H_{k,N-k}(r, z)| \leq C|z|^{2N}$, $0 \leq r \leq \delta$.

Lemma 2.7. *If $0 < |\omega| < \delta/2$, $\text{Im } \omega > 0$, then*

$$(2.23) \quad I_l = \int_0^\delta \left(\pi i - \log \frac{1 - \sqrt{\frac{r-\omega}{2r}}}{1 + \sqrt{\frac{r-\omega}{2r}}} \right) \frac{r^l dr}{\sqrt{r-\omega}} = s_l(\omega) + C_l \omega^l \sqrt{\omega},$$

where the s_l are analytic in the domain $0 < |\omega| < \delta/2$, $\text{Im } \omega > 0$, and $C_l \in \mathbb{R}$.

We shall prove this lemma in Appendix A. Now (2.21)-(2.23) imply that for $0 < |\omega| < \delta/2$, $\text{Im } \omega > 0$ we have

$$(2.24) \quad \begin{aligned} J_k(\omega, z) &= \sum_{j=0}^N b_{k,j}(z) \omega^j + \omega^k \sqrt{\omega} \sum_{j=0}^{N-k} c_{k,j}(z) \omega^j + \tilde{a}_{N,k}(\omega, z) \omega^{N+1} \\ &\quad + \omega^k \int_0^\delta \left(\pi i - \log \frac{1 - \sqrt{\frac{r-\omega}{2r}}}{1 + \sqrt{\frac{r-\omega}{2r}}} \right) \frac{H_{k,N-k}(r, z) r^{N-k} dr}{\sqrt{r-\omega}}, \end{aligned}$$

where $|b_{k,j}(z)| \leq C|z|^{2N}$, $|c_{k,j}(z)| \leq C|z|^{2(k+j)}$, and $|\tilde{a}_{N,k}(\omega, z)| \leq C|z|^{2N}$. Next,

$$(2.25) \quad \int_0^\delta \left(\pi i - \log \frac{1 - \sqrt{\frac{r-\omega}{2r}}}{1 + \sqrt{\frac{r-\omega}{2r}}} \right) \frac{H_{k,N-k}(r, z) r^{N-k} dr}{\sqrt{r-\omega}} = \int_0^{2|\omega|} + \int_{2|\omega|}^\delta = \mathcal{I}_1 + \mathcal{I}_2.$$

In $\mathcal{I}_1$ we put $r = |\omega|\tau$, obtaining

$$(2.26) \quad \begin{aligned} |\mathcal{I}_1| &= |\omega|^{N-k} \sqrt{|\omega|} \left| \int_0^2 \left(\pi i - \log \frac{1 - \sqrt{\frac{\tau-\omega/|\omega|}{2\tau}}}{1 + \sqrt{\frac{\tau-\omega/|\omega|}{2\tau}}} \right) \frac{H_{k,N-k}(|\omega|\tau, z) \tau^{N-k} d\tau}{\sqrt{\tau - \omega/|\omega|}} \right| \\ &\leq C|z|^{2N} |\omega|^{N-k} \sqrt{|\omega|}. \end{aligned}$$

We expand $\sqrt{r-\omega}$ and the function in brackets in finite Taylor series with respect to ω/r :

$$(2.27) \quad \begin{aligned} \mathcal{I}_2 &= \int_{2|\omega|}^\delta H_{k,N-k}(r, z) r^{N-k-1/2} \\ &\quad \times \left(d_0 + d_1 \frac{\omega}{r} + \cdots + d_{N-k} \frac{\omega^{N-k}}{r^{N-k}} + \widehat{d}_{N-k}(\omega/r) \frac{\omega^{N-k}}{r^{N-k}} \right) dr \\ &= \int_{2|\omega|}^\delta H_{k,N-k}(r, z) \\ &\quad \times \left(d_0 r^{N-k-1/2} + d_1 \omega r^{N-k-3/2} + \cdots + d_{N-k} \omega^{N-k} r^{-1/2} \right) dr + \tilde{u}_{N-k}(\omega, z) \\ &= \int_0^\delta H_{k,N-k}(r, z) \\ &\quad \times \left(d_0 r^{N-k-1/2} + d_1 \omega r^{N-k-3/2} + \cdots + d_{N-k} \omega^{N-k} r^{-1/2} \right) dr + \widehat{u}_{N-k}(\omega, z) \\ &= \sum_{j=0}^{N-k} u_j(z) \omega^j + \widehat{u}_{N-k}(\omega, z), \end{aligned}$$

and we arrive at the estimates $|\widehat{d}_{N-k}(\omega/r)| \leq C$, $|u_j(z)| \leq C|z|^{2N}$, as well as $|\tilde{u}_{N-k}(\omega, z)|, |\widehat{u}_{N-k}(\omega, z)| \leq C|z|^{2N} |\omega|^{N-k}$. Now (2.24)–(2.27) imply that

$$(2.28) \quad J_k(\omega, z) = \sum_{j=0}^N d_{k,j}(z) \omega^j + \omega^k \sqrt{\omega} \sum_{j=0}^{N-k} c_{k,j}(z) \omega^j + \tilde{d}_{N,k}(\omega, z),$$

where $|d_{k,j}(z)| \leq C|z|^{2N}$, $|c_{k,j}(z)| \leq C|z|^{2(k+j)}$, and $|\tilde{d}_{N,k}(\omega, z)| \leq C|z|^{2N} |\omega|^N$.

Step ii. It remains to consider the term $\tilde{J}_N(\omega, z)$ on the right-hand side of (2.18):

$$\begin{aligned}
 \tilde{J}_N(\omega, z) &= \int_{\Pi} \frac{H_N(r, \rho_1, z) \sqrt{r + \rho_1}}{r} \\
 &\quad \times \left(\rho_1^{N-1} + \omega \rho_1^{N-2} + \cdots + \omega^{N-1} + \frac{\omega^N}{\rho_1 - \omega} \right) d\rho_1 d\rho_2 \\
 (2.29) \quad &= \sum_{j=0}^{N-1} w_j(z) \omega^j + \omega^N \int_{\Pi} \frac{H_N(r, \rho_1, z) \sqrt{r + \rho_1}}{(\rho_1 - \omega)r} d\rho_1 d\rho_2,
 \end{aligned}$$

where $|w_j(z)| \leq C|z|^{2N}$.

Lemma 2.8. *If $0 < |\omega| < \delta/2$, $\text{Im } \omega > 0$, then*

$$(2.30) \quad \left| \int_{\Pi} \frac{H_N(r, \rho_1, z) \sqrt{r + \rho_1}}{(\rho_1 - \omega)r} d\rho_1 d\rho_2 \right| \leq C|z|^{2N} \log^2 |z|, \quad |z| > 1.$$

This lemma is proved in Appendix B.

Step iii. Finally, (2.18) and (2.28)–(2.30) imply that

$$Q_1(\omega, z) = \sum_{k=0}^N q_k(z) \omega^k + \sqrt{\omega} \sum_{k=0}^N p_k(z) \omega^k + \hat{q}_N(\omega, z), \quad |\omega| \rightarrow 0,$$

where $|\hat{q}_N(\omega, z)| \leq C|z|^{2N} \log^2 |z| |\omega|^N$. Next, we have $p_k(z) = \mathcal{O}(|z|^{2k})$ and $q_k(z) = \mathcal{O}(|z|^{2N})$ for $0 \leq k \leq N$. Therefore, $q_k(z) = \mathcal{O}(|z|^{2k})$, because the $q_k(z)$ do not depend on N . $\square$

Corollary 2.9. *Let $\sigma > 3/2$. Then in $\mathcal{B}(\sigma, -\sigma)$ we have*

$$(2.31) \quad R_0(4 + \omega) = \mathcal{O}(1), \quad |\omega| \rightarrow 0, \quad \text{Im } \omega > 0.$$

Corollary 2.10. *The expansion (2.13) can be differentiated. More precisely,*

$$(2.32) \quad \partial_{\omega} R_0(4 + \omega) = \frac{B_0}{2\sqrt{\omega}} + \mathcal{O}(1), \quad |\omega| \rightarrow 0, \quad \text{Im } \omega > 0,$$

in $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 7/2$,

$$(2.33) \quad \partial_{\omega}^2 R_0(4 + \omega) = -\frac{B_0}{4\omega\sqrt{\omega}} + \mathcal{O}(\omega^{-1/2}), \quad |\omega| \rightarrow 0, \quad \text{Im } \omega > 0,$$

in $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 11/2$.

Proof. It suffices to obtain asymptotics of the form (2.32) and (2.33) for Q_1 (see (2.14)). Formula (2.14) yields

$$\begin{aligned}
 (2.34) \quad \partial_{\omega} Q_1(\omega, z) &= \int_0^{\infty} \int_0^{\infty} \frac{F_1(z, \tau^2, s^2) \zeta_2(\tau^2 + s^2) \tau d\tau ds}{(\tau^2 - s^2 - \omega)^2} \\
 &= \int_0^{\infty} ds \int_0^{\infty} F_1(z, \tau^2, s^2) \zeta_2(\tau^2 + s^2) \partial_{\tau} \frac{-1/2}{\tau^2 - s^2 - \omega} d\tau \\
 &= -\frac{1}{2} \int_0^{\infty} \frac{F_1(z, 0, s^2) \zeta_2(s^2)}{s^2 + \omega} ds + \int_0^{\infty} \int_0^{\infty} \frac{\partial_{\tau}^2 (F_1(z, \tau^2, s^2) \zeta_2(\tau^2 + s^2)) \tau}{\tau^2 - s^2 - \omega} d\tau ds \\
 &= S_1(\omega, z) + S_2(\omega, z).
 \end{aligned}$$

The asymptotics of $S_2(\omega, z)$ is similar to (2.31):

$$(2.35) \quad S_2(\omega) = \mathcal{O}(1), \quad |\omega| \rightarrow 0, \quad \text{Im } \omega > 0,$$

in $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 7/2$. Observe that differentiation with respect to τ^2 gives extra factors of $|z|^2$, so that the value of σ increases by two in comparison with (2.31).

Consider $S_1(\omega, z)$:

$$(2.36) \quad S_1(\omega, z) = \int_0^\delta \frac{(F_2(z, s^2) - F_2(z, |\omega|)) ds}{s^2 + \omega} + \int_0^\delta \frac{F_2(z, |\omega|) ds}{s^2 + \omega}.$$

For the function $F_2(z, s^2) = -F_1(z, 0, s^2)\zeta_2(s^2)/2$ we have

$$(2.37) \quad |F_2(z, s^2)| \leq C, \quad |\partial_{s^2} F_2(z, s^2)| \leq C|z|^2, \quad |\partial_{s^2}^2 F_2(z, s^2)| \leq C|z|^4.$$

We estimate the first integral on the right in (2.36) using the second bound in (2.37):

$$\left| \int_0^\delta \frac{(F_2(z, s^2) - F_2(z, |\omega|)) ds}{s^2 + \omega} \right| \leq C|z|^2 \int_0^\delta \frac{|s^2 - |\omega||}{|s^2 + \omega|} ds \leq C|z|^2.$$

For the second integral on the right in (2.36) we have

$$(2.38) \quad \int_0^\delta \frac{F_2(z, |\omega|) ds}{s^2 + \omega} = F_2(z, |\omega|) \frac{1}{2\sqrt{-\omega}} \left(\log \frac{\delta - \sqrt{-\omega}}{\delta + \sqrt{-\omega}} - \pi i \right) = p(z) \frac{1}{\sqrt{\omega}} + q(\omega, z),$$

where

$$|p(z)| + |q(\omega, z)| \leq C, \quad 0 < |\omega| < \delta/2, \quad \text{Im } \omega > 0.$$

Therefore,

$$(2.39) \quad S_1(\omega) = \frac{P_1}{\sqrt{\omega}} + \mathcal{O}(1), \quad |\omega| \rightarrow 0, \quad \text{Im } \omega > 0,$$

in $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 7/2$. Relations (2.35) and (2.39) imply the asymptotics of the first derivative. Next, we consider the second derivative:

(2.40)

$$\begin{aligned} \partial_\omega^2 Q_1(\omega, z) &= \int_0^\infty \int_0^\infty \frac{2F_1(z, \tau^2, s^2)\zeta_2(\tau^2 + s^2)\tau d\tau ds}{(\tau^2 - s^2 - \omega)^3} \\ &= \int_0^\infty ds \int_0^\infty F_1(z, \tau^2, s^2)\zeta_2(\tau^2 + s^2)\partial_\tau \frac{-1/2}{(\tau^2 - s^2 - \omega)^2} d\tau \\ &= \frac{1}{2} \int_0^\infty \frac{F_1(z, 0, s^2)\zeta_2(s^2)}{(s^2 + \omega)^2} ds + \int_0^\infty \int_0^\infty \frac{\partial_{\tau^2} (F_1(z, \tau^2, s^2)\zeta_2(\tau^2 + s^2))\tau}{(\tau^2 - s^2 - \omega)^2} d\tau ds \\ &= U_1(\omega, z) + U_2(\omega, z). \end{aligned}$$

The proof of the asymptotics for $U_2(\omega, z)$ is similar to that for the first derivative. We obtain

$$U_2(\omega) = \frac{P_2}{\sqrt{\omega}} + \mathcal{O}(1), \quad |\omega| \rightarrow 0, \quad \text{Im } \omega > 0,$$

in $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 11/2$. Now we consider $U_1(\omega, z)$:

$$\begin{aligned} U_1(\omega, z) &= - \int_0^\delta \frac{F_2(z, s^2)}{(s^2 + \omega)^2} ds \\ &= - \int_0^\delta \frac{F_2(z, s^2) - F_2(z, |\omega|) - F_2'(z, |\omega|)(s^2 - |\omega|)}{(s^2 + \omega)^2} ds + \int_0^\delta \frac{F_2(z, |\omega|)}{(s^2 + \omega)^2} ds \\ &\quad + \int_0^\delta \frac{F_2'(z, |\omega|)}{s^2 + \omega} ds = U_{11}(\omega, z) + U_{12}(\omega, z) + U_{13}(\omega, z). \end{aligned}$$

The last bound in (2.37) implies that

$$(2.41) \quad |U_{11}(\omega, z)| \leq C|z|^4.$$

As in (2.38), we see that

$$(2.42) \quad \begin{aligned} U_{13}(\omega, z) &= \frac{p_1(z)}{\sqrt{\omega}} + q_1(\omega, z), \\ |p_1(z)| + |q_1(\omega, z)| &\leq C|z|^2, \quad 0 < |\omega| < \frac{\delta^2}{2}, \quad \operatorname{Im} \omega > 0. \end{aligned}$$

Finally,

$$(2.43) \quad \begin{aligned} U_{12}(\omega, z) &= F_2(z, |\omega|) \left(\frac{1}{4\omega\sqrt{-\omega}} \left(\pi i - \log \frac{1 + \sqrt{-\omega}/\delta}{1 - \sqrt{-\omega}/\delta} \right) + \frac{\delta}{2\omega(\delta^2 + \omega)} \right) \\ &= F_2(z, |\omega|) \left(\frac{1}{4\omega\sqrt{-\omega}} \left(\pi i - \frac{2\sqrt{-\omega}}{\delta} + \frac{2\omega\sqrt{-\omega}}{3\delta^3} + \dots \right) + \frac{1}{2\omega\delta} \left(1 - \frac{\omega}{\delta^2} + \dots \right) \right) \\ &= \frac{s_2(z)}{\omega\sqrt{\omega}} + \frac{p_2(z)}{\sqrt{\omega}} + q_2(\omega, z), \quad |s_2(z)| + |p_2(z)| + |q_2(\omega, z)| \leq C, \\ &\quad 0 < |\omega| < \frac{\delta^2}{2}, \quad \operatorname{Im} \omega > 0. \end{aligned}$$

Formulas (2.41)–(2.43) yield the asymptotics of the second derivative. $\square$

§3. PERTURBED RESOLVENT

3.1. Limiting absorption principle. For the perturbed resolvent, the limiting absorption principle holds true.

Proposition 3.1. *Suppose $V \in \mathcal{V}$, $\sigma > 3/2$. Then the following limits as $\varepsilon \rightarrow 0+$ exist:*

$$(3.1) \quad R(\omega \pm i\varepsilon) \xrightarrow{B(\sigma, -\sigma)} R(\omega \pm i0), \quad \omega \in \Sigma \setminus \{0, 4, 8, 12\}.$$

Proof. Let $\omega \in \Sigma \setminus \{0, 4, 8, 12\}$, and let $\sigma > 3/2$. Since the potential V has finite support, Proposition 2.2 shows that

$$I + VR_0(\omega \pm i\varepsilon) \xrightarrow{B(\sigma, \sigma)} I + VR_0(\omega \pm i0), \quad \varepsilon \rightarrow 0+.$$

The kernel of the operator $I + VR_0(\omega \pm i0)$ is trivial (see [13, Theorem 10]), and the operator $VR_0(\omega \pm i0)$ is finite-dimensional. Consequently, the operator $I + VR_0(\omega \pm i0)$ is invertible, and moreover,

$$(3.2) \quad (I + VR_0(\omega \pm i\varepsilon))^{-1} \xrightarrow{B(\sigma, \sigma)} (I + VR_0(\omega \pm i0))^{-1}, \quad \varepsilon \rightarrow 0+.$$

Now, relations (2.4), (3.2) and the identity $R = R_0(I + VR_0)^{-1}$ imply the existence of the limits (3.1). $\square$

Remark 3.2. For $\omega \in \Sigma \setminus \{0, 4, 8, 12\}$ the derivatives $\partial_\omega^k R(\omega \pm i0)$ belong to $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 3/2 + k$.

Proof. This follows from Proposition 2.2 and the identity (see [6, Theorem 9.2])

$$R^{(k)} = \left[(1 - RV)R_0^{(k)} - \binom{k}{1} R'VR_0^{k-1} - \dots \right] (1 - VR), \quad k = 1, 2, \dots \quad \square$$

3.2. Asymptotics near critical points. Here we are going to obtain an asymptotic expansion for the perturbed resolvent $R(\omega)$ near the critical points ω_k , $k = 1, 2, 3, 4$.

Definition 3.3. i) A set $\mathcal{W} \subset \mathcal{V}$ is said to be generic if for each $V \in \mathcal{V}$ we have $\alpha V \in \mathcal{W}$ for all $\alpha \in \mathbb{R}$ with the possible exception of a discrete set.

ii) We say that a property is fulfilled for “generic” V if it is fulfilled for all V in a generic subset $\mathcal{W} \subset \mathcal{V}$.

Theorem 3.4. *Let $\sigma > 3/2$. Then for “generic” $V \in \mathcal{V}$ we have the following expansion:*

$$(3.3) \quad R(\omega) = D_1 + \mathcal{O}(\sqrt{\omega}), \quad |\omega| \rightarrow 0, \quad \arg \omega \in (0, 2\pi),$$

in the norm of $\mathcal{B}(\sigma, -\sigma)$.

Proof. We use the relation

$$(3.4) \quad R(\omega) = T^{-1}(\omega)R_0(\omega), \quad \text{where } T(\omega) := I + R_0(\omega)V.$$

By (2.10),

$$(3.5) \quad T(\omega) = I + A_0V + \mathcal{O}(\sqrt{\omega}), \quad |\omega| \rightarrow 0, \quad \arg \omega \in (0, 2\pi).$$

We prove that for “generic” $V \in \mathcal{V}$ the operator $T(\omega)$ is invertible in $l^2_{-\sigma}$ for all sufficiently small $|\omega| > 0$. It suffices to check that the operator $T(0) = I + A_0V$ is invertible in $l^2_{-\sigma}$, or that the operator with the kernel

$$(1+x^2)^{-\sigma/2}(\delta(x-y) + A_0V(y))(1+y^2)^{\sigma/2}$$

is invertible in l^2 . Consider the operator

$$\mathcal{A}(\alpha) = \text{Op}[(1+x^2)^{-\sigma/2}(\delta(x-y) + \alpha A_0V(y))(1+y^2)^{\sigma/2}] = 1 + \alpha\mathcal{K}, \quad \alpha \in \mathbb{C}.$$

For $\sigma > 3/2$,

$$K(x, y) = (1+x^2)^{-\sigma/2}A_0V(y)(1+y^2)^{\sigma/2} \in l^2(\mathbb{Z}^2 \times \mathbb{Z}^2).$$

Hence, $K(x, y)$ is a Hilbert–Schmidt kernel, so that the operator $\mathcal{K} = \text{Op}(K(x, y))$: $l^2 \rightarrow l^2$ is compact. Further, $\mathcal{A}(\alpha)$ is analytic in $\alpha \in \mathbb{C}$, and $\mathcal{A}(0)$ is invertible. It follows that $\mathcal{A}(\alpha)$ is invertible for all $\alpha \in \mathbb{C}$ outside of a discrete set; see [1]. Thus, we can replace the original potential V by αV with α arbitrarily close to 1, if necessary, to have $T(0)$ invertible.

Now (3.4) and (3.5) imply that

$$(3.6) \quad R(\omega) = (I + T(0) + \mathcal{O}(\sqrt{\omega}))^{-1}(A_0 + \mathcal{O}(\sqrt{\omega})) = T(0)^{-1}A_0 + \mathcal{O}(\sqrt{\omega})$$

for sufficiently small $|\omega| > 0$. □

Remark 3.5. i) The expansion of the resolvent near the second elliptic point $\omega_4 = 12$ is similar to (3.3).

ii) The expansion of the form (3.3) near the hyperbolic points $\omega_2 = 4$ and $\omega_3 = 8$ require larger values of σ . Namely, for “generic” $V \in \mathcal{V}$ we have

$$R(\omega_k + \omega) = D_k + \mathcal{O}(\sqrt{\omega}), \quad |\omega| \rightarrow 0, \quad \text{Im } \omega > 0, \quad k = 1, 2,$$

in the norm of $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 7/2$.

iii) These expansions can be differentiated twice in $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 5/2$ for elliptic points and with $\sigma > 11/2$ for hyperbolic points. In these cases, $\partial_\omega^2 R(\omega_k + \omega) = \mathcal{O}(\omega^{-3/2})$, $k = 1, 2, 3, 4$.

§4. LONG-TIME ASYMPTOTICS

Now we apply Lemma 4.2 below, which is a version of [6, Lemma 10.2], to prove the following theorem.

Theorem 4.1. *Let $\sigma > 11/2$. Then for “generic” $V \in \mathcal{V}$ the following asymptotics is valid:*

$$(4.1) \quad \left\| e^{-itH} - \sum_{j=1}^n e^{-it\mu_j} P_j \right\|_{B(\sigma, -\sigma)} = \mathcal{O}(t^{-3/2}), \quad t \rightarrow \infty.$$

Here P_j denotes the projection to the eigenspace corresponding to the eigenvalue $\mu_j \in \mathbb{R} \setminus [0, 12]$, $j = 1, \dots, n$.

Proof. Estimate (4.1) is based on the formula

$$(4.2) \quad e^{-itH} = -\frac{1}{2\pi i} \oint_{|\omega|=C} e^{-it\omega} R(\omega) d\omega, \quad C > \max\{12; |\mu_j|, j = 1, \dots, n\}.$$

The above integral is equal to the sum of the residues at the poles of $R(\omega)$ and the integral over the contour around the segment $[0, 12]$, i.e.,

$$e^{-itH} - \sum_{j=1}^n e^{-it\mu_j} P_j = \frac{1}{2\pi i} \int_{[0,12]} e^{-it\omega} (R(\omega + i0) - R(\omega - i0)) d\omega = \int_{[0,12]} e^{-it\omega} P(\omega) d\omega.$$

The main contribution to the long-time asymptotics is given by the integrals over neighborhoods of the critical points. For example, consider the integral over the neighborhood of the point $\omega_1 = 0$. The expansion (3.3) and Remark 3.5 imply that

$$(4.3) \quad \partial^k P(\omega) = \mathcal{O}(\partial^k \sqrt{\omega}), \quad \omega \rightarrow +0, \quad \omega \in \mathbb{R}, \quad k = 0, 1, 2,$$

in $B(\sigma, -\sigma)$ with $\sigma > 11/2$. The following result is a special case of [6, Lemma 10.2].

Lemma 4.2. *Assume $\mathcal{B}$ is a Banach space, $a > 0$, and $F \in C(0, a; \mathcal{B})$ satisfies $F(0) = F(a) = 0$, $F' \in L^1(0, a; \mathcal{B})$, and $F''(\omega) = \mathcal{O}(\omega^{-3/2})$ as $\omega \rightarrow 0$. Then*

$$\int_{[0,a]} e^{-it\omega} F(\omega) d\omega = \mathcal{O}(t^{-3/2}), \quad t \rightarrow \infty.$$

Set $F(\omega) = \zeta(\omega)P(\omega)$, where ζ is a smooth function such that $\zeta(\omega) = 1$ for $\omega \in [-1/2, 1/2]$ and $\text{supp } \zeta \subset (-1, 1)$, and let $a = 1$ and $\mathcal{B} = B(\sigma, -\sigma)$ with $\sigma > 11/2$. Then estimate (4.3) allows us to apply Lemma 4.2, obtaining

$$\int_{[0,1]} e^{-it\omega} \zeta(\omega) P(\omega) d\omega = \mathcal{O}(t^{-3/2}), \quad t \rightarrow \infty,$$

in $B(\sigma, -\sigma)$ with $\sigma > 11/2$.

The integrals over neighborhoods of the other critical points can be treated similarly. $\square$

§5. KLEIN–GORDON EQUATION

Now we extend the results of §§3–4 to the case of the Klein–Gordon equation (1.4). Denote $\Psi(t) \equiv (\psi(\cdot, t), \dot{\psi}(\cdot, t))$, $\Psi_0 \equiv (\psi_0, \pi_0)$. Then (1.4) becomes

$$i\dot{\Psi}(t) = \mathbf{H}\Psi(t) = \begin{pmatrix} 0 & i \\ i(\Delta - m^2 - V) & 0 \end{pmatrix} \Psi(t), \quad t \in \mathbb{R}; \quad \Psi(0) = \Psi_0.$$

The resolvent $\mathbf{R}(\omega)$ can be expressed in terms of the resolvent $R(\omega)$ as

$$(5.1) \quad \mathbf{R}(\omega) = \begin{pmatrix} \omega R(\omega^2 - m^2) & iR(\omega^2 - m^2) \\ -i(1 + \omega^2 R(\omega^2 - m^2)) & \omega R(\omega^2 - m^2) \end{pmatrix}, \quad \omega^2 - m^2 \in \mathbb{C} \setminus [0, 12].$$

The representation (5.1) and the properties of $R(\omega)$ imply the following long-time asymptotics.

Suppose $\sigma > 11/2$ and $\Psi_0 \in l_\sigma^2 \oplus l_\sigma^2$. Then for “generic” $V \in \mathcal{V}$ we have

$$\left\| e^{-it\mathbf{H}} \Psi_0 - \sum_{\pm} \sum_{j=1}^n e^{-it\nu_j^\pm} \mathbf{P}_j^\pm \Psi_0 \right\|_{l_{-\sigma}^2 \oplus l_{-\sigma}^2} = \mathcal{O}(t^{-3/2}), \quad t \rightarrow \infty.$$

Here the $\mathbf{P}_j^\pm$ are the projections onto the eigenspaces corresponding to the eigenvalues $\nu_j^\pm = \pm \sqrt{m^2 + \mu_j}$, $j = 1, \dots, n$.

§6. APPENDIX A

Lemma 2.7 will be proved by induction. For $l = 0$ we get

$$\begin{aligned}
I_0 &= \int_0^\delta \left(\pi i - \log \frac{1 - \sqrt{\frac{r-\omega}{2r}}}{1 + \sqrt{\frac{r-\omega}{2r}}} \right) \frac{dr}{\sqrt{r-\omega}} \\
&= 2\sqrt{r-\omega} \left(\pi i - \log \frac{1 - \sqrt{\frac{r-\omega}{2r}}}{1 + \sqrt{\frac{r-\omega}{2r}}} \right) \Big|_0^\delta + 2\sqrt{2}\omega \int_0^\delta \frac{dr}{\sqrt{r}(r+\omega)} \\
&= 2\sqrt{\delta-\omega} \left(\pi i - \log \frac{1 - \sqrt{\frac{\delta-\omega}{2\delta}}}{1 + \sqrt{\frac{\delta-\omega}{2\delta}}} \right) - i2\sqrt{2}\omega \log \frac{\sqrt{r}-i\sqrt{\omega}}{\sqrt{r}+i\sqrt{\omega}} \Big|_0^\delta \\
&= \tilde{s}_0(\omega) - i2\sqrt{2}\omega \log \frac{1 - i\sqrt{\frac{\omega}{\delta}}}{1 + i\sqrt{\frac{\omega}{\delta}}} - \pi\sqrt{2}\omega = s_0(\omega) + C_0\sqrt{\omega},
\end{aligned}$$

where $\tilde{s}_0$ and s_0 are analytic functions of ω , $C_0 = -\pi\sqrt{2}$.

Next, for $l \geq 1$ we have

$$\begin{aligned}
I_l &= \int_0^\delta \left(\pi i - \log \frac{1 - \sqrt{\frac{r-\omega}{2r}}}{1 + \sqrt{\frac{r-\omega}{2r}}} \right) \frac{r^l dr}{\sqrt{r-\omega}} = 2\delta^l \sqrt{\delta-\omega} \left(\pi i - \log \frac{1 - \sqrt{\frac{\delta-\omega}{2\delta}}}{1 + \sqrt{\frac{\delta-\omega}{2\delta}}} \right) \\
&\quad - 2l \int_0^\delta \frac{r^{l-1}(r-\omega)}{\sqrt{r-\omega}} \left(\pi i - \log \frac{1 - \sqrt{\frac{r-\omega}{2r}}}{1 + \sqrt{\frac{r-\omega}{2r}}} \right) dr + 2\sqrt{2}\omega \int_0^\delta \frac{r^l dr}{\sqrt{r}(r+\omega)} \\
&= \tilde{s}_l(\omega) - 2lI_l + 2l\omega I_{l-1} + 2\sqrt{2}\omega \int_0^\delta \frac{dr}{\sqrt{r}} \left(r^{l-1} - \omega r^{l-2} + \cdots + (-\omega)^{l-1} + \frac{(-\omega)^l}{r+\omega} \right) \\
&= \tilde{s}_l(\omega) - 2lI_l + 2l\omega I_{l-1} + \tilde{\tilde{s}}_l(\omega) - i2\sqrt{2}\omega(-\omega)^l \log \frac{1 - i\sqrt{\frac{\omega}{\delta}}}{1 + i\sqrt{\frac{\omega}{\delta}}} - 2\pi\sqrt{2}\omega(-\omega)^l,
\end{aligned}$$

where $\tilde{s}_l$ and $\tilde{\tilde{s}}_l$ are analytic functions of ω . Consequently, $I_l = s_l(\omega) + C_l\omega^l\sqrt{\omega}$, where the s_l are analytic functions of ω and $C_l \in \mathbb{R}$.

§7. APPENDIX B

Here we prove Lemma 2.8. We estimate only the integral over $\Pi_+ = \{0 \leq \rho_1, \rho_2 \leq \delta\}$. The integral over $\Pi \setminus \Pi_+$ can be estimated similarly. We split the integral over Π_+ into two integrals:

$$\begin{aligned}
\int_{\Pi_+} \frac{H_N(r, \rho_1, z) \sqrt{r+\rho_1} d\rho_1 d\rho_2}{(\rho_1 - \omega)r} &= \int_{\Pi_+} \frac{(H_N(r, \rho_1, z) - H_N(r, |\omega|, z)) \sqrt{r+\rho_1} d\rho_1 d\rho_2}{(\rho_1 - \omega)r} \\
&\quad + \int_{\Pi_+} \frac{H_N(r, |\omega|, z) \sqrt{r+\rho_1} d\rho_1 d\rho_2}{(\rho_1 - \omega)r} = J_1 + J_2.
\end{aligned}$$

As in (2.20), we obtain

$$J_2 = \int_0^\delta \left(\pi i - \log \frac{1 - \sqrt{\frac{r-\omega}{2r}}}{1 + \sqrt{\frac{r-\omega}{2r}}} \right) \frac{H_N(r, |\omega|, z) dr}{\sqrt{2}(r-\omega)}.$$

Note that

$$\begin{aligned} \left| \log \left| \frac{\sqrt{2r} - \sqrt{r - \omega}}{\sqrt{2r} + \sqrt{r - \omega}} \right| \right| &= \left| \log \left| \frac{r + \omega}{(\sqrt{2r} + \sqrt{r - \omega})^2} \right| \right| \\ &\leq |\log |r + \omega|| + 2|\log |\sqrt{2r} + \sqrt{r - \omega}|| \leq 2|\log |r - |\omega|||. \end{aligned}$$

Then (2.8) implies that

$$|J_2| \leq C|z|^{2N} \int_0^\delta \frac{1 + |\log |r - |\omega||}{|\sqrt{r - |\omega||}} dr \leq C|z|^{2N}.$$

Next, for $|z| > 1$ we split J_1 as follows:

$$J_1 = J_{11} + J_{12} + J_{13},$$

where J_{1j} is an integral over Π_j , $j = 1, 2, 3$, with

$$\begin{aligned} \Pi_1 &= \{(\rho_1, \rho_2) \in \Pi_+ : |r| < 1/|z|^{4/3}\}, \\ \Pi_2 &= \{(\rho_1, \rho_2) \in \Pi_+ \setminus \Pi_1 : |\rho_1 - |\omega|| < 1/|z|^{8/3}\}, \\ \Pi_3 &= \Pi_+ \setminus (\Pi_1 \cup \Pi_2) \end{aligned}$$

(see Figure 1).

FIGURE 1. Case where $|\omega| - 1/|z|^{8/3} < 0$.

Using (2.17) and the inequality $|\rho_1 - \omega| \geq |\rho_1 - |\omega||$, we get

$$\begin{aligned} |J_{11}| &\leq C|z|^{2N+2} \int_{\Pi_1} \frac{|\rho_1 - |\omega|| \sqrt{r + \rho_1} d\rho_1 d\rho_2}{|\rho_1 - \omega| r} \leq C|z|^{2N+2} \int_{\Pi_1} \frac{\sqrt{r + \rho_1} d\rho_1 d\rho_2}{r} \\ &\leq C|z|^{2N+2} \int_0^{\pi/2} d\psi \int_0^{1/|z|^{4/3}} \sqrt{r + r \cos \psi} dr \\ &\leq C|z|^{2N+2} \int_0^{\pi/2} \cos \frac{\psi}{2} d\psi \int_0^{1/|z|^{4/3}} \sqrt{r} dr \leq C|z|^{2N}. \end{aligned}$$

Similarly, for the second integral we obtain

$$\begin{aligned} |J_{12}| &\leq C|z|^{2N+2} \int_{\Pi_2} \frac{\sqrt{r+\rho_1} d\rho_1 d\rho_2}{r} \\ &\leq C|z|^{2N+2} \int_{\Pi_2} \frac{d\rho_1 d\rho_2}{\sqrt{r}} \leq C|z|^{2N+2} |z|^{2/3} \frac{\delta}{|z|^{8/3}} \leq C|z|^{2N}, \end{aligned}$$

because $1/\sqrt{r} \leq |z|^{2/3}$ for $(\rho_1, \rho_2) \in \Pi_2$, and $|\Pi_2| \leq 2\delta/|z|^{8/3}$. Finally, (2.17) implies that

$$|J_{13}| \leq C|z|^{2N} \int_{\Pi_3} \frac{d\rho_1 d\rho_2}{|\rho_1 - |\omega||\sqrt{\rho_1^2 + \rho_2^2}} \leq C|z|^{2N} \log^2 |z|,$$

because for any vertical interval $I \in \Pi_3$ we have

$$\int_I \frac{d\rho_2}{\sqrt{\rho_1^2 + \rho_2^2}} = \log(\rho_2 + \sqrt{\rho_1^2 + \rho_2^2}) \leq C \log |z|.$$

§8. APPENDIX C. ASYMPTOTIC COMPLETENESS

We apply the results obtained above to construct the asymptotic scattering states. Let u_k be eigenfunctions of the operator H corresponding to the eigenvalues μ_k , and let $U_0(t)$ be the dynamical group of the free Schrödinger equation.

Theorem 8.1. *Suppose $\sigma > 11/2$ and $\psi_0 \in l_\sigma^2$. Then, for “generic” $V \in \mathcal{V}$, the solution of (1.1) admits the following long-time asymptotics:*

$$(8.1) \quad \psi(\cdot, t) = \sum_{k=1}^n C_k e^{-it\mu_k} u_k + U_0(t)\phi_\pm + r_\pm(t), \quad t \rightarrow \pm\infty,$$

where the $\phi_\pm \in l^2$ are the corresponding scattering states, and

$$\|r_\pm(t)\|_{l^2} = \mathcal{O}(|t|^{-1/2}).$$

Proof. For definiteness, we consider the case where $t \rightarrow +\infty$. We apply the projection P^c onto the continuous spectrum of the operator H to both sides of (1.1):

$$(8.2) \quad iP^c \dot{\psi} = P^c H \psi = H_0 P^c \psi + V P^c \psi,$$

because P^c and H commute. Applying the Duhamel representation to equation (8.2), we obtain

$$(8.3) \quad P^c \psi(t) = U_0(t) P^c \psi(0) + \int_0^t U_0(t-\tau) V P^c \psi(\tau) d\tau, \quad t \in \mathbb{R}.$$

This formula can be rewritten as

$$\begin{aligned} P^c \psi(t) &= U_0(t) \left(P^c \psi(0) + \int_0^\infty U_0(-\tau) V P^c \psi(\tau) d\tau \right) - \int_t^\infty U_0(t-\tau) V P^c \psi(\tau) d\tau \\ &= U_0(t) \phi + r_+(t). \end{aligned}$$

We show that the integrals converge, and the function

$$\phi_+ = P^c \psi(0) + \int_0^\infty U_0(-\tau) V P^c \psi(\tau) d\tau$$

belongs to l^2 . Indeed, (4.1) implies that

$$\begin{aligned} \int_0^\infty \|U_0(-\tau) V P^c \psi(\tau)\|_{l^2} d\tau &= \int_0^\infty \|V P^c \psi(\tau)\|_{l^2} d\tau \leq C \int_0^\infty \|P^c \psi(\tau)\|_{l_{-\sigma}^2} d\tau \\ &\leq C \int_0^\infty t^{-3/2} \|\psi(0)\|_{l_\sigma^2} d\tau \leq C. \end{aligned}$$

Here we have used the fact that $U_0(t)$ is unitary in l^2 and the identity $P^c = I - P^d$, where P^d is the projection onto the subspace corresponding to the discrete spectrum, which consists of functions that decay exponentially. The estimate for $r_+(t)$ is proved similarly. $\square$

For the Klein–Gordon equation, an asymptotics of the form (8.1) is also valid.

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