HOMOGENIZATION IN THE SOBOLEV CLASS $H^1(\mathbb{R}^d)$ FOR SECOND ORDER PERIODIC ELLIPTIC OPERATORS WITH THE INCLUSION OF FIRST ORDER TERMS

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ABSTRACT. Matrix periodic elliptic second order differential operators $\mathcal{B}_\varepsilon$ in $\mathbb{R}^d$ with rapidly oscillating coefficients (depending on $\mathbf{x}/\varepsilon$) are studied. The principal part of the operator is given in a factorized form $b(\mathbf{D})^*g(\varepsilon^{-1}\mathbf{x})b(\mathbf{D})$, where g is a periodic, bounded and positive definite matrix-valued function and $b(\mathbf{D})$ is a matrix first order operator whose symbol is a matrix of maximal rank. The operator also has zero and first order terms with unbounded coefficients. The problem of homogenization in the small period limit is considered. Approximation for the generalized resolvent of the operator $\mathcal{B}_\varepsilon$ is obtained in the operator norm in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ with error term $O(\varepsilon)$. Also, approximation for this resolvent is obtained in the norm of operators acting from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to $H^1(\mathbb{R}^d; \mathbb{C}^n)$ with error term of order ε and with the corrector taken into account. The general results are applied to homogenization problems for the Schrödinger operator and the two-dimensional Pauli operator with potentials involving singular terms.

§0. INTRODUCTION

0.1. Main results. We study the problem of homogenization in the small period limit for matrix differential operators (DO's) $\mathcal{B}_\varepsilon$, $\varepsilon > 0$, acting in the space $L_2(\mathbb{R}^d; \mathbb{C}^n)$. Let Γ be a lattice in $\mathbb{R}^d$, and let Ω be the cell of Γ . We use the notation $\varphi^\varepsilon(\mathbf{x}) = \varphi(\varepsilon^{-1}\mathbf{x})$ for any measurable Γ -periodic function $\varphi(\mathbf{x})$ in $\mathbb{R}^d$.

The principal part $\mathcal{A}_\varepsilon$ of the operator $\mathcal{B}_\varepsilon$ is given in a factorized form

$$\mathcal{A}_\varepsilon = b(\mathbf{D})^*g^\varepsilon(\mathbf{x})b(\mathbf{D}),$$

where $b(\mathbf{D})$ is a matrix homogeneous first order DO and $g(\mathbf{x})$ is a bounded and positive definite matrix-valued function in $\mathbb{R}^d$, periodic with respect to the lattice Γ . (The precise assumptions on $b(\mathbf{D})$ and $g(\mathbf{x})$ are given below in §5.) Homogenization problems for the operator $\mathcal{A}_\varepsilon$ have been studied in detail in a series of papers [BSu1, BSu2, BSu3, BSu4, BSu5]. Now we study more general operators $\mathcal{B}_\varepsilon$: we add zero and first order terms to $\mathcal{A}_\varepsilon$:

$$(0.1) \quad \mathcal{B}_\varepsilon \mathbf{u} = \mathcal{A}_\varepsilon \mathbf{u} + \sum_{j=1}^d (a_j^\varepsilon(\mathbf{x})D_j \mathbf{u} + D_j(a_j^\varepsilon(\mathbf{x}))^* \mathbf{u}) + Q^\varepsilon(\mathbf{x})\mathbf{u}.$$

Here the $a_j(\mathbf{x})$, $j = 1, \dots, d$, are Γ -periodic $(n \times n)$ -matrix-valued functions such that $a_j \in L_\rho(\Omega)$, $\rho = 2$ for $d = 1$, and $\rho > d$ for $d \geq 2$. In general, the potential $Q^\varepsilon(\mathbf{x})$ is a distribution (with values in the class of Hermitian matrices) generated by a rapidly oscillating matrix-valued measure. It is only assumed that the generalized potential Q^ε

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is subordinate in a sense to the principal part of the operator. In particular, if the measure mentioned above is absolutely continuous with respect to Lebesgue measure, then $Q^\varepsilon(\mathbf{x}) = Q(\varepsilon^{-1}\mathbf{x})$, where $Q(\mathbf{x})$ is a Γ -periodic Hermitian $(n \times n)$ -matrix-valued function such that $Q \in L_s(\Omega)$, $s = 1$ for $d = 1$ and $s > d/2$ for $d \geq 2$. The coefficients of the operator (0.1) oscillate rapidly as $\varepsilon \rightarrow 0$.

A typical problem of homogenization theory for the operator (0.1) is to approximate the resolvent $(\mathcal{B}_\varepsilon + \lambda I)^{-1}$ or the generalized resolvent $(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1}$ as $\varepsilon \rightarrow 0$. Here $Q_0(\mathbf{x})$ is a positive definite and bounded Γ -periodic matrix-valued function. (We study the generalized resolvent from the very beginning; then, as a particular case, we get the results for the usual resolvent.) The parameter λ is subject to a condition that ensures that the operator $\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon$ is positive definite. It turns out that, as $\varepsilon \rightarrow 0$, the operators $(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1}$ converge to $(\mathcal{B}^0 + \lambda \overline{Q_0})^{-1}$ in the operator norm in $L_2(\mathbb{R}^d; \mathbb{C}^n)$. Here $\mathcal{B}^0$ is the so-called *effective operator* with constant coefficients, given by

$$(0.2) \quad \mathcal{B}^0 = \mathcal{A}^0 + \mathcal{Y}^0 + (\mathcal{Y}^0)^* + \overline{Q} - W.$$

The operator $\mathcal{A}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ is the effective operator for $\mathcal{A}_\varepsilon$; g^0 is a constant positive “effective” matrix. The effective matrix is constructed by a well-known procedure; under our assumptions, this construction is described below in §6. The first order operator $\mathcal{Y}^0$ and the matrix W are defined in a nontrivial way (see (7.2), (7.3), and (8.2)). Next, $\overline{Q}$ is the mean value of $Q(\mathbf{x})$ over Ω , and $\overline{Q_0}$ is defined similarly.

The following estimate is our *first main result*:

$$(0.3) \quad \|(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - (\mathcal{B}^0 + \lambda \overline{Q_0})^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon, \quad 0 < \varepsilon \leq 1.$$

Estimate (0.3) is order-sharp; the constant C is controlled explicitly in terms of the problem data. Earlier, estimate (0.3) was obtained in [Su] under the assumptions that the coefficients $a_j(\mathbf{x})$ and $Q(\mathbf{x})$ are bounded.

The present paper continues the investigation started in [Su]. Besides estimate (0.3), we find an approximation for the operator $(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1}$ in the norm of operators acting from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to $H^1(\mathbb{R}^d; \mathbb{C}^n)$ with order ε error term:

$$(0.4) \quad \|(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - (\mathcal{B}^0 + \lambda \overline{Q_0})^{-1} - \varepsilon K(\varepsilon)\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C\varepsilon, \quad 0 < \varepsilon \leq 1.$$

Here $K(\varepsilon)$ is what is called the *corrector*. The corrector has zero order with respect to ε , but it involves rapidly oscillating factors. In general, the corrector involves the smoothing operator Π_ε (see (9.17)). We carefully analyze conditions under which it is possible to get rid of the smoothing operator.

It is easily seen that estimate (0.4) reduces to the inequality

$$(0.5) \quad \|(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{1/2} ((\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - (\mathcal{B}^0 + \lambda \overline{Q_0})^{-1} - \varepsilon K(\varepsilon))\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon, \\ 0 < \varepsilon \leq 1.$$

Estimates (0.3) and (0.5) are obtained by one and the same method.

Note that the operator

$$(0.6) \quad \mathcal{B}_\varepsilon = \mathcal{A}_\varepsilon + \varepsilon^{-1} v^\varepsilon + Q^\varepsilon$$

with a singular potential $\varepsilon^{-1} v^\varepsilon(\mathbf{x})$ can be reduced to the form (0.1) (see §11). Here $v(\mathbf{x})$ is a Γ -periodic Hermitian matrix-valued function with zero mean value. In general, the corresponding homogenized operator has nontrivial first order terms.

Besides the approximation (0.4), we obtain an approximation for the so-called “flux” in the L_2 -norm. Also, we study the homogenization problem for more general operators of the form $\tilde{\mathcal{B}}_\varepsilon = (f^\varepsilon)^* \mathcal{B}_\varepsilon f^\varepsilon$ and consider examples of applications of the general results. In §13, we study the Schrödinger operator with rapidly oscillating coefficients: the metric g^ε , the magnetic potential $\mathbf{A}^\varepsilon$, and the electric potential $\varepsilon^{-2} \tilde{v}^\varepsilon + \varepsilon^{-1} v^\varepsilon + \mathcal{V}^\varepsilon$ oscillate

rapidly; the electric potential has singular terms. In §14, our general results are applied to the two-dimensional periodic Pauli operator with the magnetic potential $\varepsilon^{-1}\check{\mathbf{A}}^\varepsilon + \mathbf{A}^\varepsilon$ (with a singular term) perturbed by a matrix-valued electric potential $\varepsilon^{-1}v^\varepsilon + \mathcal{V}^\varepsilon$.

0.2. If $\mathcal{B}_\varepsilon = \mathcal{A}_\varepsilon$ (i.e., in the absence of lower order terms), an estimate of the form (0.3) was obtained for the first time in the papers [BSu1, BSu2] by the spectral method. Estimate (0.4) for $\mathcal{B}_\varepsilon = \mathcal{A}_\varepsilon$ was obtained in [BSu5]. By a different method suggested by V. V. Zhikov, estimates such as (0.3) and (0.4) were obtained in the papers [Zh, ZhPas] for the acoustics operator and the operator of elasticity theory in the absence of lower order terms and for $Q_0 = I$.

Homogenization problems for a scalar periodic elliptic second order equation involving first order terms were studied in the book [BeLP, Chapter 1, §13]. The convergence of solutions was obtained in a certain weak sense; no error estimates were studied.

In the presence of zero and first order terms, the homogenization problem for the operator (0.1) was studied in the paper [Bo] by D. I. Borisov. In [Bo], expression (0.2) for the effective operator $\mathcal{B}^0$ was found and error estimates of the forms (0.3) and (0.4) were proved. Moreover, it was assumed that the coefficients of the operator depend not only on the rapid variable, but also on the slow variable. However, in [Bo] it was assumed that the coefficients of the operator are sufficiently smooth. The constants in estimates obtained in [Bo] depend on the “smooth” norms of the coefficients.

In the paper [Su], estimate (0.3) was obtained in the case where the coefficients a_j and Q are bounded. Now, we lift this restriction. Estimates (0.3) and (0.4) are the main results of the paper. The constants in estimates are controlled in terms of the problem data.

0.3. The method of investigation is a further development of the method suggested in [BSu1, BSu2] (and developed in [BSu3, BSu4, BSu5]). Using the scaling transformation, we reduce the problem to the study of the operator pencil

$$(0.7) \quad \mathcal{B}(\varepsilon) = \mathcal{A} + \varepsilon \sum_{j=1}^d (a_j D_j + D_j a_j^*) + \varepsilon^2 Q$$

acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and depending on the parameter ε . The coefficients of this operator depend on $\mathbf{x}$ (but not on $\varepsilon^{-1}\mathbf{x}$). Next, by the Floquet-Bloch theory, the operator (0.7) expands in the direct integral of operators $\mathcal{B}(\mathbf{k}, \varepsilon)$ acting in $L_2(\Omega; \mathbb{C}^n)$ and depending on the parameter $\mathbf{k} \in \mathbb{R}^d$ (the quasimomentum). The operator $\mathcal{B}(\mathbf{k}, \varepsilon)$ is given by the expression

$$\mathcal{B}(\mathbf{k}, \varepsilon) = \mathcal{A}(\mathbf{k}) + \varepsilon \sum_{j=1}^d (a_j (D_j + k_j) + (D_j + k_j) a_j^*) + \varepsilon^2 Q,$$

where $\mathcal{A}(\mathbf{k}) = b(\mathbf{D} + \mathbf{k})^* g b(\mathbf{D} + \mathbf{k})$, with periodic boundary conditions. As in [BSu1, BSu2], we distinguish the one-dimensional parameter $t = |\mathbf{k}|$ and view the operator family $\mathcal{B}(\mathbf{k}, \varepsilon)$ as a quadratic operator pencil with respect to the parameters t and ε . Here, a considerable part of the constructions can be done in the framework of the abstract operator theory.

Under this abstract approach, we study a family of selfadjoint operators

$$(0.8) \quad B(t, \varepsilon) = A(t) + \varepsilon(Y_2^* Y(t) + Y(t)^* Y_2) + \varepsilon^2 Q,$$

acting in some Hilbert space $\mathfrak{H}$. It is assumed that the operator $A(t)$ admits a factorization of the form $A(t) = X(t)^* X(t)$, where $X(t) = X_0 + tX_1$. The main condition is that the point $\lambda_0 = 0$ is an eigenvalue of finite multiplicity n for the operator $A(0)$. We denote $\mathfrak{N} = \text{Ker } A(0)$. We also assume that the operator $Y(t) = Y_0 + tY_1$ is “dominated”

by the operator $X(t)$ (see §1). The operator Y_2 is assumed to be strongly subordinate to the operator $X(t)$. The operator Q corresponds to the quadratic form $\mathbf{q}[u, u]$, which is subject to some subordination conditions with respect to the form $\|X(t)u\|^2$.

We introduce the *one-dimensional parameter* $\tau = (t^2 + \varepsilon^2)^{1/2}$ and study the operator (0.8) as a quadratic operator pencil with respect to the parameter τ by methods of analytic perturbation theory (with respect to τ). We distinguish a selfadjoint operator $\mathcal{S}$ acting in the subspace $\mathfrak{N}$ and call it the *spectral germ* of the operator family $B(t, \varepsilon)$. The germ is defined in terms of the threshold characteristics of the operator (i.e., the spectral characteristics of the operator $B(t, \varepsilon)$ near the bottom of its spectrum). The principal term of approximation for the generalized resolvent of $B(t, \varepsilon)$ is given in terms of the germ. We also obtain a sharper approximation for the generalized resolvent in terms of appropriate finite rank operators (this corresponds to taking the corrector into account). These approximations are the main results of the abstract part of the paper (Theorems 3.1 and 3.2). By the method described above, on the basis of these abstract theorems we find the required approximations for the generalized resolvent of the operator $\mathcal{B}_\varepsilon$.

Note that in [BSu1, BSu2, BSu3, BSu4, BSu5] the starting point was the study of the operator pencil $A(t)$ with the help of the analytic perturbation theory with respect to the parameter t , and only then was the resolvent $(A(t) + \varepsilon^2 I)^{-1}$ or the generalized resolvent $(A(t) + \varepsilon^2 Q)^{-1}$ studied. Thus, the parameters t and ε were “separated”. That approach allows one to take into account the zero order terms, but not first order terms. Moreover, the zero order terms must be bounded. Now we introduce the parameter $\tau = (t^2 + \varepsilon^2)^{1/2}$ and treat the operator family (0.8) as a whole by methods of analytic perturbation theory with respect to the parameter τ . This allows us to take into account not only the zero order but also the first order terms; moreover, the lower order terms may have unbounded coefficients. It turns out that the “genuine” parameter of the problem is τ .

0.4. The layout of the paper. The paper consists of three chapters. Chapter 1 (§§1–3) is devoted to the abstract operator-theoretic method. In §1, the main objects are introduced. In §2, estimates for the difference of resolvents on some contour in the complex plane are obtained, and the required threshold approximations for the operator $B(t, \varepsilon)$ are established. On the basis of the results of §2, the main theorems of the abstract part of the paper (Theorems 3.1 and 3.2) are proved in §3. These theorems deal with approximation of the generalized resolvent of $B(t, \varepsilon)$.

Chapter 2 (§§4–8) is devoted to periodic differential operators acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$. In the short §4, preliminaries are given and the Gelfand transformation is described. In §5, the operator family $\mathcal{B}(\varepsilon)$ is introduced; the operator $\mathcal{B}(\varepsilon)$ is expanded in the direct integral of operators $\mathcal{B}(\mathbf{k}, \varepsilon)$ acting in $L_2(\Omega; \mathbb{C}^n)$. In §6, the operator family $\mathcal{B}(\mathbf{k}, \varepsilon)$ is incorporated in the abstract approach of Chapter 1, the effective characteristics are calculated, and the effective operator family $\mathcal{B}^0(\mathbf{k}, \varepsilon)$ is defined. In §7, on the basis of Theorems 3.1 and 3.2, an approximation for the generalized resolvent of the operator $\mathcal{B}(\mathbf{k}, \varepsilon)$ is obtained in terms of the generalized resolvent of $\mathcal{B}^0(\mathbf{k}, \varepsilon)$ (Theorem 7.1). Also, a sharper approximation for this resolvent is found, with corrector taken into account (Theorem 7.4). In §8, an approximation of the generalized resolvent of the operator $\mathcal{B}(\varepsilon)$ is deduced from Theorems 7.1 and 7.4 by using the direct integral expansion. This approximation involves the smoothing operator Π . In the principal term of approximation, Π can always be replaced by the identity operator. Next, we carefully analyze the possibility of getting rid of the smoothing operator in the corrector. Theorems 8.1, 8.2, and 8.13 on approximation of the generalized resolvent of the operator $\mathcal{B}(\varepsilon)$ are basic for further applications to the homogenization problems.

Chapter 3 (§§9–14) is devoted to homogenization problems. In §9, the homogenization problem for the operator $\mathcal{B}_\varepsilon$ is considered. By the scaling transformation, Theorems 8.1, 8.2, and 8.13 imply the *main results of the paper*. These are Theorem 9.2 on approximation of the generalized resolvent of $\mathcal{B}_\varepsilon$ in the operator norm in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ with order ε error term and Theorems 9.7, 9.8 on approximation of this generalized resolvent (with corrector taken into account) in the norm of operators acting from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to $H^1(\mathbb{R}^d; \mathbb{C}^n)$, again with order ε error term. In §10, we obtain an approximation for the so-called flux in the L_2 -norm with error term $O(\varepsilon)$. In §11, it is shown that the homogenization problem for the operator (0.6) with a singular potential can be solved with the help of the results of §9. In §12, the homogenization problem for the more general operator $\tilde{\mathcal{B}}_\varepsilon = (f^\varepsilon)^* \mathcal{B}_\varepsilon f^\varepsilon$ is considered. §§13 and 14 are devoted to applications of general results. The Schrödinger operator is studied in §13 and the two-dimensional Pauli operator is studied in §14.

0.5. Notation. Let $\mathcal{G}$ and $\mathcal{G}_*$ be separable Hilbert spaces. The symbols $(\cdot, \cdot)_{\mathcal{G}}$ and $\|\cdot\|_{\mathcal{G}}$ stand for the inner product and the norm in $\mathcal{G}$, respectively; the symbol $\|\cdot\|_{\mathcal{G} \rightarrow \mathcal{G}_*}$ denotes the norm of a continuous linear operator acting from $\mathcal{G}$ to $\mathcal{G}_*$. Sometimes we omit indices if this does not lead to confusion. By $I = I_{\mathcal{G}}$ we denote the identity operator in $\mathcal{G}$. If $A : \mathcal{G} \rightarrow \mathcal{G}_*$ is a linear operator, then $\text{Dom } A$ denotes its domain. If $\mathfrak{N}$ is a subspace in $\mathcal{G}$, then $\mathfrak{N}^\perp := \mathcal{G} \ominus \mathfrak{N}$. If P is the orthogonal projection of $\mathcal{G}$ onto $\mathfrak{N}$, then $P^\perp$ is the orthogonal projection onto $\mathfrak{N}^\perp$. The spectrum of a closed operator T in $\mathcal{G}$ is denoted by $\text{spec } T$. The symbol $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_n$ stands for the standard inner product in $\mathbb{C}^n$, $|\cdot|$ denotes the norm of a vector in $\mathbb{C}^n$; finally, $\mathbf{1} = \mathbf{1}_n$ is the identity $(n \times n)$ -matrix. For $z \in \mathbb{C}$, we write z^* for the complex conjugate number; this nonstandard notation is employed because $\bar{g}$ denotes the mean value of a matrix-valued function g . If a is an $(m \times n)$ -matrix, then $|a|$ is the norm of the corresponding linear operator from $\mathbb{C}^n$ to $\mathbb{C}^m$; a^t denotes the transpose of a , and a^* denotes the $(n \times m)$ -matrix adjoint to a .

The L_p -classes of $\mathbb{C}^n$ -valued functions on a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $L_p(\mathcal{O}; \mathbb{C}^n)$, $1 \leq p \leq \infty$. The Sobolev classes of $\mathbb{C}^n$ -valued functions (in a domain $\mathcal{O} \subseteq \mathbb{R}^d$) of order s and with summability index p are denoted by $W_p^s(\mathcal{O}; \mathbb{C}^n)$. If $p = 2$, we use the notation $H^s(\mathcal{O}; \mathbb{C}^n)$, $s \in \mathbb{R}$. If $n = 1$, we write simply $L_p(\mathcal{O})$, $W_p^s(\mathcal{O})$, $H^s(\mathcal{O})$, etc., but (if this does not lead to confusion) we use this short notation also for spaces of vector-valued or matrix-valued functions.

Next, $\mathbf{x} = (x^1, \dots, x^d) \in \mathbb{R}^d$, $iD_j = \partial_j = \partial/\partial x^j$, $j = 1, \dots, d$, $\nabla = \text{grad} = (\partial_1, \dots, \partial_d)$, $\mathbf{D} = -i\nabla = (D_1, \dots, D_d)$.

By C , c , $\mathcal{C}$, $\mathfrak{C}$ (probably, with indices and marks) we denote various constants in the estimates.

CHAPTER 1. ABSTRACT OPERATOR-THEORETIC METHOD

§1. QUADRATIC TWO-PARAMETRIC OPERATOR PENCILS

We study the operator family $B(t, \varepsilon)$ depending on two real-valued parameters ε and t ,

$$(1.1) \quad 0 \leq \varepsilon \leq 1, \quad t \in \mathbb{R}.$$

In Chapter 1, *it is always assumed that the restrictions (1.1) are satisfied*.

1.1. The operators $X(t)$ and $A(t)$. Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be complex separable Hilbert spaces. Let $X_0 : \mathfrak{H} \rightarrow \mathfrak{H}_*$ be a densely defined and closed linear operator, and let $X_1 : \mathfrak{H} \rightarrow \mathfrak{H}_*$ be a bounded linear operator. Then the operator $X(t) = X_0 + tX_1 : \mathfrak{H} \rightarrow \mathfrak{H}_*$ is closed on the domain $\text{Dom } X(t) = \text{Dom } X_0$. The operator $A(t) = X(t)^* X(t)$ is selfadjoint in $\mathfrak{H}$. The operator $A(t)$ is generated by the closed quadratic form $\|X(t)u\|_{\mathfrak{H}_*}^2$, $u \in \text{Dom } X_0$.

We put $A_0 := A(0) = X_0^* X_0$ and $\mathfrak{N} := \text{Ker } A_0 = \text{Ker } X_0$, and assume that the following condition is satisfied.

Condition 1.1. The point $\lambda_0 = 0$ is an isolated point of the spectrum of A_0 , and $0 < n := \dim \mathfrak{N} < \infty$.

Let d^0 be the distance from the point $\lambda_0 = 0$ to the rest of the spectrum of A_0 . We put $\mathfrak{N}_* = \text{Ker } X_0^*$ and $n_* := \dim \mathfrak{N}_*$. Assume that $n \leq n_* \leq \infty$. Let P and P_* be the orthogonal projections of $\mathfrak{H}$ onto $\mathfrak{N}$ and of $\mathfrak{H}_*$ onto $\mathfrak{N}_*$, respectively.

The operator family $A(t)$ has been studied in detail in [BSu2, Chapter 1], [BSu3], and [BSu5, Chapter 1].

1.2. The operators $Y(t)$ and Y_2 . Let $\tilde{\mathfrak{H}}$ be yet another separable Hilbert space. Let $Y_0 : \mathfrak{H} \rightarrow \tilde{\mathfrak{H}}$ be a densely defined linear operator such that $\text{Dom } X_0 \subset \text{Dom } Y_0$, and let $Y_1 : \mathfrak{H} \rightarrow \tilde{\mathfrak{H}}$ be a bounded linear operator. We put $Y(t) = Y_0 + tY_1$, $\text{Dom } Y(t) = \text{Dom } Y_0$. Suppose that the following condition is satisfied.

Condition 1.2. There exists a constant $c_1 > 0$ such that

$$(1.2) \quad \|Y(t)u\|_{\tilde{\mathfrak{H}}} \leq c_1 \|X(t)u\|_{\mathfrak{H}_*}, \quad u \in \text{Dom } X_0, \quad t \in \mathbb{R}.$$

Estimate (1.2) with $t = 0$ implies that $\text{Ker } X_0 \subset \text{Ker } Y_0$, i.e., $Y_0 P = 0$.

Let $Y_2 : \mathfrak{H} \rightarrow \tilde{\mathfrak{H}}$ be a densely defined linear operator such that $\text{Dom } X_0 \subset \text{Dom } Y_2$. We assume that the operator Y_2 is strongly subordinate to $X(t)$, i.e., that the following condition is satisfied.

Condition 1.3. For any $\nu > 0$ there exists a number $C(\nu) > 0$ such that

$$(1.3) \quad \|Y_2 u\|_{\tilde{\mathfrak{H}}}^2 \leq \nu \|X(t)u\|_{\mathfrak{H}_*}^2 + C(\nu) \|u\|_{\mathfrak{H}}^2, \quad u \in \text{Dom } X_0, \quad t \in \mathbb{R}.$$

1.3. The form $\mathfrak{q}$. Let $\mathfrak{q}[u, v]$ be a densely defined Hermitian sesquilinear form in $\mathfrak{H}$ such that $\text{Dom } X_0 \subset \text{Dom } \mathfrak{q}$. The form $\mathfrak{q}$ is subject to the following condition.

Condition 1.4. There exist constants $0 < \kappa \leq 1$, $c_0 \in \mathbb{R}$, $c_2 \geq 0$, and $c_3 \geq 0$ such that

$$(1.4) \quad -(1 - \kappa) \|X(t)u\|_{\mathfrak{H}_*}^2 - c_0 \|u\|_{\mathfrak{H}}^2 \leq \mathfrak{q}[u, u] \leq c_2 \|X(t)u\|_{\mathfrak{H}_*}^2 + c_3 \|u\|_{\mathfrak{H}}^2, \\ u \in \text{Dom } X_0, \quad t \in \mathbb{R}.$$

1.4. The operator $B(t, \varepsilon)$. In the space $\mathfrak{H}$, we consider the following quadratic form:

$$(1.5) \quad \mathfrak{b}(t, \varepsilon)[u, u] = \|X(t)u\|_{\mathfrak{H}_*}^2 + 2\varepsilon \text{Re}(Y(t)u, Y_2 u)_{\tilde{\mathfrak{H}}} + \varepsilon^2 \mathfrak{q}[u, u], \quad u \in \text{Dom } X_0.$$

We estimate this form from below. By (1.2), we have

$$(1.6) \quad 2\varepsilon |\text{Re}(Y(t)u, Y_2 u)_{\tilde{\mathfrak{H}}}| \leq 2\varepsilon c_1 \|X(t)u\|_{\mathfrak{H}_*} \|Y_2 u\|_{\tilde{\mathfrak{H}}} \\ \leq \frac{\kappa}{4} \|X(t)u\|_{\mathfrak{H}_*}^2 + \frac{4c_1^2}{\kappa} \varepsilon^2 \|Y_2 u\|_{\tilde{\mathfrak{H}}}^2, \quad u \in \text{Dom } X_0.$$

Using estimate (1.3) with $\nu = \kappa^2(16c_1^2)^{-1}$ and denoting

$$(1.7) \quad c_4 := 4\kappa^{-1}c_1^2 C(\nu) \quad \text{for } \nu = \kappa^2(16c_1^2)^{-1},$$

we obtain

$$(1.8) \quad \frac{4c_1^2}{\kappa} \|Y_2 u\|_{\tilde{\mathfrak{H}}}^2 \leq \frac{\kappa}{4} \|X(t)u\|_{\mathfrak{H}_*}^2 + c_4 \|u\|_{\mathfrak{H}}^2, \quad u \in \text{Dom } X_0.$$

Finally, relations (1.5), (1.6), (1.8), the lower estimate (1.4), and the restriction (1.1) imply that

$$(1.9) \quad \mathfrak{b}(t, \varepsilon)[u, u] \geq \frac{\kappa}{2} \|X(t)u\|_{\mathfrak{H}_*}^2 - (c_0 + c_4) \varepsilon^2 \|u\|_{\mathfrak{H}}^2, \quad u \in \text{Dom } X_0.$$

Hence, the form (1.5) is closed and lower semibounded.

The selfadjoint operator in $\mathfrak{H}$ generated by the form (1.5) is denoted by $B(t, \varepsilon)$. The operator $B(t, \varepsilon)$ is our *main object* in Chapter 1. Formally, we can write

$$(1.10) \quad B(t, \varepsilon) = A(t) + \varepsilon(Y_2^* Y(t) + Y(t)^* Y_2) + \varepsilon^2 Q.$$

Relations (1.2) and (1.3) with $\nu = 1$, the upper estimate (1.4), and (1.1) imply the following upper estimate for the form (1.5):

$$(1.11) \quad \mathfrak{b}(t, \varepsilon)[u, u] \leq (2 + c_1^2 + c_2) \|X(t)u\|_{\mathfrak{H}_*}^2 + (C(1) + c_3) \varepsilon^2 \|u\|_{\mathfrak{H}}^2, \quad u \in \text{Dom } X_0.$$

Our goal in Chapter 1 is to study the resolvent $(B(t, \varepsilon) + \lambda \varepsilon^2 I)^{-1}$ or the generalized resolvent $(B(t, \varepsilon) + \lambda \varepsilon^2 Q_0)^{-1}$ of the operator (1.10). Here $Q_0 : \mathfrak{H} \rightarrow \mathfrak{H}$ is a bounded and positive definite operator. It is convenient to consider the generalized resolvent from the very beginning; then in the particular case where $Q_0 = I$ we shall obtain results for the usual resolvent. The parameter $\lambda \in \mathbb{R}$ is subject to the following restriction:

$$(1.12) \quad \begin{aligned} \lambda &\geq \|Q_0^{-1}\| (c_0 + 2c_4) \quad \text{if } \lambda \geq 0, \\ \lambda &\geq \|Q_0\|^{-1} (c_0 + 2c_4) \quad \text{if } \lambda < 0 \quad (\text{and } c_0 + 2c_4 < 0), \end{aligned}$$

where c_0 is the constant from Condition 1.4, and c_4 is defined by (1.7). Condition (1.12) ensures that

$$(1.13) \quad \lambda(Q_0 u, u)_{\mathfrak{H}} \geq (c_0 + 2c_4) \|u\|_{\mathfrak{H}}^2, \quad u \in \mathfrak{H}.$$

By (1.9) and (1.13), we have

$$(1.14) \quad \mathfrak{b}(t, \varepsilon)[u, u] + \lambda \varepsilon^2 (Q_0 u, u)_{\mathfrak{H}} \geq \frac{\kappa}{2} \|X(t)u\|_{\mathfrak{H}_*}^2 + c_4 \varepsilon^2 \|u\|_{\mathfrak{H}}^2, \quad u \in \text{Dom } X_0.$$

Thus, under the above assumptions, the operator $B(t, \varepsilon) + \lambda \varepsilon^2 Q_0$ is positive. Inequality (1.11) implies the following estimate for $u \in \text{Dom } X_0$:

$$(1.15) \quad \begin{aligned} \mathfrak{b}(t, \varepsilon)[u, u] + \lambda \varepsilon^2 (Q_0 u, u)_{\mathfrak{H}} \\ \leq (2 + c_1^2 + c_2) \|X(t)u\|_{\mathfrak{H}_*}^2 + (C(1) + c_3 + |\lambda| \|Q_0\|) \varepsilon^2 \|u\|_{\mathfrak{H}}^2. \end{aligned}$$

1.5. Passage to the parameter τ . The family $B(t, \varepsilon)$ is an analytic operator family with respect to the parameters t and ε . If $t = \varepsilon = 0$, then the operator (1.10) coincides with A_0 and has an isolated eigenvalue $\lambda_0 = 0$ of multiplicity n . We wish to apply arguments of analytic perturbation theory. However, if $n > 1$, then such arguments cannot be applied directly, because we deal with a multiple eigenvalue and a multidimensional parameter. In order to overcome this difficulty, we introduce the *one-dimensional parameter* $\tau = (t^2 + \varepsilon^2)^{1/2}$ and also the additional parameters $\vartheta_1 = t\tau^{-1}$, $\vartheta_2 = \varepsilon\tau^{-1}$, $\vartheta = (\vartheta_1, \vartheta_2)$. Now the operator $B(t, \varepsilon)$ will be redenoted by $B(\tau; \vartheta)$. Then the quadratic form of the operator

$$\mathfrak{B}_\lambda(\tau; \vartheta) := B(t, \varepsilon) + \lambda \varepsilon^2 Q_0 = B(\tau; \vartheta) + \lambda \tau^2 \vartheta_2^2 Q_0$$

can be written as

$$(1.16) \quad \begin{aligned} \mathfrak{b}_\lambda(\tau; \vartheta)[u, u] &:= \mathfrak{b}(t, \varepsilon)[u, u] + \lambda \varepsilon^2 (Q_0 u, u)_{\mathfrak{H}} \\ &= \|(X_0 + \tau \vartheta_1 X_1)u\|_{\mathfrak{H}_*}^2 + 2\tau \vartheta_2 \text{Re}(Y_0 u, Y_2 u)_{\mathfrak{H}} \\ &\quad + 2\tau^2 \vartheta_1 \vartheta_2 \text{Re}(Y_1 u, Y_2 u)_{\mathfrak{H}} + \tau^2 \vartheta_2^2 (\mathfrak{q}[u, u] + \lambda(Q_0 u, u)_{\mathfrak{H}}), \\ &\quad u \in \text{Dom } X_0. \end{aligned}$$

Formally, the corresponding operator can be written as

$$(1.17) \quad \begin{aligned} \mathfrak{B}_\lambda(\tau; \vartheta) &= (X_0^* + \tau \vartheta_1 X_1^*)(X_0 + \tau \vartheta_1 X_1) + \tau \vartheta_2 (Y_2^* Y_0 + Y_0^* Y_2) \\ &\quad + \tau^2 \vartheta_1 \vartheta_2 (Y_2^* Y_1 + Y_1^* Y_2) + \tau^2 \vartheta_2^2 (Q + \lambda Q_0). \end{aligned}$$

We study the operator family $\mathfrak{B}_\lambda(\tau; \vartheta)$ by methods of analytic perturbation theory with respect to the one-dimensional parameter τ . At the same time, it is necessary to make our constructions and estimates uniform with respect to the additional parameter ϑ , taking into account the fact that $\vartheta_1^2 + \vartheta_2^2 = 1$.

Let $F(\tau; \vartheta; s)$ be the spectral projection of the operator $\mathfrak{B}_\lambda(\tau; \vartheta)$ for the interval $[0, s]$. We fix a number $\delta \in (0, \kappa d^0/13)$ and put

$$(1.18) \quad \tau_0 = \delta^{1/2} \left((2 + c_1^2 + c_2) \|X_1\|^2 + C(1) + c_3 + |\lambda| \|Q_0\| \right)^{-1/2}.$$

The next statement is similar to [Su, Proposition 1.3].

Proposition 1.5. *For $|\tau| \leq \tau_0$ we have*

$$(1.19) \quad F(\tau; \vartheta; \delta) = F(\tau; \vartheta; 3\delta), \quad \text{rank } F(\tau; \vartheta; \delta) = n, \quad |\tau| \leq \tau_0.$$

Proof. Assume that $\text{rank } F(\tau; \vartheta; \delta) < n$. Then there exists $f \in \mathfrak{N}$ such that $\|f\|_{\mathfrak{H}} = 1$ and $f \perp F(\tau; \vartheta; \delta)\mathfrak{H}$. Hence, $\mathfrak{b}_\lambda(\tau; \vartheta)[f, f] > \delta$. On the other hand, estimate (1.15) implies that

$$\begin{aligned} \mathfrak{b}_\lambda(\tau; \vartheta)[f, f] &\leq (2 + c_1^2 + c_2) \|\tau \vartheta_1 X_1 f\|_{\mathfrak{H}_*}^2 + (C(1) + c_3 + |\lambda| \|Q_0\|) \tau^2 \vartheta_2^2 \|f\|_{\mathfrak{H}}^2 \\ &\leq \tau^2 ((2 + c_1^2 + c_2) \|X_1\|^2 + C(1) + c_3 + |\lambda| \|Q_0\|) \leq \delta, \quad |\tau| \leq \tau_0. \end{aligned}$$

This contradiction shows that

$$(1.20) \quad \text{rank } F(\tau; \vartheta; \delta) \geq n, \quad |\tau| \leq \tau_0.$$

Now, assume that $\text{rank } F(\tau; \vartheta; 3\delta) > n$. Then there exists an element $f \in F(\tau; \vartheta; 3\delta)\mathfrak{H}$ such that $\|f\|_{\mathfrak{H}} = 1$ and $f \perp \mathfrak{N}$. Hence, $\|X_0 f\|_{\mathfrak{H}_*}^2 \geq d^0$. But

$$\|X_0 f\|_{\mathfrak{H}_*} \leq \|X_0 f + \tau \vartheta_1 X_1 f\|_{\mathfrak{H}_*} + \tau_0 \|X_1\| \|f\|_{\mathfrak{H}}, \quad |\tau| \leq \tau_0,$$

and, by (1.14) (recalling that $\kappa \leq 1$), we obtain

$$\begin{aligned} \|X_0 f\|_{\mathfrak{H}_*}^2 &\leq 2 \|X_0 f + \tau \vartheta_1 X_1 f\|_{\mathfrak{H}_*}^2 + 2\tau_0^2 \|X_1\|^2 \|f\|_{\mathfrak{H}}^2 \leq \frac{4}{\kappa} \mathfrak{b}_\lambda(\tau; \vartheta)[f, f] + 2\tau_0^2 \|X_1\|^2 \|f\|_{\mathfrak{H}}^2 \\ &\leq \frac{12\delta}{\kappa} \|f\|_{\mathfrak{H}}^2 + \frac{2\delta}{2 + c_1^2 + c_2} \|f\|_{\mathfrak{H}}^2 \leq \frac{13\delta}{\kappa} < d^0. \end{aligned}$$

We arrive at a contradiction. Consequently,

$$(1.21) \quad \text{rank } F(\tau; \vartheta; 3\delta) \leq n, \quad |\tau| \leq \tau_0.$$

Comparing (1.20) and (1.21), we see that

$$\text{rank } F(\tau; \vartheta; \delta) = \text{rank } F(\tau; \vartheta; 3\delta) = n$$

for $|\tau| \leq \tau_0$. This is equivalent to (1.19). $\square$

We shall often write $F(\tau; \vartheta)$ instead of $F(\tau; \vartheta; \delta)$.

1.6. The operators Z and $\tilde{Z}$. In the present and the next subsections, we introduce some operators that arise in perturbation theory considerations. We denote $\mathcal{D} := \text{Dom } X_0 \cap \mathfrak{N}^\perp$. Since the point $\lambda_0 = 0$ is isolated in the spectrum of A_0 , the form $(X_0 \varphi, X_0 \zeta)$, $\varphi, \zeta \in \mathcal{D}$, determines an inner product in $\mathcal{D}$, converting $\mathcal{D}$ into a Hilbert space. Let $\omega \in \mathfrak{N}$. Let $\phi \in \mathcal{D}$ be a solution of the equation

$$(1.22) \quad X_0^*(X_0 \phi + X_1 \omega) = 0,$$

which is understood in the weak sense. In other words, the element $\phi \in \mathcal{D}$ satisfies the identity

$$(1.23) \quad (X_0 \phi, X_0 \zeta)_{\mathfrak{H}_*} = -(X_1 \omega, X_0 \zeta)_{\mathfrak{H}_*}, \quad \zeta \in \mathcal{D}.$$

Since the right-hand side of (1.23) is a continuous antilinear functional of $\zeta \in \mathcal{D}$, the Riesz theorem shows that there exists a unique solution. This solution satisfies the estimate $\|X_0\phi\|_{\mathfrak{H}_*} \leq \|X_1\omega\|_{\mathfrak{H}_*}$. We introduce the operator $Z : \mathfrak{N} \rightarrow \mathfrak{H}$ that takes an element $\omega \in \mathfrak{N}$ to the solution $\phi \in \mathcal{D}$ of equation (1.22): $Z\omega = \phi$.

Similarly, given $\omega \in \mathfrak{N}$, we consider a solution $\psi \in \mathcal{D}$ of the equation

$$(1.24) \quad X_0^*X_0\psi + Y_0^*Y_2\omega = 0,$$

which is understood in the weak sense. In other words, ψ satisfies the identity

$$(1.25) \quad (X_0\psi, X_0\zeta)_{\mathfrak{H}_*} = -(Y_2\omega, Y_0\zeta)_{\tilde{\mathfrak{H}}}, \quad \zeta \in \mathcal{D}.$$

By Condition 1.2, the right-hand side of (1.25) is a continuous antilinear functional of $\zeta \in \mathcal{D}$. Therefore, by the Riesz theorem, there exists a unique solution, which satisfies the estimate $\|X_0\psi\|_{\mathfrak{H}_*} \leq c_1\|Y_2\omega\|_{\tilde{\mathfrak{H}}}$. We introduce the operator $\tilde{Z} : \mathfrak{N} \rightarrow \mathfrak{H}$ that takes an element $\omega \in \mathfrak{N}$ to the solution $\psi \in \mathcal{D}$ of equation (1.24): $\tilde{Z}\omega = \psi$.

1.7. The operators R and S . We introduce the operator $R := X_0Z + X_1 : \mathfrak{N} \rightarrow \mathfrak{N}_*$. As was shown in [BSu2, Chapter 1, (1.11)], the operator R can also be defined by the relation $R = P_*X_1|_{\mathfrak{N}}$. In accordance with [BSu2, Chapter 1, Subsection 1.3], the operator $S = R^*R : \mathfrak{N} \rightarrow \mathfrak{N}$ is called the *spectral germ of the operator family $A(t)$ at $t = 0$* . The germ S can be written as $S = PX_1^*P_*X_1|_{\mathfrak{N}}$. The spectral germ plays an important role in the study of the spectral properties of $A(t)$ near the bottom of the spectrum (the threshold properties). The properties of the spectral germ were studied in detail in [BSu2, Chapter 1, Subsections 1.3, 1.4, 1.6]. In the next subsection, we shall introduce the notion of the germ for the more general operator family (1.17).

1.8. Analytic branches of eigenvalues and eigenvectors of $\mathfrak{B}_\lambda(\tau; \vartheta)$. The general facts of analytic perturbation theory (see [K]) show that for $|\tau| \leq \tau_0$ there exist real-analytic (in τ) functions $\lambda_l(\tau; \vartheta)$ (branches of eigenvalues) and real-analytic $\mathfrak{H}$ -valued functions $\varphi_l(\tau; \vartheta)$ (branches of eigenvectors) such that

$$(1.26) \quad \mathfrak{B}_\lambda(\tau; \vartheta)\varphi_l(\tau; \vartheta) = \lambda_l(\tau; \vartheta)\varphi_l(\tau; \vartheta), \quad |\tau| \leq \tau_0, \quad l = 1, \dots, n,$$

and the elements $\varphi_l(\tau; \vartheta)$, $l = 1, \dots, n$, form an orthonormal basis in the eigenspace $F(\tau; \vartheta)\mathfrak{H}$. Moreover, for sufficiently small τ_* and $|\tau| \leq \tau_*$ (where $\tau_* \leq \tau_0$), we have the following convergent power series expansions:

$$(1.27) \quad \lambda_l(\tau; \vartheta) = \gamma_l(\vartheta)\tau^2 + \mu_l(\vartheta)\tau^3 + \dots, \quad \gamma_l(\vartheta) \geq 0, \quad l = 1, \dots, n,$$

$$(1.28) \quad \varphi_l(\tau; \vartheta) = \omega_l(\vartheta) + \tau\varphi_l^{(1)}(\vartheta) + \tau^2\varphi_l^{(2)}(\vartheta) + \dots, \quad l = 1, \dots, n.$$

The elements $\omega_l(\vartheta)$, $l = 1, \dots, n$, form an orthonormal basis in $\mathfrak{N}$, whence

$$(1.29) \quad P = \sum_{l=1}^n (\cdot, \omega_l(\vartheta))_{\mathfrak{H}} \omega_l(\vartheta).$$

Relation (1.26) means that

$$\mathfrak{b}_\lambda(\tau; \vartheta)[\varphi_l(\tau; \vartheta), \zeta] = \lambda_l(\tau; \vartheta)(\varphi_l(\tau; \vartheta), \zeta)_{\mathfrak{H}}, \quad \zeta \in \text{Dom } X_0.$$

In more detail, we have

$$(1.30) \quad \begin{aligned} & ((X_0 + \tau\vartheta_1X_1)\varphi_l(\tau; \vartheta), (X_0 + \tau\vartheta_1X_1)\zeta)_{\mathfrak{H}_*} + \tau\vartheta_2((Y_0 + \tau\vartheta_1Y_1)\varphi_l(\tau; \vartheta), Y_2\zeta)_{\tilde{\mathfrak{H}}} \\ & + \tau\vartheta_2(Y_2\varphi_l(\tau; \vartheta), (Y_0 + \tau\vartheta_1Y_1)\zeta)_{\tilde{\mathfrak{H}}} + \tau^2\vartheta_2^2(\mathfrak{q}[\varphi_l(\tau; \vartheta), \zeta] + \lambda(Q_0\varphi_l(\tau; \vartheta), \zeta)_{\mathfrak{H}}) \\ & = \lambda_l(\tau; \vartheta)(\varphi_l(\tau; \vartheta), \zeta)_{\mathfrak{H}}, \quad \zeta \in \text{Dom } X_0. \end{aligned}$$

We plug (1.27) and (1.28) into (1.30) and compare coefficients of the same powers of τ . The identity for the terms of order of τ^0 is ensured because $\omega_l(\vartheta) \in \mathfrak{N}$. Comparing the terms of order of τ^1 , we obtain

$$(X_0\varphi_l^{(1)}(\vartheta) + \vartheta_1 X_1 \omega_l(\vartheta), X_0 \zeta)_{\mathfrak{H}_*} + \vartheta_2 (Y_2 \omega_l(\vartheta), Y_0 \zeta)_{\tilde{\mathfrak{H}}} = 0, \quad \zeta \in \text{Dom } X_0.$$

This means that $\varphi_l^{(1)}(\vartheta)$ is a solution of the equation

$$(1.31) \quad X_0^*(X_0\varphi_l^{(1)}(\vartheta) + \vartheta_1 X_1 \omega_l(\vartheta)) + \vartheta_2 Y_0^* Y_2 \omega_l(\vartheta) = 0,$$

which is understood in the weak sense. In terms of the operators Z and $\tilde{Z}$ introduced in Subsection 1.6, we have

$$(1.32) \quad \varphi_l^{(1)}(\vartheta) = \vartheta_1 Z \omega_l(\vartheta) + \vartheta_2 \tilde{Z} \omega_l(\vartheta) + \tilde{\omega}_l(\vartheta), \quad \tilde{\omega}_l(\vartheta) \in \mathfrak{N}.$$

At this step of the perturbation arguments, the elements $\tilde{\omega}_l(\vartheta) \in \mathfrak{N}$ remain unknown, but it is easy to check that

$$(1.33) \quad (\tilde{\omega}_k(\vartheta), \omega_l(\vartheta))_{\mathfrak{H}} + (\omega_k(\vartheta), \tilde{\omega}_l(\vartheta))_{\mathfrak{H}} = 0, \quad k, l = 1, \dots, n.$$

Indeed, we substitute (1.28) in the identities $(\varphi_k(\tau; \vartheta), \varphi_l(\tau; \vartheta))_{\mathfrak{H}} = \delta_{kl}$. Up to terms of zero order (in τ), these relations mean that the system $\omega_l(\vartheta)$, $l = 1, \dots, n$, is orthonormal. Comparing the terms of order of τ^1 , we see that

$$(1.34) \quad (\varphi_k^{(1)}(\vartheta), \omega_l(\vartheta))_{\mathfrak{H}} + (\omega_k(\vartheta), \varphi_l^{(1)}(\vartheta))_{\mathfrak{H}} = 0, \quad k, l = 1, \dots, n.$$

Substituting (1.32) in (1.34) and taking into account that $\text{Ran } Z \subset \mathfrak{N}^\perp$, $\text{Ran } \tilde{Z} \subset \mathfrak{N}^\perp$, we arrive at (1.33).

Comparing the terms of order of τ^2 in (1.30), we arrive at the relation

$$\begin{aligned} & (X_0\varphi_l^{(2)} + \vartheta_1 X_1 \varphi_l^{(1)}, X_0 \zeta)_{\mathfrak{H}_*} + (X_0\varphi_l^{(1)} + \vartheta_1 X_1 \omega_l, \vartheta_1 X_1 \zeta)_{\mathfrak{H}_*} \\ & + \vartheta_2 \left((Y_0\varphi_l^{(1)}, Y_2 \zeta)_{\tilde{\mathfrak{H}}} + (Y_2\varphi_l^{(1)}, Y_0 \zeta)_{\tilde{\mathfrak{H}}} \right) \\ & + \vartheta_1 \vartheta_2 \left((Y_1 \omega_l, Y_2 \zeta)_{\tilde{\mathfrak{H}}} + (Y_2 \omega_l, Y_1 \zeta)_{\tilde{\mathfrak{H}}} \right) \\ & + \vartheta_2^2 (\mathbf{q}[\omega_l, \zeta] + \lambda(Q_0 \omega_l, \zeta)_{\mathfrak{H}}) = \gamma_l(\omega_l, \zeta)_{\mathfrak{H}}, \quad \zeta \in \text{Dom } X_0. \end{aligned}$$

For brevity, we have dropped the argument ϑ in the coefficients in (1.27), (1.28). Hence, for $\zeta \in \mathfrak{N}$ we have

$$\begin{aligned} & (X_0\varphi_l^{(1)} + \vartheta_1 X_1 \omega_l, \vartheta_1 X_1 \zeta)_{\mathfrak{H}_*} + \vartheta_2 (Y_0\varphi_l^{(1)}, Y_2 \zeta)_{\tilde{\mathfrak{H}}} + \vartheta_1 \vartheta_2 \left((Y_1 \omega_l, Y_2 \zeta)_{\tilde{\mathfrak{H}}} + (Y_2 \omega_l, Y_1 \zeta)_{\tilde{\mathfrak{H}}} \right) \\ & + \vartheta_2^2 (\mathbf{q}[\omega_l, \zeta] + \lambda(Q_0 \omega_l, \zeta)_{\mathfrak{H}}) = \gamma_l(\omega_l, \zeta)_{\mathfrak{H}}, \quad \zeta \in \mathfrak{N}. \end{aligned}$$

Substituting (1.32) in this relation, we arrive at the formula

$$\begin{aligned} & \vartheta_1^2 (X_0 Z \omega_l + X_1 \omega_l, X_1 \zeta)_{\mathfrak{H}_*} + \vartheta_1 \vartheta_2 (X_0 \tilde{Z} \omega_l, X_1 \zeta)_{\mathfrak{H}_*} \\ & + \vartheta_1 \vartheta_2 (Y_0 Z \omega_l, Y_2 \zeta)_{\tilde{\mathfrak{H}}} + \vartheta_2^2 (Y_0 \tilde{Z} \omega_l, Y_2 \zeta)_{\tilde{\mathfrak{H}}} \\ & + \vartheta_1 \vartheta_2 \left((Y_1 \omega_l, Y_2 \zeta)_{\tilde{\mathfrak{H}}} + (Y_2 \omega_l, Y_1 \zeta)_{\tilde{\mathfrak{H}}} \right) \\ & + \vartheta_2^2 (\mathbf{q}[\omega_l, \zeta] + \lambda(Q_0 \omega_l, \zeta)_{\mathfrak{H}}) = \gamma_l(\omega_l, \zeta)_{\mathfrak{H}}, \quad \zeta \in \mathfrak{N}. \end{aligned} \quad (1.35)$$

By the equation of the form (1.22) for $Z \omega_l$ and the definition of the operators R and S (see Subsection 1.7), we have

$$\begin{aligned} & (X_0 Z \omega_l + X_1 \omega_l, X_1 \zeta)_{\mathfrak{H}_*} = (X_0 Z \omega_l + X_1 \omega_l, X_0 Z \zeta + X_1 \zeta)_{\mathfrak{H}_*} \\ & = (R \omega_l, R \zeta)_{\mathfrak{H}_*} = (S \omega_l, \zeta)_{\mathfrak{H}}. \end{aligned} \quad (1.36)$$

Next, the definition of the operator Z (see (1.23)) shows that

$$(1.37) \quad (X_0 \tilde{Z} \omega_l, X_1 \zeta)_{\mathfrak{H}_*} = -(X_0 \tilde{Z} \omega_l, X_0 Z \zeta)_{\mathfrak{H}_*}.$$

Using the definition of the operator $\tilde{Z}$ (see (1.25)), we see that

$$(1.38) \quad \begin{aligned} (Y_0 Z \omega_l, Y_2 \zeta)_{\tilde{\mathfrak{H}}} &= -(X_0 Z \omega_l, X_0 \tilde{Z} \zeta)_{\mathfrak{H}_*}, \\ (Y_0 \tilde{Z} \omega_l, Y_2 \zeta)_{\tilde{\mathfrak{H}}} &= -(X_0 \tilde{Z} \omega_l, X_0 \tilde{Z} \zeta)_{\mathfrak{H}_*}. \end{aligned}$$

By (1.36)–(1.38), identity (1.35) can be rewritten as

$$(1.39) \quad \begin{aligned} &\vartheta_1^2(S \omega_l, \zeta)_{\mathfrak{H}} - \vartheta_1 \vartheta_2(X_0 \tilde{Z} \omega_l, X_0 Z \zeta)_{\mathfrak{H}_*} - \vartheta_1 \vartheta_2(X_0 Z \omega_l, X_0 \tilde{Z} \zeta)_{\mathfrak{H}_*} \\ &- \vartheta_2^2(X_0 \tilde{Z} \omega_l, X_0 \tilde{Z} \zeta)_{\mathfrak{H}_*} + \vartheta_1 \vartheta_2((Y_1 \omega_l, Y_2 \zeta)_{\tilde{\mathfrak{H}}} + (Y_2 \omega_l, Y_1 \zeta)_{\tilde{\mathfrak{H}}}) \\ &+ \vartheta_2^2(\mathfrak{q}[\omega_l, \zeta] + \lambda(Q_0 \omega_l, \zeta)_{\mathfrak{H}}) = \gamma_l(\omega_l, \zeta)_{\mathfrak{H}}, \quad \zeta \in \mathfrak{N}. \end{aligned}$$

The left-hand side of (1.39) is a sesquilinear form of the following selfadjoint operator in $\mathfrak{N}$:

$$(1.40) \quad \begin{aligned} \mathcal{S}(\vartheta) &= \vartheta_1^2 S - \vartheta_1 \vartheta_2(X_0 Z)^*(X_0 \tilde{Z}) - \vartheta_1 \vartheta_2(X_0 \tilde{Z})^*(X_0 Z) \\ &- \vartheta_2^2(X_0 \tilde{Z})^*(X_0 \tilde{Z}) + \vartheta_1 \vartheta_2 P(Y_2^* Y_1 + Y_1^* Y_2)|_{\mathfrak{N}} + \vartheta_2^2(Q_{\mathfrak{N}} + \lambda Q_{0\mathfrak{N}}). \end{aligned}$$

Here $Q_{\mathfrak{N}}$ is the selfadjoint operator in $\mathfrak{N}$ generated by the form $\mathfrak{q}[u, u]$, $u \in \mathfrak{N}$, and $Q_{0\mathfrak{N}} = P Q_0|_{\mathfrak{N}}$.

The operator (1.40) is called the spectral germ of the operator family $\mathfrak{B}_{\lambda}(\tau; \vartheta)$ at $\tau = 0$.

Now, (1.39) can be written as

$$(\mathcal{S}(\vartheta) \omega_l(\vartheta), \zeta)_{\mathfrak{H}} = \gamma_l(\vartheta)(\omega_l(\vartheta), \zeta)_{\mathfrak{H}}, \quad \zeta \in \mathfrak{N}.$$

We have proved the following statement.

Proposition 1.6. *The numbers $\gamma_l(\vartheta)$ and the elements $\omega_l(\vartheta)$ defined by (1.26)–(1.28) are eigenvalues and eigenvectors of the selfadjoint operator $\mathcal{S}(\vartheta) : \mathfrak{N} \rightarrow \mathfrak{N}$ defined by (1.40):*

$$(1.41) \quad \mathcal{S}(\vartheta) \omega_l(\vartheta) = \gamma_l(\vartheta) \omega_l(\vartheta), \quad l = 1, \dots, n.$$

From (1.29) and (1.41) it follows that

$$(1.42) \quad \mathcal{S}(\vartheta) P = \sum_{l=1}^n \gamma_l(\vartheta)(\cdot, \omega_l(\vartheta))_{\mathfrak{H}} \omega_l(\vartheta).$$

Proposition 1.6 was proved earlier in [Su, Proposition 1.5] under more restrictive assumptions on the operator $\mathfrak{B}_{\lambda}(\tau; \vartheta)$.

1.9. The operator-valued functions $F(\tau; \vartheta)$ and $\mathfrak{B}_{\lambda}(\tau; \vartheta)F(\tau; \vartheta)$ are real-analytic in τ for $|\tau| \leq \tau_0$. By (1.19) and (1.26) we have

$$(1.43) \quad F(\tau; \vartheta) = \sum_{l=1}^n (\cdot, \varphi_l(\tau; \vartheta))_{\mathfrak{H}} \varphi_l(\tau; \vartheta), \quad |\tau| \leq \tau_0.$$

By (1.28), (1.29), and (1.43), for sufficiently small τ_* we have the following power series expansion:

$$(1.44) \quad F(\tau; \vartheta) = P + \tau F_1(\vartheta) + \dots, \quad |\tau| \leq \tau_*,$$

where

$$(1.45) \quad F_1(\vartheta) = \sum_{l=1}^n \left((\cdot, \omega_l(\vartheta))_{\mathfrak{H}} \varphi_l^{(1)}(\vartheta) + (\cdot, \varphi_l^{(1)}(\vartheta))_{\mathfrak{H}} \omega_l(\vartheta) \right).$$

Next, relations (1.26) and (1.19) imply that

$$(1.46) \quad \mathfrak{B}_{\lambda}(\tau; \vartheta) F(\tau; \vartheta) = \sum_{l=1}^n \lambda_l(\tau; \vartheta) (\cdot, \varphi_l(\tau; \vartheta))_{\mathfrak{H}} \varphi_l(\tau; \vartheta), \quad |\tau| \leq \tau_0.$$

Combining (1.46) with (1.27), (1.28), and (1.42), we arrive at the expansion

$$(1.47) \quad \mathfrak{B}_\lambda(\tau; \vartheta)F(\tau; \vartheta) = \tau^2 \mathcal{S}(\vartheta)P + \cdots, \quad |\tau| \leq \tau_*.$$

Now, we obtain an invariant representation for the operator (1.45). We substitute (1.32) in (1.45) and use (1.29), obtaining

$$\begin{aligned} F_1(\vartheta) &= \sum_{l=1}^n \left((\cdot, \omega_l(\vartheta))_{\mathfrak{H}} (\vartheta_1 Z \omega_l(\vartheta) + \vartheta_2 \tilde{Z} \omega_l(\vartheta)) + (\cdot, \vartheta_1 Z \omega_l(\vartheta) + \vartheta_2 \tilde{Z} \omega_l(\vartheta))_{\mathfrak{H}} \omega_l(\vartheta) \right) + J(\vartheta) \\ &= (\vartheta_1 Z + \vartheta_2 \tilde{Z})P + P(\vartheta_1 Z^* + \vartheta_2 \tilde{Z}^*) + J(\vartheta), \end{aligned}$$

where

$$J(\vartheta) = \sum_{l=1}^n ((\cdot, \tilde{\omega}_l(\vartheta))_{\mathfrak{H}} \omega_l(\vartheta) + (\cdot, \omega_l(\vartheta))_{\mathfrak{H}} \tilde{\omega}_l(\vartheta)).$$

It is clear that $J(\vartheta)|_{\mathfrak{H}^\perp} = 0$. From (1.33) it directly follows that $(J(\vartheta)\omega_j(\vartheta), \omega_k(\vartheta))_{\mathfrak{H}} = 0$, $j, k = 1, \dots, n$, whence $J(\vartheta)|_{\mathfrak{H}} = 0$. Consequently, $J(\vartheta) = 0$. Thus, we have

$$(1.48) \quad F_1(\vartheta) = (\vartheta_1 Z + \vartheta_2 \tilde{Z})P + P(\vartheta_1 Z^* + \vartheta_2 \tilde{Z}^*).$$

The power series expansions (1.44) and (1.47) are not quite suitable for our purposes. We only need estimates for $F(\tau; \vartheta) - P$ and $\mathfrak{B}_\lambda(\tau; \vartheta)F(\tau; \vartheta) - \tau^2 \mathcal{S}(\vartheta)P$, and also for the operator $\mathfrak{B}_\lambda(\tau; \vartheta)^{1/2}(F(\tau; \vartheta) - P - \tau F_1(\vartheta))$, but on the entire interval $|\tau| \leq \tau_0$. Such estimates can be obtained by integrating the resolvent of $\mathfrak{B}_\lambda(\tau; \vartheta)$ over an appropriate contour in the complex plane (see §2).

§2. INTEGRATING THE DIFFERENCE OF RESOLVENTS OVER A CONTOUR. THRESHOLD APPROXIMATIONS

2.1. The resolvent identity. We denote

$$(2.1) \quad R_z(0) = (A_0 - zI)^{-1}, \quad R_z(\tau; \vartheta) = (\mathfrak{B}_\lambda(\tau; \vartheta) - zI)^{-1},$$

where $z \in \mathbb{C}$ is a regular point both for A_0 and for $\mathfrak{B}_\lambda(\tau; \vartheta)$. The usual resolvent identity is not applicable now. We need a version of the resolvent identity for operators whose quadratic forms have one and the same domain (see [BSu2, Chapter 1, §2]).

Let $\mathfrak{d}$ denote the space $\text{Dom } X_0$ with the metric form

$$(2.2) \quad \|u\|_{\mathfrak{d}}^2 = \|X_0 u\|_{\mathfrak{H}_*}^2 + \delta \|u\|_{\mathfrak{H}}^2,$$

and let $T(\tau; \vartheta)$ be the operator in $\mathfrak{d}$ generated by the quadratic form

$$(2.3) \quad \mathfrak{t}(\tau; \vartheta)[u, u] = \mathfrak{b}_\lambda(\tau; \vartheta)[u, u] - \|X_0 u\|_{\mathfrak{H}_*}^2, \quad u \in \mathfrak{d}.$$

We put

$$(2.4) \quad \Omega_z(0) = I + (z + \delta)R_z(0), \quad \Omega_z(\tau; \vartheta) = I + (z + \delta)R_z(\tau; \vartheta).$$

In accordance with [BSu2, Chapter 1, (2.13)], the following resolvent identity is true:

$$(2.5) \quad R_z(\tau; \vartheta) - R_z(0) = -\Omega_z(0)T(\tau; \vartheta)R_z(\tau; \vartheta).$$

We put

$$\alpha^2 = \alpha^2(\tau; \vartheta) = \sup_{0 \neq u \in \mathfrak{d}} \frac{\|X_0 u\|_{\mathfrak{H}_*}^2 + \delta \|u\|_{\mathfrak{H}}^2}{\mathfrak{b}_\lambda(\tau; \vartheta)[u, u] + \delta \|u\|_{\mathfrak{H}}^2}$$

and estimate the number $\alpha(\tau; \vartheta)$. By (1.14) and (1.18), for $|\tau| \leq \tau_0$ we have

$$\begin{aligned} \|X_0 u\|_{\mathfrak{H}_*}^2 + \delta \|u\|_{\mathfrak{H}}^2 &\leq 2\|X(t)u\|_{\mathfrak{H}_*}^2 + 2\tau_0^2 \|X_1\|^2 \|u\|_{\mathfrak{H}}^2 + \delta \|u\|_{\mathfrak{H}}^2 \\ &\leq \frac{4}{\kappa} \mathbf{b}_\lambda(\tau; \vartheta)[u, u] + (2\tau_0^2 \|X_1\|^2 + \delta) \|u\|_{\mathfrak{H}}^2 \leq \frac{4}{\kappa} \mathbf{b}_\lambda(\tau; \vartheta)[u, u] + 2\delta \|u\|_{\mathfrak{H}}^2 \\ &\leq \frac{4}{\kappa} (\mathbf{b}_\lambda(\tau; \vartheta)[u, u] + \delta \|u\|_{\mathfrak{H}}^2). \end{aligned}$$

Hence,

$$(2.6) \quad \alpha = \alpha(\tau; \vartheta) \leq 2\kappa^{-1/2}, \quad |\tau| \leq \tau_0.$$

The following estimates will be needed below (see [BSu2, Chapter 1, (2.15)–(2.17)]):

$$(2.7) \quad \|LR_z(\tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq \alpha \delta^{-1} \|\Omega_z(\tau; \vartheta)\| \|L\|_{\mathfrak{D} \rightarrow \mathfrak{D}},$$

$$(2.8) \quad \|LR_z(0)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq \delta^{-1} \|\Omega_z(0)\| \|L\|_{\mathfrak{D} \rightarrow \mathfrak{D}},$$

$$(2.9) \quad \begin{aligned} &\|L_1 \Omega_z(0) L_2 R_z(\tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \\ &\leq \alpha \delta^{-1} \|\Omega_z(\tau; \vartheta)\| (1 + |z| + \delta \|\Omega_z(0)\|) \|L_1\|_{\mathfrak{D} \rightarrow \mathfrak{D}} \|L_2\|_{\mathfrak{D} \rightarrow \mathfrak{D}}, \end{aligned}$$

where L, L_1, L_2 are bounded operators in $\mathfrak{D}$, and $\|\Omega_z(\tau; \vartheta)\| = \|\Omega_z(\tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}}$.

2.2. Estimates for the operator $T(\tau; \vartheta)$. By (1.16), the form (2.3) can be written as

$$(2.10) \quad \mathbf{t}(\tau; \vartheta)[u, u] = \tau \mathbf{t}_1(\vartheta)[u, u] + \tau^2 \mathbf{t}_2(\vartheta)[u, u],$$

where

$$(2.11) \quad \mathbf{t}_1(\vartheta)[u, u] = 2\vartheta_1 \operatorname{Re}(X_0 u, X_1 u)_{\mathfrak{H}_*} + 2\vartheta_2 \operatorname{Re}(Y_0 u, Y_2 u)_{\mathfrak{H}}, \quad u \in \mathfrak{D},$$

$$(2.12) \quad \begin{aligned} \mathbf{t}_2(\vartheta)[u, u] &= \vartheta_1^2 \|X_1 u\|_{\mathfrak{H}_*}^2 + 2\vartheta_1 \vartheta_2 \operatorname{Re}(Y_1 u, Y_2 u)_{\mathfrak{H}} \\ &\quad + \vartheta_2^2 (\mathbf{q}[u, u] + \lambda(Q_0 u, u)_{\mathfrak{H}}), \quad u \in \mathfrak{D}. \end{aligned}$$

Then the operator $T(\tau; \vartheta)$ generated in $\mathfrak{D}$ by the form (2.10) can be written as

$$(2.13) \quad T(\tau; \vartheta) = \tau T_1(\vartheta) + \tau^2 T_2(\vartheta),$$

where the operators $T_1(\vartheta)$, $T_2(\vartheta)$ are generated by the forms (2.11) and (2.12), respectively. We estimate the corresponding norms. By Condition 1.2 and estimate (1.3) with $t = 0$ and $\nu = 1$, we have

$$\begin{aligned} \mathbf{t}_1(\vartheta)[u, u] &\leq 2\|X_0 u\|_{\mathfrak{H}_*} \|X_1 u\|_{\mathfrak{H}_*} + 2c_1 \|X_0 u\|_{\mathfrak{H}_*} \|Y_2 u\|_{\mathfrak{H}} \\ &\leq (2 + c_1^2) \|X_0 u\|_{\mathfrak{H}_*}^2 + \|X_1 u\|_{\mathfrak{H}_*}^2 + C(1) \|u\|_{\mathfrak{H}}^2 \\ &\leq \max\{2 + c_1^2, (\|X_1\|^2 + C(1))\delta^{-1}\} (\|X_0 u\|_{\mathfrak{H}_*}^2 + \delta \|u\|_{\mathfrak{H}}^2). \end{aligned}$$

Hence,

$$(2.14) \quad \|T_1(\vartheta)\|_{\mathfrak{D} \rightarrow \mathfrak{D}} \leq C_T^{(1)} = \max\{2 + c_1^2, (\|X_1\|^2 + C(1))\delta^{-1}\}.$$

By Condition (1.3) with $t = 0$ and $\nu = 1$ and the upper estimate (1.4), the form (2.12) satisfies the inequality

$$\begin{aligned} \mathbf{t}_2(\vartheta)[u, u] &\leq \|X_1 u\|_{\mathfrak{H}_*}^2 + 2\|Y_1 u\|_{\mathfrak{H}} \|Y_2 u\|_{\mathfrak{H}} + \mathbf{q}[u, u] + |\lambda| (Q_0 u, u)_{\mathfrak{H}} \\ &\leq (c_2 + 1) \|X_0 u\|_{\mathfrak{H}_*}^2 + (\|X_1\|^2 + \|Y_1\|^2 + C(1) + c_3 + |\lambda| \|Q_0\|) \|u\|_{\mathfrak{H}}^2 \\ &\leq \max\{c_2 + 1, (\|X_1\|^2 + \|Y_1\|^2 + C(1) + c_3 + |\lambda| \|Q_0\|)\delta^{-1}\} (\|X_0 u\|_{\mathfrak{H}_*}^2 + \delta \|u\|_{\mathfrak{H}}^2). \end{aligned}$$

Consequently,

$$(2.15) \quad \|T_2(\vartheta)\|_{\mathfrak{D} \rightarrow \mathfrak{D}} \leq C_T^{(2)} = \max\{c_2 + 1, (\|X_1\|^2 + \|Y_1\|^2 + C(1) + c_3 + |\lambda| \|Q_0\|)\delta^{-1}\}.$$

For the norm of the operator (2.13) we have

$$\|T(\tau; \vartheta)\|_{\mathfrak{d} \rightarrow \mathfrak{d}} \leq |\tau|(\|T_1(\vartheta)\|_{\mathfrak{d} \rightarrow \mathfrak{d}} + \tau_0 \|T_2(\vartheta)\|_{\mathfrak{d} \rightarrow \mathfrak{d}}), \quad |\tau| \leq \tau_0.$$

Combining this with (2.14) and (2.15), we obtain

$$(2.16) \quad \|T(\tau; \vartheta)\|_{\mathfrak{d} \rightarrow \mathfrak{d}} \leq C_T |\tau|, \quad |\tau| \leq \tau_0,$$

$$(2.17) \quad C_T = C_T^{(1)} + \tau_0 C_T^{(2)}.$$

2.3. The contour Γ_δ . Estimates for the difference of resolvents. Let Γ_δ denote the contour on the complex plane that envelops the real interval $[0, \delta]$ equidistantly at the distance δ .

We assume that $|\tau| \leq \tau_0$, where the number τ_0 is defined by (1.18). By (1.19), the distance from Γ_δ to the spectrum of $\mathfrak{B}_\lambda(\tau; \vartheta)$ is at least δ , whence

$$(2.18) \quad \|R_z(\tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq \delta^{-1}, \quad z \in \Gamma_\delta, \quad |\tau| \leq \tau_0.$$

Since

$$(2.19) \quad |z| \leq 2\delta \text{ for } z \in \Gamma_\delta,$$

we see that the operator (2.4) satisfies the estimate

$$(2.20) \quad \|\Omega_z(\tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq 4, \quad z \in \Gamma_\delta, \quad |\tau| \leq \tau_0.$$

If $\tau = 0$, estimates (2.18) and (2.20) remain true (for the operators $R_z(0)$ and $\Omega_z(0)$, respectively).

By (2.5)–(2.7), (2.16), and (2.20), we have

$$(2.21) \quad \|R_z(\tau; \vartheta) - R_z(0)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq 16\alpha\delta^{-1} \|T(\tau; \vartheta)\|_{\mathfrak{d} \rightarrow \mathfrak{d}} \leq 32\kappa^{-1/2} \delta^{-1} C_T |\tau|, \\ |\tau| \leq \tau_0, \quad z \in \Gamma_\delta.$$

Estimate (2.21) does not suffice for our goals. We separate the terms of order of τ and τ^2 in the difference of resolvents and prove an estimate of order of τ^3 for the remainder term. By (2.5) and (2.13), we obtain

$$(2.22) \quad R_z(\tau; \vartheta) - R_z(0) = -\Omega_z(0)T(\tau; \vartheta)R_z(\tau; \vartheta) = \tau \mathcal{I}_1(z; \tau; \vartheta) + \tau^2 \mathcal{I}_2(z; \tau; \vartheta),$$

where

$$(2.23) \quad \mathcal{I}_j(z; \tau; \vartheta) = -\Omega_z(0)T_j(\vartheta)R_z(\tau; \vartheta), \quad j = 1, 2.$$

We start with the operator $\mathcal{I}_2(z; \tau; \vartheta)$. Iterating (2.5), we see that

$$(2.24) \quad \mathcal{I}_2(z; \tau; \vartheta) = -\Omega_z(0)T_2(\vartheta)(R_z(0) - \Omega_z(0)T(\tau; \vartheta)R_z(\tau; \vartheta)) \\ = \mathcal{I}_2^0(z; \vartheta) + \tilde{\mathcal{I}}_2(z; \tau; \vartheta),$$

where

$$(2.25) \quad \mathcal{I}_2^0(z; \vartheta) = -\Omega_z(0)T_2(\vartheta)R_z(0),$$

$$(2.26) \quad \tilde{\mathcal{I}}_2(z; \tau; \vartheta) = \Omega_z(0)T_2(\vartheta)\Omega_z(0)T(\tau; \vartheta)R_z(\tau; \vartheta).$$

The term $\mathcal{I}_2^0(z; \vartheta)$ does not depend on τ . By (2.20) and (2.9), the operator (2.26) satisfies

$$\|\tilde{\mathcal{I}}_2(z; \tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq 4\|T_2(\vartheta)\Omega_z(0)T(\tau; \vartheta)R_z(\tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \\ \leq 16\alpha\delta^{-1}(1 + 4|z + \delta|\delta^{-1})\|T_2(\vartheta)\|_{\mathfrak{d} \rightarrow \mathfrak{d}}\|T(\tau; \vartheta)\|_{\mathfrak{d} \rightarrow \mathfrak{d}}.$$

Combining this with (2.6), (2.15), (2.16), and (2.19), we arrive at the estimate

$$(2.27) \quad \|\tilde{\mathcal{I}}_2(z; \tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq 416\kappa^{-1/2} \delta^{-1} C_T^{(2)} C_T |\tau|, \quad |\tau| \leq \tau_0, \quad z \in \Gamma_\delta.$$

The study of the term $\mathcal{I}_1(z; \tau; \vartheta)$ is more cumbersome. By (2.5) and (2.13), we have

$$(2.28) \quad \begin{aligned} \mathcal{I}_1(z; \tau; \vartheta) &= -\Omega_z(0)T_1(\vartheta)(R_z(0) - \Omega_z(0)T(\tau; \vartheta)R_z(\tau; \vartheta)) \\ &= \mathcal{I}_1^0(z; \vartheta) + \tilde{\mathcal{I}}_1(z; \tau; \vartheta), \end{aligned}$$

where

$$(2.29) \quad \begin{aligned} \mathcal{I}_1^0(z; \vartheta) &= -\Omega_z(0)T_1(\vartheta)R_z(0), \\ \tilde{\mathcal{I}}_1(z; \tau; \vartheta) &= \Omega_z(0)T_1(\vartheta)\Omega_z(0)T(\tau; \vartheta)R_z(\tau; \vartheta) \\ &= \tau\Omega_z(0)T_1(\vartheta)\Omega_z(0)T_1(\vartheta)R_z(\tau; \vartheta) + \tau^2\Omega_z(0)T_1(\vartheta)\Omega_z(0)T_2(\vartheta)R_z(\tau; \vartheta). \end{aligned}$$

The term $\mathcal{I}_1^0(z; \vartheta)$ does not depend on τ . To analyze the term $\tilde{\mathcal{I}}_1(z; \tau; \vartheta)$, we apply (2.5) once again:

$$\begin{aligned} \tilde{\mathcal{I}}_1(z; \tau; \vartheta) &= \tau\Omega_z(0)T_1(\vartheta)\Omega_z(0)T_1(\vartheta)(R_z(0) - \Omega_z(0)T(\tau; \vartheta)R_z(\tau; \vartheta)) \\ &\quad + \tau^2\Omega_z(0)T_1(\vartheta)\Omega_z(0)T_2(\vartheta)R_z(\tau; \vartheta). \end{aligned}$$

As a result, we obtain

$$(2.30) \quad \tilde{\mathcal{I}}_1(z; \tau; \vartheta) = \tau\mathcal{I}_2^{00}(z; \vartheta) + \hat{\mathcal{I}}_1(z; \tau; \vartheta),$$

where the term

$$(2.31) \quad \mathcal{I}_2^{00}(z; \vartheta) = \Omega_z(0)T_1(\vartheta)\Omega_z(0)T_1(\vartheta)R_z(0)$$

does not depend on τ , and the term $\hat{\mathcal{I}}_1(z; \tau; \vartheta)$ is given by

$$\begin{aligned} \hat{\mathcal{I}}_1(z; \tau; \vartheta) &= -\tau\Omega_z(0)T_1(\vartheta)\Omega_z(0)T_1(\vartheta)\Omega_z(0)T(\tau; \vartheta)R_z(\tau; \vartheta) \\ &\quad + \tau^2\Omega_z(0)T_1(\vartheta)\Omega_z(0)T_2(\vartheta)R_z(\tau; \vartheta) \\ &= -\tau\Omega_z(0)(T_1(\vartheta))^2\Omega_z(0)T(\tau; \vartheta)R_z(\tau; \vartheta) \\ &\quad - \tau(z + \delta)\Omega_z(0)T_1(\vartheta)R_z(0)T_1(\vartheta)\Omega_z(0)T(\tau; \vartheta)R_z(\tau; \vartheta) \\ &\quad + \tau^2\Omega_z(0)T_1(\vartheta)\Omega_z(0)T_2(\vartheta)R_z(\tau; \vartheta). \end{aligned}$$

We have used (2.4) in the last identity. Combining this with (2.8), (2.9), and (2.20), we obtain

$$\begin{aligned} \|\hat{\mathcal{I}}_1(z; \tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} &\leq 16|\tau|\alpha\delta^{-1}(1 + 4|z + \delta|\delta^{-1})^2\|T_1(\vartheta)\|_{\mathfrak{d} \rightarrow \mathfrak{d}}^2\|T(\tau; \vartheta)\|_{\mathfrak{d} \rightarrow \mathfrak{d}} \\ &\quad + 16\tau^2\alpha\delta^{-1}(1 + 4|z + \delta|\delta^{-1})\|T_1(\vartheta)\|_{\mathfrak{d} \rightarrow \mathfrak{d}}\|T_2(\vartheta)\|_{\mathfrak{d} \rightarrow \mathfrak{d}}, \\ &\quad |\tau| \leq \tau_0, \quad z \in \Gamma_\delta. \end{aligned}$$

Together with (2.6), (2.19), and (2.14)–(2.16), this yields the estimate

$$(2.32) \quad \|\hat{\mathcal{I}}_1(z; \tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq \check{C}_T\delta^{-1}\tau^2, \quad |\tau| \leq \tau_0, \quad z \in \Gamma_\delta,$$

where

$$(2.33) \quad \check{C}_T = 32 \cdot 13^2\kappa^{-1/2}(C_T^{(1)})^2C_T + 32 \cdot 13\kappa^{-1/2}C_T^{(1)}C_T^{(2)}.$$

Combining relations (2.22), (2.24), (2.27), (2.28), (2.30), and (2.32), we arrive at the following result.

Theorem 2.1. *For $|\tau| \leq \tau_0$ and $z \in \Gamma_\delta$, we have*

$$(2.34) \quad R_z(\tau; \vartheta) - R_z(0) = \tau\mathcal{I}_1^0(z; \vartheta) + \tau^2\mathcal{I}_2^+(z; \vartheta) + \Psi^0(z; \tau; \vartheta),$$

where $\mathcal{I}_2^+(z; \vartheta) = \mathcal{I}_2^0(z; \vartheta) + \mathcal{I}_2^{00}(z; \vartheta)$. The operators $\mathcal{I}_1^0(z; \vartheta)$, $\mathcal{I}_2^0(z; \vartheta)$, and $\mathcal{I}_2^{00}(z; \vartheta)$ are defined by (2.29), (2.25), and (2.31), respectively. The operator

$$\Psi^0(z; \tau; \vartheta) = \tau\hat{\mathcal{I}}_1(z; \tau; \vartheta) + \tau^2\tilde{\mathcal{I}}_2(z; \tau; \vartheta)$$

admits the estimate

$$(2.35) \quad \|\Psi^0(z; \tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_T^0 \delta^{-1} |\tau|^3, \quad |\tau| \leq \tau_0, \quad z \in \Gamma_\delta,$$

where

$$(2.36) \quad C_T^0 = \check{C}_T + 416\kappa^{-1/2} C_T^{(2)} C_T.$$

2.4. Threshold approximations. We have (see [Su, (3.2), (3.4)])

$$(2.37) \quad F(\tau; \vartheta) - P = \frac{1}{2\pi i} \int_{\Gamma_\delta} (R_z(0) - R_z(\tau; \vartheta)) dz,$$

$$(2.38) \quad \mathfrak{B}_\lambda(\tau; \vartheta) F(\tau; \vartheta) = \frac{1}{2\pi i} \int_{\Gamma_\delta} z(R_z(0) - R_z(\tau; \vartheta)) dz.$$

Using (2.37) and (2.21), and estimating the length of the contour Γ_δ by $2\delta + 2\pi\delta$, we see that

$$(2.39) \quad \|F(\tau; \vartheta) - P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq 32(1 + \pi^{-1})\kappa^{-1/2} C_T |\tau|, \quad |\tau| \leq \tau_0.$$

Relations (2.38) and (2.34) imply that

$$(2.40) \quad \mathfrak{B}_\lambda(\tau; \vartheta) F(\tau; \vartheta) = \tau I_1(\vartheta) + \tau^2 I_2(\vartheta) + \Psi(\tau; \vartheta),$$

where

$$(2.41) \quad \begin{aligned} I_1(\vartheta) &= -\frac{1}{2\pi i} \int_{\Gamma_\delta} z \mathcal{I}_1^0(z; \vartheta) dz, \quad I_2(\vartheta) = -\frac{1}{2\pi i} \int_{\Gamma_\delta} z \mathcal{I}_2^+(z; \vartheta) dz, \\ \Psi(\tau; \vartheta) &= -\frac{1}{2\pi i} \int_{\Gamma_\delta} z \Psi^0(z; \tau; \vartheta) dz. \end{aligned}$$

By (2.35), the term $\Psi(\tau; \vartheta)$ satisfies the estimate

$$(2.42) \quad \|\Psi(\tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq 2\delta(1 + \pi^{-1}) C_T^0 |\tau|^3, \quad |\tau| \leq \tau_0.$$

Comparing (2.40) and (1.47), we conclude that the operators $I_1(\vartheta)$ and $I_2(\vartheta)$ defined by (2.41) satisfy the identities

$$(2.43) \quad I_1(\vartheta) = 0, \quad I_2(\vartheta) = \mathcal{S}(\vartheta)P.$$

Relations (2.39), (2.40), (2.42), and (2.43) imply the following result.

Theorem 2.2. *We have*

$$(2.44) \quad \|F(\tau; \vartheta) - P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_1 |\tau|, \quad |\tau| \leq \tau_0,$$

$$(2.45) \quad \|\mathfrak{B}_\lambda(\tau; \vartheta) F(\tau; \vartheta) - \tau^2 \mathcal{S}(\vartheta)P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_2 |\tau|^3, \quad |\tau| \leq \tau_0.$$

Here

$$(2.46) \quad C_1 = 32(1 + \pi^{-1})\kappa^{-1/2} C_T, \quad C_2 = 2\delta(1 + \pi^{-1}) C_T^0.$$

The number τ_0 is defined by (1.18), C_T is defined in accordance with (2.14), (2.15), (2.17), and C_T^0 is defined as in (2.33) and (2.36).

2.5. Representation for $F(\tau; \vartheta)$. Besides estimate (2.44), we need to separate the term of order of τ^1 in the difference $F(\tau; \vartheta) - P$ and estimate the remainder term. By (2.22) and (2.28), we have

$$(2.47) \quad R_z(\tau; \vartheta) - R_z(0) = \tau \mathcal{I}_1^0(z; \vartheta) + \Phi(z; \tau; \vartheta),$$

where

$$(2.48) \quad \begin{aligned} \Phi(z; \tau; \vartheta) &= \tau \tilde{\mathcal{I}}_1(z; \tau; \vartheta) + \tau^2 \mathcal{I}_2(z; \tau; \vartheta) \\ &= \tau \Omega_z(0) T_1(\vartheta) \Omega_z(0) T(\tau; \vartheta) R_z(\tau; \vartheta) - \tau^2 \Omega_z(0) T_2(\vartheta) R_z(\tau; \vartheta). \end{aligned}$$

From (2.7), (2.9), and (2.20) it follows that

$$\begin{aligned} \|\Phi(z; \tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} &\leq 16|\tau| \alpha \delta^{-1} (1 + 4|z + \delta| \delta^{-1}) \|T_1(\vartheta)\|_{\mathfrak{d} \rightarrow \mathfrak{d}} \|T(\tau; \vartheta)\|_{\mathfrak{d} \rightarrow \mathfrak{d}} \\ &\quad + 16\tau^2 \alpha \delta^{-1} \|T_2(\vartheta)\|_{\mathfrak{d} \rightarrow \mathfrak{d}}. \end{aligned}$$

Combining this with (2.6), (2.14)–(2.16), and (2.19), we obtain

$$(2.49) \quad \begin{aligned} \|\Phi(z; \tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} &\leq \check{C} \delta^{-1} \tau^2, \quad |\tau| \leq \tau_0, \quad z \in \Gamma_\delta, \\ \check{C} &= 32\kappa^{-1/2} (13C_T^{(1)} C_T + C_T^{(2)}). \end{aligned}$$

Substituting (2.47) in (2.37) yields

$$(2.50) \quad F(\tau; \vartheta) - P = \tau F_1(\vartheta) + F_2(\tau; \vartheta),$$

where

$$(2.51) \quad F_1(\vartheta) = -\frac{1}{2\pi i} \int_{\Gamma_\delta} \mathcal{I}_1^0(z; \vartheta) dz,$$

$$(2.52) \quad F_2(\tau; \vartheta) = -\frac{1}{2\pi i} \int_{\Gamma_\delta} \Phi(z; \tau; \vartheta) dz.$$

By (2.49), the operator (2.52) is $O(\tau^2)$. Comparing (2.50), (2.51) and (1.44), (1.48), we conclude that the operator (2.51) can be written in the form (1.48).

2.6. Estimation of the term $\mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} F_2(\tau; \vartheta)$. Below we shall need an estimate for the norm of the operator $\mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} F_2(\tau; \vartheta)$ in the space $\mathfrak{H}$. We start with four lemmas.

Lemma 2.3. *We have*

$$(2.53) \quad \|\mathfrak{B}_\lambda(\tau; \vartheta)^{1/2}\|_{\mathfrak{d} \rightarrow \mathfrak{H}} \leq C_3, \quad |\tau| \leq \tau_0,$$

where

$$(2.54) \quad C_3^2 = 2(2 + c_1^2 + c_2).$$

Proof. By (1.15), for $u \in \mathfrak{d}$ we have

$$\begin{aligned} \|\mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} u\|_{\mathfrak{H}}^2 &= \mathfrak{b}_\lambda(\tau; \vartheta)[u, u] \\ &\leq (2 + c_1^2 + c_2) \|(X_0 + \tau \vartheta_1 X_1) u\|_{\mathfrak{H}_*}^2 + (C(1) + c_3 + |\lambda| \|Q_0\|) \tau^2 \vartheta_2^2 \|u\|_{\mathfrak{H}}^2. \end{aligned}$$

Combining this with (1.18), for $|\tau| \leq \tau_0$ we see that

$$\begin{aligned} \|\mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} u\|_{\mathfrak{H}}^2 &\leq (2 + c_1^2 + c_2) (2 \|X_0 u\|_{\mathfrak{H}_*}^2 + 2\tau_0^2 \|X_1\|^2 \|u\|_{\mathfrak{H}}^2) \\ &\quad + (C(1) + c_3 + |\lambda| \|Q_0\|) \tau_0^2 \|u\|_{\mathfrak{H}}^2 \\ &\leq 2(2 + c_1^2 + c_2) \|X_0 u\|_{\mathfrak{H}_*}^2 + 2\delta \|u\|_{\mathfrak{H}}^2 \leq C_3^2 \|u\|_{\mathfrak{d}}^2, \end{aligned}$$

where the constant C_3 is defined by (2.54). $\square$

Lemma 2.4. *The operators (2.1) satisfy the estimates*

$$(2.55) \quad \|R_z(0)\|_{\mathfrak{H} \rightarrow \mathfrak{D}} \leq 2\delta^{-1/2}, \quad z \in \Gamma_\delta,$$

$$(2.56) \quad \|R_z(\tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{D}} \leq (12\kappa^{-1} + 2)^{1/2}\delta^{-1/2}, \quad z \in \Gamma_\delta, \quad |\tau| \leq \tau_0.$$

Proof. By (2.2), we have

$$\begin{aligned} \|R_z(0)u\|_{\mathfrak{D}}^2 &= \|X_0 R_z(0)u\|_{\mathfrak{H}_*}^2 + \delta \|R_z(0)u\|_{\mathfrak{H}}^2 \\ &= (A_0(A_0 - zI)^{-1}u, R_z(0)u)_{\mathfrak{H}} + \delta \|R_z(0)u\|_{\mathfrak{H}}^2 \\ &= (u, R_z(0)u)_{\mathfrak{H}} + (z + \delta) \|R_z(0)u\|_{\mathfrak{H}}^2. \end{aligned}$$

Combining this with (2.18) (with $\tau = 0$) and (2.19), we obtain

$$\|R_z(0)u\|_{\mathfrak{D}}^2 \leq 4\delta^{-1} \|u\|_{\mathfrak{H}}^2,$$

which is equivalent to (2.55).

Next, using (2.2), the identity $X_0 = X(t) - tX_1$, and the restriction $|\tau| \leq \tau_0$, we get

$$\begin{aligned} \|R_z(\tau; \vartheta)u\|_{\mathfrak{D}}^2 &= \|X_0 R_z(\tau; \vartheta)u\|_{\mathfrak{H}_*}^2 + \delta \|R_z(\tau; \vartheta)u\|_{\mathfrak{H}}^2 \\ &\leq 2\|X(t)R_z(\tau; \vartheta)u\|_{\mathfrak{H}_*}^2 + 2\tau_0^2 \|X_1\|^2 \|R_z(\tau; \vartheta)u\|_{\mathfrak{H}}^2 + \delta \|R_z(\tau; \vartheta)u\|_{\mathfrak{H}}^2. \end{aligned}$$

Hence, taking (1.14) into account, we obtain

$$(2.57) \quad \|R_z(\tau; \vartheta)u\|_{\mathfrak{D}}^2 \leq \frac{4}{\kappa} \mathfrak{b}_\lambda(\tau; \vartheta) [R_z(\tau; \vartheta)u, R_z(\tau; \vartheta)u] + (2\tau_0^2 \|X_1\|^2 + \delta) \|R_z(\tau; \vartheta)u\|_{\mathfrak{H}}^2.$$

We transform the first term on the right-hand side of (2.57):

$$\begin{aligned} \mathfrak{b}_\lambda(\tau; \vartheta) [R_z(\tau; \vartheta)u, R_z(\tau; \vartheta)u] &= (\mathfrak{B}_\lambda(\tau; \vartheta)(\mathfrak{B}_\lambda(\tau; \vartheta) - zI)^{-1}u, R_z(\tau; \vartheta)u)_{\mathfrak{H}} \\ (2.58) \quad &= (u, R_z(\tau; \vartheta)u)_{\mathfrak{H}} + z \|R_z(\tau; \vartheta)u\|_{\mathfrak{H}}^2 \leq 3\delta^{-1} \|u\|_{\mathfrak{H}}^2, \quad |\tau| \leq \tau_0, \quad z \in \Gamma_\delta \end{aligned}$$

(we have used estimates (2.18) and (2.19)).

The second term on the right-hand side of (2.57) is estimated with the help of (2.18) and (1.18):

$$(2.59) \quad (2\tau_0^2 \|X_1\|^2 + \delta) \|R_z(\tau; \vartheta)u\|_{\mathfrak{H}}^2 \leq 2\delta^{-1} \|u\|_{\mathfrak{H}}^2, \quad |\tau| \leq \tau_0, \quad z \in \Gamma_\delta.$$

Relations (2.57)–(2.59) show that

$$\|R_z(\tau; \vartheta)u\|_{\mathfrak{D}}^2 \leq \delta^{-1} (12\kappa^{-1} + 2) \|u\|_{\mathfrak{H}}^2, \quad |\tau| \leq \tau_0, \quad z \in \Gamma_\delta,$$

which implies (2.56). $\square$

Lemma 2.5. *The operator $\Omega_z(0)$ defined by (2.4) satisfies the estimate*

$$(2.60) \quad \|\Omega_z(0)\|_{\mathfrak{D} \rightarrow \mathfrak{D}} \leq 7, \quad z \in \Gamma_\delta.$$

Proof. By (2.4), (2.19), and (2.55), we have

$$(2.61) \quad \|\Omega_z(0)u\|_{\mathfrak{D}} \leq \|u\|_{\mathfrak{D}} + |z + \delta| \|R_z(0)u\|_{\mathfrak{D}} \leq \|u\|_{\mathfrak{D}} + 6\delta^{1/2} \|u\|_{\mathfrak{H}}, \quad z \in \Gamma_\delta.$$

Using (2.2), we see that $\|u\|_{\mathfrak{H}} \leq \delta^{-1/2} \|u\|_{\mathfrak{D}}$, and combining this with (2.61), we arrive at (2.60). $\square$

Lemma 2.6. *Let $\Phi(z; \tau; \vartheta)$ denote the operator (2.48). Then*

$$(2.62) \quad \|\mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} \Phi(z; \tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_4 \delta^{-1/2} \tau^2, \quad z \in \Gamma_\delta, \quad |\tau| \leq \tau_0,$$

where

$$(2.63) \quad C_4 = C_3(12\kappa^{-1} + 2)^{1/2} (49C_T^{(1)} C_T + 7C_T^{(2)}).$$

Proof. By (2.48), we have

$$\begin{aligned} \mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} \Phi(z; \tau; \vartheta) &= \tau \mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} \Omega_z(0) T_1(\vartheta) \Omega_z(0) T(\tau; \vartheta) R_z(\tau; \vartheta) \\ &\quad - \tau^2 \mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} \Omega_z(0) T_2(\vartheta) R_z(\tau; \vartheta). \end{aligned}$$

Hence,

$$(2.64) \quad \begin{aligned} \|\mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} \Phi(z; \tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} &\leq \|\mathfrak{B}_\lambda(\tau; \vartheta)^{1/2}\|_{\mathfrak{D} \rightarrow \mathfrak{H}} \|R_z(\tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{D}} \\ &\quad \times (|\tau| \|\Omega_z(0)\|_{\mathfrak{D} \rightarrow \mathfrak{D}}^2 \|T_1(\vartheta)\|_{\mathfrak{D} \rightarrow \mathfrak{D}} \|T(\tau; \vartheta)\|_{\mathfrak{D} \rightarrow \mathfrak{D}} + \tau^2 \|\Omega_z(0)\|_{\mathfrak{D} \rightarrow \mathfrak{D}} \|T_2(\vartheta)\|_{\mathfrak{D} \rightarrow \mathfrak{D}}). \end{aligned}$$

Using estimates (2.53), (2.56), (2.60), and also (2.14)–(2.16), from (2.64) we deduce that

$$\begin{aligned} \|\mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} \Phi(z; \tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} &\leq \tau^2 \delta^{-1/2} C_3 (12\kappa^{-1} + 2)^{1/2} (49C_T^{(1)} C_T + 7C_T^{(2)}), \quad z \in \Gamma_\delta, \quad |\tau| \leq \tau_0, \end{aligned}$$

which is equivalent to (2.62). $\square$

Now, (2.52) and (2.62) imply the following statement.

Proposition 2.7. *Let $F_2(\tau; \vartheta)$ be the operator defined by (2.52) and (2.48). Then*

$$(2.65) \quad \|\mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} F_2(\tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_4 \delta^{1/2} (1 + \pi^{-1}) \tau^2, \quad |\tau| \leq \tau_0.$$

§3. APPROXIMATIONS FOR THE OPERATOR $\mathfrak{B}_\lambda(\tau; \vartheta)^{-1}$

3.1. The leading term of approximation for the operator $\mathfrak{B}_\lambda(\tau; \vartheta)^{-1}$. *It is assumed (cf. [Su, Subsection 3.3]) that*

$$(3.1) \quad A(t) \geq c_* t^2 I, \quad c_* > 0, \quad |t| \leq \tau_0,$$

for some $c_* > 0$. Then, by (1.14), we have

$$(3.2) \quad \mathfrak{B}_\lambda(\tau; \vartheta) \geq \check{c}_* \tau^2 I, \quad |\tau| \leq \tau_0, \quad \check{c}_* = \frac{1}{2} \min\{\kappa c_*, 2c_4\}.$$

From (3.2) and (1.26) it follows that the eigenvalues $\lambda_l(\tau; \vartheta)$ of the operator $\mathfrak{B}_\lambda(\tau; \vartheta)$ satisfy the estimates

$$(3.3) \quad \lambda_l(\tau; \vartheta) \geq \check{c}_* \tau^2, \quad l = 1, \dots, n, \quad |\tau| \leq \tau_0.$$

Relations (3.3) and (1.27) imply that $\gamma_l(\vartheta) \geq \check{c}_*$, $l = 1, \dots, n$, whence (see (1.41))

$$(3.4) \quad \mathcal{S}(\vartheta) \geq \check{c}_* I_{\mathfrak{N}}.$$

Now, we obtain the leading term of approximation for the operator $\mathfrak{B}_\lambda(\tau; \vartheta)^{-1}$.

Theorem 3.1. *Suppose that the assumptions of Subsections 1.1–1.4 are satisfied. Let $\mathfrak{B}_\lambda(\tau; \vartheta)$ be the operator (1.17), and let P be the orthogonal projection of $\mathfrak{H}$ onto the subspace $\mathfrak{N} = \text{Ker } X_0$. Let $\mathcal{S}(\vartheta)$ be the spectral germ of the operator $\mathfrak{B}_\lambda(\tau; \vartheta)$ at $\tau = 0$, defined by (1.40). Let τ_0 be the number defined by (1.18). Suppose also that condition (3.1) is satisfied. Then*

$$(3.5) \quad \|\mathfrak{B}_\lambda(\tau; \vartheta)^{-1} - (\tau^2 \mathcal{S}(\vartheta))^{-1} P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_0 |\tau|^{-1}, \quad |\tau| \leq \tau_0,$$

where

$$(3.6) \quad C_0 = \tau_0 (3\delta)^{-1} + 2C_1 (\check{c}_*)^{-1} + C_2 (\check{c}_*)^{-2}.$$

Proof. The operator under the norm sign in (3.5) can be written as

$$(3.7) \quad \begin{aligned} & \mathfrak{B}_\lambda(\tau; \vartheta)^{-1} - (\tau^2 \mathcal{S}(\vartheta))^{-1} P \\ &= \mathfrak{B}_\lambda(\tau; \vartheta)^{-1} F(\tau; \vartheta)^\perp + \mathfrak{B}_\lambda(\tau; \vartheta)^{-1} F(\tau; \vartheta) - (\tau^2 \mathcal{S}(\vartheta))^{-1} P. \end{aligned}$$

By Proposition 1.5,

$$(3.8) \quad \|\mathfrak{B}_\lambda(\tau; \vartheta)^{-1} F(\tau; \vartheta)^\perp\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq (3\delta)^{-1}, \quad |\tau| \leq \tau_0.$$

Next, we consider the operator

$$(3.9) \quad G(\tau; \vartheta) := \mathfrak{B}_\lambda(\tau; \vartheta)^{-1} F(\tau; \vartheta) - (\tau^2 \mathcal{S}(\vartheta))^{-1} P = G_1(\tau; \vartheta) + G_2(\tau; \vartheta) + G_3(\tau; \vartheta),$$

where

$$\begin{aligned} G_1(\tau; \vartheta) &= \mathfrak{B}_\lambda(\tau; \vartheta)^{-1} F(\tau; \vartheta)(F(\tau; \vartheta) - P), \\ G_2(\tau; \vartheta) &= (F(\tau; \vartheta) - P)(\tau^2 \mathcal{S}(\vartheta))^{-1} P, \\ G_3(\tau; \vartheta) &= F(\tau; \vartheta) \mathfrak{B}_\lambda(\tau; \vartheta)^{-1} P - F(\tau; \vartheta)(\tau^2 \mathcal{S}(\vartheta))^{-1} P. \end{aligned}$$

By (2.44) and (3.2),

$$(3.10) \quad \|G_1(\tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_1(\check{c}_*)^{-1} |\tau|^{-1}, \quad |\tau| \leq \tau_0.$$

Similarly, inequalities (2.44) and (3.4) show that

$$(3.11) \quad \|G_2(\tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_1(\check{c}_*)^{-1} |\tau|^{-1}, \quad |\tau| \leq \tau_0.$$

By (3.2), (3.4), and (2.45), the operator

$$G_3(\tau; \vartheta) = F(\tau; \vartheta) \mathfrak{B}_\lambda(\tau; \vartheta)^{-1} (\tau^2 \mathcal{S}(\vartheta) P - \mathfrak{B}_\lambda(\tau; \vartheta)) (\tau^2 \mathcal{S}(\vartheta))^{-1} P$$

satisfies the estimate

$$(3.12) \quad \|G_3(\tau; \vartheta)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_2(\check{c}_*)^{-2} |\tau|^{-1}, \quad |\tau| \leq \tau_0.$$

As a result, from (3.7)–(3.12) we obtain

$$\|\mathfrak{B}_\lambda(\tau; \vartheta)^{-1} - (\tau^2 \mathcal{S}(\vartheta))^{-1} P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq (3\delta)^{-1} + (2C_1(\check{c}_*)^{-1} + C_2(\check{c}_*)^{-2}) |\tau|^{-1}, \quad |\tau| \leq \tau_0,$$

which implies (3.5). $\square$

3.2. Approximation for the operator $\mathfrak{B}_\lambda(\tau; \vartheta)^{-1}$ with a corrector taken into account.

Theorem 3.2. *Under the assumptions of Theorem 3.1, let Z and $\tilde{Z}$ be the operators defined in Subsection 1.6. Then for $|\tau| \leq \tau_0$ we have*

$$(3.13) \quad \left\| \mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} \left(\mathfrak{B}_\lambda(\tau; \vartheta)^{-1} - (\tau^2 \mathcal{S}(\vartheta))^{-1} P - \tau(\vartheta_1 Z + \vartheta_2 \tilde{Z})(\tau^2 \mathcal{S}(\vartheta))^{-1} P \right) \right\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_5.$$

The constant C_5 is given by

$$(3.14) \quad C_5 = (3\delta)^{-1/2} + C_1(\check{c}_*)^{-1/2} + C_2(\check{c}_*)^{-3/2} + C_4(1 + \pi^{-1})\delta^{1/2}(\check{c}_*)^{-1}.$$

Proof. We have

$$(3.15) \quad \begin{aligned} \mathfrak{B}_\lambda(\tau; \vartheta)^{-1/2} &= \mathfrak{B}_\lambda(\tau; \vartheta)^{-1/2} F(\tau; \vartheta)^\perp + \mathfrak{B}_\lambda(\tau; \vartheta)^{-1/2} F(\tau; \vartheta) \\ &= \mathfrak{B}_\lambda(\tau; \vartheta)^{-1/2} F(\tau; \vartheta)^\perp + \mathfrak{B}_\lambda(\tau; \vartheta)^{-1/2} F(\tau; \vartheta)(F(\tau; \vartheta) - P) \\ &\quad + \mathfrak{B}_\lambda(\tau; \vartheta)^{-1/2} F(\tau; \vartheta) P. \end{aligned}$$

The first term on the right in (3.15) is estimated with the help of (1.19):

$$(3.16) \quad \|\mathfrak{B}_\lambda(\tau; \vartheta)^{-1/2} F(\tau; \vartheta)^\perp\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq (3\delta)^{-1/2}, \quad |\tau| \leq \tau_0.$$

By (3.2), we have

$$(3.17) \quad \|\mathfrak{B}_\lambda(\tau; \vartheta)^{-1/2}\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq (\check{c}_*)^{-1/2} |\tau|^{-1}, \quad |\tau| \leq \tau_0.$$

Together with (2.44), this implies an estimate for the second term:

$$(3.18) \quad \|\mathfrak{B}_\lambda(\tau; \vartheta)^{-1/2} F(\tau; \vartheta) (F(\tau; \vartheta) - P)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_1 (\check{c}_*)^{-1/2}, \quad |\tau| \leq \tau_0.$$

Using (3.15), (3.16), and (3.18), we obtain

$$(3.19) \quad \|\mathfrak{B}_\lambda(\tau; \vartheta)^{-1/2} - \mathfrak{B}_\lambda(\tau; \vartheta)^{-1/2} F(\tau; \vartheta) P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq (3\delta)^{-1/2} + C_1 (\check{c}_*)^{-1/2}, \quad |\tau| \leq \tau_0.$$

Next, we have the obvious identity

$$(3.20) \quad \begin{aligned} & F(\tau; \vartheta) \mathfrak{B}_\lambda(\tau; \vartheta)^{-1/2} P - F(\tau; \vartheta) \mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} (\tau^2 \mathcal{S}(\vartheta))^{-1} P \\ &= F(\tau; \vartheta) \mathfrak{B}_\lambda(\tau; \vartheta)^{-1/2} (\tau^2 \mathcal{S}(\vartheta) P - \mathfrak{B}_\lambda(\tau; \vartheta) F(\tau; \vartheta)) (\tau^2 \mathcal{S}(\vartheta))^{-1} P. \end{aligned}$$

To estimate the operator (3.20), we use (2.45), (3.17), and (3.4):

$$(3.21) \quad \|F(\tau; \vartheta) \mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} (\mathfrak{B}_\lambda(\tau; \vartheta)^{-1} - (\tau^2 \mathcal{S}(\vartheta))^{-1} P) P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_2 (\check{c}_*)^{-3/2}, \quad |\tau| \leq \tau_0.$$

Combining (3.19) and (3.21), for $|\tau| \leq \tau_0$ we obtain

$$(3.22) \quad \begin{aligned} & \|\mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} (\mathfrak{B}_\lambda(\tau; \vartheta)^{-1} - F(\tau; \vartheta) (\tau^2 \mathcal{S}(\vartheta))^{-1} P)\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \\ & \leq (3\delta)^{-1/2} + C_1 (\check{c}_*)^{-1/2} + C_2 (\check{c}_*)^{-3/2}. \end{aligned}$$

Next, by (2.50) and (1.48), we have

$$(3.23) \quad F(\tau; \vartheta) P = P + \tau F_1(\vartheta) P + F_2(\tau; \vartheta) P = P + \tau(\vartheta_1 Z + \vartheta_2 \tilde{Z}) P + F_2(\tau; \vartheta) P.$$

We have used the fact that $Z^* P = \tilde{Z}^* P = 0$. Relations (3.22) and (3.23) imply the inequality

$$(3.24) \quad \begin{aligned} & \left\| \mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} \left(\mathfrak{B}_\lambda(\tau; \vartheta)^{-1} - (\tau^2 \mathcal{S}(\vartheta))^{-1} P - \tau(\vartheta_1 Z + \vartheta_2 \tilde{Z})(\tau^2 \mathcal{S}(\vartheta))^{-1} P \right) \right\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \\ & \leq (3\delta)^{-1/2} + C_1 (\check{c}_*)^{-1/2} + C_2 (\check{c}_*)^{-3/2} + \left\| \mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} F_2(\tau; \vartheta) (\tau^2 \mathcal{S}(\vartheta))^{-1} P \right\|_{\mathfrak{H} \rightarrow \mathfrak{H}}. \end{aligned}$$

An estimate for the last term on the right-hand side of (3.24) follows from (2.65) and (3.4):

$$(3.25) \quad \left\| \mathfrak{B}_\lambda(\tau; \vartheta)^{1/2} F_2(\tau; \vartheta) (\tau^2 \mathcal{S}(\vartheta))^{-1} P \right\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_4 (1 + \pi^{-1}) \delta^{1/2} (\check{c}_*)^{-1}, \quad |\tau| \leq \tau_0.$$

Now, estimate (3.13) with the constant (3.14) follows directly from (3.24) and (3.25). $\square$

3.3. By (1.40), the operator $\tau^2 \mathcal{S}(\vartheta)$ can be written in terms of the initial parameters t and ε :

$$(3.26) \quad \begin{aligned} \tau^2 \mathcal{S}(\vartheta) &= t^2 S + t\varepsilon \left(- (X_0 Z)^* (X_0 \tilde{Z}) - (X_0 \tilde{Z})^* (X_0 Z) + P(Y_2^* Y_1 + Y_1^* Y_2) \right)_{|\mathfrak{N}} \\ &+ \varepsilon^2 \left(- (X_0 \tilde{Z})^* (X_0 \tilde{Z}) + Q_{\mathfrak{N}} + \lambda Q_{0\mathfrak{N}} \right) =: L(t, \varepsilon). \end{aligned}$$

Note that, by (3.4),

$$(3.27) \quad L(t, \varepsilon) \geq \check{c}_* (t^2 + \varepsilon^2) I_{\mathfrak{N}}.$$

Now, we rewrite estimates (3.5) and (3.13) in the form convenient for further applications to differential operators:

$$(3.28) \quad \left\| (B(t, \varepsilon) + \varepsilon^2 \lambda Q_0)^{-1} - L(t, \varepsilon)^{-1} P \right\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_0 (t^2 + \varepsilon^2)^{-1/2}, \quad t^2 + \varepsilon^2 \leq \tau_0^2;$$

$$(3.29) \quad \left\| (B(t, \varepsilon) + \varepsilon^2 \lambda Q_0)^{1/2} \left((B(t, \varepsilon) + \varepsilon^2 \lambda Q_0)^{-1} - (I + tZ + \varepsilon \tilde{Z}) L(t, \varepsilon)^{-1} P \right) \right\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_5, \quad t^2 + \varepsilon^2 \leq \tau_0^2.$$

CHAPTER 2. PERIODIC DIFFERENTIAL OPERATORS IN $L_2(\mathbb{R}^d; \mathbb{C}^n)$.
THRESHOLD APPROXIMATIONS FOR THE GENERALIZED RESOLVENT

§4. PRELIMINARIES

4.1. Lattices in $\mathbb{R}^d$. Let Γ be a lattice in $\mathbb{R}^d$ generated by a basis $\mathbf{a}_1, \dots, \mathbf{a}_d$, i.e.,

$$\Gamma = \left\{ \mathbf{a} \in \mathbb{R}^d : \mathbf{a} = \sum_{j=1}^d n^j \mathbf{a}_j, \quad n^j \in \mathbb{Z} \right\}.$$

Let Ω denote the elementary cell of the lattice Γ :

$$\Omega = \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{x} = \sum_{j=1}^d \xi^j \mathbf{a}_j, \quad 0 \leq \xi^j < 1 \right\}.$$

The basis $\mathbf{b}^1, \dots, \mathbf{b}^d$ dual to $\mathbf{a}_1, \dots, \mathbf{a}_d$ is defined by the relations $\langle \mathbf{b}^l, \mathbf{a}_j \rangle = 2\pi \delta_j^l$. This basis gives rise to the *lattice $\tilde{\Gamma}$ dual to Γ* :

$$\tilde{\Gamma} = \left\{ \mathbf{b} \in \mathbb{R}^d : \mathbf{b} = \sum_{i=1}^d \mu_i \mathbf{b}^i, \quad \mu_i \in \mathbb{Z} \right\}.$$

We denote by $\tilde{\Omega}$ the central *Brillouin zone* of the lattice $\tilde{\Gamma}$:

$$(4.1) \quad \tilde{\Omega} = \{ \mathbf{k} \in \mathbb{R}^d : |\mathbf{k}| < |\mathbf{k} - \mathbf{b}|, \quad 0 \neq \mathbf{b} \in \tilde{\Gamma} \}.$$

Note that $\tilde{\Omega}$ is a fundamental domain for $\tilde{\Gamma}$. We shall use the notation $|\Omega| = \text{meas } \Omega$, $|\tilde{\Omega}| = \text{meas } \tilde{\Omega}$. By r_0 we denote the radius of the ball inscribed in $\text{clos } \tilde{\Omega}$. Then

$$(4.2) \quad 2r_0 = \min |\mathbf{b}|, \quad 0 \neq \mathbf{b} \in \tilde{\Gamma}.$$

4.2. The Fourier series. The following Fourier series expansion for functions $\mathbf{u}(\mathbf{x})$, $\mathbf{x} \in \Omega$, is related to the lattice Γ :

$$(4.3) \quad \mathbf{u}(\mathbf{x}) = |\Omega|^{-1/2} \sum_{\mathbf{b} \in \tilde{\Gamma}} \hat{\mathbf{u}}_{\mathbf{b}} \exp(i\langle \mathbf{b}, \mathbf{x} \rangle).$$

The corresponding discrete Fourier transformation $\{\hat{\mathbf{u}}_{\mathbf{b}}\} \mapsto \mathbf{u}$ is a unitary mapping of $l_2(\tilde{\Gamma}; \mathbb{C}^n)$ onto $L_2(\Omega; \mathbb{C}^n)$:

$$(4.4) \quad \int_{\Omega} |\mathbf{u}(\mathbf{x})|^2 d\mathbf{x} = \sum_{\mathbf{b} \in \tilde{\Gamma}} |\hat{\mathbf{u}}_{\mathbf{b}}|^2.$$

Let $\tilde{H}^1(\Omega; \mathbb{C}^n)$ denote the subspace of all functions in $H^1(\Omega; \mathbb{C}^n)$ whose Γ -periodic extension to $\mathbb{R}^d$ belongs to the class $H_{\text{loc}}^1(\mathbb{R}^d; \mathbb{C}^n)$. We have

$$(4.5) \quad \int_{\Omega} |(\mathbf{D} + \mathbf{k})\mathbf{u}|^2 d\mathbf{x} = \sum_{\mathbf{b} \in \tilde{\Gamma}} |\mathbf{b} + \mathbf{k}|^2 |\hat{\mathbf{u}}_{\mathbf{b}}|^2, \quad \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n), \quad \mathbf{k} \in \mathbb{R}^d,$$

and the convergence of the series on the right-hand side of (4.5) is equivalent to the fact that $\mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n)$. By (4.1), (4.4), and (4.5), we have

$$(4.6) \quad \int_{\Omega} |(\mathbf{D} + \mathbf{k})\mathbf{u}|^2 d\mathbf{x} \geq \sum_{\mathbf{b} \in \tilde{\Gamma}} |\mathbf{k}|^2 |\hat{\mathbf{u}}_{\mathbf{b}}|^2 = |\mathbf{k}|^2 \int_{\Omega} |\mathbf{u}|^2 d\mathbf{x}, \quad \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n), \quad \mathbf{k} \in \tilde{\Omega}.$$

4.3. The Gelfand transformation. Initially, the Gelfand transformation $\mathcal{U}$ is defined on the functions $\mathbf{v} \in \mathcal{S}(\mathbb{R}^d; \mathbb{C}^n)$ of Schwartz class by the formula

$$\tilde{\mathbf{v}}(\mathbf{k}, \mathbf{x}) = (\mathcal{U}\mathbf{v})(\mathbf{k}, \mathbf{x}) = |\tilde{\Omega}|^{-1/2} \sum_{\mathbf{a} \in \Gamma} \exp(-i\langle \mathbf{k}, \mathbf{x} + \mathbf{a} \rangle) \mathbf{v}(\mathbf{x} + \mathbf{a}), \quad \mathbf{x} \in \Omega, \quad \mathbf{k} \in \tilde{\Omega}.$$

The operator $\mathcal{U}$ extends by continuity to a unitary mapping

$$(4.7) \quad \mathcal{U} : L_2(\mathbb{R}^d; \mathbb{C}^n) \rightarrow \int_{\tilde{\Omega}} \oplus L_2(\Omega; \mathbb{C}^n) d\mathbf{k} =: \mathcal{H}.$$

The relation $\mathbf{v} \in H^1(\mathbb{R}^d; \mathbb{C}^n)$ is equivalent to the fact that $\tilde{\mathbf{v}}(\mathbf{k}, \cdot) \in \tilde{H}^1(\Omega; \mathbb{C}^n)$ for almost every $\mathbf{k} \in \tilde{\Omega}$, and

$$\int_{\tilde{\Omega}} \int_{\Omega} (|(\mathbf{D} + \mathbf{k})\tilde{\mathbf{v}}(\mathbf{k}, \mathbf{x})|^2 + |\tilde{\mathbf{v}}(\mathbf{k}, \mathbf{x})|^2) d\mathbf{x} d\mathbf{k} < \infty.$$

Under the Gelfand transformation $\mathcal{U}$, the operator of multiplication by a bounded periodic matrix-valued function in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ turns into multiplication by the same function on the fibers of the direct integral $\mathcal{H}$. On these fibers, the first order differential operator $b(\mathbf{D})$ applied to $\mathbf{v} \in H^1(\mathbb{R}^d; \mathbb{C}^n)$ turns into the operator $b(\mathbf{D} + \mathbf{k})$ applied to $\tilde{\mathbf{v}}(\mathbf{k}, \cdot) \in \tilde{H}^1(\Omega; \mathbb{C}^n)$.

§5. PERIODIC OPERATORS IN $L_2(\mathbb{R}^d; \mathbb{C}^n)$. DIRECT INTEGRAL EXPANSION

5.1. Factorized second order operators $\mathcal{A}$. Let

$$b(\mathbf{D}) : L_2(\mathbb{R}^d; \mathbb{C}^n) \rightarrow L_2(\mathbb{R}^d; \mathbb{C}^m)$$

be a homogeneous first order DO with constant coefficients. *We assume that $m \geq n$. The symbol $b(\boldsymbol{\xi})$ of the operator $b(\mathbf{D})$ is an $(m \times n)$ -matrix-valued linear homogeneous function of $\boldsymbol{\xi} \in \mathbb{R}^d$. It is assumed that*

$$(5.1) \quad \text{rank } b(\boldsymbol{\xi}) = n, \quad 0 \neq \boldsymbol{\xi} \in \mathbb{R}^d.$$

Condition (5.1) is equivalent to the following:

$$(5.2) \quad \alpha_0 \mathbf{1}_n \leq b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \leq \alpha_1 \mathbf{1}_n, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}, \quad 0 < \alpha_0 \leq \alpha_1 < \infty,$$

with some constants α_0, α_1 .

Let $h(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, be a Γ -periodic $(m \times m)$ -matrix-valued function such that

$$(5.3) \quad h, h^{-1} \in L_\infty.$$

We consider the DO

$$(5.4) \quad \mathcal{X} = hb(\mathbf{D}) : L_2(\mathbb{R}^d; \mathbb{C}^n) \rightarrow L_2(\mathbb{R}^d; \mathbb{C}^m), \quad \text{Dom } \mathcal{X} = H^1(\mathbb{R}^d; \mathbb{C}^n).$$

The operator (5.4) is closed. The selfadjoint operator $\mathcal{A} = \mathcal{X}^* \mathcal{X}$ in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ is generated by the quadratic form

$$\mathfrak{a}[\mathbf{u}, \mathbf{u}] = \|\mathcal{X}\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Formally, $\mathcal{A}$ is given by the differential expression

$$(5.5) \quad \mathcal{A} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}), \quad g(\mathbf{x}) = h(\mathbf{x})^* h(\mathbf{x}).$$

Using the Fourier transformation and conditions (5.2), (5.3), it is easy to show that

$$(5.6) \quad \alpha_0 \|g^{-1}\|_{L_\infty}^{-1} \|\mathbf{D}\mathbf{u}\|_{L_2}^2 \leq \mathfrak{a}[\mathbf{u}, \mathbf{u}] \leq \alpha_1 \|g\|_{L_\infty} \|\mathbf{D}\mathbf{u}\|_{L_2}^2, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

5.2. The operators $\mathcal{Y}$ and $\mathcal{Y}_2$. We consider the closed operator

$$\mathcal{Y} : L_2(\mathbb{R}^d; \mathbb{C}^n) \rightarrow L_2(\mathbb{R}^d; \mathbb{C}^{dn}),$$

defined by

$$(5.7) \quad \mathcal{Y}\mathbf{u} = \mathbf{D}\mathbf{u} = \text{col}\{D_1\mathbf{u}, \dots, D_d\mathbf{u}\}, \quad \text{Dom } \mathcal{Y} = H^1(\mathbb{R}^d; \mathbb{C}^n).$$

The lower estimate (5.6) means that

$$(5.8) \quad \|\mathcal{Y}\mathbf{u}\|_{L_2(\mathbb{R}^d)} \leq c_1 \|\mathcal{X}\mathbf{u}\|_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n),$$

where

$$(5.9) \quad c_1 = \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2}.$$

Let $a_j(\mathbf{x})$, $j = 1, \dots, d$, be bounded Γ -periodic $(n \times n)$ -matrix-valued functions in $\mathbb{R}^d$ such that

$$(5.10) \quad a_j \in L_\rho(\Omega), \quad \rho = 2 \text{ for } d = 1, \quad \rho > d \text{ for } d \geq 2; \quad j = 1, \dots, d.$$

We consider the operator $\mathcal{Y}_2 : L_2(\mathbb{R}^d; \mathbb{C}^n) \rightarrow L_2(\mathbb{R}^d; \mathbb{C}^{dn})$ that acts as multiplication by the $(dn \times n)$ -matrix-valued function consisting of the matrices $a_j(\mathbf{x})^*$, $j = 1, \dots, d$. In other words,

$$\mathcal{Y}_2\mathbf{u} = \text{col}\{a_1(\mathbf{x})^*\mathbf{u}, \dots, a_d(\mathbf{x})^*\mathbf{u}\}, \quad \text{Dom } \mathcal{Y}_2 = H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Formally, we can write

$$(\mathcal{Y}_2^*\mathcal{Y} + \mathcal{Y}^*\mathcal{Y}_2)\mathbf{u} = \sum_{j=1}^d (a_j(\mathbf{x})D_j\mathbf{u} + D_j(a_j(\mathbf{x})^*\mathbf{u})).$$

For $d \geq 2$, by the Hölder inequality and condition (5.10), we have

$$(5.11) \quad \|a_j^*\mathbf{u}\|_{L_2(\Omega)}^2 \leq \|a_j\|_{L_\rho(\Omega)}^2 \|\mathbf{u}\|_{L_p(\Omega)}^2, \quad p = \frac{2\rho}{\rho-2}.$$

Since the embedding $H^1(\Omega) \subset L_p(\Omega)$ is compact, it follows that for any $\varkappa > 0$ there exists a constant $\mathcal{C}(\varkappa) > 0$ such that

$$(5.12) \quad \|\mathbf{u}\|_{L_p(\Omega)}^2 \leq \varkappa \int_\Omega |\mathbf{D}\mathbf{u}|^2 d\mathbf{x} + \mathcal{C}(\varkappa) \int_\Omega |\mathbf{u}|^2 d\mathbf{x}, \quad \mathbf{u} \in H^1(\Omega; \mathbb{C}^n).$$

Inequalities (5.11) and (5.12) remain valid for $d = 1$ with $\rho = 2$ and $p = \infty$. If $\varkappa$ is fixed, then $\mathcal{C}(\varkappa)$ is completely determined by d , ρ , and Ω . From (5.11) and (5.12), it follows that for any $\nu > 0$ there exists a constant $C_j(\nu) > 0$ such that

$$(5.13) \quad \|a_j^*\mathbf{u}\|_{L_2(\Omega)}^2 \leq \nu \|\mathbf{D}\mathbf{u}\|_{L_2(\Omega)}^2 + C_j(\nu) \|\mathbf{u}\|_{L_2(\Omega)}^2, \quad \mathbf{u} \in H^1(\Omega; \mathbb{C}^n), \quad j = 1, \dots, d.$$

For ν fixed, the constant $C_j(\nu)$ is controlled in terms of $\|a_j\|_{L_\rho(\Omega)}$, d , ρ , and the parameters of the lattice Γ . For $\mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n)$, we write inequalities (5.13) for the shifted cells $\Omega + \mathbf{a}$, $\mathbf{a} \in \Gamma$, and sum. This yields

$$(5.14) \quad \|a_j^*\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 \leq \nu \|\mathbf{D}\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 + C_j(\nu) \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n), \quad j = 1, \dots, d.$$

Summing over j and taking the lower estimate (5.6) into account, we see that for any $\nu > 0$ there exists a constant $C(\nu) > 0$ such that

$$(5.15) \quad \|\mathcal{Y}_2\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 \leq \nu \|\mathcal{X}\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 + C(\nu) \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

For ν fixed, the constant $C(\nu)$ depends only on the norms $\|a_j\|_{L_\rho(\Omega)}$, $j = 1, \dots, d$, and $\|g^{-1}\|_{L_\infty}$, on α_0 , d , ρ , and on the parameters of the lattice Γ .

Remark 5.1. Note that if $a_j \in L_\infty$, $j = 1, \dots, d$, then condition (5.14) is satisfied with $\nu = 0$ and $C_j(0) = \|a_j\|_{L_\infty}^2$, and (5.15) is satisfied with $\nu = 0$, $C(0) = \sum_{j=1}^d \|a_j\|_{L_\infty}^2$.

5.3. The form $q[\mathbf{u}, \mathbf{u}]$. Suppose that $d\mu(\mathbf{x})$ is a Γ -periodic Borel σ -finite measure in $\mathbb{R}^d$ with values in the class of Hermitian $(n \times n)$ -matrices. Then $d\mu(\mathbf{x}) = \{d\mu_{jl}(\mathbf{x})\}$, $j, l = 1, \dots, n$. In other words, $d\mu_{jl}(\mathbf{x})$ is a complex-valued Γ -periodic measure in $\mathbb{R}^d$, and $d\mu_{jl} = d\mu_{lj}^*$. Suppose that the measure $d\mu$ is such that the function $|u(\mathbf{x})|^2$ is integrable with respect to each measure $d\mu_{jl}$ for any $u \in H^1(\mathbb{R}^d)$.

In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we consider the quadratic form

$$(5.16) \quad q[\mathbf{u}, \mathbf{u}] = \int_{\mathbb{R}^d} \langle d\mu(\mathbf{x}) \mathbf{u}, \mathbf{u} \rangle = \sum_{j,l=1}^n \int_{\mathbb{R}^d} u_l u_j^* d\mu_{jl}(\mathbf{x}), \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

The measure $d\mu$ is subject to the following condition.

Condition 5.2. We have

$$(5.17) \quad -\tilde{c} \|\mathbf{D}\mathbf{u}\|_{L_2(\Omega)}^2 - c_0 \|\mathbf{u}\|_{L_2(\Omega)}^2 \leq \int_{\Omega} \langle d\mu(\mathbf{x}) \mathbf{u}, \mathbf{u} \rangle \leq \tilde{c}_2 \|\mathbf{D}\mathbf{u}\|_{L_2(\Omega)}^2 + c_3 \|\mathbf{u}\|_{L_2(\Omega)}^2, \\ \mathbf{u} \in H^1(\Omega; \mathbb{C}^n),$$

where $c_0 \in \mathbb{R}$, $\tilde{c}_2 \geq 0$, $c_3 \geq 0$, and

$$(5.18) \quad 0 \leq \tilde{c} < \alpha_0 \|g^{-1}\|_{L_\infty}^{-1}.$$

For $\mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n)$, we write inequality (5.17) over the shifted cells $\Omega + \mathbf{a}$, $\mathbf{a} \in \Gamma$, and sum; then

$$-\tilde{c} \|\mathbf{D}\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 - c_0 \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 \leq q[\mathbf{u}, \mathbf{u}] \leq \tilde{c}_2 \|\mathbf{D}\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 + c_3 \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2, \\ \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

By (5.6), this implies the estimate

$$(5.19) \quad -(1 - \kappa) \|\mathcal{X}\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 - c_0 \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 \leq q[\mathbf{u}, \mathbf{u}] \leq c_2 \|\mathcal{X}\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 + c_3 \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2, \\ \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n),$$

where

$$(5.20) \quad c_2 = \tilde{c}_2 \alpha_0^{-1} \|g^{-1}\|_{L_\infty}, \quad \kappa = 1 - \tilde{c} \alpha_0^{-1} \|g^{-1}\|_{L_\infty}, \quad 0 < \kappa \leq 1.$$

Examples of the quadratic forms (5.16) are considered below in Subsection 5.5.

5.4. The operator pencil $\mathcal{B}(\varepsilon)$. In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we consider the quadratic form

$$(5.21) \quad \mathbf{b}(\varepsilon)[\mathbf{u}, \mathbf{u}] = \mathbf{a}[\mathbf{u}, \mathbf{u}] + 2\varepsilon \operatorname{Re}(\mathcal{Y}\mathbf{u}, \mathcal{Y}_2\mathbf{u})_{L_2(\mathbb{R}^d)} + \varepsilon^2 q[\mathbf{u}, \mathbf{u}], \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n),$$

where $0 \leq \varepsilon \leq 1$. To estimate the quadratic form (5.21) from below, we use (5.8), obtaining

$$(5.22) \quad 2\varepsilon |\operatorname{Re}(\mathcal{Y}\mathbf{u}, \mathcal{Y}_2\mathbf{u})_{L_2}| \leq 2c_1 \varepsilon \|\mathcal{X}\mathbf{u}\|_{L_2} \|\mathcal{Y}_2\mathbf{u}\|_{L_2} \leq \frac{\kappa}{4} \|\mathcal{X}\mathbf{u}\|_{L_2}^2 + \frac{4c_1^2}{\kappa} \varepsilon^2 \|\mathcal{Y}_2\mathbf{u}\|_{L_2}^2, \\ \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Put

$$(5.23) \quad c_4 = 4\kappa^{-1} c_1^2 C(\nu) \quad \text{for } \nu = \kappa^2 (16c_1^2)^{-1}.$$

Then, by (5.15), (5.22), and (5.23),

$$(5.24) \quad 2\varepsilon |\operatorname{Re}(\mathcal{Y}\mathbf{u}, \mathcal{Y}_2\mathbf{u})_{L_2}| \leq \frac{\kappa}{2} \|\mathcal{X}\mathbf{u}\|_{L_2}^2 + c_4 \varepsilon^2 \|\mathbf{u}\|_{L_2}^2, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Now, the lower estimate (5.19), relations (5.21) and (5.24), and the restriction $0 \leq \varepsilon \leq 1$ imply that

$$(5.25) \quad \mathbf{b}(\varepsilon)[\mathbf{u}, \mathbf{u}] \geq \frac{\kappa}{2} \mathbf{a}[\mathbf{u}, \mathbf{u}] - (c_0 + c_4) \varepsilon^2 \|\mathbf{u}\|_{L_2}^2, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Estimate (5.25) shows that (5.21) is closed and lower semibounded. Let $\mathcal{B}(\varepsilon)$ denote the selfadjoint operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ generated by this quadratic form. Formally, we can write

$$(5.26) \quad \begin{aligned} \mathcal{B}(\varepsilon) &= \mathcal{A} + \varepsilon(\mathcal{Y}_2^* \mathcal{Y} + \mathcal{Y}^* \mathcal{Y}_2) + \varepsilon^2 Q \\ &= b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) + \varepsilon \sum_{j=1}^d (a_j(\mathbf{x}) D_j + D_j a_j(\mathbf{x})^*) + \varepsilon^2 Q(\mathbf{x}), \end{aligned}$$

where $Q(\mathbf{x})$ can be interpreted as a generalized matrix-valued potential generated by the measure $d\mu(\mathbf{x})$.

Our goal is to study the generalized resolvent $(\mathcal{B}(\varepsilon) + \varepsilon^2 \lambda Q_0)^{-1}$. Here Q_0 is the operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ acting as multiplication by the Γ -periodic, positive definite, and bounded $(n \times n)$ -matrix-valued function $Q_0(\mathbf{x})$. We assume that the parameter $\lambda \in \mathbb{R}$ satisfies the following restriction:

$$(5.27) \quad \begin{aligned} \lambda &\geq \|Q_0^{-1}\|_{L_\infty} (c_0 + 2c_4) \quad \text{if } \lambda \geq 0, \\ \lambda &\geq \|Q_0\|_{L_\infty}^{-1} (c_0 + 2c_4) \quad \text{if } \lambda < 0 \text{ (and } c_0 + 2c_4 < 0), \end{aligned}$$

which ensures that

$$(5.28) \quad \lambda(Q_0 \mathbf{u}, \mathbf{u})_{L_2(\mathbb{R}^d)} \geq (c_0 + 2c_4) \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2, \quad \mathbf{u} \in L_2(\mathbb{R}^d; \mathbb{C}^n).$$

By (5.25) and (5.28),

$$\begin{aligned} \mathbf{b}_\lambda(\varepsilon)[\mathbf{u}, \mathbf{u}] &:= \mathbf{b}(\varepsilon)[\mathbf{u}, \mathbf{u}] + \varepsilon^2 \lambda(Q_0 \mathbf{u}, \mathbf{u})_{L_2(\mathbb{R}^d)} \geq \frac{\kappa}{2} \mathbf{a}[\mathbf{u}, \mathbf{u}] + c_4 \varepsilon^2 \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2, \\ &\quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n). \end{aligned}$$

Thus, the operator

$$(5.29) \quad \mathcal{B}_\lambda(\varepsilon) := \mathcal{B}(\varepsilon) + \varepsilon^2 \lambda Q_0$$

is positive. Relations (5.6), (5.8), (5.15) (with $\nu = 1$) and the upper estimate (5.19) imply the following upper estimate for the quadratic form of this operator:

$$(5.30) \quad \begin{aligned} \mathbf{b}_\lambda(\varepsilon)[\mathbf{u}, \mathbf{u}] &\leq (2 + c_1^2 + c_2) \mathbf{a}[\mathbf{u}, \mathbf{u}] + (C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty}) \varepsilon^2 \|\mathbf{u}\|_{L_2}^2 \\ &\leq (2 + c_1^2 + c_2) \alpha_1 \|g\|_{L_\infty} \|\mathbf{D}\mathbf{u}\|_{L_2}^2 + (C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty}) \varepsilon^2 \|\mathbf{u}\|_{L_2}^2, \\ &\quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n). \end{aligned}$$

5.5. Examples. Now, we consider examples of the quadratic forms (5.16) that satisfy Condition 5.2 *a priori*.

Example 5.3. Suppose that the measure $d\mu$ is absolutely continuous with respect to Lebesgue measure: $d\mu(\mathbf{x}) = Q(\mathbf{x}) d\mathbf{x}$, where $Q(\mathbf{x})$ is a Γ -periodic Hermitian $(n \times n)$ -matrix-valued function in $\mathbb{R}^d$ such that

$$(5.31) \quad Q \in L_s(\Omega), \quad s = 1 \quad \text{for } d = 1; \quad s > \frac{d}{2} \quad \text{for } d \geq 2.$$

Then

$$q[\mathbf{u}, \mathbf{u}] = \int_{\mathbb{R}^d} \langle Q(\mathbf{x}) \mathbf{u}, \mathbf{u} \rangle d\mathbf{x}, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Under condition (5.31), for any $\nu > 0$ there exists a constant $C_Q(\nu)$ such that

$$\int_{\Omega} |\langle Q(\mathbf{x}) \mathbf{u}, \mathbf{u} \rangle| d\mathbf{x} \leq \nu \int_{\Omega} |\mathbf{D}\mathbf{u}|^2 d\mathbf{x} + C_Q(\nu) \int_{\Omega} |u|^2 d\mathbf{x}, \quad \mathbf{u} \in H^1(\Omega; \mathbb{C}^n);$$

see the proof of estimate (5.13). If ν is fixed, the constant $C_Q(\nu)$ is controlled in terms of d, s , the norm $\|Q\|_{L_s(\Omega)}$, and the parameters of the lattice. Putting $2\nu = \alpha_0 \|g^{-1}\|_{L_\infty}^{-1}$, we see that condition (5.17) is satisfied with $\tilde{c} = \nu$ (this ensures (5.18)), $c_0 = C_Q(\nu)$,

$\tilde{c}_2 = 1$, and $c_3 = C_Q(1)$. Then inequalities (5.19) are satisfied with $\kappa = 1/2$ and $c_2 = \alpha_0^{-1} \|g^{-1}\|_{L_\infty}$. The constant c_3 is controlled in terms of d, s , the norm $\|Q\|_{L_s(\Omega)}$, and the parameters of the lattice. The constant c_0 is controlled in terms of $d, s, \alpha_0, \|g^{-1}\|_{L_\infty}, \|Q\|_{L_s(\Omega)}$, and the parameters of Γ .

Example 5.3'. The case where $Q \in L_\infty$ is simpler. In this case, condition (5.17) is satisfied with $\tilde{c} = \tilde{c}_2 = 0$, $c_0 = c_3 = \|Q\|_{L_\infty}$, and (5.19) is valid with $c_2 = 0$ and $\kappa = 1$.

Example 5.4. Assume that $d \geq 2$ and that Σ is a Γ -periodic system of piecewise-smooth $(d-1)$ -dimensional surfaces in $\mathbb{R}^d$. To avoid stipulations, we assume that the intersection of Σ with Ω is a piecewise-smooth surface of class C^1 (a union of a finite number of smooth pieces). Suppose that the measure $d\mu$ is supported on Σ and is absolutely continuous with respect to the $(d-1)$ -dimensional Lebesgue measure dS_Σ : $d\mu(\mathbf{x}) = \sigma(\mathbf{x}) dS_\Sigma(\mathbf{x})$, where $\sigma(\mathbf{x})$ is a Γ -periodic Hermitian $(n \times n)$ -matrix-valued function on Σ such that

$$(5.32) \quad \sigma \in L_{r,\text{loc}}(\Sigma), \quad r > d-1.$$

Under condition (5.32), for any $\nu > 0$ there exists a constant $C_\Sigma(\nu)$ such that

$$\int_{\Sigma \cap \Omega} |\langle \sigma(\mathbf{x}) \mathbf{u}, \mathbf{u} \rangle| dS_\Sigma(\mathbf{x}) \leq \nu \int_\Omega |\mathbf{D}\mathbf{u}|^2 d\mathbf{x} + C_\Sigma(\nu) \int_\Omega |\mathbf{u}|^2 d\mathbf{x}, \quad \mathbf{u} \in H^1(\Omega; \mathbb{C}^n).$$

For ν fixed, the constant $C_\Sigma(\nu)$ is controlled in terms of d, r , the norm $\|\sigma\|_{L_r(\Sigma \cap \Omega)}$, and the parameters of the lattice Γ and the system of surfaces Σ . Putting $2\nu = \alpha_0 \|g^{-1}\|_{L_\infty}^{-1}$, we see that condition (5.17) is satisfied with $\tilde{c} = \nu$ (this ensures (5.18)), $c_0 = C_\Sigma(\nu)$, $\tilde{c}_2 = 1$, and $c_3 = C_\Sigma(1)$. Then inequalities (5.19) are valid with $\kappa = 1/2$ and $c_2 = \alpha_0^{-1} \|g^{-1}\|_{L_\infty}$. The constant c_3 depends only on d, r , the norm $\|\sigma\|_{L_r(\Sigma \cap \Omega)}$, the system of surfaces Σ , and the parameters of the lattice. The constant c_0 is controlled in terms of the same objects and also α_0 and $\|g^{-1}\|_{L_\infty}$.

Example 5.4'. Let $d = 1$, and, for definiteness, let $\Gamma = \mathbb{Z}$. Suppose that $x_1, \dots, x_M \in [0, 1]$ is a given set of points, and $\sigma_j, j = 1, \dots, M$, is a given set of Hermitian $(n \times n)$ -matrices. Let μ be a sum of delta-functions supported at the points $x_j + n, j = 1, \dots, M, n \in \mathbb{Z}$, with the “densities” σ_j . Then

$$q[\mathbf{u}, \mathbf{u}] = \sum_{n \in \mathbb{Z}} \sum_{j=1}^M \langle \sigma_j \mathbf{u}(x_j + n), \mathbf{u}(x_j + n) \rangle, \quad \mathbf{u} \in H^1(\mathbb{R}; \mathbb{C}^n).$$

In the one-dimensional case, for any $\nu > 0$ there exists a constant $C(\nu)$ such that

$$\sum_{j=1}^M |\langle \sigma_j \mathbf{u}(x_j), \mathbf{u}(x_j) \rangle| \leq \nu \|D\mathbf{u}\|_{L_2(0,1)}^2 + C(\nu) \|\mathbf{u}\|_{L_2(0,1)}^2, \quad \mathbf{u} \in H^1((0, 1); \mathbb{C}^n).$$

This ensures the required conditions. Condition (5.19) is valid with $\kappa = 1/2$ and $c_2 = \alpha_0^{-1} \|g^{-1}\|_{L_\infty}$. The constant c_3 depends only on $|\sigma_j|, j = 1, \dots, M$, and c_0 is controlled in terms of the same parameters and also α_0 and $\|g^{-1}\|_{L_\infty}$.

Of course, the role of $q[\mathbf{u}, \mathbf{u}]$ can be played by the sum of the forms considered in Examples 5.3 and 5.4.

We return to the general assumptions. *For convenience of further references, by the “initial data” we shall mean the following set of parameters:*

$$(5.33) \quad \begin{aligned} & d, m, n, \rho; \quad \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|a_j\|_{L_\rho(\Omega)}, j = 1, \dots, d; \\ & \tilde{c}, c_0, \tilde{c}_2, c_3 \text{ from Condition 5.2; } |\lambda|, \|Q_0\|_{L_\infty}. \end{aligned}$$

We shall trace the dependence of constants in estimates on the initial data and the parameters of the lattice. Note that the constants $c_1, C(1), \kappa, c_2$, and c_4 (see (5.9),

(5.15), (5.20), (5.23)) are determined by the initial data and the lattice. In the case of Example 5.3, the set of constants $\tilde{c}$, c_0 , $\tilde{c}_2$, c_3 in the initial data can be replaced by the norm $\|Q\|_{L_s(\Omega)}$, and in the case of Example 5.4, it can be replaced by the norm $\|\sigma\|_{L_r(\Sigma \cap \Omega)}$ and the parameters of the system of surfaces Σ .

5.6. The operators $\mathcal{A}(\mathbf{k})$. We put

$$(5.34) \quad \mathfrak{H} = L_2(\Omega; \mathbb{C}^n), \quad \mathfrak{H}_* = L_2(\Omega; \mathbb{C}^m), \quad \tilde{\mathfrak{H}} = L_2(\Omega; \mathbb{C}^{dn}),$$

and consider the closed operator $\mathcal{X}(\mathbf{k}) : \mathfrak{H} \rightarrow \mathfrak{H}_*$, $\mathbf{k} \in \mathbb{R}^d$, defined by the relations

$$(5.35) \quad \mathcal{X}(\mathbf{k}) = hb(\mathbf{D} + \mathbf{k}), \quad \text{Dom } \mathcal{X}(\mathbf{k}) = \tilde{H}^1(\Omega; \mathbb{C}^n).$$

The selfadjoint operator

$$(5.36) \quad \mathcal{A}(\mathbf{k}) = \mathcal{X}(\mathbf{k})^* \mathcal{X}(\mathbf{k}) : \mathfrak{H} \rightarrow \mathfrak{H}$$

is generated by the quadratic form

$$\mathfrak{a}(\mathbf{k})[\mathbf{u}, \mathbf{u}] = \|\mathcal{X}(\mathbf{k})\mathbf{u}\|_{\mathfrak{H}_*}^2, \quad \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n).$$

By using the Fourier series expansion (4.3) for a function $\mathbf{u}$ and conditions (5.2), (5.3), it is easy to check that

$$(5.37) \quad \alpha_0 \|g^{-1}\|_{L_\infty}^{-1} \|(\mathbf{D} + \mathbf{k})\mathbf{u}\|_{L_2(\Omega)}^2 \leq \mathfrak{a}(\mathbf{k})[\mathbf{u}, \mathbf{u}] \leq \alpha_1 \|g\|_{L_\infty} \|(\mathbf{D} + \mathbf{k})\mathbf{u}\|_{L_2(\Omega)}^2, \\ \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n).$$

Estimates (5.37) and (4.6) imply that

$$(5.38) \quad \mathcal{A}(\mathbf{k}) \geq c_* |\mathbf{k}|^2 I, \quad \mathbf{k} \in \tilde{\Omega}, \quad c_* = \alpha_0 \|g^{-1}\|_{L_\infty}^{-1}.$$

We put

$$(5.39) \quad \mathfrak{N} := \text{Ker } \mathcal{A}(0) = \text{Ker } \mathcal{X}(0).$$

Relations (5.37) with $\mathbf{k} = 0$ show that the kernel (5.39) consists of constant vector-valued functions:

$$(5.40) \quad \mathfrak{N} = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n) : \mathbf{u} = \mathbf{c} \in \mathbb{C}^n\}, \quad \dim \mathfrak{N} = n.$$

From (4.2) and (4.5) with $\mathbf{k} = 0$ it can be seen that if $\mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n)$ and $\int_\Omega \mathbf{u} d\mathbf{x} = 0$ (i.e., $\hat{\mathbf{u}}_0 = 0$), then

$$(5.41) \quad \|\mathbf{D}\mathbf{u}\|_{L_2(\Omega)}^2 \geq 4r_0^2 \|\mathbf{u}\|_{L_2(\Omega)}^2, \quad \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n), \quad \mathbf{u} \perp \mathfrak{N}.$$

Relation (5.41) and the lower estimate (5.37) with $\mathbf{k} = 0$ show that for the distance d^0 from the point $\lambda_0 = 0$ to the rest of the spectrum of the operator $\mathcal{A}(0)$ we have

$$(5.42) \quad d^0 \geq 4c_* r_0^2.$$

5.7. The operators $\mathcal{Y}(\mathbf{k})$ and Y_2 . Consider the operator $\mathcal{Y}(\mathbf{k}) : \mathfrak{H} \rightarrow \tilde{\mathfrak{H}}$ defined by the relations

$$(5.43) \quad \mathcal{Y}(\mathbf{k})\mathbf{u} = (\mathbf{D} + \mathbf{k})\mathbf{u} = \text{col}\{(D_1 + k_1)\mathbf{u}, \dots, (D_d + k_d)\mathbf{u}\}, \\ \text{Dom } \mathcal{Y}(\mathbf{k}) = \tilde{H}^1(\Omega; \mathbb{C}^n).$$

Using (5.37), we see that

$$(5.44) \quad \|\mathcal{Y}(\mathbf{k})\mathbf{u}\|_{\tilde{\mathfrak{H}}} \leq c_1 \|\mathcal{X}(\mathbf{k})\mathbf{u}\|_{\mathfrak{H}_*}, \quad \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n),$$

where the constant c_1 is given by (5.9).

Consider the operator $Y_2 : \mathfrak{H} \rightarrow \tilde{\mathfrak{H}}$ acting as multiplication by the $(dn \times n)$ -matrix-valued function composed of the matrices $a_j(\mathbf{x})^*$, $j = 1, \dots, d$. In other words,

$$(5.45) \quad Y_2 \mathbf{u} = \text{col}\{a_1(\mathbf{x})^* \mathbf{u}, \dots, a_d(\mathbf{x})^* \mathbf{u}\}, \quad \text{Dom } Y_2 = \tilde{H}^1(\Omega; \mathbb{C}^n).$$

Replacing $\mathbf{u}(\mathbf{x})$ by $\mathbf{u}(\mathbf{x}) \exp(i\langle \mathbf{k}, \mathbf{x} \rangle)$ in (5.13) (these functions belong to $H^1(\Omega; \mathbb{C}^n)$ simultaneously), we see that

$$\|a_j^* \mathbf{u}\|_{L_2(\Omega)}^2 \leq \nu \|(\mathbf{D} + \mathbf{k})\mathbf{u}\|_{L_2(\Omega)}^2 + C_j(\nu) \|\mathbf{u}\|_{L_2(\Omega)}^2, \quad \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n), \quad j = 1, \dots, d.$$

(Certainly, this inequality is true also on the entire class $H^1(\Omega; \mathbb{C}^n)$.) Summing over j and using (5.37), we see that for any $\nu > 0$ there exists a constant $C(\nu) > 0$ such that

$$(5.46) \quad \|Y_2 \mathbf{u}\|_{\mathfrak{H}}^2 \leq \nu \|\mathcal{X}(\mathbf{k})\mathbf{u}\|_{\mathfrak{H}_*}^2 + C(\nu) \|\mathbf{u}\|_{\mathfrak{H}}^2, \quad \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n), \quad \mathbf{k} \in \mathbb{R}^d.$$

Here the constant $C(\nu)$ is the same as in (5.15).

5.8. The form $q_\Omega[\mathbf{u}, \mathbf{u}]$. In the space $\mathfrak{H}$, we consider the quadratic form

$$(5.47) \quad q_\Omega[\mathbf{u}, \mathbf{u}] = \int_\Omega \langle d\mu(\mathbf{x})\mathbf{u}, \mathbf{u} \rangle = \sum_{j,l=1}^n \int_\Omega u_j u_l^* d\mu_{jl}(\mathbf{x}), \quad \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n).$$

Replacing $\mathbf{u}(\mathbf{x})$ by $\mathbf{u}(\mathbf{x}) \exp(i\langle \mathbf{k}, \mathbf{x} \rangle)$ in (5.17) (these functions belong to $H^1(\Omega; \mathbb{C}^n)$ simultaneously), we get

$$-\tilde{c} \|(\mathbf{D} + \mathbf{k})\mathbf{u}\|_{L_2(\Omega)}^2 - c_0 \|\mathbf{u}\|_{L_2(\Omega)}^2 \leq q_\Omega[\mathbf{u}, \mathbf{u}] \leq \tilde{c}_2 \|(\mathbf{D} + \mathbf{k})\mathbf{u}\|_{L_2(\Omega)}^2 + c_3 \|\mathbf{u}\|_{L_2(\Omega)}^2, \\ \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n), \quad \mathbf{k} \in \mathbb{R}^d.$$

(Obviously, these inequalities are valid also on the entire class $H^1(\Omega; \mathbb{C}^n)$.)

By (5.37), this implies the estimate

$$(5.48) \quad -(1 - \kappa) \|\mathcal{X}(\mathbf{k})\mathbf{u}\|_{\mathfrak{H}_*}^2 - c_0 \|\mathbf{u}\|_{\mathfrak{H}}^2 \leq q_\Omega[\mathbf{u}, \mathbf{u}] \leq c_2 \|\mathcal{X}(\mathbf{k})\mathbf{u}\|_{\mathfrak{H}_*}^2 + c_3 \|\mathbf{u}\|_{\mathfrak{H}}^2, \\ \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n),$$

where the constants κ , c_0 , c_2 , and c_3 are the same as in (5.19).

5.9. The operator pencil $\mathcal{B}(\mathbf{k}, \varepsilon)$. In the space $\mathfrak{H}$, we consider the quadratic form

$$(5.49) \quad \mathbf{b}(\mathbf{k}, \varepsilon)[\mathbf{u}, \mathbf{u}] = \mathbf{a}(\mathbf{k})[\mathbf{u}, \mathbf{u}] + 2\varepsilon \operatorname{Re}(\mathcal{Y}(\mathbf{k})\mathbf{u}, Y_2 \mathbf{u})_{\mathfrak{H}} + \varepsilon^2 q_\Omega[\mathbf{u}, \mathbf{u}], \quad \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n).$$

To estimate the form (5.49) from below, we use (5.44), (5.46), and (5.48). It is easily seen that (cf. the proof of (5.25))

$$(5.50) \quad \mathbf{b}(\mathbf{k}; \varepsilon)[\mathbf{u}, \mathbf{u}] \geq \frac{\kappa}{2} \mathbf{a}(\mathbf{k})[\mathbf{u}, \mathbf{u}] - (c_0 + c_4) \varepsilon^2 \|\mathbf{u}\|_{\mathfrak{H}}^2, \quad \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n).$$

Inequality (5.50) shows that the quadratic form (5.49) is closed and lower semibounded. The selfadjoint operator in $\mathfrak{H}$ generated by this form is denoted by $\mathcal{B}(\mathbf{k}, \varepsilon)$. Formally, we have

$$(5.51) \quad \begin{aligned} \mathcal{B}(\mathbf{k}, \varepsilon) &= \mathcal{A}(\mathbf{k}) + \varepsilon(Y_2^* \mathcal{Y}(\mathbf{k}) + \mathcal{Y}(\mathbf{k})^* Y_2) + \varepsilon^2 Q \\ &= b(\mathbf{D} + \mathbf{k})^* g(\mathbf{x}) b(\mathbf{D} + \mathbf{k}) \\ &\quad + \varepsilon \sum_{j=1}^d (a_j(\mathbf{x})(D_j + k_j) + (D_j + k_j)a_j(\mathbf{x})^*) + \varepsilon^2 Q(\mathbf{x}). \end{aligned}$$

We study the generalized resolvent $(\mathcal{B}(\mathbf{k}; \varepsilon) + \varepsilon^2 \lambda Q_0)^{-1}$. Here Q_0 is the operator in $L_2(\Omega; \mathbb{C}^n)$ acting as multiplication by the matrix-valued function $Q_0(\mathbf{x})$. Condition (5.27) ensures that

$$(5.52) \quad \lambda(Q_0 \mathbf{u}, \mathbf{u})_{L_2(\Omega)} \geq (c_0 + 2c_4) \|\mathbf{u}\|_{L_2(\Omega)}^2, \quad \mathbf{u} \in L_2(\Omega; \mathbb{C}^n).$$

By (5.50) and (5.52),

$$(5.53) \quad \mathbf{b}_\lambda(\mathbf{k}; \varepsilon)[\mathbf{u}, \mathbf{u}] := \mathbf{b}(\mathbf{k}; \varepsilon)[\mathbf{u}, \mathbf{u}] + \varepsilon^2 \lambda(Q_0 \mathbf{u}, \mathbf{u})_{\mathfrak{H}} \geq \frac{\kappa}{2} \mathbf{a}(\mathbf{k})[\mathbf{u}, \mathbf{u}] + c_4 \varepsilon^2 \|\mathbf{u}\|_{\mathfrak{H}}^2, \\ \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n).$$

Thus, the operator

$$(5.54) \quad \mathcal{B}_\lambda(\mathbf{k}; \varepsilon) := \mathcal{B}(\mathbf{k}; \varepsilon) + \varepsilon^2 \lambda Q_0$$

is positive. Next, relations (5.37), (5.44), (5.46) (with $\nu = 1$) and the upper estimate (5.48) imply the following upper bound for the quadratic form of this operator for $\mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^n)$:

$$(5.55) \quad \begin{aligned} \mathbf{b}_\lambda(\mathbf{k}; \varepsilon)[\mathbf{u}, \mathbf{u}] &\leq (2 + c_1^2 + c_2)\alpha(\mathbf{k})[\mathbf{u}, \mathbf{u}] + (C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty})\varepsilon^2 \|\mathbf{u}\|_{L_2(\Omega)}^2 \\ &\leq (2 + c_1^2 + c_2)\alpha_1 \|g\|_{L_\infty} \|(\mathbf{D} + \mathbf{k})\mathbf{u}\|_{L_2(\Omega)}^2 \\ &\quad + (C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty})\varepsilon^2 \|\mathbf{u}\|_{L_2(\Omega)}^2. \end{aligned}$$

5.10. Direct integral expansion for the operator $\mathcal{B}_\lambda(\varepsilon)$. Under the Gelfand transformation $\mathcal{U}$, the operator (5.29) acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ expands in the direct integral of operators (5.54) acting in $L_2(\Omega; \mathbb{C}^n) = \mathfrak{H}$:

$$(5.56) \quad \mathcal{U}\mathcal{B}_\lambda(\varepsilon)\mathcal{U}^{-1} = \int_{\tilde{\Omega}} \oplus \mathcal{B}_\lambda(\mathbf{k}, \varepsilon) d\mathbf{k}.$$

In more detail, this means the following. For $\mathbf{v} \in H^1(\mathbb{R}^d; \mathbb{C}^n)$, we have

$$(5.57) \quad \tilde{\mathbf{v}}(\mathbf{k}, \cdot) \in \tilde{H}^1(\Omega; \mathbb{C}^n) \text{ for a.e. } \mathbf{k} \in \tilde{\Omega},$$

$$(5.58) \quad \mathbf{b}_\lambda(\varepsilon)[\mathbf{v}, \mathbf{v}] = \int_{\tilde{\Omega}} \mathbf{b}_\lambda(\mathbf{k}, \varepsilon)[\tilde{\mathbf{v}}(\mathbf{k}, \cdot), \tilde{\mathbf{v}}(\mathbf{k}, \cdot)] d\mathbf{k}.$$

Conversely, if $\tilde{\mathbf{v}} \in \mathcal{H}$ satisfies (5.57) and the integral in (5.58) is finite, then $\mathbf{v} \in H^1(\mathbb{R}^d; \mathbb{C}^n)$ and (5.58) is valid.

§6. INCORPORATION OF THE OPERATORS $\mathcal{B}(\mathbf{k}, \varepsilon)$ IN THE ABSTRACT SCHEME

6.1. For $d > 1$, the operators $\mathcal{B}(\mathbf{k}, \varepsilon)$ depend on the multidimensional parameter $\mathbf{k}$. As in [BSu2, Chapter 2], we distinguish a one-dimensional parameter t by putting

$$\mathbf{k} = t\boldsymbol{\theta}, \quad t = |\mathbf{k}|, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}.$$

We apply the method of Chapter 1. Now, all the objects will depend on the additional parameter $\boldsymbol{\theta}$. We must make our constructions and estimates uniform in $\boldsymbol{\theta}$. The spaces $\mathfrak{H}$, $\mathfrak{H}_*$, and $\tilde{\mathfrak{H}}$ are defined by (5.34). We put $X(t) = X(t; \boldsymbol{\theta}) := \mathcal{X}(t\boldsymbol{\theta})$. By (5.35), we have $X(t; \boldsymbol{\theta}) = X_0 + tX_1(\boldsymbol{\theta})$, where

$$(6.1) \quad X_0 = h(\mathbf{x})b(\mathbf{D}), \quad \text{Dom } X_0 = \tilde{H}^1(\Omega; \mathbb{C}^n); \quad X_1(\boldsymbol{\theta}) = h(\mathbf{x})b(\boldsymbol{\theta}).$$

Next, we put $A(t) = A(t; \boldsymbol{\theta}) := \mathcal{A}(t\boldsymbol{\theta})$. By (5.36), $A(t; \boldsymbol{\theta}) = X(t; \boldsymbol{\theta})^* X(t; \boldsymbol{\theta})$. In accordance with (5.39) and (5.40), the kernel $\mathfrak{N} = \text{Ker } \mathcal{A}(0) = \text{Ker } X_0$ is n -dimensional and consists of constant vectors. Condition 1.1 is satisfied, and d^0 obeys (5.42). As was shown in [BSu2, Chapter 2, §3], the condition $n \leq n_* = \dim \text{Ker } X_0^*$ is also satisfied; here, either $n_* = n$ (if $m = n$) or $n_* = \infty$ (if $m > n$). The orthogonal projection P of the space $\mathfrak{H} = L_2(\Omega; \mathbb{C}^n)$ onto the subspace $\mathfrak{N} = \mathbb{C}^n$ is the operator of averaging over the cell Ω :

$$P\mathbf{u} = |\Omega|^{-1} \int_{\Omega} \mathbf{u}(\mathbf{x}) d\mathbf{x}.$$

The role of $Y(t)$ is played by the operator $Y(t; \boldsymbol{\theta}) := \mathcal{Y}(t\boldsymbol{\theta})$. By (5.43), we have $Y(t; \boldsymbol{\theta}) = Y_0 + tY_1(\boldsymbol{\theta})$, where

$$(6.2) \quad \begin{aligned} Y_0\mathbf{u} &= \mathbf{D}\mathbf{u} = \text{col}\{D_1\mathbf{u}, \dots, D_d\mathbf{u}\}, \quad \text{Dom } Y_0 = \tilde{H}^1(\Omega; \mathbb{C}^n); \\ Y_1(\boldsymbol{\theta})\mathbf{u} &= \text{col}\{\theta_1\mathbf{u}, \dots, \theta_d\mathbf{u}\}. \end{aligned}$$

Condition 1.2 is ensured by (5.44). The operator Y_2 is defined by (5.45). By (5.46), Condition 1.3 is fulfilled. The role of the form $\mathbf{q}$ from Subsection 1.3 is played by the form $q_\Omega[\mathbf{u}, \mathbf{u}]$ defined by (5.47). Relation (5.48) shows that Condition 1.4 is also fulfilled.

Finally, the role of the operator pencil $B(t, \varepsilon)$ (see (1.10)) is played by the operator family (5.51): $B(t, \varepsilon; \boldsymbol{\theta}) := \mathcal{B}(t\boldsymbol{\theta}, \varepsilon)$. We have

$$B(t, \varepsilon; \boldsymbol{\theta}) = A(t; \boldsymbol{\theta}) + \varepsilon(Y_2^* Y(t; \boldsymbol{\theta}) + Y(t; \boldsymbol{\theta})^* Y_2) + \varepsilon^2 Q.$$

The role of the operator Q_0 occurring in Subsection 1.4 belongs to the operator of multiplication by the matrix-valued function $Q_0(\mathbf{x})$. By (5.27), the parameter λ satisfies (1.12). Estimate (1.14) corresponds to (5.53).

Thus, all the assumptions of the abstract method described in §1 are now satisfied.

6.2. In accordance with Subsection 1.5, we should fix a positive number δ such that $\delta < \kappa d^0/13$. Taking (5.42) into account, we fix δ as follows:

$$(6.3) \quad \delta = \frac{1}{4} \kappa c_* r_0^2 = \frac{1}{4} \kappa \alpha_0 \|g^{-1}\|_{L_\infty}^{-1} r_0^2.$$

Relations (6.1), (5.2), and (5.3) show that

$$(6.4) \quad \|X_1(\boldsymbol{\theta})\| \leq \alpha_1^{1/2} \|g\|_{L_\infty}^{1/2}, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}.$$

Obviously, (6.2) implies that

$$(6.5) \quad \|Y_1(\boldsymbol{\theta})\| = 1, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}.$$

Instead of the sharp value of the constant (1.18), which is equal to

$$\delta^{1/2} ((2 + c_1^2 + c_2) \|X_1(\boldsymbol{\theta})\|^2 + C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty})^{-1/2}$$

and depends on $\boldsymbol{\theta}$, we take the following value, which is suitable for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$:

$$(6.6) \quad \tau_0 = \delta^{1/2} ((2 + c_1^2 + c_2) \alpha_1 \|g\|_{L_\infty} + C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty})^{-1/2}.$$

From (6.3), (6.6), and (5.20) it is clear that the number δ is controlled in terms of α_0 , $\|g^{-1}\|_{L_\infty}$, $\tilde{c}$, and r_0 only, while τ_0 depends only on the initial data (5.33) and the parameters of the lattice Γ .

Condition (3.1) is satisfied due to (5.38). Then, by (5.53), the operator $B(t, \varepsilon; \boldsymbol{\theta}) + \varepsilon^2 \lambda Q_0$ satisfies a condition of the form (3.2):

$$(6.7) \quad B(t, \varepsilon; \boldsymbol{\theta}) + \varepsilon^2 \lambda Q_0 \geq \check{c}_*(t^2 + \varepsilon^2)I, \quad \mathbf{k} = t\boldsymbol{\theta} \in \tilde{\Omega}, \quad 0 \leq \varepsilon \leq 1,$$

$$(6.8) \quad \check{c}_* = \min\{\kappa c_*/2, c_4\}.$$

6.3. The operators $Z(\boldsymbol{\theta})$ and $\tilde{Z}$. The operator Z defined in Subsection 1.6 now depends on $\boldsymbol{\theta}$. Let $\mathbf{c} \in \mathfrak{N} = \mathbb{C}^n$. Equation (1.22) takes the form

$$(6.9) \quad b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\mathbf{v}_\theta + b(\boldsymbol{\theta})\mathbf{c}) = 0.$$

Let $\mathbf{v}_\theta \in \tilde{H}^1(\Omega; \mathbb{C}^n)$ be the solution of (6.9) satisfying $\int_\Omega \mathbf{v}_\theta d\mathbf{x} = 0$. (Equation (6.9) is understood in the weak sense.) Then the operator $Z(\boldsymbol{\theta}) : \mathfrak{N} \rightarrow \mathfrak{H}$ takes a vector $\mathbf{c}$ to the element $\mathbf{v}_\theta$, i.e., $Z(\boldsymbol{\theta})\mathbf{c} = \mathbf{v}_\theta$.

We introduce the Γ -periodic $(n \times m)$ -matrix-valued function $\Lambda(\mathbf{x})$ satisfying the equation

$$(6.10) \quad b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0, \quad \int_\Omega \Lambda(\mathbf{x}) d\mathbf{x} = 0.$$

Obviously, $Z(\boldsymbol{\theta})$ is the operator of multiplication by the matrix $\Lambda(\mathbf{x})b(\boldsymbol{\theta})$.

Now we construct the operator $\tilde{Z}$ (see Subsection 1.6). Let $\mathbf{c} \in \mathfrak{N} = \mathbb{C}^n$. Equation (1.24) takes the form

$$(6.11) \quad b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) \mathbf{w} + \sum_{j=1}^d D_j (a_j(\mathbf{x})^* \mathbf{c}) = 0.$$

Let $\mathbf{w} \in \tilde{H}^1(\Omega; \mathbb{C}^n)$ be the (weak) solution of equation (6.11) satisfying $\int_{\Omega} \mathbf{w} d\mathbf{x} = 0$. Then the operator $\tilde{Z} : \mathfrak{N} \rightarrow \mathfrak{H}$ takes a vector $\mathbf{c}$ to the element $\mathbf{w}$, i.e., $\tilde{Z}\mathbf{c} = \mathbf{w}$. Let $\tilde{\Lambda}(\mathbf{x})$ be the periodic $(n \times n)$ -matrix-valued function satisfying the equation

$$(6.12) \quad b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) \tilde{\Lambda}(\mathbf{x}) + \sum_{j=1}^d D_j a_j(\mathbf{x})^* = 0, \quad \int_{\Omega} \tilde{\Lambda}(\mathbf{x}) d\mathbf{x} = 0.$$

Obviously, the operator $\tilde{Z}$ acts as multiplication by the matrix $\tilde{\Lambda}(\mathbf{x})$. Thus,

$$(6.13) \quad Z(\boldsymbol{\theta})\mathbf{c} = \Lambda(\mathbf{x})b(\boldsymbol{\theta})\mathbf{c}, \quad \tilde{Z}\mathbf{c} = \tilde{\Lambda}(\mathbf{x})\mathbf{c}, \quad \mathbf{c} \in \mathfrak{N} = \mathbb{C}^n.$$

6.4. The operator $S(\boldsymbol{\theta})$. The effective matrix. The operator S defined in Subsection 1.7 now depends on $\boldsymbol{\theta}$. In accordance with [BSu2, Chapter 3, §1], the operator $S(\boldsymbol{\theta}) : \mathfrak{N} \rightarrow \mathfrak{N}$ acts as multiplication by the matrix $b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta})$:

$$(6.14) \quad S(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta}), \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}.$$

Here g^0 is a constant positive $(m \times m)$ -matrix called the *effective matrix*. The matrix g^0 can be defined in terms of the matrix $\Lambda(\mathbf{x})$ introduced above:

$$(6.15) \quad g^0 = |\Omega|^{-1} \int_{\Omega} g(\mathbf{x}) (b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) d\mathbf{x}.$$

The effective matrix has the following properties (see [BSu2, Chapter 3, §1]).

Proposition 6.1. *The effective matrix g^0 satisfies the estimates*

$$(6.16) \quad \underline{g} \leq g^0 \leq \bar{g},$$

where

$$\bar{g} = |\Omega|^{-1} \int_{\Omega} g(\mathbf{x}) d\mathbf{x}, \quad \underline{g}^{-1} = |\Omega|^{-1} \int_{\Omega} g(\mathbf{x})^{-1} d\mathbf{x}.$$

If $m = n$, then we always have $g^0 = \underline{g}$.

Estimates (6.16) are known in homogenization theory as the Voigt–Reuss bracketing.

Proposition 6.2. *The identity $g^0 = \bar{g}$ is equivalent to the relations*

$$(6.17) \quad b(\mathbf{D})^* \mathbf{g}_k(\mathbf{x}) = 0, \quad k = 1, \dots, m,$$

where the $\mathbf{g}_k(\mathbf{x})$, $k = 1, \dots, m$, are the columns of the matrix $g(\mathbf{x})$.

Proposition 6.3. *The identity $g^0 = \underline{g}$ is equivalent to the representations*

$$(6.18) \quad \mathbf{l}_k(\mathbf{x}) = \mathbf{l}_k^0 + b(\mathbf{D})\mathbf{w}_k, \quad \mathbf{l}_k^0 \in \mathbb{C}^m, \quad \mathbf{w}_k \in \tilde{H}^1(\Omega; \mathbb{C}^n), \quad k = 1, \dots, m,$$

where the $\mathbf{l}_k(\mathbf{x})$, $k = 1, \dots, m$, are the columns of the matrix $g(\mathbf{x})^{-1}$.

§7. APPROXIMATION FOR THE GENERALIZED RESOLVENT
OF THE OPERATOR $\mathcal{B}(\mathbf{k}, \varepsilon)$

7.1. The operator $L(t, \varepsilon)$. Now, the operator $L(t, \varepsilon)$ defined by (3.26) and acting in the space $\mathfrak{N} = \mathbb{C}^n$ (viewed as a subspace of $L_2(\Omega; \mathbb{C}^n)$) depends on $\boldsymbol{\theta}$ and is represented as

$$(7.1) \quad \begin{aligned} L(t, \varepsilon; \boldsymbol{\theta}) = & t^2 S(\boldsymbol{\theta}) - t\varepsilon (X_0 Z(\boldsymbol{\theta}))^* (X_0 \tilde{Z}) - t\varepsilon (X_0 \tilde{Z})^* (X_0 Z(\boldsymbol{\theta})) + t\varepsilon P(Y_2^* Y_1(\boldsymbol{\theta}) \\ & + Y_1(\boldsymbol{\theta})^* Y_2)|_{\mathfrak{N}} + \varepsilon^2 (- (X_0 \tilde{Z})^* (X_0 \tilde{Z}) + Q_{\mathfrak{N}} + \lambda Q_{0\mathfrak{N}}). \end{aligned}$$

By (6.1) and (6.13), the operator $X_0 Z(\boldsymbol{\theta}) : \mathfrak{N} \rightarrow \mathfrak{H}$ acts as multiplication of a constant vector $\mathbf{c} \in \mathfrak{N}$ by the matrix $h(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}))b(\boldsymbol{\theta})$, and the operator $X_0 \tilde{Z} : \mathfrak{N} \rightarrow \mathfrak{H}$ acts as multiplication by the matrix $h(\mathbf{x})(b(\mathbf{D})\tilde{\Lambda}(\mathbf{x}))$. Then

$$(X_0 Z(\boldsymbol{\theta}))^* (X_0 \tilde{Z}) = Pb(\boldsymbol{\theta})^* (b(\mathbf{D})\Lambda)^* g(b(\mathbf{D})\tilde{\Lambda})|_{\mathfrak{N}};$$

i.e., the operator $(X_0 Z(\boldsymbol{\theta}))^* (X_0 \tilde{Z}) : \mathfrak{N} \rightarrow \mathfrak{N}$ is multiplication by the matrix $b(\boldsymbol{\theta})^* V$, where

$$(7.2) \quad V := |\Omega|^{-1} \int_{\Omega} (b(\mathbf{D})\Lambda(\mathbf{x}))^* g(\mathbf{x})(b(\mathbf{D})\tilde{\Lambda}(\mathbf{x})) d\mathbf{x}.$$

The adjoint operator $(X_0 \tilde{Z})^* (X_0 Z(\boldsymbol{\theta})) : \mathfrak{N} \rightarrow \mathfrak{N}$ corresponds to the matrix $V^* b(\boldsymbol{\theta})$. Finally, $(X_0 \tilde{Z})^* (X_0 \tilde{Z}) : \mathfrak{N} \rightarrow \mathfrak{N}$ acts as multiplication by the matrix

$$(7.3) \quad W := |\Omega|^{-1} \int_{\Omega} (b(\mathbf{D})\tilde{\Lambda}(\mathbf{x}))^* g(\mathbf{x})(b(\mathbf{D})\tilde{\Lambda}(\mathbf{x})) d\mathbf{x}.$$

By (6.2) and (5.45), the operator $P(Y_2^* Y_1(\boldsymbol{\theta}) + Y_1(\boldsymbol{\theta})^* Y_2)|_{\mathfrak{N}}$ acts as multiplication by the matrix

$$(7.4) \quad \sum_{j=1}^d (\overline{a_j + a_j^*}) \theta_j = \sum_{j=1}^d \theta_j |\Omega|^{-1} \int_{\Omega} (a_j(\mathbf{x}) + a_j(\mathbf{x})^*) d\mathbf{x}.$$

Obviously, the operator $Q_{\mathfrak{N}}$ generated by the form $q_{\Omega}[\mathbf{c}, \mathbf{c}]$, $\mathbf{c} \in \mathfrak{N}$, acts as multiplication by the matrix

$$(7.5) \quad \overline{Q} := |\Omega|^{-1} \int_{\Omega} d\mu(\mathbf{x}).$$

Finally, the operator $Q_{0\mathfrak{N}}$ corresponds to the matrix

$$(7.6) \quad \overline{Q}_0 = |\Omega|^{-1} \int_{\Omega} Q_0(\mathbf{x}) d\mathbf{x}.$$

As a result, taking (6.14) into account, we see that the operator (7.1) takes the form

$$(7.7) \quad \begin{aligned} L(t, \varepsilon; \boldsymbol{\theta}) = & t^2 b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta}) + t\varepsilon \left(-b(\boldsymbol{\theta})^* V - V^* b(\boldsymbol{\theta}) + \sum_{j=1}^d (\overline{a_j + a_j^*}) \theta_j \right) \\ & + \varepsilon^2 (-W + \overline{Q} + \lambda \overline{Q}_0). \end{aligned}$$

Since the symbol b is first order homogeneous, we can rewrite (7.7) in terms of the parameter $\mathbf{k} = t\boldsymbol{\theta}$:

$$(7.8) \quad \begin{aligned} L(t, \varepsilon; \boldsymbol{\theta}) = & L(\mathbf{k}, \varepsilon) = b(\mathbf{k})^* g^0 b(\mathbf{k}) + \varepsilon \left(-b(\mathbf{k})^* V - V^* b(\mathbf{k}) + \sum_{j=1}^d (\overline{a_j + a_j^*}) k_j \right) \\ & + \varepsilon^2 (-W + \overline{Q} + \lambda \overline{Q}_0). \end{aligned}$$

By (3.27), the operator $L(\mathbf{k}, \varepsilon)$ satisfies the estimate

$$(7.9) \quad L(\mathbf{k}, \varepsilon) \geq \check{c}_*(|\mathbf{k}|^2 + \varepsilon^2)I_{\mathfrak{H}}.$$

Note that, due to the presence of the projection P , $\mathbf{k}$ can be replaced by $(\mathbf{D} + \mathbf{k})$ in the expression for the operator $L(\mathbf{k}, \varepsilon)P : \mathfrak{H} \rightarrow \mathfrak{H}$ (because $\mathbf{D}P = 0$). We put

$$(7.10) \quad \mathcal{A}^0(\mathbf{k}) = b(\mathbf{D} + \mathbf{k})^* g^0 b(\mathbf{D} + \mathbf{k}),$$

$$(7.11) \quad \mathcal{Y}^0(\mathbf{k}) = -b(\mathbf{D} + \mathbf{k})^* V + \sum_{j=1}^d \overline{a_j}(D_j + k_j),$$

$$(7.12) \quad \mathcal{B}^0(\mathbf{k}, \varepsilon) = \mathcal{A}^0(\mathbf{k}) + \varepsilon(\mathcal{Y}^0(\mathbf{k}) + \mathcal{Y}^0(\mathbf{k})^*) + \varepsilon^2(\overline{Q} - W),$$

$$(7.13) \quad \mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon) = \mathcal{B}^0(\mathbf{k}, \varepsilon) + \varepsilon^2 \lambda \overline{Q}_0.$$

Then

$$(7.14) \quad L(\mathbf{k}, \varepsilon)P = \mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon)P.$$

7.2. Approximation for the generalized resolvent of the operator $\mathcal{B}(\mathbf{k}, \varepsilon)$ for $|\mathbf{k}|^2 + \varepsilon^2 \leq \tau_0^2$. We apply the abstract Theorem 3.1 (precisely, estimate (3.28)) to the operator family $\mathcal{B}_\lambda(\mathbf{k}, \varepsilon) = \mathcal{B}(\mathbf{k}, \varepsilon) + \varepsilon^2 \lambda \overline{Q}_0 = B(t, \varepsilon; \boldsymbol{\theta}) + \varepsilon^2 \lambda \overline{Q}_0$. First, we need to specify the constants in the estimates. Now, the constants occurring in Chapter 1 will depend on $\boldsymbol{\theta}$, but they can be replaced by other constants that are suitable for all $\boldsymbol{\theta}$. Let δ be given by (6.3), τ_0 by (6.6), and $\check{c}_*$ by (6.8). Using (6.3)–(6.5) and (5.9), instead of the sharp values of the constants $C_T^{(1)}$, $C_T^{(2)}$ (see (2.14), (2.15)), which now depend on $\boldsymbol{\theta}$, we take

$$(7.15) \quad C_T^{(1)} = \max\{2 + \alpha_0^{-1} \|g^{-1}\|_{L_\infty}, (\alpha_1 \|g\|_{L_\infty} + C(1)) 4\kappa^{-1} \alpha_0^{-1} \|g^{-1}\|_{L_\infty} r_0^{-2}\},$$

$$(7.16) \quad C_T^{(2)} = \max\{c_2 + 1, (\alpha_1 \|g\|_{L_\infty} + 1 + C(1) + c_3 + |\lambda| \|\overline{Q}_0\|_{L_\infty}) 4\kappa^{-1} \alpha_0^{-1} \|g^{-1}\|_{L_\infty} r_0^{-2}\}.$$

Next, in accordance with (2.17), we put

$$(7.17) \quad C_T = C_T^{(1)} + \tau_0 C_T^{(2)}.$$

Using (2.33) and (2.36), we find

$$(7.18) \quad \check{C}_T = 32 \cdot 13^2 \kappa^{-1/2} (C_T^{(1)})^2 C_T + 32 \cdot 13 \kappa^{-1/2} C_T^{(1)} C_T^{(2)},$$

$$(7.19) \quad C_T^0 = \check{C}_T + 416 \kappa^{-1/2} C_T^{(2)} C_T.$$

Finally, the constants C_1 and C_2 are defined in accordance with (2.46):

$$(7.20) \quad C_1 = 32(1 + \pi^{-1}) \kappa^{-1/2} C_T, \quad C_2 = 2\delta(1 + \pi^{-1}) C_T^0,$$

while C_0 is defined as in (3.6):

$$(7.21) \quad C_0 = \tau_0(3\delta)^{-1} + 2C_1(\check{c}_*)^{-1} + C_2(\check{c}_*)^{-2}.$$

Here δ is defined by (6.3), τ_0 by (6.6), $\check{c}_*$ by (6.8), and C_1, C_2 are defined in accordance with (7.15)–(7.21).

The values of the constants (7.15)–(7.21) are controlled in terms of the “initial data” (5.33) and the parameters of the lattice Γ .

Applying estimate (3.28) and taking (7.14) into account, we arrive at the following estimate:

$$(7.22) \quad \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{-1} - (\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_0(|\mathbf{k}|^2 + \varepsilon^2)^{-1/2}, \quad |\mathbf{k}|^2 + \varepsilon^2 \leq \tau_0^2.$$

7.3. Estimates for $|\mathbf{k}|^2 + \varepsilon^2 > \tau_0^2$ are trivial. By (6.7), we have

$$(7.23) \quad \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{-1}\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq (\check{c}_*)^{-1}(|\mathbf{k}|^2 + \varepsilon^2)^{-1} \leq (\check{c}_*)^{-1} \tau_0^{-1}(|\mathbf{k}|^2 + \varepsilon^2)^{-1/2},$$

$$\mathbf{k} \in \tilde{\Omega}, \quad |\mathbf{k}|^2 + \varepsilon^2 > \tau_0^2.$$

From (7.9) and (7.14) it follows that

$$(7.24) \quad \|(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq (\check{c}_*)^{-1}(|\mathbf{k}|^2 + \varepsilon^2)^{-1},$$

whence

$$(7.25) \quad \|(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq (\check{c}_*)^{-1} \tau_0^{-1}(|\mathbf{k}|^2 + \varepsilon^2)^{-1/2}, \quad \mathbf{k} \in \tilde{\Omega}, \quad |\mathbf{k}|^2 + \varepsilon^2 > \tau_0^2.$$

Now, relations (7.23) and (7.25) imply that

$$(7.26) \quad \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{-1} - (\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq 2(\check{c}_*)^{-1} \tau_0^{-1}(|\mathbf{k}|^2 + \varepsilon^2)^{-1/2},$$

$$\mathbf{k} \in \tilde{\Omega}, \quad |\mathbf{k}|^2 + \varepsilon^2 > \tau_0^2.$$

Combining (7.22) and (7.26), we arrive at the following result.

Theorem 7.1. *Under the assumptions of Subsections 5.1–5.4, let $\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)$ be the operator defined by (5.51), (5.54), and let $\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon)$ be the operator defined as in (7.10)–(7.13). Let P be the orthogonal projection of the space $\mathfrak{H} = L_2(\Omega; \mathbb{C}^n)$ onto the subspace $\mathfrak{N} = \mathbb{C}^n$. Then*

$$(7.27) \quad \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{-1} - (\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq \hat{C}_0(|\mathbf{k}|^2 + \varepsilon^2)^{-1/2}, \quad \mathbf{k} \in \tilde{\Omega}, \quad 0 < \varepsilon \leq 1.$$

The constant

$$\hat{C}_0 = \max\{C_0, 2(\check{c}_*)^{-1} \tau_0^{-1}\}$$

is controlled in terms of the initial data (5.33) and the parameters of the lattice Γ .

7.4. Approximation with corrector for the generalized resolvent of the operator $\mathcal{B}(\mathbf{k}, \varepsilon)$ for $|\mathbf{k}|^2 + \varepsilon^2 \leq \tau_0^2$. Now, we apply Theorem 3.2 (estimate (3.29)) to the operator family $\mathcal{B}(t, \varepsilon; \boldsymbol{\theta}) + \varepsilon^2 \lambda Q_0$. We only need to specify the values of the constants. Let δ be the number defined by (6.3), let $\check{c}_*$ be defined by (6.8), and let C_1, C_2 be defined as in (7.15)–(7.20). By (2.54) and (5.9), the constant C_3 is given by

$$C_3^2 = 2(2 + \alpha_0^{-1} \|g^{-1}\|_{L_\infty} + c_2).$$

Using (2.63), we find C_4 :

$$C_4 = C_3(12\kappa^{-1} + 2)^{1/2}(49C_T^{(1)}C_T + 7C_T^{(2)}),$$

where $C_T^{(1)}, C_T^{(2)}, C_T$ are defined by (7.15)–(7.17).

Finally, the constant C_5 is defined in accordance with (3.14):

$$(7.28) \quad C_5 = (3\delta)^{-1/2} + C_1(\check{c}_*)^{-1/2} + C_2(\check{c}_*)^{-3/2} + C_4(1 + \pi^{-1})\delta^{1/2}(\check{c}_*)^{-1}.$$

We see that C_5 depends only on the initial data (5.33) and the parameters of the lattice Γ .

We apply (3.29) and use relations (7.14), (6.13) and the identity $tZ(\boldsymbol{\theta})P = \Lambda b(\mathbf{k})P = \Lambda b(\mathbf{D} + \mathbf{k})P$, which follows from (6.13). This yields

$$(7.29) \quad \left\| \mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2} \left(\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{-1} - (I + \Lambda b(\mathbf{D} + \mathbf{k}) + \varepsilon \tilde{\Lambda})(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P \right) \right\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_5,$$

$$|\mathbf{k}|^2 + \varepsilon^2 \leq \tau_0^2.$$

7.5. Estimates with corrector for $|\mathbf{k}|^2 + \varepsilon^2 > \tau_0^2$. If $\mathbf{k} \in \tilde{\Omega}$ and $|\mathbf{k}|^2 + \varepsilon^2 > \tau_0^2$, then each term under the norm sign in (7.29) can be estimated separately. By (6.7), we have

$$(7.30) \quad \begin{aligned} \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{-1/2}\|_{\mathfrak{H} \rightarrow \mathfrak{H}} &\leq (\check{c}_*)^{-1/2}(|\mathbf{k}|^2 + \varepsilon^2)^{-1/2} \leq (\check{c}_*)^{-1/2}\tau_0^{-1}, \\ \mathbf{k} &\in \tilde{\Omega}, \quad |\mathbf{k}|^2 + \varepsilon^2 > \tau_0^2. \end{aligned}$$

We estimate the operator $\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2}(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P$:

$$(7.31) \quad \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2}(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \|(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}}.$$

Using (5.55) and the identity $\mathbf{D}P = 0$, we obtain

$$\begin{aligned} \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2}P\mathbf{u}\|_{\mathfrak{H}}^2 &= \mathbf{b}_\lambda(\mathbf{k}, \varepsilon)[P\mathbf{u}, P\mathbf{u}] \\ &\leq ((2 + c_1^2 + c_2)\alpha_1\|g\|_{L_\infty}|\mathbf{k}|^2 + (C(1) + c_3 + |\lambda|\|Q_0\|_{L_\infty})\varepsilon^2) \|P\mathbf{u}\|_{\mathfrak{H}}^2. \end{aligned}$$

Hence,

$$(7.32) \quad \begin{aligned} \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \\ \leq ((2 + c_1^2 + c_2)\alpha_1\|g\|_{L_\infty}|\mathbf{k}|^2 + (C(1) + c_3 + |\lambda|\|Q_0\|_{L_\infty})\varepsilon^2)^{1/2}. \end{aligned}$$

Combining (7.25), (7.31), and (7.32), we obtain the estimate

$$(7.33) \quad \begin{aligned} \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2}(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} &\leq (\check{c}_*)^{-1}\tau_0^{-1}(|\mathbf{k}|^2 + \varepsilon^2)^{-1/2} \\ &\times ((2 + c_1^2 + c_2)\alpha_1\|g\|_{L_\infty}|\mathbf{k}|^2 + (C(1) + c_3 + |\lambda|\|Q_0\|_{L_\infty})\varepsilon^2)^{1/2} \\ &\leq (\check{c}_*)^{-1}\tau_0^{-1}((2 + c_1^2 + c_2)\alpha_1\|g\|_{L_\infty} + C(1) + c_3 + |\lambda|\|Q_0\|_{L_\infty})^{1/2}, \\ &\quad |\mathbf{k}|^2 + \varepsilon^2 > \tau_0^2. \end{aligned}$$

Now we estimate the operator $\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2}(\Lambda b(\mathbf{D} + \mathbf{k}) + \varepsilon\tilde{\Lambda})(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P$:

$$(7.34) \quad \begin{aligned} \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2}(\Lambda b(\mathbf{D} + \mathbf{k}) + \varepsilon\tilde{\Lambda})(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \\ \leq \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2}\Lambda P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \|b(\mathbf{D} + \mathbf{k})(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \\ + \varepsilon\|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2}\tilde{\Lambda}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \|(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}}. \end{aligned}$$

The norm of the operator $(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P$ was estimated in (7.24). From (7.24) and (5.2) it follows that

$$(7.35) \quad \begin{aligned} \|b(\mathbf{D} + \mathbf{k})(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \\ = \|b(\mathbf{k})(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq \alpha_1^{1/2}|\mathbf{k}|(\check{c}_*)^{-1}(|\mathbf{k}|^2 + \varepsilon^2)^{-1}. \end{aligned}$$

It remains to estimate the operators $\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2}\Lambda P$ and $\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2}\tilde{\Lambda}P$.

Lemma 7.2. *We have*

$$(7.36) \quad \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2}\Lambda P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_\Lambda(\mathbf{k}, \varepsilon),$$

where $C_\Lambda(\mathbf{k}, \varepsilon)$ is defined below in (7.44).

Proof. By (6.10),

$$(7.37) \quad \|g^{1/2}b(\mathbf{D})\Lambda\|_{L_2(\Omega)} \leq m^{1/2}\|g\|_{L_\infty}^{1/2}|\Omega|^{1/2}.$$

Then, using (5.2) and the discrete Fourier transformation, we obtain

$$\|\mathbf{D}\Lambda\|_{L_2(\Omega)} \leq \alpha_0^{-1/2}\|b(\mathbf{D})\Lambda\|_{L_2(\Omega)} \leq m^{1/2}\alpha_0^{-1/2}\|g^{-1}\|_{L_\infty}^{1/2}\|g\|_{L_\infty}^{1/2}|\Omega|^{1/2}.$$

Since Λ is a periodic matrix-valued function with zero mean, we can apply the Fourier series expansion and identity (4.2), obtaining

$$(7.38) \quad \|\Lambda\|_{L_2(\Omega)} \leq (2r_0)^{-1}\|\mathbf{D}\Lambda\|_{L_2(\Omega)} \leq m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\|g^{-1}\|_{L_\infty}^{1/2}\|g\|_{L_\infty}^{1/2}|\Omega|^{1/2}.$$

Next, by (5.2), we have

$$(7.39) \quad \|g^{1/2}b(\mathbf{k})\Lambda\|_{L_2(\Omega)} \leq \|g\|_{L_\infty}^{1/2} \alpha_1^{1/2} |\mathbf{k}| \|\Lambda\|_{L_2(\Omega)}.$$

From (7.37)–(7.39) it follows that

$$(7.40) \quad \begin{aligned} & \|g^{1/2}b(\mathbf{D} + \mathbf{k})\Lambda\|_{L_2(\Omega)} \\ & \leq m^{1/2} \|g\|_{L_\infty}^{1/2} |\Omega|^{1/2} + |\mathbf{k}| m^{1/2} (2r_0)^{-1} \alpha_1^{1/2} \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2} \|g\|_{L_\infty} |\Omega|^{1/2}. \end{aligned}$$

By (5.55), we have

$$(7.41) \quad \begin{aligned} \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2} \Lambda P \mathbf{u}\|_{\mathfrak{H}}^2 &= \mathbf{b}_\lambda(\mathbf{k}, \varepsilon) [\Lambda P \mathbf{u}, \Lambda P \mathbf{u}] \\ &\leq (2 + c_1^2 + c_2) \|g^{1/2}b(\mathbf{D} + \mathbf{k})\Lambda P \mathbf{u}\|_{\mathfrak{H}_*}^2 \\ &\quad + \varepsilon^2 (C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty}) \|\Lambda P \mathbf{u}\|_{L_2(\Omega)}^2. \end{aligned}$$

The first term on the right-hand side of (7.41) is estimated by using (7.40):

$$(7.42) \quad \begin{aligned} & \|g^{1/2}b(\mathbf{D} + \mathbf{k})\Lambda P \mathbf{u}\|_{\mathfrak{H}_*}^2 \\ & \leq \|g^{1/2}b(\mathbf{D} + \mathbf{k})\Lambda\|_{L_2(\Omega)}^2 \|P \mathbf{u}\|^2 = \|g^{1/2}b(\mathbf{D} + \mathbf{k})\Lambda\|_{L_2(\Omega)}^2 |\Omega|^{-1} \|P \mathbf{u}\|_{\mathfrak{H}}^2 \\ & \leq m (\|g\|_{L_\infty}^{1/2} + |\mathbf{k}| (2r_0)^{-1} \alpha_1^{1/2} \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2} \|g\|_{L_\infty})^2 \|P \mathbf{u}\|_{\mathfrak{H}}^2. \end{aligned}$$

The second term on the right-hand side of (7.41) can be estimated with the help of (7.38). We have

$$(7.43) \quad \|\Lambda P \mathbf{u}\|_{L_2(\Omega)}^2 = \|\Lambda\|_{L_2(\Omega)}^2 |\Omega|^{-1} \|P \mathbf{u}\|_{\mathfrak{H}}^2 \leq m (2r_0)^{-2} \alpha_0^{-1} \|g^{-1}\|_{L_\infty} \|g\|_{L_\infty} \|P \mathbf{u}\|_{\mathfrak{H}}^2.$$

Now, relations (7.41)–(7.43) imply (7.36) with

$$(7.44) \quad \begin{aligned} C_\Lambda(\mathbf{k}, \varepsilon)^2 &= (2 + c_1^2 + c_2) m (\|g\|_{L_\infty}^{1/2} + |\mathbf{k}| (2r_0)^{-1} \alpha_1^{1/2} \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2} \|g\|_{L_\infty})^2 \\ &\quad + \varepsilon^2 (C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty}) m (2r_0)^{-2} \alpha_0^{-1} \|g^{-1}\|_{L_\infty} \|g\|_{L_\infty}. \end{aligned} \quad \square$$

From (7.35) and (7.36) it follows that

$$(7.45) \quad \begin{aligned} & \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2} \Lambda P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \|b(\mathbf{D} + \mathbf{k})(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1} P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \\ & \leq C_\Lambda(\mathbf{k}, \varepsilon) \alpha_1^{1/2} (\check{c}_*)^{-1} |\mathbf{k}| (|\mathbf{k}|^2 + \varepsilon^2)^{-1} \leq \hat{C}_\Lambda, \quad |\mathbf{k}|^2 + \varepsilon^2 > \tau_0^2, \end{aligned}$$

where

$$(7.46) \quad \begin{aligned} \hat{C}_\Lambda^2 &= \alpha_1 (\check{c}_*)^{-2} (2 + c_1^2 + c_2) m (\|g\|_{L_\infty}^{1/2} \tau_0^{-1} + (2r_0)^{-1} \alpha_1^{1/2} \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2} \|g\|_{L_\infty})^2 \\ &\quad + \alpha_1 (\check{c}_*)^{-2} (C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty}) m (4r_0)^{-2} \alpha_0^{-1} \|g^{-1}\|_{L_\infty} \|g\|_{L_\infty}. \end{aligned}$$

Lemma 7.3. *We have*

$$(7.47) \quad \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2} \tilde{\Lambda} P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_{\tilde{\Lambda}}(\mathbf{k}, \varepsilon),$$

where $C_{\tilde{\Lambda}}(\mathbf{k}, \varepsilon)$ is defined below in (7.58).

Proof. By (6.12),

$$(7.48) \quad \|g^{1/2}b(\mathbf{D})\tilde{\Lambda}\|_{L_2(\Omega)}^2 \leq C_a n^{1/2} \|\mathbf{D}\tilde{\Lambda}\|_{L_2(\Omega)},$$

where

$$C_a^2 = \sum_{j=1}^d \int_{\Omega} |a_j(\mathbf{x})|^2 d\mathbf{x}.$$

Using (5.2) and the Fourier series expansion, we see that

$$(7.49) \quad \|\mathbf{D}\tilde{\Lambda}\|_{L_2(\Omega)} \leq \alpha_0^{-1/2} \|b(\mathbf{D})\tilde{\Lambda}\|_{L_2(\Omega)} \leq \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2} \|g^{1/2}b(\mathbf{D})\tilde{\Lambda}\|_{L_2(\Omega)}.$$

Inequalities (7.48) and (7.49) show that

$$(7.50) \quad \|g^{1/2}b(\mathbf{D})\tilde{\Lambda}\|_{L_2(\Omega)} \leq C_a n^{1/2} \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2}.$$

By (7.49) and (7.50), we have

$$(7.51) \quad \|\mathbf{D}\tilde{\Lambda}\|_{L_2(\Omega)} \leq C_a n^{1/2} \alpha_0^{-1} \|g^{-1}\|_{L_\infty}.$$

Since $\tilde{\Lambda}$ is a periodic matrix-valued function with zero mean, we can use the Fourier series expansion to obtain

$$(7.52) \quad \|\tilde{\Lambda}\|_{L_2(\Omega)} \leq (2r_0)^{-1} \|\mathbf{D}\tilde{\Lambda}\|_{L_2(\Omega)} \leq (2r_0)^{-1} C_a n^{1/2} \alpha_0^{-1} \|g^{-1}\|_{L_\infty}.$$

Next, by (5.2),

$$(7.53) \quad \|g^{1/2}b(\mathbf{k})\tilde{\Lambda}\|_{L_2(\Omega)} \leq \|g\|_{L_\infty}^{1/2} \alpha_1^{1/2} |\mathbf{k}| \|\tilde{\Lambda}\|_{L_2(\Omega)}.$$

From (7.50), (7.52), and (7.53) it follows that

$$(7.54) \quad \begin{aligned} & \|g^{1/2}b(\mathbf{D} + \mathbf{k})\tilde{\Lambda}\|_{L_2(\Omega)} \\ & \leq C_a n^{1/2} \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2} + |\mathbf{k}| (2r_0)^{-1} C_a n^{1/2} \alpha_0^{-1} \alpha_1^{1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}. \end{aligned}$$

Using (5.55), we get

$$(7.55) \quad \begin{aligned} \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2} \tilde{\Lambda} P \mathbf{u}\|_{\mathfrak{H}}^2 &= \mathfrak{b}_\lambda(\mathbf{k}, \varepsilon) [\tilde{\Lambda} P \mathbf{u}, \tilde{\Lambda} P \mathbf{u}] \\ &\leq (2 + c_1^2 + c_2) \|g^{1/2}b(\mathbf{D} + \mathbf{k})\tilde{\Lambda} P \mathbf{u}\|_{\mathfrak{H}_*}^2 \\ &\quad + \varepsilon^2 (C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty}) \|\tilde{\Lambda} P \mathbf{u}\|_{L_2(\Omega)}^2. \end{aligned}$$

The first term on the right-hand side of (7.55) is estimated by using (7.54):

$$(7.56) \quad \begin{aligned} & \|g^{1/2}b(\mathbf{D} + \mathbf{k})\tilde{\Lambda} P \mathbf{u}\|_{\mathfrak{H}_*}^2 \\ & \leq n \left(C_a \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2} + |\mathbf{k}| (2r_0)^{-1} C_a \alpha_0^{-1} \alpha_1^{1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty} \right)^2 |\Omega|^{-1} \|P \mathbf{u}\|_{\mathfrak{H}}^2. \end{aligned}$$

By (7.52), for the second term we have

$$(7.57) \quad \|\tilde{\Lambda} P \mathbf{u}\|_{L_2(\Omega)}^2 \leq (2r_0)^{-2} C_a^2 n \alpha_0^{-2} \|g^{-1}\|_{L_\infty}^2 |\Omega|^{-1} \|P \mathbf{u}\|_{\mathfrak{H}}^2.$$

Now, relations (7.55)–(7.57) imply estimate (7.47) with

$$(7.58) \quad \begin{aligned} C_{\tilde{\Lambda}}(\mathbf{k}, \varepsilon)^2 &= (2 + c_1^2 + c_2) n (C_a \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2} \\ &\quad + |\mathbf{k}| (2r_0)^{-1} C_a \alpha_0^{-1} \alpha_1^{1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty})^2 |\Omega|^{-1} \\ &\quad + \varepsilon^2 (C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty}) (2r_0)^{-2} C_a^2 n \alpha_0^{-2} \|g^{-1}\|_{L_\infty}^2 |\Omega|^{-1}. \quad \square \end{aligned}$$

From (7.24) and (7.47) it follows that

$$(7.59) \quad \begin{aligned} & \varepsilon \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2} \tilde{\Lambda} P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \|(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1} P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \\ & \leq \varepsilon C_{\tilde{\Lambda}}(\mathbf{k}, \varepsilon) (\tilde{c}_*)^{-1} (|\mathbf{k}|^2 + \varepsilon^2)^{-1} \leq \hat{C}_{\tilde{\Lambda}}, \quad |\mathbf{k}|^2 + \varepsilon^2 > \tau_0^2, \end{aligned}$$

where

$$(7.60) \quad \begin{aligned} (\hat{C}_{\tilde{\Lambda}})^2 &= (\tilde{c}_*)^{-2} (2 + c_1^2 + c_2) n C_a^2 \alpha_0^{-1} \|g^{-1}\|_{L_\infty} \\ &\quad \times \left(\tau_0^{-1} + (4r_0)^{-1} \alpha_0^{-1/2} \alpha_1^{1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2} \right)^2 |\Omega|^{-1} \\ &\quad + (\tilde{c}_*)^{-2} (C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty}) (2r_0)^{-2} C_a^2 n \alpha_0^{-2} \|g^{-1}\|_{L_\infty}^2 |\Omega|^{-1}. \end{aligned}$$

We summarize the results. Relations (7.34), (7.45), and (7.59) imply that

$$(7.61) \quad \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2} (\Lambda b(\mathbf{D} + \mathbf{k}) + \varepsilon \tilde{\Lambda}) (\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1} P\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq \hat{C}_\Lambda + \hat{C}_{\tilde{\Lambda}}, \quad |\mathbf{k}|^2 + \varepsilon^2 > \tau_0^2.$$

Combining (7.30), (7.33), and (7.61), we obtain the following estimate for the operator under the norm sign in (7.29) for $|\mathbf{k}|^2 + \varepsilon^2 > \tau_0^2$:

$$(7.62) \quad \left\| \mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2} \left(\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{-1} - (I + \Lambda b(\mathbf{D} + \mathbf{k}) + \varepsilon \tilde{\Lambda})(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1} P \right) \right\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_6, \\ \mathbf{k} \in \tilde{\Omega}, \quad |\mathbf{k}|^2 + \varepsilon^2 > \tau_0^2,$$

where

$$(7.63) \quad C_6 = (\check{c}_*)^{-1/2} \tau_0^{-1} \\ + (\check{c}_*)^{-1} \tau_0^{-1} \left((2 + c_1^2 + c_2) \alpha_1 \|g\|_{L_\infty} + C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty} \right)^{1/2} \\ + \hat{C}_\Lambda + \hat{C}_{\tilde{\Lambda}}.$$

Here τ_0 is defined by (6.6), $\check{c}_*$ by (6.8), $\hat{C}_\Lambda$ by (7.46), and $\hat{C}_{\tilde{\Lambda}}$ by (7.60). Clearly, the constant C_6 depends only on the initial data (5.33) and the parameters of the lattice Γ .

Relations (7.29) and (7.62) imply the following result.

Theorem 7.4. *Under the assumptions of Theorem 7.1, suppose that the matrix-valued function $\Lambda(\mathbf{x})$ is the periodic solution of problem (6.10) and that $\tilde{\Lambda}(\mathbf{x})$ is the periodic solution of problem (6.12). Then for $\mathbf{k} \in \tilde{\Omega}$ and $0 < \varepsilon \leq 1$ we have*

$$(7.64) \quad \left\| \mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{1/2} \left(\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{-1} - (I + \Lambda b(\mathbf{D} + \mathbf{k}) + \varepsilon \tilde{\Lambda})(\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1} P \right) \right\|_{\mathfrak{H} \rightarrow \mathfrak{H}} \leq C_7,$$

where $C_7 = \max\{C_5, C_6\}$, the constant C_5 is defined by (7.28), and C_6 is defined by (7.63). The constant C_7 depends only on the initial data (5.33) and the parameters of the lattice Γ .

§8. APPROXIMATION FOR THE GENERALIZED RESOLVENT OF THE OPERATOR $\mathcal{B}(\varepsilon)$

8.1. The leading term of approximation for the generalized resolvent of the operator $\mathcal{B}(\varepsilon)$. We return to the study of the operator pencil $\mathcal{B}(\varepsilon)$ acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$; these operators were introduced in Subsection 5.4. In accordance with [BSu2, Chapter 3, §1], the operator

$$(8.1) \quad \mathcal{A}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$$

is called the *effective* operator for $\mathcal{A} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$. The effective matrix g^0 is defined by (6.15). Next, we put

$$(8.2) \quad \mathcal{Y}^0 = -b(\mathbf{D})^* V + \sum_{j=1}^d \overline{a_j} D_j,$$

where the matrix V is defined by (7.2), and $\overline{a_j}$ is the mean value of a_j over the cell Ω . Let W be the matrix defined by (7.3). The matrices $\overline{Q}$ and $\overline{Q}_0$ are defined by (7.5) and (7.6), respectively. In the space $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we consider the operator

$$(8.3) \quad \mathcal{B}^0(\varepsilon) = \mathcal{A}^0 + \varepsilon(\mathcal{Y}^0 + (\mathcal{Y}^0)^*) + \varepsilon^2(\overline{Q} - W).$$

The operator

$$(8.4) \quad \mathcal{B}_\lambda^0(\varepsilon) := \mathcal{B}^0(\varepsilon) + \varepsilon^2 \lambda \overline{Q}_0$$

is the second order DO with constant coefficients and with the symbol $L(\boldsymbol{\xi}, \varepsilon)$ (see (7.8)):

$$(8.5) \quad L(\boldsymbol{\xi}, \varepsilon) = b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) + \varepsilon \left(-b(\boldsymbol{\xi})^* V - V^* b(\boldsymbol{\xi}) + \sum_{j=1}^d (\overline{a_j} + \overline{a_j^*}) \xi_j \right) \\ + \varepsilon^2 (-W + \overline{Q} + \lambda \overline{Q}_0).$$

By (7.9), the matrix $L(\boldsymbol{\xi}, \varepsilon)$ satisfies the estimate

$$(8.6) \quad L(\boldsymbol{\xi}, \varepsilon) \geq \check{c}_*(|\boldsymbol{\xi}|^2 + \varepsilon^2)\mathbf{1}_n.$$

With the help of the Gelfand transformation (see (5.56)), the operator $\mathcal{B}_\lambda(\varepsilon)$ expands in the direct integral of the operators $\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)$. A similar expansion is valid for the operator $\mathcal{B}_\lambda^0(\varepsilon)$.

We put

$$(8.7) \quad \Pi = \mathcal{U}^{-1}[P]\mathcal{U},$$

where $[P]$ is the operator in $\mathcal{H}$ (see (4.7)) that acts layerwise as the operator P of averaging over the cell Ω . It is easily seen (see [BSu4, §6]) that the operator (8.7) is a pseudodifferential operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ with the symbol $\chi_{\tilde{\Omega}}(\boldsymbol{\xi})$, where $\chi_{\tilde{\Omega}}(\boldsymbol{\xi})$ is the characteristic function of the set $\tilde{\Omega}$. Thus,

$$(8.8) \quad (\Pi \mathbf{f})(\mathbf{x}) = (2\pi)^{-d/2} \int_{\tilde{\Omega}} e^{i(\mathbf{x}, \boldsymbol{\xi})} \hat{\mathbf{f}}(\boldsymbol{\xi}) d\boldsymbol{\xi},$$

where $\hat{\mathbf{f}}(\boldsymbol{\xi})$ is the Fourier image of $\mathbf{f}(\mathbf{x})$.

So, under the Gelfand transformation, the operator $\mathcal{B}_\lambda(\varepsilon)^{-1} - (\mathcal{B}_\lambda^0(\varepsilon))^{-1}\Pi$ expands in the direct integral of operators standing under the norm sign in (7.27). Since the Gelfand transformation is unitary, we obtain

$$\|\mathcal{B}_\lambda(\varepsilon)^{-1} - (\mathcal{B}_\lambda^0(\varepsilon))^{-1}\Pi\|_{L_2(\mathbb{R}^d; \mathbb{C}^n) \rightarrow L_2(\mathbb{R}^d; \mathbb{C}^n)} = \sup_{\mathbf{k} \in \tilde{\Omega}} \|\mathcal{B}_\lambda(\mathbf{k}, \varepsilon)^{-1} - (\mathcal{B}_\lambda^0(\mathbf{k}, \varepsilon))^{-1}P\|_{\mathfrak{H} \rightarrow \mathfrak{H}}.$$

By (7.27), this implies that

$$(8.9) \quad \|\mathcal{B}_\lambda(\varepsilon)^{-1} - (\mathcal{B}_\lambda^0(\varepsilon))^{-1}\Pi\|_{L_2(\mathbb{R}^d; \mathbb{C}^n) \rightarrow L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq \hat{C}_0 \varepsilon^{-1}, \quad 0 < \varepsilon \leq 1.$$

We want to show that in (8.9) the pseudodifferential operator Π can be replaced by the identity operator I . For this, we estimate the norm of the operator

$$(8.10) \quad \Sigma(\varepsilon) := (\mathcal{B}_\lambda^0(\varepsilon))^{-1}(I - \Pi).$$

The operator (8.10) is a pseudodifferential operator with the symbol $L(\boldsymbol{\xi}, \varepsilon)^{-1}(1 - \chi_{\tilde{\Omega}}(\boldsymbol{\xi}))$, where the matrix $L(\boldsymbol{\xi}, \varepsilon)$ is defined by (8.5) and satisfies estimate (8.6). We have

$$(8.11) \quad \begin{aligned} \varepsilon \|\Sigma(\varepsilon)\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} &= \varepsilon \sup_{\boldsymbol{\xi} \in \mathbb{R}^d} |L(\boldsymbol{\xi}, \varepsilon)^{-1}(1 - \chi_{\tilde{\Omega}}(\boldsymbol{\xi}))| \\ &\leq \sup_{|\boldsymbol{\xi}| \geq r_0} (\check{c}_*)^{-1} \varepsilon (|\boldsymbol{\xi}|^2 + \varepsilon^2)^{-1} \leq (\check{c}_*)^{-1} r_0^{-1}. \end{aligned}$$

Inequalities (8.9) and (8.11) imply the following result.

Theorem 8.1. *Under the assumptions of Subsections 5.1–5.4, let $\mathcal{B}_\lambda(\varepsilon)$ be the operator defined by (5.26), (5.29), and let $\mathcal{B}_\lambda^0(\varepsilon)$ be the operator defined by (8.3), (8.4). Then*

$$(8.12) \quad \|\mathcal{B}_\lambda(\varepsilon)^{-1} - (\mathcal{B}_\lambda^0(\varepsilon))^{-1}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n) \rightarrow L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq \check{C}_0 \varepsilon^{-1}, \quad 0 < \varepsilon \leq 1,$$

where $\check{C}_0 = \hat{C}_0 + (\check{c}_*)^{-1} r_0^{-1}$. The constant $\check{C}_0$ depends only on the initial data (5.33) and the parameters of the lattice Γ .

8.2. Approximation with corrector for the generalized resolvent of the operator $\mathcal{B}(\varepsilon)$. Now we obtain a sharper approximation for the operator $\mathcal{B}_\lambda(\varepsilon)^{-1}$ on the basis of Theorem 7.4. Note that the operator $b(\mathbf{D})$ acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ expands in the direct integral of operators $b(\mathbf{D} + \mathbf{k})$. Under the Gelfand transformation, the operator of multiplication by the periodic matrix-valued function $\Lambda(\mathbf{x})$ ($\tilde{\Lambda}(\mathbf{x})$, respectively) turns into the operator of multiplication by the same matrix $\Lambda(\mathbf{x})$ ($\tilde{\Lambda}(\mathbf{x})$, respectively) on the

fibers of the direct integral $\mathcal{H}$ (see (4.7)). Thus, under the Gelfand transformation, the operator

$$\mathcal{B}_\lambda(\varepsilon)^{1/2} \left(\mathcal{B}_\lambda(\varepsilon)^{-1} - (I + \Lambda b(\mathbf{D}) + \varepsilon \tilde{\Lambda})(\mathcal{B}_\lambda^0(\varepsilon))^{-1} \Pi \right)$$

expands in the direct integral of operators standing under the norm sign in (7.64). Since the Gelfand transformation is unitary, estimate (7.64) yields

$$(8.13) \quad \left\| \mathcal{B}_\lambda(\varepsilon)^{1/2} \left(\mathcal{B}_\lambda(\varepsilon)^{-1} - (I + \Lambda b(\mathbf{D}) + \varepsilon \tilde{\Lambda})(\mathcal{B}_\lambda^0(\varepsilon))^{-1} \Pi \right) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_7, \\ 0 < \varepsilon \leq 1.$$

8.3. Elimination of the operator Π from the leading term. We show that in the leading term of approximation the pseudodifferential operator Π can be replaced by the identity operator I . For this, we estimate the norm of the operator

$$\mathcal{B}_\lambda(\varepsilon)^{1/2} (\mathcal{B}_\lambda^0(\varepsilon))^{-1} (I - \Pi) = \mathcal{B}_\lambda(\varepsilon)^{1/2} \Sigma(\varepsilon).$$

By (5.30),

$$(8.14) \quad \begin{aligned} \|\mathcal{B}_\lambda(\varepsilon)^{1/2} \Sigma(\varepsilon) \mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 &= \mathbf{b}_\lambda(\varepsilon) [\Sigma(\varepsilon) \mathbf{u}, \Sigma(\varepsilon) \mathbf{u}] \\ &\leq (2 + c_1^2 + c_2) \alpha_1 \|g\|_{L_\infty} \|\mathbf{D} \Sigma(\varepsilon) \mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 \\ &\quad + \varepsilon^2 (C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty}) \|\Sigma(\varepsilon) \mathbf{u}\|_{L_2(\mathbb{R}^d)}^2. \end{aligned}$$

As in (8.11), by (8.6) we have

$$(8.15) \quad \|\mathbf{D} \Sigma(\varepsilon)\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \sup_{|\boldsymbol{\xi}| \geq r_0} (\check{c}_*)^{-1} |\boldsymbol{\xi}| (|\boldsymbol{\xi}|^2 + \varepsilon^2)^{-1} \leq (\check{c}_*)^{-1} r_0^{-1}.$$

Substituting (8.11) and (8.15) into (8.14), we obtain

$$(8.16) \quad \begin{aligned} \|\mathcal{B}_\lambda(\varepsilon)^{1/2} \Sigma(\varepsilon)\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)}^2 &\leq (\check{c}_*)^{-2} r_0^{-2} ((2 + c_1^2 + c_2) \alpha_1 \|g\|_{L_\infty} + C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty}). \end{aligned}$$

Relations (8.13) and (8.16) imply the following statement.

Theorem 8.2. *Under the assumptions of Theorem 8.1, let the matrix $\Lambda(\mathbf{x})$ be the Γ -periodic solution of problem (6.10), and let $\tilde{\Lambda}(\mathbf{x})$ be the Γ -periodic solution of problem (6.12). Let Π be the operator (8.8). Then for $0 < \varepsilon \leq 1$ we have*

$$(8.17) \quad \left\| \mathcal{B}_\lambda(\varepsilon)^{1/2} \left(\mathcal{B}_\lambda(\varepsilon)^{-1} - (I + \Lambda b(\mathbf{D}) \Pi + \varepsilon \tilde{\Lambda} \Pi)(\mathcal{B}_\lambda^0(\varepsilon))^{-1} \right) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \tilde{C}_7,$$

where

$$\tilde{C}_7 = C_7 + (\check{c}_*)^{-1} r_0^{-1} ((2 + c_1^2 + c_2) \alpha_1 \|g\|_{L_\infty} + C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty})^{1/2}.$$

The constant $\tilde{C}_7$ depends only on the initial data (5.33) and the parameters of the lattice Γ .

8.4. On the possibility of elimination of the operator Π from the corrector. **Sufficient conditions on Λ .** In order to eliminate the pseudodifferential operator Π from the corrector, we need to make additional assumptions on $\Lambda(\mathbf{x})$ and $\tilde{\Lambda}(\mathbf{x})$. In some cases, these assumptions are valid automatically. In order to eliminate Π from the term of the corrector containing Λ , it suffices to impose the following condition.

Condition 8.3. Suppose that the matrix $\Lambda(\mathbf{x})$ that is the periodic solution of problem (6.10) is bounded:

$$(8.18) \quad \Lambda \in L_\infty.$$

Let $[\Lambda]$ denote the operator of multiplication by $\Lambda(\mathbf{x})$ acting from $L_2(\mathbb{R}^d; \mathbb{C}^m)$ to $L_2(\mathbb{R}^d; \mathbb{C}^n)$. The following statement was proved in [BSu5, Lemma 8.3].

Proposition 8.4. *Suppose that Condition 8.3 is satisfied. Then the operator $g^{1/2}b(\mathbf{D})[\Lambda]$ is a continuous mapping of $H^1(\mathbb{R}^d; \mathbb{C}^m)$ to $L_2(\mathbb{R}^d; \mathbb{C}^m)$, and*

$$(8.19) \quad \|g^{1/2}b(\mathbf{D})[\Lambda]\|_{H^1(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \mathcal{C}_\Lambda,$$

where $\mathcal{C}_\Lambda$ depends only on $d, m, \alpha_1, \|g\|_{L_\infty}$, and $\|\Lambda\|_{L_\infty}$.

The following statement is a consequence of Proposition 8.4.

Proposition 8.5. *Under Condition 8.3, we have*

$$(8.20) \quad \|\mathcal{B}_\lambda(\varepsilon)^{1/2}\Lambda b(\mathbf{D})\Sigma(\varepsilon)\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C'_\Lambda, \quad 0 < \varepsilon \leq 1,$$

where

$$(C'_\Lambda)^2 = \mathcal{C}_\Lambda^2(2 + c_1^2 + c_2)\alpha_1(\check{c}_*)^{-2}(1 + r_0^{-2}) + (C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty})\|\Lambda\|_{L_\infty}^2\alpha_1(2\check{c}_*)^{-2}.$$

Proof. By (5.30), for $\mathbf{v} \in H^1(\mathbb{R}^d; \mathbb{C}^m)$ we have

$$\begin{aligned} \|\mathcal{B}_\lambda(\varepsilon)^{1/2}\Lambda\mathbf{v}\|_{L_2(\mathbb{R}^d)}^2 &= \mathbf{b}_\lambda(\varepsilon)[\Lambda\mathbf{v}, \Lambda\mathbf{v}] \leq (2 + c_1^2 + c_2)\|g^{1/2}b(\mathbf{D})\Lambda\mathbf{v}\|_{L_2(\mathbb{R}^d)}^2 \\ &\quad + \varepsilon^2(C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty})\|\Lambda\mathbf{v}\|_{L_2(\mathbb{R}^d)}^2. \end{aligned}$$

By (8.18) and (8.19), this implies that

$$(8.21) \quad \begin{aligned} &\|\mathcal{B}_\lambda(\varepsilon)^{1/2}\Lambda\mathbf{v}\|_{L_2}^2 \\ &\leq (2 + c_1^2 + c_2)\mathcal{C}_\Lambda^2\|\mathbf{v}\|_{H^1}^2 + \varepsilon^2(C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty})\|\Lambda\|_{L_\infty}^2\|\mathbf{v}\|_{L_2}^2. \end{aligned}$$

We substitute $\mathbf{v} = b(\mathbf{D})\Sigma(\varepsilon)\mathbf{u}$ in (8.21), where $\mathbf{u} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$. Using (8.6) and (5.2), we obtain

$$(8.22) \quad \begin{aligned} \|\mathbf{v}\|_{H^1} &= \|b(\mathbf{D})\Sigma(\varepsilon)\mathbf{u}\|_{H^1} \\ &\leq \sup_{\boldsymbol{\xi} \in \mathbb{R}^d} (1 + |\boldsymbol{\xi}|^2)^{1/2} |b(\boldsymbol{\xi})L(\boldsymbol{\xi}, \varepsilon)^{-1}| (1 - \chi_{\tilde{\Omega}}(\boldsymbol{\xi})) \|\mathbf{u}\|_{L_2} \\ &\leq (\check{c}_*)^{-1} \alpha_1^{1/2} \sup_{|\boldsymbol{\xi}| \geq r_0} (1 + |\boldsymbol{\xi}|^2)^{1/2} |\boldsymbol{\xi}| (|\boldsymbol{\xi}|^2 + \varepsilon^2)^{-1} \|\mathbf{u}\|_{L_2} \\ &\leq (\check{c}_*)^{-1} \alpha_1^{1/2} (1 + r_0^{-2})^{1/2} \|\mathbf{u}\|_{L_2}. \end{aligned}$$

Similarly,

$$(8.23) \quad \begin{aligned} \varepsilon\|\mathbf{v}\|_{L_2} &= \varepsilon\|b(\mathbf{D})\Sigma(\varepsilon)\mathbf{u}\|_{L_2} \\ &\leq (\check{c}_*)^{-1} \alpha_1^{1/2} \sup_{|\boldsymbol{\xi}| \geq r_0} \varepsilon |\boldsymbol{\xi}| (|\boldsymbol{\xi}|^2 + \varepsilon^2)^{-1} \|\mathbf{u}\|_{L_2} \leq (2\check{c}_*)^{-1} \alpha_1^{1/2} \|\mathbf{u}\|_{L_2}. \end{aligned}$$

Relations (8.21)–(8.23) imply inequality (8.20). $\square$

Proposition 8.5 makes it possible to replace the operator Π by the identity operator in the term of the corrector containing Λ , provided Condition 8.3 is satisfied. In its turn, in [BSu5, Lemma 8.7], it was shown that Condition 8.3 is satisfied *a fortiori* if the following condition is fulfilled.

Condition 8.6. Suppose that at least one of the following assumptions is satisfied:

- 1° $d \leq 2$;
- 2° the dimension d is arbitrary, and the operator $\mathcal{A}$ is of the form $\mathcal{A} = \mathbf{D}^*g(\mathbf{x})\mathbf{D}$, where the matrix $g(\mathbf{x})$ has real entries;
- 3° the dimension d is arbitrary, and the effective matrix is of the form $g^0 = \underline{g}$ (which is equivalent to the representations (6.18)).

8.5. On the possibility of elimination of the operator Π from the corrector.
Sufficient conditions on $\tilde{\Lambda}$. In order to eliminate the operator Π from the term of the corrector containing $\tilde{\Lambda}$, it suffices to impose the following condition.

Condition 8.7. Suppose that the matrix-valued function $\tilde{\Lambda}(\mathbf{x})$ that is the periodic solution of problem (6.12) is such that

$$(8.24) \quad \tilde{\Lambda} \in L_p(\Omega), \quad p = 2 \text{ for } d = 1, \quad p > 2 \text{ for } d = 2, \quad p = d \text{ for } d \geq 3.$$

Proposition 8.8. Under Condition 8.7, the operator $g^{1/2}b(\mathbf{D})[\tilde{\Lambda}]$ is a continuous mapping of $H^2(\mathbb{R}^d; \mathbb{C}^n)$ to $L_2(\mathbb{R}^d; \mathbb{C}^m)$, and

$$(8.25) \quad \|g^{1/2}b(\mathbf{D})[\tilde{\Lambda}]\|_{H^2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \mathcal{C}_{\tilde{\Lambda}},$$

where

$$\mathcal{C}_{\tilde{\Lambda}}^2 = (20\|g\|_{L_\infty}\alpha_1 d + 8)C_\Omega^2 \left(\sum_{i=1}^n \|\mathbf{v}_i\|_{L_p(\Omega)}^2 \right) + n(2 + 16\alpha_0^{-1}\|g^{-1}\|_{L_\infty})C_{\Omega,\rho}^2 \left(\sum_{j=1}^d \|a_j\|_{L_\rho(\Omega)}^2 \right).$$

Here the $\mathbf{v}_i(\mathbf{x})$, $i = 1, \dots, n$, are the columns of the matrix $\tilde{\Lambda}(\mathbf{x})$, C_Ω is the norm of the embedding of $H^1(\Omega)$ in $L_{2p/(p-2)}(\Omega)$, and $C_{\Omega,\rho}$ is the norm of the embedding of $H^1(\Omega)$ in $L_{2\rho/(\rho-2)}(\Omega)$.

Proof. The vector-valued function $\mathbf{v}_i(\mathbf{x})$ is the Γ -periodic solution of the problem

$$(8.26) \quad b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) \mathbf{v}_i + \sum_{j=1}^d D_j a_j(\mathbf{x})^* \mathbf{e}_i = 0, \quad \int_{\Omega} \mathbf{v}_i d\mathbf{x} = 0.$$

Here $\{\mathbf{e}_i\}_{i=1,\dots,n}$ is the standard orthonormal basis in $\mathbb{C}^n$. We have $b(\mathbf{D}) = \sum_{l=1}^d b_l D_l$, where the b_l are constant $(m \times n)$ -matrices. From (5.2) it follows that $|b_l| \leq \alpha_1^{1/2}$, $l = 1, \dots, d$. Then

$$(8.27) \quad \sum_{l=1}^d |b_l|^2 \leq \alpha_1 d.$$

Let $u \in H^2(\mathbb{R}^d)$. Then

$$(8.28) \quad g^{1/2}b(\mathbf{D})(\mathbf{v}_i u) = g^{1/2}(b(\mathbf{D})\mathbf{v}_i)u + \sum_{l=1}^d g^{1/2}b_l(D_l u)\mathbf{v}_i.$$

We estimate the second term on the right-hand side of (8.28):

$$(8.29) \quad \left\| \sum_{l=1}^d g^{1/2}b_l(D_l u)\mathbf{v}_i \right\|_{L_2(\mathbb{R}^d)} \leq \|g\|_{L_\infty}^{1/2} \alpha_1^{1/2} d^{1/2} \left(\int_{\mathbb{R}^d} |\mathbf{D}u|^2 |\mathbf{v}_i|^2 d\mathbf{x} \right)^{1/2}.$$

Next, we have

$$(8.30) \quad \int_{\mathbb{R}^d} |\mathbf{D}u|^2 |\mathbf{v}_i|^2 d\mathbf{x} = \sum_{\mathbf{a} \in \Gamma} \int_{\Omega + \mathbf{a}} |\mathbf{D}u|^2 |\mathbf{v}_i|^2 d\mathbf{x}.$$

Let $d \geq 3$. We apply the Hölder inequality with indices $s = d/2$ and $s' = d/(d-2)$. Then

$$(8.31) \quad \int_{\Omega + \mathbf{a}} |\mathbf{D}u|^2 |\mathbf{v}_i|^2 d\mathbf{x} \leq \left(\int_{\Omega} |\mathbf{v}_i|^d d\mathbf{x} \right)^{2/d} \left(\int_{\Omega + \mathbf{a}} |\mathbf{D}u|^{2d/(d-2)} d\mathbf{x} \right)^{(d-2)/d}.$$

By the continuous embedding $H^1(\Omega) \subset L_{2d/(d-2)}(\Omega)$, we have

$$(8.32) \quad \left(\int_{\Omega+\mathbf{a}} |\mathbf{D}u|^{2d/(d-2)} d\mathbf{x} \right)^{(d-2)/2d} \leq C_\Omega \|\mathbf{D}u\|_{H^1(\Omega+\mathbf{a})}.$$

The embedding constant C_Ω depends only on d and Ω (i.e., on the lattice Γ). From (8.30)–(8.32) it follows that

$$(8.33) \quad \int_{\mathbb{R}^d} |\mathbf{D}u|^2 |\mathbf{v}_i|^2 d\mathbf{x} \leq C_\Omega^2 \|\mathbf{v}_i\|_{L_p(\Omega)}^2 \|u\|_{H^2(\mathbb{R}^d)}^2,$$

where $p = d \geq 3$. By (8.29), this implies the estimate

$$(8.34) \quad \left\| \sum_{l=1}^d g^{1/2} b_l(D_l u) \mathbf{v}_i \right\|_{L_2(\mathbb{R}^d)} \leq \|g\|_{L_\infty}^{1/2} \alpha_1^{1/2} d^{1/2} C_\Omega \|\mathbf{v}_i\|_{L_p(\Omega)} \|u\|_{H^2(\mathbb{R}^d)}.$$

In the case where $d = 2$, instead of (8.31) one should apply the Hölder inequality with the indices $s = p/2 > 1$ and $s' = p/(p-2) < \infty$ and employ the embedding $H^1(\Omega) \subset L_\sigma(\Omega)$ for any $\sigma < \infty$. Let C_Ω be the corresponding embedding constant for $\sigma = 2p/(p-2)$. If $d = 1$, the left-hand side of (8.31) is dominated by $\|\mathbf{v}_i\|_{L_2(\Omega)}^2 \|\mathbf{D}u\|_{L_\infty(\Omega+\mathbf{a})}^2$ and the embedding $H^1(\Omega) \subset L_\infty(\Omega)$ can be used. Let C_Ω be the corresponding embedding constant. Then estimates (8.33) and (8.34) remain true also for $d = 1, 2$.

Next, relation (8.26) implies that

$$(8.35) \quad \int_{\mathbb{R}^d} \left(\langle g(\mathbf{x}) b(\mathbf{D}) \mathbf{v}_i, b(\mathbf{D}) \mathbf{w} \rangle + \sum_{j=1}^d \langle a_j(\mathbf{x})^* \mathbf{e}_i, D_j \mathbf{w} \rangle \right) d\mathbf{x} = 0$$

for any $\mathbf{w} \in H^1(\mathbb{R}^d; \mathbb{C}^n)$ such that $\mathbf{w}(\mathbf{x}) = 0$ for $|\mathbf{x}| > R$ (with some $R > 0$).

Let $u \in C_0^\infty(\mathbb{R}^d)$. We put $\mathbf{w}(\mathbf{x}) = |u(\mathbf{x})|^2 \mathbf{v}_i(\mathbf{x})$. Then

$$b(\mathbf{D}) \mathbf{w} = |u|^2 b(\mathbf{D}) \mathbf{v}_i + \sum_{l=1}^d b_l(D_l |u|^2) \mathbf{v}_i.$$

Substituting this expression into (8.35), we obtain

$$\int_{\mathbb{R}^d} \left(\langle g(\mathbf{x}) b(\mathbf{D}) \mathbf{v}_i, |u|^2 b(\mathbf{D}) \mathbf{v}_i + \sum_{l=1}^d b_l(D_l |u|^2) \mathbf{v}_i \rangle + \sum_{j=1}^d \langle a_j(\mathbf{x})^* \mathbf{e}_i, D_j (|u|^2 \mathbf{v}_i) \rangle \right) d\mathbf{x} = 0.$$

Hence,

$$(8.36) \quad J_0 := \int_{\mathbb{R}^d} |g^{1/2} b(\mathbf{D}) \mathbf{v}_i|^2 |u|^2 d\mathbf{x} = J_1 + J_2,$$

where

$$\begin{aligned} J_1 &= - \int_{\mathbb{R}^d} \langle g^{1/2} b(\mathbf{D}) \mathbf{v}_i, \sum_{l=1}^d g^{1/2} b_l(D_l |u|^2) \mathbf{v}_i \rangle d\mathbf{x}, \\ J_2 &= - \int_{\mathbb{R}^d} \sum_{j=1}^d \langle a_j^* \mathbf{e}_i, D_j (|u|^2 \mathbf{v}_i) \rangle d\mathbf{x} = - \int_{\mathbb{R}^d} \sum_{j=1}^d \langle a_j^* \mathbf{e}_i, (D_j (\mathbf{v}_i u)) \bar{u} + \mathbf{v}_i u (D_j \bar{u}) \rangle d\mathbf{x}. \end{aligned}$$

The term J_1 is estimated with the help of (8.27):

$$\begin{aligned} |J_1| &\leq \|g\|_{L_\infty}^{1/2} \alpha_1^{1/2} d^{1/2} \int_{\mathbb{R}^d} 2 |g^{1/2} b(\mathbf{D}) \mathbf{v}_i| |u| |\mathbf{D}u| |\mathbf{v}_i| d\mathbf{x} \\ &\leq \frac{1}{2} \int_{\mathbb{R}^d} |g^{1/2} b(\mathbf{D}) \mathbf{v}_i|^2 |u|^2 d\mathbf{x} + 2 \|g\|_{L_\infty} \alpha_1 d \int_{\mathbb{R}^d} |\mathbf{D}u|^2 |\mathbf{v}_i|^2 d\mathbf{x}. \end{aligned}$$

Combining this with (8.33), we see that

$$(8.37) \quad |J_1| \leq \frac{1}{2}J_0 + 2\|g\|_{L_\infty} \alpha_1 d C_\Omega^2 \|\mathbf{v}_i\|_{L_\rho(\Omega)}^2 \|u\|_{H^2(\mathbb{R}^d)}^2.$$

Now, we proceed to estimating the term J_2 . By condition (5.10) on the coefficients a_j , we have

$$(8.38) \quad \int_{\mathbb{R}^d} |a_j(\mathbf{x})|^2 |u|^2 d\mathbf{x} \leq C_{\Omega,\rho}^2 \|a_j\|_{L_\rho(\Omega)}^2 \|u\|_{H^1(\mathbb{R}^d)}^2;$$

cf. the proof of (8.33). Here $C_{\Omega,\rho}$ is the norm of the embedding $H^1(\Omega) \subset L_{2\rho/(\rho-2)}(\Omega)$. We have

$$\begin{aligned} |J_2| &\leq \sum_{j=1}^d \int_{\mathbb{R}^d} (|D_j(\mathbf{v}_i u)| |a_j| |u| + |\mathbf{v}_i| |D_j u| |a_j| |u|) d\mathbf{x} \\ &\leq \mu \int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_i u)|^2 d\mathbf{x} + \int_{\mathbb{R}^d} |\mathbf{v}_i|^2 |\mathbf{D}u|^2 d\mathbf{x} + \left(\frac{1}{4\mu} + \frac{1}{4}\right) \sum_{j=1}^d \int_{\mathbb{R}^d} |a_j|^2 |u|^2 d\mathbf{x} \end{aligned}$$

for any $\mu > 0$. Combining this with (8.33) and (8.38), we arrive at the estimate

$$(8.39) \quad \begin{aligned} |J_2| &\leq \mu \int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_i u)|^2 d\mathbf{x} \\ &\quad + \left(C_\Omega^2 \|\mathbf{v}_i\|_{L_\rho(\Omega)}^2 + \left(\frac{1}{4} + \frac{1}{4\mu}\right) C_{\Omega,\rho}^2 \sum_{j=1}^d \|a_j\|_{L_\rho(\Omega)}^2 \right) \|u\|_{H^2(\mathbb{R}^d)}^2. \end{aligned}$$

Now, relations (8.36), (8.37), and (8.39) imply that

$$(8.40) \quad \begin{aligned} \frac{1}{2}J_0 &\leq \mu \int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_i u)|^2 d\mathbf{x} \\ &\quad + \left((2\|g\|_{L_\infty} \alpha_1 d + 1) C_\Omega^2 \|\mathbf{v}_i\|_{L_\rho(\Omega)}^2 + \left(\frac{1}{4} + \frac{1}{4\mu}\right) C_{\Omega,\rho}^2 \sum_{j=1}^d \|a_j\|_{L_\rho(\Omega)}^2 \right) \|u\|_{H^2(\mathbb{R}^d)}^2. \end{aligned}$$

Comparing (8.28), (8.34), and (8.40), we obtain

$$(8.41) \quad \begin{aligned} &\|g^{1/2} b(\mathbf{D})(\mathbf{v}_i u)\|_{L_2(\mathbb{R}^d)}^2 \\ &\leq 2J_0 + 2 \left\| \sum_{l=1}^d g^{1/2} b_l(D_l u) \mathbf{v}_i \right\|_{L_2(\mathbb{R}^d)}^2 \\ &\leq 4\mu \int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_i u)|^2 d\mathbf{x} \\ &\quad + \left((10\|g\|_{L_\infty} \alpha_1 d + 4) C_\Omega^2 \|\mathbf{v}_i\|_{L_\rho(\Omega)}^2 + (1 + \mu^{-1}) C_{\Omega,\rho}^2 \sum_{j=1}^d \|a_j\|_{L_\rho(\Omega)}^2 \right) \|u\|_{H^2(\mathbb{R}^d)}^2. \end{aligned}$$

By (5.6),

$$\begin{aligned} 4\mu \int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_i u)|^2 d\mathbf{x} &\leq 4\mu \alpha_0^{-1} \|g^{-1}\|_{L_\infty} \|g^{1/2} b(\mathbf{D})(\mathbf{v}_i u)\|_{L_2(\mathbb{R}^d)}^2 \\ &= \frac{1}{2} \|g^{1/2} b(\mathbf{D})(\mathbf{v}_i u)\|_{L_2(\mathbb{R}^d)}^2 \quad \text{for } \mu = \frac{1}{8} \alpha_0 \|g^{-1}\|_{L_\infty}^{-1}. \end{aligned}$$

Together with (8.41), this yields the estimate

$$\|g^{1/2} b(\mathbf{D})(\mathbf{v}_i u)\|_{L_2(\mathbb{R}^d)}^2 \leq C_i^2 \|u\|_{H^2(\mathbb{R}^d)}^2,$$

where

$$C_i^2 = (20\|g\|_{L_\infty}\alpha_1 d + 8)C_\Omega^2\|\mathbf{v}_i\|_{L_p(\Omega)}^2 + (2 + 16\alpha_0^{-1}\|g^{-1}\|_{L_\infty})C_{\Omega,\rho}^2\sum_{j=1}^d\|a_j\|_{L_\rho(\Omega)}^2.$$

Thus,

$$\|g^{1/2}b(\mathbf{D})[\mathbf{v}_i]\|_{H^2(\mathbb{R}^d)\rightarrow L_2(\mathbb{R}^d)} \leq C_i, \quad i = 1, \dots, n,$$

whence

$$\|g^{1/2}b(\mathbf{D})[\tilde{\Lambda}]\|_{H^2(\mathbb{R}^d)\rightarrow L_2(\mathbb{R}^d)} \leq \left(\sum_{i=1}^n C_i^2\right)^{1/2};$$

i.e., (8.25) is true. $\square$

The following statement is a consequence of Proposition 8.8.

Proposition 8.9. *Under Condition 8.7, we have*

$$(8.42) \quad \|\mathcal{B}_\lambda(\varepsilon)^{1/2}\tilde{\Lambda}\Sigma(\varepsilon)\|_{L_2(\mathbb{R}^d)\rightarrow L_2(\mathbb{R}^d)} \leq C'_\Lambda, \quad 0 < \varepsilon \leq 1,$$

where

$$\begin{aligned} (C'_\Lambda)^2 &= (2 + c_1^2 + c_2)\mathcal{C}_\Lambda^2(\check{c}_*)^{-2}(1 + r_0^{-2})^2 \\ &\quad + (C(1) + c_3 + |\lambda|\|Q_0\|_{L_\infty})C_\Omega^2\|\tilde{\Lambda}\|_{L_p(\Omega)}^2(\check{c}_*)^{-2}(1 + r_0^{-2}). \end{aligned}$$

Proof. By (5.30), for $\mathbf{v} \in H^2(\mathbb{R}^d; \mathbb{C}^n)$ we have

$$(8.43) \quad \begin{aligned} \|\mathcal{B}_\lambda(\varepsilon)^{1/2}\tilde{\Lambda}\mathbf{v}\|_{L_2}^2 &= \mathfrak{b}_\lambda(\varepsilon)[\tilde{\Lambda}\mathbf{v}, \tilde{\Lambda}\mathbf{v}] \\ &\leq (2 + c_1^2 + c_2)\|g^{1/2}b(\mathbf{D})\tilde{\Lambda}\mathbf{v}\|_{L_2}^2 + \varepsilon^2(C(1) + c_3 + |\lambda|\|Q_0\|_{L_\infty})\|\tilde{\Lambda}\mathbf{v}\|_{L_2}^2. \end{aligned}$$

Next, as in (8.33), we get

$$(8.44) \quad \|\tilde{\Lambda}\mathbf{v}\|_{L_2(\mathbb{R}^d)} \leq C_\Omega\|\tilde{\Lambda}\|_{L_p(\Omega)}\|\mathbf{v}\|_{H^1(\mathbb{R}^d)}.$$

From (8.43), (8.44), and (8.25) it follows that

$$(8.45) \quad \begin{aligned} \|\mathcal{B}_\lambda(\varepsilon)^{1/2}\tilde{\Lambda}\mathbf{v}\|_{L_2}^2 &\leq (2 + c_1^2 + c_2)\mathcal{C}_\Lambda^2\|\mathbf{v}\|_{H^2}^2 \\ &\quad + \varepsilon^2(C(1) + c_3 + |\lambda|\|Q_0\|_{L_\infty})C_\Omega^2\|\tilde{\Lambda}\|_{L_p(\Omega)}^2\|\mathbf{v}\|_{H^1}^2. \end{aligned}$$

We substitute $\mathbf{v} = \Sigma(\varepsilon)\mathbf{u}$, where $\mathbf{u} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$. By (8.6), we have

$$(8.46) \quad \begin{aligned} \|\mathbf{v}\|_{H^2} &= \|\Sigma(\varepsilon)\mathbf{u}\|_{H^2} \leq \sup_{\boldsymbol{\xi} \in \mathbb{R}^d} (1 + |\boldsymbol{\xi}|^2)L(\boldsymbol{\xi}, \varepsilon)^{-1}(1 - \chi_{\tilde{\Omega}}(\boldsymbol{\xi}))\|\mathbf{u}\|_{L_2} \\ &\leq (\check{c}_*)^{-1} \sup_{|\boldsymbol{\xi}| \geq r_0} (1 + |\boldsymbol{\xi}|^2)(|\boldsymbol{\xi}|^2 + \varepsilon^2)^{-1}\|\mathbf{u}\|_{L_2} \leq (\check{c}_*)^{-1}(1 + r_0^{-2})\|\mathbf{u}\|_{L_2}. \end{aligned}$$

Similarly,

$$(8.47) \quad \begin{aligned} \varepsilon\|\mathbf{v}\|_{H^1} &= \varepsilon\|\Sigma(\varepsilon)\mathbf{u}\|_{H^1} \leq (\check{c}_*)^{-1} \sup_{|\boldsymbol{\xi}| \geq r_0} \varepsilon(1 + |\boldsymbol{\xi}|^2)^{1/2}(|\boldsymbol{\xi}|^2 + \varepsilon^2)^{-1}\|\mathbf{u}\|_{L_2} \\ &\leq (\check{c}_*)^{-1}(1 + r_0^{-2})^{1/2}\|\mathbf{u}\|_{L_2}. \end{aligned}$$

Now, relations (8.45)–(8.47) imply (8.42). $\square$

Proposition 8.9 makes it possible to replace the operator Π by the identity operator in the term of the corrector containing $\tilde{\Lambda}$, provided Condition 8.7 is satisfied. In its turn, Condition 8.7 is satisfied *a fortiori* if the following condition is fulfilled.

Condition 8.10. Suppose that at least one of the following assumptions is satisfied:

1° $d \leq 4$;

2° the dimension d is arbitrary, and the operator $\mathcal{A}$ is of the form $\mathcal{A} = \mathbf{D}^*g(\mathbf{x})\mathbf{D}$, where the matrix $g(\mathbf{x})$ has real entries.

Proposition 8.11. *Condition 8.10 implies Condition 8.7.*

Proof. 1°. Let $d \leq 4$. The matrix-valued function $\tilde{\Lambda}(\mathbf{x})$, which is the periodic solution of the problem (6.12), belongs to the class $\tilde{H}^1(\Omega)$. We use the embedding $H^1(\Omega) \subset L_s(\Omega)$, where $s = 2d/(d-2)$ for $d \geq 3$, $s < \infty$ for $d = 2$, and $s = \infty$ for $d = 1$. Hence, $\tilde{\Lambda} \in L_s(\Omega)$. Obviously, if $d \leq 4$, this implies (8.24).

2°. In the case where $\mathcal{A} = \mathbf{D}^*g(\mathbf{x})\mathbf{D}$ and $g(\mathbf{x})$ has real entries, Theorem 13.1 of Chapter III of the book [LaU] shows that $\tilde{\Lambda} \in L_\infty$. $\square$

Remark 8.12. From (7.51) and (7.52), it is seen that the norm $\|\tilde{\Lambda}\|_{H^1(\Omega)}$ is dominated by a constant that depends only on n , α_0 , $\|g^{-1}\|_{L_\infty}$, C_a , and r_0 . If $d \leq 4$, then $\|\tilde{\Lambda}\|_{L_p(\Omega)} \leq C_p(\Omega)\|\tilde{\Lambda}\|_{H^1(\Omega)}$, where p is as in (8.24), and $C_p(\Omega)$ is the norm of the embedding $H^1(\Omega) \subset L_p(\Omega)$. Therefore, for $d \leq 4$, the norm $\|\tilde{\Lambda}\|_{L_p(\Omega)}$ is controlled in terms of d , p , n , α_0 , $\|g^{-1}\|_{L_\infty}$, $\|a_j\|_{L_p(\Omega)}$, $j = 1, \dots, d$, and the parameters of the lattice Γ .

8.6. Comparing Theorem 8.2 and Propositions 8.5, 8.9, we arrive at the following result.

Theorem 8.13. *Suppose that the assumptions of Theorem 8.2 are satisfied.*

1°. *If Condition 8.3 (or the more restrictive Condition 8.6) is satisfied, then for $0 < \varepsilon \leq 1$ we have*

$$(8.48) \quad \left\| \mathcal{B}_\lambda(\varepsilon)^{1/2} \left(\mathcal{B}_\lambda(\varepsilon)^{-1} - (I + \Lambda b(\mathbf{D}) + \varepsilon \tilde{\Lambda} \Pi)(\mathcal{B}_\lambda^0(\varepsilon))^{-1} \right) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \tilde{C}_7 + C'_\Lambda.$$

The constant on the right-hand side of (8.48) depends only on the initial data (5.33), the parameters of the lattice Γ , and $\|\Lambda\|_{L_\infty}$.

2°. *If Condition 8.7 (or the more restrictive Condition 8.10) is satisfied, then for $0 < \varepsilon \leq 1$ we have*

$$(8.49) \quad \left\| \mathcal{B}_\lambda(\varepsilon)^{1/2} \left(\mathcal{B}_\lambda(\varepsilon)^{-1} - (I + \Lambda b(\mathbf{D}) \Pi + \varepsilon \tilde{\Lambda})(\mathcal{B}_\lambda^0(\varepsilon))^{-1} \right) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \tilde{C}_7 + C'_\Lambda.$$

The constant on the right-hand side of (8.49) depends only on the initial data (5.33), the parameters of the lattice Γ , and the norm $\|\tilde{\Lambda}\|_{L_p(\Omega)}$.

3°. *If both Condition 8.3 (or 8.6) and Condition 8.7 (or 8.10) are satisfied, then for $0 < \varepsilon \leq 1$ we have*

$$(8.50) \quad \left\| \mathcal{B}_\lambda(\varepsilon)^{1/2} \left(\mathcal{B}_\lambda(\varepsilon)^{-1} - (I + \Lambda b(\mathbf{D}) + \varepsilon \tilde{\Lambda})(\mathcal{B}_\lambda^0(\varepsilon))^{-1} \right) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \tilde{C}_7 + C'_\Lambda + C'_\tilde{\Lambda}.$$

The constant on the right-hand side of (8.50) depends only on the initial data (5.33), the parameters of the lattice Γ , and the norms $\|\Lambda\|_{L_\infty}$, $\|\tilde{\Lambda}\|_{L_p(\Omega)}$.

Remark 8.14. Conditions 8.3 and 8.7 are also satisfied if the matrix-valued functions $g(\mathbf{x})$ and $a_j(\mathbf{x})$, $j = 1, \dots, d$, are sufficiently smooth.

Remark 8.15. In the case where $\mathcal{A} = \mathbf{D}^*g(\mathbf{x})\mathbf{D}$ and the matrix $g(\mathbf{x})$ has real entries, Theorem 13.1 of Chapter III of the book [LaU] shows that the norm $\|\Lambda\|_{L_\infty}$ is dominated by a constant that depends only on $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, d , and Ω , and the norm $\|\tilde{\Lambda}\|_{L_\infty}$ is controlled in terms of d , $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|a_j\|_{L_p(\Omega)}$, $j = 1, \dots, d$, and Ω . Conditions 8.6 and 8.10 are satisfied. Since in this case $n = 1$, $m = d$, and $\alpha_0 = \alpha_1 = 1$, the initial data (5.33) is reduced to the following:

$$(8.51) \quad d, \rho; \quad \|g\|_{L_\infty}, \quad \|g^{-1}\|_{L_\infty}, \quad \|a_j\|_{L_p(\Omega)}, \quad j = 1, \dots, d, \\ \tilde{c}, \quad c_0, \quad \tilde{c}_2, c_3; \quad |\lambda|, \quad \|Q_0\|_{L_\infty}.$$

Then Theorem 8.13 (statement 3°) implies the following.

Proposition 8.16. *Under the assumptions of Theorem 8.2, suppose that the operator $\mathcal{A}$ is of the form $\mathcal{A} = \mathbf{D}^* g(\mathbf{x}) \mathbf{D}$, where the matrix $g(\mathbf{x})$ has real entries. Then estimate (8.50) is valid. The constant on the right-hand side of (8.50) depends only on the initial data (8.51) and the parameters of the lattice Γ .*

8.7. The case of the zero corrector. Let $g^0 = \bar{g}$, which is equivalent to (6.17). Then the periodic solution of problem (6.10) is equal to zero: $\Lambda(\mathbf{x}) = 0$. If

$$(8.52) \quad \sum_{j=1}^d D_j a_j(\mathbf{x})^* = 0,$$

then the periodic solution of problem (6.12) is equal to zero: $\tilde{\Lambda}(\mathbf{x}) = 0$. In this case, relation (8.17) simplifies. Theorem 8.2 implies the following statement.

Proposition 8.17. *Under the assumptions of Theorem 8.1, suppose that $g^0 = \bar{g}$ and that relations (8.52) are satisfied. Then*

$$\|\mathcal{B}_\lambda(\varepsilon)^{1/2} (\mathcal{B}_\lambda(\varepsilon)^{-1} - (\mathcal{B}_\lambda^0(\varepsilon))^{-1})\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \tilde{C}_7, \quad 0 < \varepsilon \leq 1.$$

The constant $\tilde{C}_7$ depends only on the initial data (5.33) and the parameters of the lattice Γ .

CHAPTER 3. HOMOGENIZATION OF PERIODIC DIFFERENTIAL OPERATORS

§9. HOMOGENIZATION PROBLEM FOR THE OPERATOR $\mathcal{B}_\varepsilon$

9.1. Statement of the problem. For any Γ -periodic function $\phi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, we denote $\phi^\varepsilon(\mathbf{x}) = \phi(\varepsilon^{-1}\mathbf{x})$, $\varepsilon > 0$. In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, consider the operator

$$\mathcal{A}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$$

generated by the quadratic form

$$\mathbf{a}_\varepsilon[\mathbf{u}, \mathbf{u}] = \int_{\mathbb{R}^d} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D}) \mathbf{u}, b(\mathbf{D}) \mathbf{u} \rangle d\mathbf{x}, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

The form $\mathbf{a}_\varepsilon$ satisfies the following estimates similar to (5.6):

$$(9.1) \quad \alpha_0 \|g^{-1}\|_{L_\infty}^{-1} \|\mathbf{D}\mathbf{u}\|_{L_2}^2 \leq \mathbf{a}_\varepsilon[\mathbf{u}, \mathbf{u}] \leq \alpha_1 \|g\|_{L_\infty} \|\mathbf{D}\mathbf{u}\|_{L_2}^2, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Next, let $\mathcal{Y}$ be the operator defined by (5.7), and let $\mathcal{Y}_{2,\varepsilon} : L_2(\mathbb{R}^d; \mathbb{C}^n) \rightarrow L_2(\mathbb{R}^d; \mathbb{C}^{dn})$ be the operator of multiplication by the $(dn \times n)$ -matrix-valued function composed of the matrices $(a_j^\varepsilon(\mathbf{x}))^*$, $j = 1, \dots, d$. Thus,

$$\mathcal{Y}_{2,\varepsilon} \mathbf{u} = \text{col} \{ (a_1^\varepsilon(\mathbf{x}))^* \mathbf{u}, \dots, (a_d^\varepsilon(\mathbf{x}))^* \mathbf{u} \}, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Let $d\mu(\mathbf{x})$ be the matrix-valued measure in $\mathbb{R}^d$ defined in Subsection 5.3. We define the measure $d\mu^\varepsilon(\mathbf{x})$ as follows. For any Borel set $\Delta \subset \mathbb{R}^d$, we consider the set $\varepsilon^{-1}\Delta = \{\mathbf{y} = \varepsilon^{-1}\mathbf{x} : \mathbf{x} \in \Delta\}$ and put

$$\mu^\varepsilon(\Delta) = \varepsilon^d \mu(\varepsilon^{-1}\Delta).$$

Consider the quadratic form q_ε defined by

$$(9.2) \quad q_\varepsilon[\mathbf{u}, \mathbf{u}] = \int_{\mathbb{R}^d} \langle d\mu^\varepsilon(\mathbf{x}) \mathbf{u}, \mathbf{u} \rangle, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

If $q[\mathbf{u}, \mathbf{u}]$ is the form given in Example 5.3, then $d\mu^\varepsilon(\mathbf{x}) = Q^\varepsilon(\mathbf{x}) d\mathbf{x}$ and the form (9.2) is given by

$$q_\varepsilon[\mathbf{u}, \mathbf{u}] = \int_{\mathbb{R}^d} \langle Q^\varepsilon(\mathbf{x}) \mathbf{u}, \mathbf{u} \rangle d\mathbf{x}, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

If $q[\mathbf{u}, \mathbf{u}]$ is the form given in Example 5.4, then

$$q_\varepsilon[\mathbf{u}, \mathbf{u}] = \varepsilon \int_{\varepsilon\Sigma} \langle \sigma^\varepsilon(\mathbf{x}) \mathbf{u}, \mathbf{u} \rangle dS_{\varepsilon\Sigma}(\mathbf{x}), \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Suppose that all assumptions of Subsections 5.1–5.4 are satisfied. In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we consider the following quadratic form;

$$(9.3) \quad \begin{aligned} \mathbf{b}_\varepsilon[\mathbf{u}, \mathbf{u}] &= \mathbf{a}_\varepsilon[\mathbf{u}, \mathbf{u}] + 2 \operatorname{Re}(\mathcal{Y}\mathbf{u}, \mathcal{Y}_{2,\varepsilon}\mathbf{u})_{L_2(\mathbb{R}^d)} + q_\varepsilon[\mathbf{u}, \mathbf{u}] \\ &= \int_{\mathbb{R}^d} \left(\langle g^\varepsilon(\mathbf{x}) b(\mathbf{D}) \mathbf{u}, b(\mathbf{D}) \mathbf{u} \rangle + 2 \operatorname{Re} \sum_{j=1}^d \langle a_j^\varepsilon(\mathbf{x}) D_j \mathbf{u}, \mathbf{u} \rangle \right) d\mathbf{x} + q_\varepsilon[\mathbf{u}, \mathbf{u}], \\ &\quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n). \end{aligned}$$

Let T_ε be the unitary scaling transformation in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ defined by

$$(9.4) \quad (T_\varepsilon \mathbf{u})(\mathbf{y}) = \varepsilon^{d/2} \mathbf{u}(\varepsilon \mathbf{y}).$$

Obviously,

$$(9.5) \quad \mathbf{a}_\varepsilon[\mathbf{u}, \mathbf{u}] = \varepsilon^{-2} \mathbf{a}[T_\varepsilon \mathbf{u}, T_\varepsilon \mathbf{u}], \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n),$$

$$(9.6) \quad \mathbf{b}_\varepsilon[\mathbf{u}, \mathbf{u}] = \varepsilon^{-2} \mathbf{b}(\varepsilon)[T_\varepsilon \mathbf{u}, T_\varepsilon \mathbf{u}], \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n),$$

where $\mathbf{a}$ is the form defined in Subsection 5.1, and $\mathbf{b}(\varepsilon)$ is the form (5.21).

From (5.25) and (9.5), (9.6) it follows that

$$(9.7) \quad \mathbf{b}_\varepsilon[\mathbf{u}, \mathbf{u}] \geq \frac{\kappa}{2} \mathbf{a}_\varepsilon[\mathbf{u}, \mathbf{u}] - (c_0 + c_4) \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Consequently, the form (9.3) is closed and lower semibounded.

The selfadjoint operator $\mathcal{B}_\varepsilon$ generated in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ by the quadratic form (9.3) is our main object. Formally, we can write

$$(9.8) \quad \mathcal{B}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) + \sum_{j=1}^d (a_j^\varepsilon(\mathbf{x}) D_j + D_j (a_j^\varepsilon(\mathbf{x}))^*) + Q^\varepsilon(\mathbf{x}),$$

where $Q^\varepsilon(\mathbf{x})$ should be viewed as the generalized potential generated by the measure $d\mu^\varepsilon$. The coefficients of the operator (9.8) oscillate rapidly as $\varepsilon \rightarrow 0$.

The homogenization problem for the operator (9.8) consists in approximating the resolvent $(\mathcal{B}_\varepsilon + \lambda I)^{-1}$ or the generalized resolvent $(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1}$ as $\varepsilon \rightarrow 0$. Here Q_0^ε is the operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ acting as multiplication by the matrix-valued function $Q_0^\varepsilon(\mathbf{x})$. As above, it is assumed that the parameter λ is subject to (5.27). Then (9.7) implies that

$$(9.9) \quad \mathbf{b}_\varepsilon[\mathbf{u}, \mathbf{u}] + \lambda (Q_0^\varepsilon \mathbf{u}, \mathbf{u})_{L_2(\mathbb{R}^d)} \geq \frac{\kappa}{2} \mathbf{a}_\varepsilon[\mathbf{u}, \mathbf{u}] + c_4 \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Thus, the operator $\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon$ is positive definite.

9.2. The leading term of approximation for the generalized resolvent of the operator $\mathcal{B}_\varepsilon$. Suppose that the operator $\mathcal{A}^0$ is defined by (8.1) and that $\mathcal{Y}^0$ is defined by (8.2). Let W , $\overline{Q}$, and $\overline{Q}_0$ be the matrices defined by (7.3), (7.5), and (7.6), respectively. The operator

$$(9.10) \quad \mathcal{B}^0 = \mathcal{A}^0 + \mathcal{Y}^0 + (\mathcal{Y}^0)^* + \overline{Q} - W$$

is called the *effective operator* for $\mathcal{B}_\varepsilon$. In more detail, the effective operator can be written as

$$(9.11) \quad \mathcal{B}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}) - b(\mathbf{D})^* V - V^* b(\mathbf{D}) + \sum_{j=1}^d (\overline{a_j + a_j^*}) D_j + \overline{Q} - W.$$

Here $\overline{g^0}$ is the effective matrix (6.15), the matrix V is defined by (7.2), and the matrices $\overline{a_j + a_j^*}$ are defined by (7.4). The operator (9.11) is a second order DO with constant coefficients. Note that the matrix W is nonnegative (see (7.3)): $W \geq 0$. Hence, the lowest term in (9.11) does not exceed $\overline{Q}$.

Remark 9.1. Under condition (8.52), we have $\tilde{\Lambda}(\mathbf{x}) = 0$ and $V = W = 0$ (see (6.12) and (7.2), (7.3)). Then the expression for the effective operator $\mathcal{B}^0$ simplifies:

$$(9.12) \quad \mathcal{B}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}) + \sum_{j=1}^d \overline{(a_j + a_j^*)} D_j + \overline{Q};$$

i.e., the lower terms are homogenized in a trivial way.

By the scaling transformation, Theorem 8.1 implies our first main result.

Theorem 9.2. *Suppose that all assumptions of Subsections 5.1–5.4 are fulfilled. Let $\mathcal{B}_\varepsilon$ be the operator (9.8), and let $\mathcal{B}^0$ be the operator (9.10). Then*

$$(9.13) \quad \|(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \check{C}_0 \varepsilon, \quad 0 < \varepsilon \leq 1.$$

The constant $\check{C}_0$ depends only on the initial data (5.33) and the parameters of the lattice Γ .

Proof. Let T_ε be the unitary operator (9.4). Identity (9.6) means that

$$(9.14) \quad \mathcal{B}_\varepsilon = \varepsilon^{-2} T_\varepsilon^* \mathcal{B}(\varepsilon) T_\varepsilon,$$

whence

$$(9.15) \quad (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} = \varepsilon^2 T_\varepsilon^* (\mathcal{B}(\varepsilon) + \varepsilon^2 \lambda Q_0)^{-1} T_\varepsilon.$$

The effective operator satisfies a similar identity:

$$(9.16) \quad (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} = \varepsilon^2 T_\varepsilon^* (\mathcal{B}^0(\varepsilon) + \varepsilon^2 \lambda \overline{Q}_0)^{-1} T_\varepsilon.$$

Subtracting (9.16) from (9.15) and using the fact that the operator T_ε is unitary, we obtain

$$\begin{aligned} & \|(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \\ &= \varepsilon^2 \|(\mathcal{B}(\varepsilon) + \varepsilon^2 \lambda Q_0)^{-1} - (\mathcal{B}^0(\varepsilon) + \varepsilon^2 \lambda \overline{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)}. \end{aligned}$$

Now, estimate (9.13) follows from (8.12). $\square$

9.3. Results involving the corrector. Now, using Theorem 8.2, we obtain an approximation with corrector for the generalized resolvent of the operator $\mathcal{B}_\varepsilon$.

Let Π_ε denote the pseudodifferential operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ with the symbol $\chi_{\tilde{\Omega}/\varepsilon}(\boldsymbol{\xi})$:

$$(9.17) \quad (\Pi_\varepsilon \mathbf{f})(\mathbf{x}) = (2\pi)^{-d/2} \int_{\tilde{\Omega}/\varepsilon} e^{i\langle \mathbf{x}, \boldsymbol{\xi} \rangle} \hat{\mathbf{f}}(\boldsymbol{\xi}) d\boldsymbol{\xi}.$$

Theorem 9.3. *Under the assumptions of Subsections 5.1–5.4, let $\mathcal{B}_\varepsilon$ be the operator (9.8), and let $\mathcal{B}^0$ be the operator (9.10). Suppose that the matrix $\Lambda(\mathbf{x})$ is the periodic solution of the problem (6.10) and that the matrix $\tilde{\Lambda}(\mathbf{x})$ is the periodic solution of problem (6.12). Put $\Lambda^\varepsilon(\mathbf{x}) = \Lambda(\varepsilon^{-1}\mathbf{x})$ and $\tilde{\Lambda}^\varepsilon(\mathbf{x}) = \tilde{\Lambda}(\varepsilon^{-1}\mathbf{x})$. Let Π_ε be the operator defined by (9.17). Then for $0 < \varepsilon \leq 1$ we have*

$$(9.18) \quad \left\| (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{1/2} \left((\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - \left(I + \varepsilon (\Lambda^\varepsilon b(\mathbf{D}) + \tilde{\Lambda}^\varepsilon) \Pi_\varepsilon \right) \mathcal{R}^0 \right) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \check{C}_7 \varepsilon,$$

where $\mathcal{R}^0 = (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1}$. The constant $\check{C}_7$ depends only on the initial data (5.33) and the parameters of the lattice Γ .

Proof. We apply the scaling transformation and use identities (9.14)–(9.16) and also the identities

$$\Lambda^\varepsilon b(\mathbf{D}) = \varepsilon^{-1} T_\varepsilon^* \Lambda b(\mathbf{D}) T_\varepsilon, \quad [\tilde{\Lambda}^\varepsilon] = T_\varepsilon^* [\tilde{\Lambda}] T_\varepsilon, \quad \Pi_\varepsilon = T_\varepsilon^* \Pi T_\varepsilon.$$

This yields the identity

$$(9.19) \quad \begin{aligned} & (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{1/2} \left((\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - \left(I + \varepsilon (\Lambda^\varepsilon b(\mathbf{D}) + \tilde{\Lambda}^\varepsilon) \Pi_\varepsilon \right) \mathcal{R}^0 \right) \\ &= \varepsilon T_\varepsilon^* \mathcal{B}_\lambda(\varepsilon)^{1/2} \left(\mathcal{B}_\lambda(\varepsilon)^{-1} - \left(I + (\Lambda b(\mathbf{D}) + \varepsilon \tilde{\Lambda}) \Pi \right) (\mathcal{B}_\lambda^0(\varepsilon))^{-1} \right) T_\varepsilon. \end{aligned}$$

Since the operator T_ε is unitary, estimate (9.18) follows directly from (8.17) and (9.19). $\square$

Similarly, by the scaling transformation, Theorem 8.13 implies the following result.

Theorem 9.4. *Suppose that the assumptions of Theorem 9.3 are satisfied.*

1°. *If Condition 8.3 (or 8.6) is satisfied, then for $0 < \varepsilon \leq 1$ we have*

$$(9.20) \quad \begin{aligned} & \left\| (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{1/2} \left((\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - \left(I + \varepsilon (\Lambda^\varepsilon b(\mathbf{D}) + \tilde{\Lambda}^\varepsilon) \Pi_\varepsilon \right) \mathcal{R}^0 \right) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \\ & \leq (\tilde{C}_7 + C'_\Lambda) \varepsilon. \end{aligned}$$

The constant on the right-hand side of (9.20) depends only on the initial data (5.33), the parameters of the lattice Γ , and the norm $\|\Lambda\|_{L_\infty}$.

2°. *If Condition 8.7 (or 8.10) is satisfied, then for $0 < \varepsilon \leq 1$ we have*

$$(9.21) \quad \begin{aligned} & \left\| (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{1/2} \left((\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - \left(I + \varepsilon (\Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon + \tilde{\Lambda}^\varepsilon) \right) \mathcal{R}^0 \right) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \\ & \leq (\tilde{C}_7 + C'_\Lambda) \varepsilon. \end{aligned}$$

The constant on the right-hand side of (9.21) depends only on the initial data (5.33), the parameters of the lattice Γ , and the norm $\|\tilde{\Lambda}\|_{L_p(\Omega)}$.

3°. *If both Condition 8.3 (or 8.6) and Condition 8.7 (or 8.10) are satisfied, then for $0 < \varepsilon \leq 1$ we have*

$$(9.22) \quad \begin{aligned} & \left\| (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{1/2} \left((\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - \left(I + \varepsilon (\Lambda^\varepsilon b(\mathbf{D}) + \tilde{\Lambda}^\varepsilon) \right) \mathcal{R}^0 \right) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \\ & \leq (\tilde{C}_7 + C'_\Lambda + C'_\Lambda) \varepsilon. \end{aligned}$$

The constant on the right-hand side of (9.22) depends only on the initial data (5.33), the parameters of the lattice Γ , and the norms $\|\Lambda\|_{L_\infty}$, $\|\tilde{\Lambda}\|_{L_p(\Omega)}$.

Proposition 8.16 implies the following statement.

Proposition 9.5. *Under the assumptions of Theorem 9.3, let $\mathcal{A} = \mathbf{D}^* g(\mathbf{x}) \mathbf{D}$, where the matrix $g(\mathbf{x})$ has real entries. Then estimate (9.22) is valid, and the constant on the right-hand side of (9.22) depends only on the initial data (8.51) and the parameters of the lattice Γ .*

The following statement is deduced from Proposition 8.17; it concerns the case of the zero corrector.

Proposition 9.6. *Let $\mathcal{B}_\varepsilon$ be the operator (9.8), and let $\mathcal{B}^0$ be the operator (9.10). Suppose that $g^0 = \bar{g}$ and that (8.52) is true. Then*

$$\left\| (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{1/2} \left((\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - (\mathcal{B}^0 + \lambda \bar{Q}_0)^{-1} \right) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \tilde{C}_7 \varepsilon, \quad 0 < \varepsilon \leq 1.$$

The constant $\tilde{C}_7$ depends only on the initial data (5.33) and the parameters of the lattice Γ .

9.4. Main results. Approximation for the generalized resolvent of the operator $\mathcal{B}_\varepsilon$ in the operator norm (from L_2 to H^1). Now we use Theorem 9.3 to obtain an approximation for the operator $(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1}$ in the operator norm (from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to $H^1(\mathbb{R}^d; \mathbb{C}^n)$).

Theorem 9.7. *Under the assumptions of Theorem 9.3 we have*

$$(9.23) \quad \left\| (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - \left(I + \varepsilon(\Lambda^\varepsilon b(\mathbf{D}) + \tilde{\Lambda}^\varepsilon) \Pi_\varepsilon \right) \mathcal{R}^0 \right\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C_8 \varepsilon, \quad 0 < \varepsilon \leq 1,$$

where

$$C_8^2 = \tilde{C}_7^2 \max\{2\kappa^{-1}\alpha_0^{-1}\|g^{-1}\|_{L_\infty}, c_4^{-1}\}.$$

The constant C_8 depends only on the initial data (5.33) and the parameters of the lattice Γ .

Proof. We denote

$$\Upsilon_\varepsilon := (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - \left(I + \varepsilon(\Lambda^\varepsilon b(\mathbf{D}) + \tilde{\Lambda}^\varepsilon) \Pi_\varepsilon \right) \mathcal{R}^0.$$

By (9.9) and (9.18), we have

$$\begin{aligned} \frac{\kappa}{2} \|(g^\varepsilon)^{1/2} b(\mathbf{D}) \Upsilon_\varepsilon \mathbf{F}\|_{L_2(\mathbb{R}^d)}^2 + c_4 \|\Upsilon_\varepsilon \mathbf{F}\|_{L_2(\mathbb{R}^d)}^2 &\leq \mathbf{b}_\varepsilon[\Upsilon_\varepsilon \mathbf{F}, \Upsilon_\varepsilon \mathbf{F}] + \lambda(Q_0^\varepsilon \Upsilon_\varepsilon \mathbf{F}, \Upsilon_\varepsilon \mathbf{F})_{L_2(\mathbb{R}^d)} \\ &= \|(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{1/2} \Upsilon_\varepsilon \mathbf{F}\|_{L_2(\mathbb{R}^d)}^2 \leq \tilde{C}_7^2 \varepsilon^2 \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}^2, \quad \mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1. \end{aligned}$$

Combining this with the lower estimate (9.1), for $0 < \varepsilon \leq 1$ we obtain

$$(9.24) \quad \frac{\kappa}{2} \alpha_0 \|g^{-1}\|_{L_\infty}^{-1} \|\mathbf{D} \Upsilon_\varepsilon \mathbf{F}\|_{L_2(\mathbb{R}^d)}^2 + c_4 \|\Upsilon_\varepsilon \mathbf{F}\|_{L_2(\mathbb{R}^d)}^2 \leq \tilde{C}_7^2 \varepsilon^2 \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}^2, \\ \mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n).$$

Obviously,

$$(9.25) \quad \begin{aligned} \|\Upsilon_\varepsilon \mathbf{F}\|_{H^1(\mathbb{R}^d)}^2 &\leq \max\{2\kappa^{-1}\alpha_0^{-1}\|g^{-1}\|_{L_\infty}, c_4^{-1}\} \\ &\times \left(\frac{\kappa}{2} \alpha_0 \|g^{-1}\|_{L_\infty}^{-1} \|\mathbf{D} \Upsilon_\varepsilon \mathbf{F}\|_{L_2(\mathbb{R}^d)}^2 + c_4 \|\Upsilon_\varepsilon \mathbf{F}\|_{L_2(\mathbb{R}^d)}^2 \right). \end{aligned}$$

Relations (9.24) and (9.25) lead to (9.23). $\square$

Similarly, Theorem 9.4 implies the following result.

Theorem 9.8. *Suppose that the assumptions of Theorem 9.3 are satisfied.*

1°. *If Condition 8.3 (or 8.6) is fulfilled, then for $0 < \varepsilon \leq 1$ we have*

$$\left\| (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - \left(I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) + \varepsilon \tilde{\Lambda}^\varepsilon \Pi_\varepsilon \right) \mathcal{R}^0 \right\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C'_9 \varepsilon,$$

where

$$(C'_9)^2 = (\tilde{C}_7 + C'_\Lambda)^2 \max\{2\kappa^{-1}\alpha_0^{-1}\|g^{-1}\|_{L_\infty}, c_4^{-1}\}.$$

The constant C'_9 depends only on the initial data (5.33), the parameters of the lattice Γ , and the norm $\|\Lambda\|_{L_\infty}$.

2°. *If Condition 8.7 (or 8.10) is satisfied, then for $0 < \varepsilon \leq 1$ we have*

$$\left\| (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - \left(I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon + \varepsilon \tilde{\Lambda}^\varepsilon \right) \mathcal{R}^0 \right\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C''_9 \varepsilon,$$

where

$$(C''_9)^2 = (\tilde{C}_7 + C'_\Lambda)^2 \max\{2\kappa^{-1}\alpha_0^{-1}\|g^{-1}\|_{L_\infty}, c_4^{-1}\}.$$

The constant C''_9 depends only on the initial data (5.33), the parameters of the lattice Γ , and the norm $\|\tilde{\Lambda}\|_{L_p(\Omega)}$.

3°. If both Condition 8.3 (or 8.6) and Condition 8.7 (or 8.10) are satisfied, then for $0 < \varepsilon \leq 1$ we have

$$(9.26) \quad \left\| (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - \left(I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) + \varepsilon \tilde{\Lambda}^\varepsilon \right) \mathcal{R}^0 \right\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C_9 \varepsilon,$$

where

$$C_9^2 = (\tilde{C}_7 + C'_\Lambda + C'_\Lambda)^2 \max\{2\kappa^{-1}\alpha_0^{-1}\|g^{-1}\|_{L_\infty}, c_4^{-1}\}.$$

The constant C_9 depends only on the initial data (5.33), the parameters of the lattice Γ , and the norms $\|\Lambda\|_{L_\infty}$, $\|\tilde{\Lambda}\|_{L_p(\Omega)}$.

Proposition 9.5 implies the following statement.

Proposition 9.9. *Under the assumptions of Theorem 9.3, let $\mathcal{A} = \mathbf{D}^* g(\mathbf{x}) \mathbf{D}$, where the matrix $g(\mathbf{x})$ has real entries. Then estimate (9.26) is valid, and the constant C_9 on the right-hand side of (9.26) depends only on the initial data (8.51) and the parameters of the lattice Γ .*

The following statement is deduced from Proposition 9.6; it concerns the case of the zero corrector.

Proposition 9.10. *Let $\mathcal{B}_\varepsilon$ be the operator (9.8), and let $\mathcal{B}^0$ be the operator (9.10). Suppose that $g^0 = \bar{g}$ and that (8.52) is true. Then*

$$\|(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - (\mathcal{B}^0 + \lambda \bar{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C_8 \varepsilon, \quad 0 < \varepsilon \leq 1.$$

The constant C_8 depends only on the initial data (5.33) and the parameters of the lattice Γ .

§10. APPROXIMATION OF THE FLUX

10.1. We introduce the notation

$$(10.1) \quad \mathbf{u}_\varepsilon = (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} \mathbf{F}, \quad \mathbf{u}_0 = (\mathcal{B}^0 + \lambda \bar{Q}_0)^{-1} \mathbf{F},$$

where $\mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$; we want to obtain an approximation in $L_2(\mathbb{R}^d; \mathbb{C}^m)$ for the so-called flux

$$(10.2) \quad \mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D}) \mathbf{u}_\varepsilon.$$

It is convenient to apply estimate (8.13). By the scaling transformation, this estimate implies that

$$(10.3) \quad \|(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{1/2} \mathcal{T}_\varepsilon\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_7 \varepsilon, \quad 0 < \varepsilon \leq 1,$$

where

$$\mathcal{T}_\varepsilon := (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - (\mathcal{B}^0 + \lambda \bar{Q}_0)^{-1} \Pi_\varepsilon - \varepsilon (\Lambda^\varepsilon b(\mathbf{D}) + \tilde{\Lambda}^\varepsilon) (\mathcal{B}^0 + \lambda \bar{Q}_0)^{-1} \Pi_\varepsilon.$$

From (9.9) and (10.3) it follows that

$$\begin{aligned} \frac{\kappa}{2} \|(g^\varepsilon)^{1/2} b(\mathbf{D}) \mathcal{T}_\varepsilon \mathbf{F}\|_{L_2(\mathbb{R}^d)}^2 &\leq \mathbf{b}_\varepsilon[\mathcal{T}_\varepsilon \mathbf{F}, \mathcal{T}_\varepsilon \mathbf{F}] + \lambda (Q_0^\varepsilon \mathcal{T}_\varepsilon \mathbf{F}, \mathcal{T}_\varepsilon \mathbf{F})_{L_2(\mathbb{R}^d)} \\ &= \|(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{1/2} \mathcal{T}_\varepsilon \mathbf{F}\|_{L_2(\mathbb{R}^d)}^2 \leq C_7^2 \varepsilon^2 \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}^2, \\ &\quad \mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1. \end{aligned}$$

Hence,

$$(10.4) \quad \begin{aligned} \|g^\varepsilon b(\mathbf{D}) \mathcal{T}_\varepsilon \mathbf{F}\|_{L_2(\mathbb{R}^d)} &\leq \sqrt{2} \kappa^{-1/2} C_7 \|g\|_{L_\infty}^{1/2} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \\ &\quad \mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1. \end{aligned}$$

We transform the vector-valued function $g^\varepsilon b(\mathbf{D})\mathcal{T}_\varepsilon \mathbf{F}$ using the notation (10.1) and (10.2):

$$\begin{aligned}
 g^\varepsilon b(\mathbf{D})\mathcal{T}_\varepsilon \mathbf{F} &= \mathbf{p}_\varepsilon - g^\varepsilon \Pi_\varepsilon b(\mathbf{D})\mathbf{u}_0 - \varepsilon g^\varepsilon b(\mathbf{D})(\Lambda^\varepsilon \Pi_\varepsilon b(\mathbf{D})\mathbf{u}_0 + \tilde{\Lambda}^\varepsilon \Pi_\varepsilon \mathbf{u}_0) \\
 &= \mathbf{p}_\varepsilon - g^\varepsilon \Pi_\varepsilon b(\mathbf{D})\mathbf{u}_0 - g^\varepsilon (b(\mathbf{D})\Lambda)^\varepsilon \Pi_\varepsilon b(\mathbf{D})\mathbf{u}_0 - g^\varepsilon (b(\mathbf{D})\tilde{\Lambda})^\varepsilon \Pi_\varepsilon \mathbf{u}_0 \\
 &\quad - \varepsilon g^\varepsilon \sum_{l=1}^d b_l (\Lambda^\varepsilon \Pi_\varepsilon D_l b(\mathbf{D})\mathbf{u}_0 + \tilde{\Lambda}^\varepsilon \Pi_\varepsilon D_l \mathbf{u}_0).
 \end{aligned}
 \tag{10.5}$$

Denoting

$$\tilde{g}(\mathbf{x}) = g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m), \tag{10.6}$$

we rewrite (10.5) as

$$\begin{aligned}
 g^\varepsilon b(\mathbf{D})\mathcal{T}_\varepsilon \mathbf{F} &= \mathbf{p}_\varepsilon - \tilde{g}^\varepsilon \Pi_\varepsilon b(\mathbf{D})\mathbf{u}_0 - g^\varepsilon (b(\mathbf{D})\tilde{\Lambda})^\varepsilon \Pi_\varepsilon \mathbf{u}_0 \\
 &\quad - \varepsilon g^\varepsilon \sum_{l=1}^d b_l (\Lambda^\varepsilon \Pi_\varepsilon D_l b(\mathbf{D})\mathbf{u}_0 + \tilde{\Lambda}^\varepsilon \Pi_\varepsilon D_l \mathbf{u}_0).
 \end{aligned}
 \tag{10.7}$$

We estimate the last term on the right-hand side of (10.7). Note that the operator $\mathcal{B}^0 + \lambda \overline{Q}_0$ (see (9.11)) is a second order DO with constant coefficients. Its symbol is given by

$$L(\boldsymbol{\xi}) = L(\boldsymbol{\xi}, 1) = b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) - b(\boldsymbol{\xi})^* V - V^* b(\boldsymbol{\xi}) + \sum_{j=1}^d (\overline{a_j + a_j^*}) \xi_j + \overline{Q} - W + \lambda \overline{Q}_0; \tag{10.8}$$

cf. (8.5). By (8.6) (with $\varepsilon = 1$), the symbol (10.8) is subject to the estimate

$$L(\boldsymbol{\xi}) \geq \check{c}_*(|\boldsymbol{\xi}|^2 + 1)\mathbf{1}_n, \quad \boldsymbol{\xi} \in \mathbb{R}^d. \tag{10.9}$$

Using (10.1) and (10.9), we get

$$\begin{aligned}
 \|D_l \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} &= \|D_l (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} \mathbf{F}\|_{L_2(\mathbb{R}^d)} \\
 &\leq \sup_{\boldsymbol{\xi} \in \mathbb{R}^d} |\xi_l L(\boldsymbol{\xi})^{-1}| \|\mathbf{F}\|_{L_2(\mathbb{R}^d)} \leq (2\check{c}_*)^{-1} \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}.
 \end{aligned}
 \tag{10.10}$$

Similarly, using (5.2), we see that

$$\|D_l b(\mathbf{D})\mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \leq \sup_{\boldsymbol{\xi} \in \mathbb{R}^d} |\xi_l b(\boldsymbol{\xi}) L(\boldsymbol{\xi})^{-1}| \|\mathbf{F}\|_{L_2(\mathbb{R}^d)} \leq \alpha_1^{1/2} (\check{c}_*)^{-1} \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}. \tag{10.11}$$

Next, we estimate the norm of the operator $\tilde{\Lambda}^\varepsilon \Pi_\varepsilon$ in $L_2(\mathbb{R}^d; \mathbb{C}^n)$. This operator is unitarily equivalent to the operator $\tilde{\Lambda} \Pi$. In its turn, with the help of the unitary Gelfand transformation, the operator $\tilde{\Lambda} \Pi$ expands in the direct integral of the operators $[\tilde{\Lambda}]P$ acting in $L_2(\Omega; \mathbb{C}^n)$. Then

$$\|\tilde{\Lambda}^\varepsilon \Pi_\varepsilon\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} = \|\tilde{\Lambda} \Pi\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} = \|[\tilde{\Lambda}]P\|_{L_2(\Omega) \rightarrow L_2(\Omega)} = |\Omega|^{-1/2} \|\tilde{\Lambda}\|_{L_2(\Omega)}.$$

Combining this with (7.52), we obtain

$$\|\tilde{\Lambda}^\varepsilon \Pi_\varepsilon\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq |\Omega|^{-1/2} (2r_0)^{-1} C_a n^{1/2} \alpha_0^{-1} \|g^{-1}\|_{L_\infty}. \tag{10.12}$$

Similarly, by (7.38), we have

$$\|\Lambda^\varepsilon \Pi_\varepsilon\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} = |\Omega|^{-1/2} \|\Lambda\|_{L_2(\Omega)} \leq m^{1/2} (2r_0)^{-1} \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2} \|g\|_{L_\infty}^{1/2}. \tag{10.13}$$

Now, relations (10.10)–(10.13) and (5.2) imply the following estimate for the last term on the right-hand side of (10.7):

$$\begin{aligned}
 (10.14) \quad & \varepsilon \left\| g^\varepsilon \sum_{l=1}^d b_l (\Lambda^\varepsilon \Pi_\varepsilon D_l b(\mathbf{D}) \mathbf{u}_0 + \tilde{\Lambda}^\varepsilon \Pi_\varepsilon D_l \mathbf{u}_0) \right\|_{L_2(\mathbb{R}^d)} \\
 & \leq \varepsilon \|g\|_{L_\infty} \alpha_1^{1/2} \sum_{l=1}^d \left(\|\Lambda^\varepsilon \Pi_\varepsilon\|_{L_2 \rightarrow L_2} \|D_l b(\mathbf{D}) \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \right. \\
 & \quad \left. + \|\tilde{\Lambda}^\varepsilon \Pi_\varepsilon\|_{L_2 \rightarrow L_2} \|D_l \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \right) \\
 & \leq \varepsilon \|g\|_{L_\infty} \|g^{-1}\|_{L_\infty}^{1/2} d \alpha_1^{1/2} \alpha_0^{-1/2} (4r_0 \check{c}_*)^{-1} \\
 & \quad \times \left(2m^{1/2} \|g\|_{L_\infty}^{1/2} \alpha_1^{1/2} + C_a n^{1/2} \alpha_0^{-1/2} |\Omega|^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2} \right) \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}.
 \end{aligned}$$

Relations (10.4), (10.7), and (10.14) imply the following result.

Theorem 10.1. *Under the assumptions of Theorem 9.3, let $\mathbf{p}_\varepsilon$ be defined by (10.1), (10.2). Let $\mathbf{u}_0$ be given by (10.1), and let $\tilde{g}(\mathbf{x})$ be the matrix defined by (10.6). Put $\tilde{g}^\varepsilon(\mathbf{x}) = \tilde{g}(\varepsilon^{-1}\mathbf{x})$. Then for $\mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $0 < \varepsilon \leq 1$ we have*

$$(10.15) \quad \|\mathbf{p}_\varepsilon - \tilde{g}^\varepsilon \Pi_\varepsilon b(\mathbf{D}) \mathbf{u}_0 - g^\varepsilon (b(\mathbf{D}) \tilde{\Lambda})^\varepsilon \Pi_\varepsilon \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \leq C_{10} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)},$$

where

$$\begin{aligned}
 C_{10} = & \sqrt{2} \kappa^{-1/2} C_7 \|g\|_{L_\infty}^{1/2} + \|g\|_{L_\infty} \|g^{-1}\|_{L_\infty}^{1/2} d \alpha_1^{1/2} \alpha_0^{-1/2} (4r_0 \check{c}_*)^{-1} \\
 & \times \left(2m^{1/2} \|g\|_{L_\infty}^{1/2} \alpha_1^{1/2} + C_a n^{1/2} \alpha_0^{-1/2} |\Omega|^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2} \right).
 \end{aligned}$$

The constant C_{10} depends only on the initial data (5.33) and the parameters of the lattice Γ .

10.2. Elimination of Π_ε from the approximation of the flux. Suppose that Condition 8.3 is satisfied. Then, by Theorem 9.4 (1°), estimate (9.20) is valid. It can be rewritten as

$$\|(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{1/2} \mathcal{T}'_\varepsilon\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq (\tilde{C}_7 + C'_\Lambda) \varepsilon, \quad 0 < \varepsilon \leq 1,$$

where

$$\mathcal{T}'_\varepsilon = (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} - \varepsilon \Lambda^\varepsilon b(\mathbf{D}) (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} - \varepsilon \tilde{\Lambda}^\varepsilon (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} \Pi_\varepsilon.$$

Combining this with (9.9), we obtain

$$\begin{aligned}
 (10.16) \quad & \|g^\varepsilon b(\mathbf{D}) \mathcal{T}'_\varepsilon \mathbf{F}\|_{L_2(\mathbb{R}^d)} \leq \sqrt{2} \kappa^{-1/2} (\tilde{C}_7 + C'_\Lambda) \|g\|_{L_\infty}^{1/2} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \\
 & \mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1.
 \end{aligned}$$

We transform the vector-valued function $g^\varepsilon b(\mathbf{D}) \mathcal{T}'_\varepsilon \mathbf{F}$ (cf. (10.5), (10.7)):

$$\begin{aligned}
 (10.17) \quad & g^\varepsilon b(\mathbf{D}) \mathcal{T}'_\varepsilon \mathbf{F} = \mathbf{p}_\varepsilon - \tilde{g}^\varepsilon b(\mathbf{D}) \mathbf{u}_0 - g^\varepsilon (b(\mathbf{D}) \tilde{\Lambda})^\varepsilon \Pi_\varepsilon \mathbf{u}_0 \\
 & - \varepsilon g^\varepsilon \sum_{l=1}^d b_l (\Lambda^\varepsilon D_l b(\mathbf{D}) \mathbf{u}_0 + \tilde{\Lambda}^\varepsilon \Pi_\varepsilon D_l \mathbf{u}_0).
 \end{aligned}$$

We estimate the last term on the right-hand side of (10.17), using Condition 8.3 and also estimates (10.10)–(10.12):

$$\begin{aligned}
 (10.18) \quad & \varepsilon \left\| g^\varepsilon \sum_{l=1}^d b_l (\Lambda^\varepsilon D_l b(\mathbf{D}) \mathbf{u}_0 + \tilde{\Lambda}^\varepsilon \Pi_\varepsilon D_l \mathbf{u}_0) \right\|_{L_2(\mathbb{R}^d)} \\
 & \leq \varepsilon \|g\|_{L_\infty} \alpha_1^{1/2} \sum_{l=1}^d \left(\|\Lambda\|_{L_\infty} \|D_l b(\mathbf{D}) \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} + \|\tilde{\Lambda}^\varepsilon \Pi_\varepsilon\|_{L_2 \rightarrow L_2} \|D_l \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \right) \\
 & \leq \varepsilon \|g\|_{L_\infty} d \alpha_1^{1/2} (\check{c}_*)^{-1} \left(\|\Lambda\|_{L_\infty} \alpha_1^{1/2} + |\Omega|^{-1/2} C_a n^{1/2} \|g^{-1}\|_{L_\infty} (4r_0 \alpha_0)^{-1} \right) \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}.
 \end{aligned}$$

Relations (10.16)–(10.18) imply the following result.

Theorem 10.2. *Under the assumptions of Theorem 10.1, suppose also that Condition 8.3 (or Condition 8.6) is satisfied. Then for $\mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $0 < \varepsilon \leq 1$ we have*

$$(10.19) \quad \|\mathbf{p}_\varepsilon - \tilde{g}^\varepsilon b(\mathbf{D}) \mathbf{u}_0 - g^\varepsilon (b(\mathbf{D}) \tilde{\Lambda})^\varepsilon \Pi_\varepsilon \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \leq C'_{11} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)},$$

where

$$\begin{aligned}
 C'_{11} = & \sqrt{2} \kappa^{-1/2} (\tilde{C}_7 + C'_\Lambda) \|g\|_{L_\infty}^{1/2} \\
 & + \|g\|_{L_\infty} d \alpha_1^{1/2} (\check{c}_*)^{-1} \left(\|\Lambda\|_{L_\infty} \alpha_1^{1/2} + |\Omega|^{-1/2} C_a n^{1/2} \|g^{-1}\|_{L_\infty} (4r_0 \alpha_0)^{-1} \right).
 \end{aligned}$$

The constant C'_{11} depends only on the initial data (5.33), the parameters of the lattice Γ , and the norm $\|\Lambda\|_{L_\infty}$.

Now, assume that Condition 8.7 is satisfied. Then estimate (8.42) is valid. By the scaling transformation, from (8.13) and (8.42) we deduce that

$$\|(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{1/2} \mathcal{T}_\varepsilon''\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq (C_7 + C'_\Lambda) \varepsilon, \quad 0 < \varepsilon \leq 1,$$

where

$$\mathcal{T}_\varepsilon'' = (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} \Pi_\varepsilon - \varepsilon \Lambda^\varepsilon b(\mathbf{D}) (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} \Pi_\varepsilon - \varepsilon \tilde{\Lambda}^\varepsilon (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1}.$$

Combining this with (9.9), we get

$$\begin{aligned}
 (10.20) \quad & \|g^\varepsilon b(\mathbf{D}) \mathcal{T}_\varepsilon'' \mathbf{F}\|_{L_2(\mathbb{R}^d)} \leq \sqrt{2} \kappa^{-1/2} (C_7 + C'_\Lambda) \|g\|_{L_\infty}^{1/2} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \\
 & \mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1.
 \end{aligned}$$

Now, we transform the vector-valued function $g^\varepsilon b(\mathbf{D}) \mathcal{T}_\varepsilon'' \mathbf{F}$ (cf. (10.7) and (10.17)):

$$\begin{aligned}
 (10.21) \quad & g^\varepsilon b(\mathbf{D}) \mathcal{T}_\varepsilon'' \mathbf{F} = \mathbf{p}_\varepsilon - \tilde{g}^\varepsilon \Pi_\varepsilon b(\mathbf{D}) \mathbf{u}_0 - g^\varepsilon (b(\mathbf{D}) \tilde{\Lambda})^\varepsilon \mathbf{u}_0 \\
 & - \varepsilon g^\varepsilon \sum_{l=1}^d b_l (\Lambda^\varepsilon \Pi_\varepsilon D_l b(\mathbf{D}) \mathbf{u}_0 + \tilde{\Lambda}^\varepsilon D_l \mathbf{u}_0).
 \end{aligned}$$

The last term on the right-hand side of (10.21) is estimated as follows:

$$\begin{aligned}
 (10.22) \quad & \varepsilon \left\| g^\varepsilon \sum_{l=1}^d b_l (\Lambda^\varepsilon \Pi_\varepsilon D_l b(\mathbf{D}) \mathbf{u}_0 + \tilde{\Lambda}^\varepsilon D_l \mathbf{u}_0) \right\|_{L_2(\mathbb{R}^d)} \\
 & \leq \varepsilon \|g\|_{L_\infty} \alpha_1^{1/2} \sum_{l=1}^d \left(\|\Lambda^\varepsilon \Pi_\varepsilon\|_{L_2 \rightarrow L_2} \|D_l b(\mathbf{D}) \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} + \|\tilde{\Lambda}^\varepsilon D_l \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \right).
 \end{aligned}$$

In order to estimate the L_2 -norm of the function $\tilde{\Lambda}^\varepsilon D_l \mathbf{u}_0 = \tilde{\Lambda}^\varepsilon D_l (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} \mathbf{F}$, we apply the scaling transformation:

$$(10.23) \quad \|\tilde{\Lambda}^\varepsilon D_l (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} = \varepsilon \|\tilde{\Lambda} D_l (\mathcal{B}^0(\varepsilon) + \varepsilon^2 \lambda \overline{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)}.$$

By (8.44) and (8.6), we have

$$\begin{aligned}
& \varepsilon \|\tilde{\Lambda} D_l (\mathcal{B}^0(\varepsilon) + \varepsilon^2 \lambda \overline{Q_0})^{-1} \mathbf{F}\|_{L_2(\mathbb{R}^d)} \\
& \leq \varepsilon C_\Omega \|\tilde{\Lambda}\|_{L_p(\Omega)} \|D_l (\mathcal{B}^0(\varepsilon) + \varepsilon^2 \lambda \overline{Q_0})^{-1} \mathbf{F}\|_{H^1(\mathbb{R}^d)} \\
& \leq \varepsilon C_\Omega \|\tilde{\Lambda}\|_{L_p(\Omega)} \sup_{\boldsymbol{\xi} \in \mathbb{R}^d} (1 + |\boldsymbol{\xi}|^2)^{1/2} |\xi_l L(\boldsymbol{\xi}, \varepsilon)^{-1}| \|\mathbf{F}\|_{L_2(\mathbb{R}^d)} \\
& \leq \varepsilon C_\Omega \|\tilde{\Lambda}\|_{L_p(\Omega)} (\check{c}_*)^{-1} \sup_{\boldsymbol{\xi} \in \mathbb{R}^d} (1 + |\boldsymbol{\xi}|^2)^{1/2} |\boldsymbol{\xi}| (|\boldsymbol{\xi}|^2 + \varepsilon^2)^{-1} \|\mathbf{F}\|_{L_2(\mathbb{R}^d)} \\
& \leq C_\Omega \|\tilde{\Lambda}\|_{L_p(\Omega)} (\check{c}_*)^{-1} \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \quad \mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1.
\end{aligned}$$

Together with (10.23), this implies that

$$(10.24) \quad \|\tilde{\Lambda}^\varepsilon D_l \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \leq C_\Omega \|\tilde{\Lambda}\|_{L_p(\Omega)} (\check{c}_*)^{-1} \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \quad \mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1.$$

Combining (10.22), (10.24), (10.13), and (10.11), we arrive at the estimate

$$\begin{aligned}
(10.25) \quad & \varepsilon \left\| g^\varepsilon \sum_{l=1}^d b_l (\Lambda^\varepsilon \Pi_\varepsilon D_l b(\mathbf{D}) \mathbf{u}_0 + \tilde{\Lambda}^\varepsilon D_l \mathbf{u}_0) \right\|_{L_2(\mathbb{R}^d)} \leq \varepsilon \|g\|_{L_\infty} d \alpha_1^{1/2} (\check{c}_*)^{-1} \\
& \times \left(m^{1/2} (2r_0)^{-1} \alpha_1^{1/2} \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2} \|g\|_{L_\infty}^{1/2} + C_\Omega \|\tilde{\Lambda}\|_{L_p(\Omega)} \right) \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \\
& \mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1.
\end{aligned}$$

Relations (10.20), (10.21), and (10.25) imply the following result.

Theorem 10.3. *Suppose that the assumptions of Theorem 10.1 and also Condition 8.7 (or Condition 8.10) are satisfied. Then for $\mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $0 < \varepsilon \leq 1$ we have*

$$\|\mathbf{p}_\varepsilon - \tilde{g}^\varepsilon \Pi_\varepsilon b(\mathbf{D}) \mathbf{u}_0 - g^\varepsilon (b(\mathbf{D}) \tilde{\Lambda})^\varepsilon \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \leq C''_{11} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)},$$

where

$$\begin{aligned}
C''_{11} &= \sqrt{2} \kappa^{-1/2} (C_7 + C'_\Lambda) \|g\|_{L_\infty}^{1/2} \\
&+ \|g\|_{L_\infty} d \alpha_1^{1/2} (\check{c}_*)^{-1} \left(m^{1/2} (2r_0)^{-1} \alpha_1^{1/2} \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2} \|g\|_{L_\infty}^{1/2} + C_\Omega \|\tilde{\Lambda}\|_{L_p(\Omega)} \right).
\end{aligned}$$

The constant C''_{11} depends only on the initial data (5.33), the parameters of the lattice Γ , and the norm $\|\tilde{\Lambda}\|_{L_p(\Omega)}$.

Finally, suppose that the two Conditions 8.3 and 8.7 are satisfied. Then Theorem 9.4 (3°) implies estimate (9.22), which can be written as

$$\|(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{1/2} \tilde{\mathcal{T}}_\varepsilon\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq (\tilde{C}_7 + C'_\Lambda + C'_\Lambda) \varepsilon, \quad 0 < \varepsilon \leq 1,$$

where

$$\tilde{\mathcal{T}}_\varepsilon = (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - (\mathcal{B}^0 + \lambda \overline{Q_0})^{-1} - \varepsilon (\Lambda^\varepsilon b(\mathbf{D}) + \tilde{\Lambda}^\varepsilon) (\mathcal{B}^0 + \lambda \overline{Q_0})^{-1}.$$

Combining this with (9.9), we obtain

$$\begin{aligned}
(10.26) \quad & \|g^\varepsilon b(\mathbf{D}) \tilde{\mathcal{T}}_\varepsilon \mathbf{F}\|_{L_2(\mathbb{R}^d)} \leq \sqrt{2} \kappa^{-1/2} (\tilde{C}_7 + C'_\Lambda + C'_\Lambda) \|g\|_{L_\infty}^{1/2} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \\
& \mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1.
\end{aligned}$$

The vector-valued function $g^\varepsilon b(\mathbf{D}) \tilde{\mathcal{T}}_\varepsilon \mathbf{F}$ can be written as (cf. (10.7), (10.17), and (10.21)):

$$(10.27) \quad g^\varepsilon b(\mathbf{D}) \tilde{\mathcal{T}}_\varepsilon \mathbf{F} = \mathbf{p}_\varepsilon - \tilde{g}^\varepsilon b(\mathbf{D}) \mathbf{u}_0 - g^\varepsilon (b(\mathbf{D}) \tilde{\Lambda})^\varepsilon \mathbf{u}_0 - \varepsilon g^\varepsilon \sum_{l=1}^d b_l (\Lambda^\varepsilon D_l b(\mathbf{D}) \mathbf{u}_0 + \tilde{\Lambda}^\varepsilon D_l \mathbf{u}_0).$$

The last term on the right-hand side of (10.27) is estimated with the help of (10.11) and (10.24):

$$\begin{aligned}
 & \varepsilon \left\| g^\varepsilon \sum_{l=1}^d b_l (\Lambda^\varepsilon D_l b(\mathbf{D}) \mathbf{u}_0 + \tilde{\Lambda}^\varepsilon D_l \mathbf{u}_0) \right\|_{L_2(\mathbb{R}^d)} \\
 (10.28) \quad & \leq \varepsilon \|g\|_{L_\infty} \alpha_1^{1/2} \sum_{l=1}^d \left(\|\Lambda\|_{L_\infty} \|D_l b(\mathbf{D}) \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} + \|\tilde{\Lambda}^\varepsilon D_l \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \right) \\
 & \leq \varepsilon \|g\|_{L_\infty} d \alpha_1^{1/2} (\tilde{c}_*)^{-1} \left(\|\Lambda\|_{L_\infty} \alpha_1^{1/2} + C_\Omega \|\tilde{\Lambda}\|_{L_p(\Omega)} \right) \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \leq 1.
 \end{aligned}$$

Now, relations (10.26)–(10.28) imply the following result.

Theorem 10.4. *Suppose that the assumptions of Theorem 10.1 and also Condition 8.3 (or 8.6) and Condition 8.7 (or 8.10) are satisfied. Then for $\mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $0 < \varepsilon \leq 1$ we have*

$$(10.29) \quad \|\mathbf{p}_\varepsilon - \tilde{g}^\varepsilon b(\mathbf{D}) \mathbf{u}_0 - g^\varepsilon (b(\mathbf{D}) \tilde{\Lambda})^\varepsilon \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \leq C_{11} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)},$$

where

$$C_{11} = \sqrt{2} \kappa^{-1/2} (\tilde{C}_7 + C'_\Lambda + C'_\tilde{\Lambda}) \|g\|_{L_\infty}^{1/2} + \|g\|_{L_\infty} d \alpha_1^{1/2} (\tilde{c}_*)^{-1} \left(\|\Lambda\|_{L_\infty} \alpha_1^{1/2} + C_\Omega \|\tilde{\Lambda}\|_{L_p(\Omega)} \right).$$

The constant C_{11} depends only on the initial data (5.33), the parameters of the lattice Γ , and the norms $\|\Lambda\|_{L_\infty}$, $\|\tilde{\Lambda}\|_{L_p(\Omega)}$.

Theorem 10.4 and Remark 8.15 imply the following statement.

Proposition 10.5. *Under the assumptions of Theorem 10.1, let $\mathcal{A} = \mathbf{D}^* g(\mathbf{x}) \mathbf{D}$, where the matrix $g(\mathbf{x})$ has real entries. Then estimate (10.29) is valid, and the constant on the right-hand side of (10.29) depends only on the initial data (8.51) and the parameters of the lattice Γ .*

10.3. A special case. Suppose that $g^0 = \underline{g}$, which is equivalent to (6.18). Then the matrix (10.6) is constant and coincides with g^0 , i.e., $\tilde{g}(\mathbf{x}) = g^0 = \underline{g}$ (see [BSu4, Remark 3.5]). Then Condition 8.6 is satisfied. We have $\tilde{g}^\varepsilon b(\mathbf{D}) \mathbf{u}_0 = \mathbf{p}_0$, where

$$(10.30) \quad \mathbf{p}_0 := g^0 b(\mathbf{D}) \mathbf{u}_0.$$

Assume also that (8.52) is true. Then $\tilde{\Lambda}(\mathbf{x}) = 0$.

In this case, Theorem 10.2 applies. Moreover, the approximation (10.19) for $\mathbf{p}_\varepsilon$ can be simplified:

$$(10.31) \quad \|\mathbf{p}_\varepsilon - \mathbf{p}_0\|_{L_2(\mathbb{R}^d)} \leq C_{12} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}, \quad \mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1.$$

It is easily seen (cf. the proof of Theorem 10.2) that the expression for the constant in estimate (10.31) can be simplified, because $\tilde{\Lambda} = 0$:

$$C_{12} = \sqrt{2} \kappa^{-1/2} (\tilde{C}_7 + C'_\Lambda) \|g\|_{L_\infty}^{1/2} + \|g\|_{L_\infty} d \alpha_1 (\tilde{c}_*)^{-1} \|\Lambda\|_{L_\infty}.$$

Note that if $g^0 = \underline{g}$, the norm $\|\Lambda\|_{L_\infty}$ is controlled in terms of α_0 , α_1 , $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, m , n , d , and the parameters of the lattice Γ (see [BSu4, Proposition 6.9]).

As a result, we arrive at the following statement.

Proposition 10.6. *Let $\mathbf{p}_\varepsilon$ be defined by (10.2). Suppose that $g^0 = \underline{g}$ and that relation (8.52) is fulfilled. Let $\mathbf{p}_0$ be given by (10.30). Then estimate (10.31) is valid, and the constant C_{12} depends only on the initial data (5.33) and the parameters of the lattice Γ .*

§11. HOMOGENIZATION PROBLEM FOR THE OPERATOR
WITH A SINGULAR POTENTIAL

11.1. Let $v(\mathbf{x})$ be a Γ -periodic Hermitian $(n \times n)$ -matrix-valued function in $\mathbb{R}^d$ such that

$$(11.1) \quad v \in L_s(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > \frac{d}{2} \text{ for } d \geq 2; \quad \int_{\Omega} v(\mathbf{x}) d\mathbf{x} = 0.$$

We consider the homogenization problem for the operator $\mathcal{B}_\varepsilon$ generated by the quadratic form

$$\mathbf{b}_\varepsilon[\mathbf{u}, \mathbf{u}] = \mathbf{a}_\varepsilon[\mathbf{u}, \mathbf{u}] + \varepsilon^{-1} \int_{\mathbb{R}^d} \langle v^\varepsilon(\mathbf{x}) \mathbf{u}, \mathbf{u} \rangle d\mathbf{x} + q_\varepsilon[\mathbf{u}, \mathbf{u}], \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Formally, we can write

$$(11.2) \quad \mathcal{B}_\varepsilon = \mathcal{A}_\varepsilon + \varepsilon^{-1} v^\varepsilon + Q^\varepsilon.$$

The operator (11.2) contains a “singular” potential $\varepsilon^{-1} v(\varepsilon^{-1} \mathbf{x})$. We show that the operator (11.2) is a particular case of (9.8). Indeed, the matrix-valued function $v(\mathbf{x})$ can be (nonuniquely) represented as

$$(11.3) \quad v(\mathbf{x}) = - \sum_{j=1}^d D_j a_j(\mathbf{x}),$$

where the $a_j(\mathbf{x})$ are Γ -periodic $(n \times n)$ -matrix-valued functions such that

$$(11.4) \quad \begin{aligned} a_j &\in L_\rho(\Omega), \\ \rho &= \frac{ds}{d-s} > d \text{ for } d \geq 2 \text{ and } s < d, \quad d < \rho < \infty \text{ for } d \geq 2 \text{ and } s = d, \\ \rho &= \infty \text{ for } d \geq 2 \text{ and } s > d, \quad \rho = \infty \text{ for } d = 1; \\ a_j(\mathbf{x})^* &= -a_j(\mathbf{x}), \quad \int_{\Omega} a_j(\mathbf{x}) d\mathbf{x} = 0, \quad j = 1, \dots, d. \end{aligned}$$

We describe a possible way to find such a representation. Let $\Phi(\mathbf{x})$ be a Γ -periodic $(n \times n)$ -matrix-valued solution of the problem

$$\Delta \Phi(\mathbf{x}) = v(\mathbf{x}), \quad \int_{\Omega} \Phi(\mathbf{x}) d\mathbf{x} = 0.$$

Then the matrix $\Phi(\mathbf{x})$ is Hermitian (together with $v(\mathbf{x})$) and $\Phi \in \widetilde{W}_s^2(\Omega)$, by (11.1). (Here $\widetilde{W}_s^2(\Omega)$ is the “periodic” subspace of the Sobolev space $W_s^2(\Omega)$.) We put

$$a_j = D_j \Phi(\mathbf{x}), \quad j = 1, \dots, d.$$

Obviously, the required conditions (11.3) and (11.4) are satisfied (by the embedding $W_s^1(\Omega) \subset L_\rho(\Omega)$). Moreover, we have

$$\|a_j\|_{L_\rho(\Omega)} \leq \mathcal{C}_s \|v\|_{L_s(\Omega)}, \quad j = 1, \dots, d,$$

where the constant $\mathcal{C}_s$ depends only on d , s , and Ω .

Using (11.3) and relations $a_j^* = -a_j$, we obtain

$$\sum_{j=1}^d a_j^\varepsilon D_j \mathbf{u} + \sum_{j=1}^d D_j ((a_j^\varepsilon)^* \mathbf{u}) = \sum_{j=1}^d a_j^\varepsilon D_j \mathbf{u} - \sum_{j=1}^d D_j (a_j^\varepsilon \mathbf{u}) = \varepsilon^{-1} v^\varepsilon \mathbf{u}.$$

Thus, the operator (11.2) takes the form (9.8).

11.2. We construct the effective operator for the operator (11.2). Suppose that the matrix $\Lambda(\mathbf{x})$ is the periodic solution of the problem (6.10). Let g^0 be the effective matrix (6.15), and let $\mathcal{A}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$. By (11.3) and (11.4), equation (6.12) for $\tilde{\Lambda}(\mathbf{x})$ now takes the form

$$(11.5) \quad b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) \tilde{\Lambda}(\mathbf{x}) + v(\mathbf{x}) = 0, \quad \int_{\Omega} \tilde{\Lambda}(\mathbf{x}) d\mathbf{x} = 0.$$

The constant matrices V and W are defined as in (7.2) and (7.3). By (11.4), we have $\overline{a_j} = 0$. Then, by (8.2), $\mathcal{Y}^0 = -b(\mathbf{D})^* V$. As a result, the effective operator (see (9.10)) takes the form

$$(11.6) \quad \mathcal{B}^0 = \mathcal{A}^0 - b(\mathbf{D})^* V - V^* b(\mathbf{D}) + \overline{Q} - W.$$

Applying Theorems 9.2 and 9.7, we arrive at the following statement.

Proposition 11.1. *Under the assumptions of Subsections 5.1, 5.3, and 11.1, let $\mathcal{B}_\varepsilon$ be the operator (11.2), and let $\mathcal{B}^0$ be the operator (11.6). Suppose that the matrix $\Lambda(\mathbf{x})$ is the periodic solution of problem (6.10) and that the matrix $\tilde{\Lambda}(\mathbf{x})$ is the periodic solution of problem (11.5). Let $Q_0(\mathbf{x})$ be a Γ -periodic, positive definite, and bounded matrix-valued function. Suppose λ satisfies (5.27). Let Π_ε be the pseudodifferential operator (9.17). Then*

$$\begin{aligned} \|(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} &\leq \check{C}_0 \varepsilon, \quad 0 < \varepsilon \leq 1, \\ \left\| (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - \left(I + \varepsilon (\Lambda^\varepsilon b(\mathbf{D}) + \tilde{\Lambda}^\varepsilon \Pi_\varepsilon) (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} \right) \right\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} &\leq C_8 \varepsilon, \\ &0 < \varepsilon \leq 1. \end{aligned}$$

The constants $\check{C}_0$ and C_8 depend only on $d, m, n, s, \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|v\|_{L_s(\Omega)}, \tilde{c}, c_0, \tilde{c}_2, c_3, |\lambda|, \|Q_0\|_{L_\infty}$, and the parameters of the lattice Γ .

Under additional assumptions on Λ and $\tilde{\Lambda}$, Theorem 9.8 is also applicable.

§12. HOMOGENIZATION PROBLEM FOR THE OPERATOR $\tilde{\mathcal{B}}_\varepsilon$

12.1. The operator $\tilde{\mathcal{B}}_\varepsilon$. Now, we consider the operator $\tilde{\mathcal{B}}_\varepsilon$ acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and given formally by the expression

$$(12.1) \quad \tilde{\mathcal{B}}_\varepsilon = (f^\varepsilon)^* \mathcal{B}_\varepsilon f^\varepsilon.$$

Here $f(\mathbf{x})$ is a Γ -periodic $(n \times n)$ -matrix-valued function in $\mathbb{R}^d$ such that $f, f^{-1} \in L_\infty$. The operator $\mathcal{B}_\varepsilon$ was defined in Subsection 9.1. Strictly speaking, $\tilde{\mathcal{B}}_\varepsilon$ is the selfadjoint operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ generated by the quadratic form

$$\tilde{\mathbf{b}}_\varepsilon[\mathbf{u}, \mathbf{u}] = \mathbf{b}_\varepsilon[f^\varepsilon \mathbf{u}, f^\varepsilon \mathbf{u}], \quad f^\varepsilon \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n),$$

where $\mathbf{b}_\varepsilon$ is the quadratic form (9.3). Formally, $\tilde{\mathcal{B}}_\varepsilon$ can be written as

$$\tilde{\mathcal{B}}_\varepsilon = \tilde{\mathcal{A}}_\varepsilon + \sum_{j=1}^d (f^\varepsilon)^* (a_j^\varepsilon D_j + D_j (a_j^\varepsilon)^*) f^\varepsilon + \tilde{Q}^\varepsilon,$$

where

$$(12.2) \quad \tilde{\mathcal{A}}_\varepsilon = (f^\varepsilon)^* b(\mathbf{D})^* g^\varepsilon b(\mathbf{D}) f^\varepsilon,$$

and $\tilde{Q}^\varepsilon = (f^\varepsilon)^* Q^\varepsilon f^\varepsilon$ should be viewed as the generalized potential corresponding to the measure $d\tilde{\mu}^\varepsilon = (f^\varepsilon)^* (d\mu^\varepsilon) f^\varepsilon$.

Next, let $\tilde{Q}_0(\mathbf{x})$ be a Γ -periodic $(n \times n)$ -matrix-valued function such that

$$(12.3) \quad \tilde{Q}_0(\mathbf{x}) > 0; \quad \tilde{Q}_0, \tilde{Q}_0^{-1} \in L_\infty.$$

Let $Q_0(\mathbf{x})$ be the matrix defined by

$$(12.4) \quad Q_0(\mathbf{x}) = (f(\mathbf{x})^*)^{-1} \tilde{Q}_0(\mathbf{x}) f(\mathbf{x})^{-1}.$$

The problem consists in approximating the generalized resolvent $(\tilde{\mathcal{B}}_\varepsilon + \lambda \tilde{Q}_0^\varepsilon)^{-1}$ for small ε . The following parameters play the role of the “initial data” for this problem:

$$(12.5) \quad \begin{aligned} & d, m, n, \rho; \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|a_j\|_{L_\rho(\Omega)}, j = 1, \dots, d; \\ & \tilde{c}, c_0, \tilde{c}_2, c_3 \text{ from Condition 5.2; } |\lambda|, \|\tilde{Q}_0\|_{L_\infty}, \|f^{-1}\|_{L_\infty}. \end{aligned}$$

We assume that

$$(12.6) \quad \begin{aligned} & \lambda \geq \|\tilde{Q}_0^{-1}\|_{L_\infty} \|f\|_{L_\infty}^2 (c_0 + 2c_4) \text{ if } \lambda \geq 0, \\ & \lambda \geq \|\tilde{Q}_0\|_{L_\infty}^{-1} \|f^{-1}\|_{L_\infty}^{-2} (c_0 + 2c_4) \text{ if } \lambda < 0 \text{ (and } c_0 + 2c_4 < 0). \end{aligned}$$

This ensures relation (5.27).

12.2. Results on approximation for the generalized resolvent of the operator $\tilde{\mathcal{B}}_\varepsilon$. The results on homogenization of the operator (12.1) can be deduced from Theorems 9.2, 9.7, and 9.8. By (12.1) and (12.4), we have

$$(12.7) \quad f^\varepsilon (\tilde{\mathcal{B}}_\varepsilon + \lambda \tilde{Q}_0^\varepsilon)^{-1} = (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} ((f^\varepsilon)^*)^{-1}.$$

Multiplying the operators under the norm sign in (9.13) by $(f^\varepsilon)^{-1}$ from the left and by $((f^\varepsilon)^*)^{-1}$ from the right, we arrive at the following result.

Theorem 12.1. *Let $\mathcal{B}_\varepsilon$ be the operator (9.8), and let $\mathcal{B}^0$ be the operator (9.10). Let $\tilde{\mathcal{B}}_\varepsilon$ be the operator (12.1), where f is a Γ -periodic $(n \times n)$ -matrix-valued function such that $f, f^{-1} \in L_\infty$. Suppose that $\tilde{Q}_0(\mathbf{x})$ is a Γ -periodic $(n \times n)$ -matrix-valued function satisfying conditions (12.3). Let $Q_0(\mathbf{x})$ be the matrix defined by (12.4). Suppose that λ satisfies restriction (12.6). Then for $0 < \varepsilon \leq 1$ we have*

$$(12.8) \quad \|(\tilde{\mathcal{B}}_\varepsilon + \lambda \tilde{Q}_0^\varepsilon)^{-1} - (f^\varepsilon)^{-1} (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} ((f^\varepsilon)^*)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \check{C}_0 \|f^{-1}\|_{L_\infty}^2 \varepsilon.$$

The constant $\check{C}_0 \|f^{-1}\|_{L_\infty}^2$ on the right-hand side of (12.8) depends only on the initial data (12.5) and the parameters of the lattice Γ .

Now we multiply the operators under the norm sign in (9.23) by $((f^\varepsilon)^*)^{-1}$ from the right and use the fact that the norm of the operator of multiplication by $((f^\varepsilon)^*)^{-1}$ in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ is dominated by $\|f^{-1}\|_{L_\infty}$. Theorem 9.7 yields the following.

Theorem 12.2. *Under the assumptions of Theorem 12.1, let the matrix $\Lambda(\mathbf{x})$ be the periodic solution of problem (6.10), and let $\tilde{\Lambda}(\mathbf{x})$ be the periodic solution of problem (6.12). Let Π_ε be the operator (9.17). Then for $0 < \varepsilon \leq 1$ we have*

$$(12.9) \quad \begin{aligned} & \left\| f^\varepsilon (\tilde{\mathcal{B}}_\varepsilon + \lambda \tilde{Q}_0^\varepsilon)^{-1} - \left(I + \varepsilon (\Lambda^\varepsilon b(\mathbf{D}) + \tilde{\Lambda}^\varepsilon) \Pi_\varepsilon \right) (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} ((f^\varepsilon)^*)^{-1} \right\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \\ & \leq C_8 \|f^{-1}\|_{L_\infty} \varepsilon. \end{aligned}$$

The constant $C_8 \|f^{-1}\|_{L_\infty}$ on the right-hand side of (12.9) depends only on the initial data (12.5) and the parameters of the lattice Γ .

Similarly, Theorem 9.8 implies the following result.

Theorem 12.3. *Suppose that the assumptions of Theorem 12.2 are satisfied.*

1° *If Condition 8.3 (or 8.6) is fulfilled, then for $0 < \varepsilon \leq 1$ we have*

$$(12.10) \quad \begin{aligned} & \left\| f^\varepsilon (\tilde{\mathcal{B}}_\varepsilon + \lambda \tilde{Q}_0^\varepsilon)^{-1} - \left(I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) + \varepsilon \tilde{\Lambda}^\varepsilon \Pi_\varepsilon \right) (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} ((f^\varepsilon)^*)^{-1} \right\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \\ & \leq C'_9 \|f^{-1}\|_{L_\infty} \varepsilon. \end{aligned}$$

The constant $C'_9 \|f^{-1}\|_{L_\infty}$ on the right-hand side of (12.10) depends only on the initial data (12.5), the parameters of the lattice Γ , and the norm $\|\Lambda\|_{L_\infty}$.

2° If Condition 8.7 (or 8.10) is satisfied, then for $0 < \varepsilon \leq 1$ we have

$$(12.11) \quad \left\| f^\varepsilon (\tilde{\mathcal{B}}_\varepsilon + \lambda \tilde{Q}_0^\varepsilon)^{-1} - \left(I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon + \varepsilon \tilde{\Lambda}^\varepsilon \right) (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} ((f^\varepsilon)^*)^{-1} \right\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \\ \leq C''_9 \|f^{-1}\|_{L_\infty} \varepsilon.$$

The constant $C''_9 \|f^{-1}\|_{L_\infty}$ on the right-hand side of (12.11) depends only on the initial data (12.5), the parameters of the lattice Γ , and the norm $\|\tilde{\Lambda}\|_{L_p(\Omega)}$.

3° If both Condition 8.3 (or 8.6) and Condition 8.7 (or 8.10) are satisfied, then for $0 < \varepsilon \leq 1$ we have

$$(12.12) \quad \left\| f^\varepsilon (\tilde{\mathcal{B}}_\varepsilon + \lambda \tilde{Q}_0^\varepsilon)^{-1} - \left(I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) + \varepsilon \tilde{\Lambda}^\varepsilon \right) (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} ((f^\varepsilon)^*)^{-1} \right\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \\ \leq C_9 \|f^{-1}\|_{L_\infty} \varepsilon.$$

The constant $C_9 \|f^{-1}\|_{L_\infty}$ on the right-hand side of (12.12) depends only on the initial data (12.5), the parameters of the lattice Γ , and the norms $\|\Lambda\|_{L_\infty}$, $\|\tilde{\Lambda}\|_{L_p(\Omega)}$.

Next, Proposition 9.9 implies the following statement.

Proposition 12.4. *Under the assumptions of Theorem 12.2, let $\mathcal{A} = \mathbf{D}^* g(\mathbf{x}) \mathbf{D}$, where the matrix $g(\mathbf{x})$ has real entries. Then estimate (12.12) is valid, and the constant on the right-hand side depends only on the initial data*

$$(12.13) \quad d, \rho; \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|a_j\|_{L_\rho(\Omega)}, j = 1, \dots, d, \\ \tilde{c}, c_0, \tilde{c}_2, c_3; |\lambda|, \|\tilde{Q}_0\|_{L_\infty}, \|f^{-1}\|_{L_\infty},$$

and the parameters of the lattice Γ .

Finally, Proposition 9.10 shows that the following is true.

Proposition 12.5. *Under the assumptions of Theorem 12.1, suppose that $g^0 = \bar{g}$ and that relation (8.52) is fulfilled. Then for $0 < \varepsilon \leq 1$ we have*

$$(12.14) \quad \|f^\varepsilon (\tilde{\mathcal{B}}_\varepsilon + \lambda \tilde{Q}_0^\varepsilon)^{-1} - (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} ((f^\varepsilon)^*)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C_8 \|f^{-1}\|_{L_\infty} \varepsilon.$$

The constant $C_8 \|f^{-1}\|_{L_\infty}$ on the right-hand side of (12.14) depends only on the initial data (12.5) and the parameters of the lattice Γ .

12.3. Approximation of the flux. We introduce the notation

$$(12.15) \quad \tilde{\mathbf{u}}_\varepsilon = (\tilde{\mathcal{B}}_\varepsilon + \lambda \tilde{Q}_0^\varepsilon)^{-1} \mathbf{F}, \quad \mathbf{u}_\varepsilon^0 = (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} ((f^\varepsilon)^*)^{-1} \mathbf{F},$$

where $\mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$, and study approximation in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ for the flux

$$(12.16) \quad \tilde{\mathbf{p}}_\varepsilon = g^\varepsilon b(\mathbf{D}) f^\varepsilon \tilde{\mathbf{u}}_\varepsilon.$$

We apply Theorem 10.1. Estimate (10.15) can be rewritten in operator terms:

$$(12.17) \quad \left\| g^\varepsilon b(\mathbf{D}) (\tilde{\mathcal{B}}_\varepsilon + \lambda \tilde{Q}_0^\varepsilon)^{-1} - (\tilde{g}^\varepsilon b(\mathbf{D}) + g^\varepsilon (b(\mathbf{D}) \tilde{\Lambda})^\varepsilon) \Pi_\varepsilon (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \\ \leq C_{10} \varepsilon,$$

where $0 < \varepsilon \leq 1$. Multiplying operators under the norm sign in (12.17) by $((f^\varepsilon)^*)^{-1}$ from the right and using (12.7), we obtain

$$\left\| g^\varepsilon b(\mathbf{D}) f^\varepsilon (\tilde{\mathcal{B}}_\varepsilon + \lambda \tilde{Q}_0^\varepsilon)^{-1} - (\tilde{g}^\varepsilon b(\mathbf{D}) + g^\varepsilon (b(\mathbf{D}) \tilde{\Lambda})^\varepsilon) \Pi_\varepsilon (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1} ((f^\varepsilon)^*)^{-1} \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \\ \leq C_{10} \|f^{-1}\|_{L_\infty} \varepsilon, \quad 0 < \varepsilon \leq 1.$$

This implies the following result.

Theorem 12.6. *Under the assumptions of Theorem 12.2, let $\tilde{\mathbf{p}}_\varepsilon$ and $\mathbf{u}_\varepsilon^0$ be defined by (12.15), (12.16), respectively. Then for $\mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $0 < \varepsilon \leq 1$ we have*

$$(12.18) \quad \|\tilde{\mathbf{p}}_\varepsilon - \tilde{g}^\varepsilon \Pi_\varepsilon b(\mathbf{D}) \mathbf{u}_\varepsilon^0 - g^\varepsilon(b(\mathbf{D}) \tilde{\Lambda})^\varepsilon \Pi_\varepsilon \mathbf{u}_\varepsilon^0\|_{L_2(\mathbb{R}^d)} \leq C_{10} \|f^{-1}\|_{L_\infty} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}.$$

The constant $C_{10} \|f^{-1}\|_{L_\infty}$ on the right-hand side of (12.18) depends only on the initial data (12.5) and the parameters of the lattice Γ .

Similarly, Theorems 10.2, 10.3, 10.4 yield the following.

Theorem 12.7. *Suppose that the assumptions of Theorem 12.6 are fulfilled.*

1°. *If Condition 8.3 (or 8.6) is satisfied, then for $\mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $0 < \varepsilon \leq 1$ we have*

$$(12.19) \quad \|\tilde{\mathbf{p}}_\varepsilon - \tilde{g}^\varepsilon b(\mathbf{D}) \mathbf{u}_\varepsilon^0 - g^\varepsilon(b(\mathbf{D}) \tilde{\Lambda})^\varepsilon \Pi_\varepsilon \mathbf{u}_\varepsilon^0\|_{L_2(\mathbb{R}^d)} \leq C'_{11} \|f^{-1}\|_{L_\infty} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}.$$

The constant $C'_{11} \|f^{-1}\|_{L_\infty}$ on the right-hand side of (12.19) depends only on the initial data (12.5), the parameters of the lattice Γ , and the norm $\|\tilde{\Lambda}\|_{L_\infty}$.

2°. *If Condition 8.7 (or 8.10) is satisfied, then for $\mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $0 < \varepsilon \leq 1$ we have*

$$(12.20) \quad \|\tilde{\mathbf{p}}_\varepsilon - \tilde{g}^\varepsilon \Pi_\varepsilon b(\mathbf{D}) \mathbf{u}_\varepsilon^0 - g^\varepsilon(b(\mathbf{D}) \tilde{\Lambda})^\varepsilon \mathbf{u}_\varepsilon^0\|_{L_2(\mathbb{R}^d)} \leq C''_{11} \|f^{-1}\|_{L_\infty} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}.$$

The constant $C''_{11} \|f^{-1}\|_{L_\infty}$ on the right-hand side of (12.20) depends only on the initial data (12.5), the parameters of the lattice Γ , and the norm $\|\tilde{\Lambda}\|_{L_p(\Omega)}$.

3°. *If both Condition 8.3 (or 8.6) and Condition 8.7 (or 8.10) are satisfied, then for $\mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $0 < \varepsilon \leq 1$ we have*

$$(12.21) \quad \|\tilde{\mathbf{p}}_\varepsilon - \tilde{g}^\varepsilon b(\mathbf{D}) \mathbf{u}_\varepsilon^0 - g^\varepsilon(b(\mathbf{D}) \tilde{\Lambda})^\varepsilon \mathbf{u}_\varepsilon^0\|_{L_2(\mathbb{R}^d)} \leq C_{11} \|f^{-1}\|_{L_\infty} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}.$$

The constant $C_{11} \|f^{-1}\|_{L_\infty}$ on the right-hand side of (12.21) depends only on the initial data (12.5), the parameters of the lattice Γ , and the norms $\|\Lambda\|_{L_\infty}$, $\|\tilde{\Lambda}\|_{L_p(\Omega)}$.

Proposition 10.5 implies the following statement.

Proposition 12.8. *Under the assumptions of Theorem 12.6, let $\mathcal{A} = \mathbf{D}^* g(\mathbf{x}) \mathbf{D}$, where the matrix $g(\mathbf{x})$ has real entries. Then estimate (12.21) is valid, and the constant on the right-hand side depends only on the initial data (12.13) and the parameters of the lattice Γ .*

Finally, Proposition 10.6 leads to the following result.

Proposition 12.9. *Let $\tilde{\mathbf{p}}_\varepsilon$ and $\mathbf{u}_\varepsilon^0$ be defined by (12.15) and (12.16). Suppose that $g^0 = \underline{g}$ and that (8.52) is true. Then*

$$(12.22) \quad \|\tilde{\mathbf{p}}_\varepsilon - g^0 b(\mathbf{D}) \mathbf{u}_\varepsilon^0\|_{L_2(\mathbb{R}^d)} \leq C_{12} \|f^{-1}\|_{L_\infty} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)},$$

$$\mathbf{F} \in L_2(\mathbb{R}^d; \mathbb{C}^n), \quad 0 < \varepsilon \leq 1.$$

The constant $C_{12} \|f^{-1}\|_{L_\infty}$ on the right-hand side of (12.22) depends only on the initial data (12.5) and the parameters of the lattice Γ .

§13. APPLICATIONS OF GENERAL RESULTS.

THE SCHRÖDINGER OPERATOR

13.1. A scalar elliptic operator. We consider the case where $\mathcal{A}$ is an elliptic divergence type operator,

$$(13.1) \quad \mathcal{A} = -\operatorname{div} g(\mathbf{x}) \nabla = \mathbf{D}^* g(\mathbf{x}) \mathbf{D},$$

acting in $L_2(\mathbb{R}^d)$. Here $g(\mathbf{x})$ is a Γ -periodic, bounded, and positive definite symmetric $(d \times d)$ -matrix-valued function *with real entries*. The operator (13.1) is of the form (5.5) with $n = 1$, $m = d$, and $b(\mathbf{D}) = \mathbf{D}$. Herewith, $\alpha_0 = \alpha_1 = 1$ (see (5.2)). Then

$$(13.2) \quad \mathcal{A}_\varepsilon = -\operatorname{div} g^\varepsilon(\mathbf{x}) \nabla.$$

Next, let $\mathbf{A}(\mathbf{x}) = \operatorname{col}\{A_1(\mathbf{x}), \dots, A_d(\mathbf{x})\}$, where the $A_j(\mathbf{x})$ are Γ -periodic real-valued functions such that

$$(13.3) \quad A_j \in L_\rho(\Omega), \quad \rho = 2 \text{ for } d = 1, \quad \rho > d \text{ for } d \geq 2; \quad j = 1, \dots, d.$$

Suppose $v(\mathbf{x})$ is a real-valued Γ -periodic function in $\mathbb{R}^d$ such that

$$(13.4) \quad v \in L_s(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > \frac{d}{2} \text{ for } d \geq 2; \quad \int_\Omega v(\mathbf{x}) d\mathbf{x} = 0.$$

Finally, let $\mathcal{V}(\mathbf{x})$ be a Γ -periodic real-valued function such that

$$(13.5) \quad \mathcal{V} \in L_s(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > \frac{d}{2} \text{ for } d \geq 2.$$

We consider the operator $\mathcal{B}_\varepsilon$ given formally by the differential expression

$$(13.6) \quad \mathcal{B}_\varepsilon u = (\mathbf{D} - \mathbf{A}^\varepsilon(\mathbf{x}))^* g^\varepsilon(\mathbf{x}) (\mathbf{D} - \mathbf{A}^\varepsilon(\mathbf{x})) u + \varepsilon^{-1} v^\varepsilon(\mathbf{x}) u + \mathcal{V}^\varepsilon(\mathbf{x}) u.$$

More precisely, the operator $\mathcal{B}_\varepsilon$ is defined in terms of the quadratic form

$$\mathfrak{b}_\varepsilon[u, u] = \int_{\mathbb{R}^d} (\langle g^\varepsilon(\mathbf{D} - \mathbf{A}^\varepsilon)u, (\mathbf{D} - \mathbf{A}^\varepsilon)u \rangle + (\varepsilon^{-1} v^\varepsilon + \mathcal{V}^\varepsilon)|u|^2) d\mathbf{x}, \quad u \in H^1(\mathbb{R}^d).$$

The operator $\mathcal{B}_\varepsilon$ can be viewed as the periodic Schrödinger operator with the metric g^ε , the magnetic potential $\mathbf{A}^\varepsilon$, and the electric potential $\varepsilon^{-1} v^\varepsilon + \mathcal{V}^\varepsilon$ containing the “singular” term $\varepsilon^{-1} v^\varepsilon$.

We denote

$$(13.7) \quad g(\mathbf{x}) \mathbf{A}(\mathbf{x}) =: \boldsymbol{\eta}(\mathbf{x}) = \operatorname{col}\{\eta_1(\mathbf{x}), \dots, \eta_d(\mathbf{x})\}.$$

Then the first term in (13.6) can be rewritten as

$$(13.8) \quad (\mathbf{D} - \mathbf{A}^\varepsilon)^* g^\varepsilon(\mathbf{D} - \mathbf{A}^\varepsilon) u = -\operatorname{div} g^\varepsilon \nabla u + i \sum_{j=1}^d (\eta_j^\varepsilon \partial_j u + \partial_j (\eta_j^\varepsilon u)) + \langle g^\varepsilon \mathbf{A}^\varepsilon, \mathbf{A}^\varepsilon \rangle u.$$

Next, let $\Phi(\mathbf{x})$ be a Γ -periodic solution of the problem

$$\Delta \Phi(\mathbf{x}) = v(\mathbf{x}), \quad \int_\Omega \Phi(\mathbf{x}) d\mathbf{x} = 0,$$

and let $\zeta_j(\mathbf{x}) = -\partial_j \Phi(\mathbf{x})$ (cf. Subsection 11.1). Then

$$(13.9) \quad v(\mathbf{x}) = -\sum_{j=1}^d \partial_j \zeta_j(\mathbf{x}).$$

Note that the functions $\Phi(\mathbf{x})$ and $\zeta_j(\mathbf{x})$ are real-valued and that $\Phi \in \widetilde{W}_s^2(\Omega)$, $\zeta_j \in \widetilde{W}_s^1(\Omega) \subset L_\varkappa(\Omega)$, where $\varkappa = \infty$ for $d = 1$; $\varkappa = ds(d-s)^{-1}$ for $d \geq 2$ and $s < d$, $d < \varkappa < \infty$ for $d \geq 2$ and $s = d$, $\varkappa = \infty$ for $d \geq 2$ and $s > d$. Since $s > d/2$ (see (13.4)), we have $\varkappa > d$ for $d \geq 2$. We put

$$(13.10) \quad a_j(\mathbf{x}) = -\eta_j(\mathbf{x}) + i\zeta_j(\mathbf{x}).$$

Then, by (13.3) and (13.7),

$$(13.11) \quad a_j \in L_{\rho'}(\Omega), \quad \rho' = 2 \text{ for } d = 1, \quad \rho' = \min\{\rho, \varkappa\} > d \text{ for } d \geq 2; \\ j = 1, \dots, d.$$

Relations (13.9) and (13.10) show that

$$(13.12) \quad \sum_{j=1}^d (a_j^\varepsilon(\mathbf{x}) D_j u + D_j (a_j^\varepsilon(\mathbf{x}))^* u) = i \sum_{j=1}^d (\eta_j^\varepsilon(\mathbf{x}) \partial_j u + \partial_j (\eta_j^\varepsilon(\mathbf{x}) u)) + \varepsilon^{-1} v^\varepsilon(\mathbf{x}) u.$$

Comparing (13.2), (13.6), (13.8), and (13.12), we represent the operator $\mathcal{B}_\varepsilon$ as

$$\mathcal{B}_\varepsilon = \mathcal{A}_\varepsilon + \sum_{j=1}^d (a_j^\varepsilon D_j + D_j (a_j^\varepsilon)^*) + Q^\varepsilon,$$

where

$$(13.13) \quad Q(\mathbf{x}) = \mathcal{V}(\mathbf{x}) + \langle g(\mathbf{x}) \mathbf{A}(\mathbf{x}), \mathbf{A}(\mathbf{x}) \rangle.$$

By (13.5) and (13.3), we have

$$(13.14) \quad Q \in L_{s'}(\Omega), \quad s' = 1 \quad \text{for } d = 1, \quad s' = \min\{s, \rho/2\} > \frac{d}{2} \quad \text{for } d \geq 2.$$

Thus, the operator $\mathcal{B}_\varepsilon$ is represented in the form (9.8). Its coefficients satisfy all assumptions of Subsections 5.1–5.3. By (13.11), a condition of the form (5.10) is satisfied. The form q corresponding to the potential (13.13) is as in Example 5.3. A condition of the form (5.31) corresponds to (13.14). In the description of Example 5.3 it was mentioned that Condition 5.2 is satisfied with the constants $\tilde{c} = 2^{-1} \|g^{-1}\|_{L_\infty}^{-1}$, $\tilde{c}_2 = 1$; c_3 is controlled in terms of d , s' , $\|Q\|_{L_{s'}(\Omega)}$, and the parameters of the lattice Γ , and c_0 is controlled in terms of the same parameters and $\|g^{-1}\|_{L_\infty}$.

Let $Q_0(\mathbf{x})$ be a Γ -periodic, real-valued, bounded, and positive definite function on $\mathbb{R}^d$. We consider the generalized resolvent $(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1}$. Assume that the parameter λ is subject to the restriction (5.27), where c_0 and c_4 correspond to the operator (13.6). Note that the lower bound for λ in (5.27) is defined in terms of d , ρ' , s' , $\|g^{-1}\|_{L_\infty}$, $\|a_j\|_{L_{\rho'}(\Omega)}$, $j = 1, \dots, d$, $\|Q\|_{L_{s'}(\Omega)}$, $\|Q_0^{-1}\|_{L_\infty}$, $\|Q_0\|_{L_\infty}$, and the parameters of the lattice Γ . In this case, the initial data (see (5.33)) reduces to the following set:

$$(13.15) \quad d, \rho, s; \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|\mathbf{A}\|_{L_\rho(\Omega)}, \|v\|_{L_s(\Omega)}, \|\mathcal{V}\|_{L_s(\Omega)}, |\lambda|, \|Q_0\|_{L_\infty}.$$

Now, we calculate the effective operator. The periodic solution $\Lambda(\mathbf{x})$ of the problem (6.10) is the matrix-row

$$\Lambda(\mathbf{x}) = i\Psi(\mathbf{x}), \quad \Psi(\mathbf{x}) = (\psi_1(\mathbf{x}), \dots, \psi_d(\mathbf{x})),$$

where $\psi_j \in \tilde{H}^1(\Omega)$ is the solution of the problem

$$(13.16) \quad \operatorname{div} g(\mathbf{x})(\nabla \psi_j(\mathbf{x}) + \mathbf{e}_j) = 0, \quad \int_\Omega \psi_j(\mathbf{x}) d\mathbf{x} = 0.$$

Here $\mathbf{e}_1, \dots, \mathbf{e}_d$ is the standard orthonormal basis in $\mathbb{R}^d$. Clearly, the $\psi_j(\mathbf{x})$ are real-valued functions, and the row $\Lambda(\mathbf{x})$ has purely imaginary entries. Next, $\tilde{g}(\mathbf{x})$ is the $(d \times d)$ -matrix with the columns $g(\mathbf{x})(\nabla \psi_j(\mathbf{x}) + \mathbf{e}_j)$, $j = 1, \dots, d$. The effective matrix g^0 is given by

$$g^0 = |\Omega|^{-1} \int_\Omega \tilde{g}(\mathbf{x}) d\mathbf{x}.$$

The matrices $\tilde{g}(\mathbf{x})$ and g^0 have real entries. The operator (8.1) is given by $\mathcal{A}^0 = \mathbf{D}^* g^0 \mathbf{D} = -\operatorname{div} g^0 \nabla$.

By (13.9) and (13.10), the Γ -periodic solution $\tilde{\Lambda}(\mathbf{x})$ of problem (6.12) is represented as

$$(13.17) \quad \tilde{\Lambda}(\mathbf{x}) = \tilde{\Lambda}_1(\mathbf{x}) + i\tilde{\Lambda}_2(\mathbf{x}),$$

where the real-valued Γ -periodic functions $\tilde{\Lambda}_1, \tilde{\Lambda}_2$ are solutions of the problems

$$(13.18) \quad -\operatorname{div} g(\mathbf{x}) \nabla \tilde{\Lambda}_1(\mathbf{x}) + v(\mathbf{x}) = 0, \quad \int_{\Omega} \tilde{\Lambda}_1(\mathbf{x}) d\mathbf{x} = 0,$$

$$(13.19) \quad -\operatorname{div} g(\mathbf{x}) \nabla \tilde{\Lambda}_2(\mathbf{x}) + \operatorname{div} \boldsymbol{\eta}(\mathbf{x}) = 0, \quad \int_{\Omega} \tilde{\Lambda}_2(\mathbf{x}) d\mathbf{x} = 0.$$

The matrix-column V (see (7.2)) can be written as $V = V_1 + iV_2$, where the columns V_1, V_2 with *real* entries are given by

$$(13.20) \quad \begin{aligned} V_1 &= |\Omega|^{-1} \int_{\Omega} (\nabla \Psi(\mathbf{x}))^t g(\mathbf{x}) (\nabla \tilde{\Lambda}_2(\mathbf{x})) d\mathbf{x}, \\ V_2 &= -|\Omega|^{-1} \int_{\Omega} (\nabla \Psi(\mathbf{x}))^t g(\mathbf{x}) (\nabla \tilde{\Lambda}_1(\mathbf{x})) d\mathbf{x}. \end{aligned}$$

By (7.3) and (13.17), we have

$$(13.21) \quad W = |\Omega|^{-1} \int_{\Omega} (\nabla \tilde{\Lambda}_2(\mathbf{x}) + i \nabla \tilde{\Lambda}_1(\mathbf{x}))^t g(\mathbf{x}) (\nabla \tilde{\Lambda}_2(\mathbf{x}) - i \nabla \tilde{\Lambda}_1(\mathbf{x})) d\mathbf{x}.$$

Obviously, $W \geq 0$, and $W = 0$ only if $\tilde{\Lambda}(\mathbf{x}) = 0$.

We calculate the operator $\mathcal{Y}^0 + (\mathcal{Y}^0)^*$ (see (8.2)):

$$\begin{aligned} \mathcal{Y}^0 u + (\mathcal{Y}^0)^* u &= -\mathbf{D}^* V u - V^* \mathbf{D} u + \sum_{j=1}^d (\overline{a_j} + a_j^*) D_j u \\ &= i \operatorname{div} (V_1 + iV_2) u + (V_1^t - iV_2^t) i \nabla u + 2i \sum_{j=1}^d \overline{\eta_j} \partial_j u = 2i \langle V_1 + \overline{\boldsymbol{\eta}}, \nabla u \rangle. \end{aligned}$$

As a result, the effective operator $\mathcal{B}^0$ (see (9.10)) takes the form

$$\mathcal{B}^0 u = -\operatorname{div} g^0 \nabla u + 2i \langle V_1 + \overline{\boldsymbol{\eta}}, \nabla u \rangle + (\overline{Q} - W) u,$$

which can also be written as

$$(13.22) \quad \mathcal{B}^0 u = (\mathbf{D} - \mathbf{A}^0)^* g^0 (\mathbf{D} - \mathbf{A}^0) u + \mathcal{V}^0 u,$$

where

$$(13.23) \quad \mathbf{A}^0 = (g^0)^{-1} (V_1 + \overline{\boldsymbol{\eta}}) = (g^0)^{-1} (V_1 + \overline{g\mathbf{A}}),$$

$$(13.24) \quad \mathcal{V}^0 = \overline{Q} - W - \langle g^0 \mathbf{A}^0, \mathbf{A}^0 \rangle = \overline{\mathcal{V}} + \overline{\langle g\mathbf{A}, \mathbf{A} \rangle} - \langle g^0 \mathbf{A}^0, \mathbf{A}^0 \rangle - W.$$

Here we have taken (13.7) and (13.13) into account. The effective operator $\mathcal{B}^0$ can be viewed as the Schrödinger operator with constant effective coefficients: the metric g^0 , the magnetic potential $\mathbf{A}^0$, and the electric potential $\mathcal{V}^0$.

Theorem 9.2 is applicable, implying the following statement.

Proposition 13.1. *Let $\mathcal{B}_\varepsilon$ be the operator (13.6) whose coefficients satisfy the conditions of Subsection 13.1. Let $\mathcal{B}^0$ be the operator (13.22) with the coefficients defined by (13.20), (13.21), (13.23), and (13.24). Let $Q_0(\mathbf{x})$ be a Γ -periodic, positive definite, and bounded function. Suppose that the parameter λ is subject to the restriction (5.27), where c_0 and c_4 correspond to the operator (13.6). Then for $0 < \varepsilon \leq 1$ we have*

$$\|(\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - (\mathcal{B}^0 + \lambda \overline{Q}_0)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon,$$

where the constant C depends only on the initial data (13.15) and the parameters of the lattice Γ .

Proposition 9.9 is also applicable. The corrector takes the form

$$(\Lambda^\varepsilon \mathbf{D} + \tilde{\Lambda}^\varepsilon)(\mathcal{B}^0 + \lambda \overline{Q_0})^{-1} = \sum_{j=1}^d \psi_j^\varepsilon \partial_j (\mathcal{B}^0 + \lambda \overline{Q_0})^{-1} + \tilde{\Lambda}^\varepsilon (\mathcal{B}^0 + \lambda \overline{Q_0})^{-1}.$$

We arrive at the following result.

Proposition 13.2. *Under the assumptions of Proposition 13.1, let $\psi_j(\mathbf{x})$, $j = 1, \dots, d$, be the Γ -periodic solutions of problems (13.16), and let $\tilde{\Lambda}(\mathbf{x})$ be the Γ -periodic function defined by (13.17)–(13.19). Then for $0 < \varepsilon \leq 1$ we have*

$$\left\| (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} - \left(I + \varepsilon \sum_{j=1}^d \psi_j^\varepsilon \partial_j + \varepsilon \tilde{\Lambda}^\varepsilon \right) (\mathcal{B}^0 + \lambda \overline{Q_0})^{-1} \right\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C\varepsilon.$$

The constant C depends only on the initial data (13.15) and the parameters of the lattice Γ .

Applying Proposition 10.5, we get the following.

Proposition 13.3. *Under the assumptions of Proposition 13.2, let $u_\varepsilon = (\mathcal{B}_\varepsilon + \lambda Q_0^\varepsilon)^{-1} F$, where $F \in L_2(\mathbb{R}^d)$, and let $u_0 = (\mathcal{B}^0 + \lambda \overline{Q_0})^{-1} F$. Then for $0 < \varepsilon \leq 1$ we have*

$$\|g^\varepsilon \nabla u_\varepsilon - \tilde{g}^\varepsilon \nabla u_0 - g^\varepsilon (\nabla \tilde{\Lambda})^\varepsilon u_0\|_{L_2(\mathbb{R}^d)} \leq C\varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad F \in L_2(\mathbb{R}^d),$$

where the constant C depends only on the initial data (13.15) and the parameters of the lattice Γ .

Remark 13.4. If the magnetic potential $\mathbf{A}(\mathbf{x})$ obeys the gauge condition $\operatorname{div} g(\mathbf{x}) \mathbf{A}(\mathbf{x}) = 0$, then $\tilde{\Lambda}_2(\mathbf{x}) = 0$ and $V_1 = 0$ (see (13.19), (13.20)). In this case expression (13.23) for $\mathbf{A}^0$ simplifies: $\mathbf{A}^0 = (g^0)^{-1}(\overline{g\mathbf{A}})$. Expression (13.21) for W also simplifies: $W = \langle g \nabla \tilde{\Lambda}_1, \nabla \tilde{\Lambda}_1 \rangle$.

13.2. The periodic Schrödinger operator. In $L_2(\mathbb{R}^d)$, we consider the operator

$$(13.25) \quad \tilde{\mathcal{A}} = \mathbf{D}^* \tilde{g}(\mathbf{x}) \mathbf{D} + \tilde{v}(\mathbf{x}),$$

where $\tilde{g}(\mathbf{x})$ is a Γ -periodic, bounded, and positive definite symmetric $(d \times d)$ -matrix-valued function with real entries, and $\tilde{v}(\mathbf{x})$ is a real-valued Γ -periodic function such that

$$\tilde{v} \in L_s(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > \frac{d}{2} \text{ for } d \geq 2.$$

As usual, the precise definition of the operator $\tilde{\mathcal{A}}$ is given in terms of the quadratic form

$$(13.26) \quad \tilde{\mathfrak{a}}[u, u] = \int_{\mathbb{R}^d} (\langle \tilde{g}(\mathbf{x}) \mathbf{D}u, \mathbf{D}u \rangle + \tilde{v}(\mathbf{x}) |u|^2) d\mathbf{x}, \quad u \in H^1(\mathbb{R}^d).$$

Adding an appropriate constant to $\tilde{v}(\mathbf{x})$, we may assume that the bottom of the spectrum of the operator $\tilde{\mathcal{A}}$ is the point $\lambda_0 = 0$. Under this condition, the operator (13.25) admits a convenient factorization (see, e.g., [BSu2, Chapter 6, Subsection 1.1]). We describe this factorization. The equation

$$(13.27) \quad \mathbf{D}^* \tilde{g}(\mathbf{x}) \mathbf{D} \omega(\mathbf{x}) + \tilde{v}(\mathbf{x}) \omega(\mathbf{x}) = 0$$

admits a Γ -periodic solution $\omega \in \tilde{H}^1(\Omega)$ defined up to a factor. This factor can be fixed so as to ensure that the solution ω is positive and such that

$$(13.28) \quad \int_{\Omega} \omega^2(\mathbf{x}) d\mathbf{x} = |\Omega|.$$

Moreover, this solution is positive definite and bounded:

$$(13.29) \quad 0 < \omega_0 \leq \omega(\mathbf{x}) \leq \omega_1 < \infty,$$

and also $\omega \in C^\alpha$ with appropriate $\alpha > 0$. The norms $\|\omega\|_{L_\infty}$, $\|\omega^{-1}\|_{L_\infty}$ are controlled in terms of $\|\check{g}\|_{L_\infty}$, $\|\check{g}^{-1}\|_{L_\infty}$, and $\|\check{v}\|_{L_s(\Omega)}$. The function ω is a multiplier both in $H^1(\mathbb{R}^d)$ and in $\tilde{H}^1(\Omega)$. Substituting $u = \omega\varphi$, we represent the form (13.26) as

$$\tilde{\mathbf{a}}[u, u] = \int_{\mathbb{R}^d} \omega^2(\mathbf{x}) \langle \check{g}(\mathbf{x}) \mathbf{D}\varphi, \mathbf{D}\varphi \rangle d\mathbf{x}, \quad u = \omega\varphi, \quad \varphi \in H^1(\mathbb{R}^d).$$

This means that the operator (13.25) is factorized:

$$(13.30) \quad \tilde{\mathcal{A}} = \omega^{-1} \mathbf{D}^* g \mathbf{D} \omega^{-1}, \quad g = \omega^2 \check{g}.$$

Remark 13.5. Expression (13.30) can be viewed as a definition of the operator $\tilde{\mathcal{A}}$. Then we need to assume only that $\omega(\mathbf{x})$ is a Γ -periodic measurable function satisfying (13.28) and (13.29). We can return to (13.25) with the help of the identity $\check{v} = -\omega^{-1}(\mathbf{D}^* \check{g} \mathbf{D} \omega)$. The corresponding potential $\check{v}$ can be a singular distribution.

Now, we consider the operator

$$(13.31) \quad \tilde{\mathcal{A}}_\varepsilon = (\omega^\varepsilon)^{-1} \mathbf{D}^* g^\varepsilon \mathbf{D} (\omega^\varepsilon)^{-1}.$$

In the initial terms, expression (13.31) can be written as

$$(13.32) \quad \tilde{\mathcal{A}}_\varepsilon = \mathbf{D}^* \check{g}^\varepsilon \mathbf{D} + \varepsilon^{-2} \check{v}^\varepsilon.$$

We emphasize that the operator (13.32) involves the large factor ε^{-2} standing at the rapidly oscillating potential $\check{v}^\varepsilon$. The operator (13.31) is of the form (12.2) with $n = 1$, $m = d$, $b(\mathbf{D}) = \mathbf{D}$, $g = \check{g}\omega^2$, and $f = \omega^{-1}$.

Next, as above, let $\mathbf{A}(\mathbf{x}) = \text{col}\{A_1(\mathbf{x}), \dots, A_d(\mathbf{x})\}$, where the $A_j(\mathbf{x})$ are Γ -periodic real-valued functions satisfying (13.3). Let $\tilde{v}(\mathbf{x})$ be a real-valued Γ -periodic function such that

$$(13.33) \quad \tilde{v} \in L_s(\Omega), \quad s = 1 \quad \text{for } d = 1, \quad s > \frac{d}{2} \quad \text{for } d \geq 2; \quad \int_{\Omega} \tilde{v}(\mathbf{x}) \omega^2(\mathbf{x}) d\mathbf{x} = 0.$$

Finally, let $\tilde{\mathcal{V}}(\mathbf{x})$ be a real-valued Γ -periodic function such that

$$(13.34) \quad \tilde{\mathcal{V}} \in L_s(\Omega), \quad s = 1 \quad \text{for } d = 1, \quad s > \frac{d}{2} \quad \text{for } d \geq 2.$$

We consider the operator $\tilde{\mathcal{B}}_\varepsilon$ given formally by the expression

$$(13.35) \quad \tilde{\mathcal{B}}_\varepsilon u = (\mathbf{D} - \mathbf{A}^\varepsilon(\mathbf{x}))^* \check{g}^\varepsilon(\mathbf{x}) (\mathbf{D} - \mathbf{A}^\varepsilon(\mathbf{x})) u + \varepsilon^{-2} \check{v}^\varepsilon(\mathbf{x}) u + \varepsilon^{-1} \tilde{v}^\varepsilon(\mathbf{x}) u + \tilde{\mathcal{V}}^\varepsilon(\mathbf{x}) u.$$

The operator $\tilde{\mathcal{B}}_\varepsilon$ can be viewed as the periodic Schrödinger operator with the metric $\check{g}^\varepsilon$, the magnetic potential $\mathbf{A}^\varepsilon$, and the electric potential $\varepsilon^{-2} \check{v}^\varepsilon + \varepsilon^{-1} \tilde{v}^\varepsilon + \tilde{\mathcal{V}}^\varepsilon$ containing the singular terms $\varepsilon^{-2} \check{v}^\varepsilon$ and $\varepsilon^{-1} \tilde{v}^\varepsilon$.

Next, we put

$$(13.36) \quad \tilde{v}(\mathbf{x}) \omega^2(\mathbf{x}) =: v(\mathbf{x}),$$

$$(13.37) \quad \tilde{\mathcal{V}}(\mathbf{x}) \omega^2(\mathbf{x}) =: \mathcal{V}(\mathbf{x}).$$

Using (13.31) and (13.32), we check the identity

$$\tilde{\mathcal{B}}_\varepsilon = (\omega^\varepsilon)^{-1} \mathcal{B}_\varepsilon (\omega^\varepsilon)^{-1},$$

where the operator $\mathcal{B}_\varepsilon$ is given by the expression (13.6) with g defined by (13.30), v and $\mathcal{V}$ being defined by (13.36) and (13.37), respectively. By (13.33) and (13.34), the coefficients v and $\mathcal{V}$ satisfy the required conditions (13.4) and (13.5).

Let $\tilde{Q}_0(\mathbf{x})$ be a Γ -periodic, bounded, and positive definite real-valued function. We consider the generalized resolvent $(\tilde{\mathcal{B}}_\varepsilon + \lambda \tilde{Q}_0^\varepsilon)^{-1}$. Assume that the parameter λ is subject to the restriction (12.6), where $f = \omega^{-1}$, and c_0, c_4 correspond to the operator $\mathcal{B}_\varepsilon$. The initial data reduce to the following set:

$$(13.38) \quad \begin{aligned} & d, \rho, s; \quad \|\tilde{g}\|_{L_\infty}, \quad \|\tilde{g}^{-1}\|_{L_\infty}, \quad \|\mathbf{A}\|_{L_\rho(\Omega)}, \\ & \|\tilde{v}\|_{L_s(\Omega)}, \quad \|\tilde{v}\|_{L_s(\Omega)}, \quad \|\tilde{\mathcal{V}}\|_{L_s(\Omega)}, \quad |\lambda|, \quad \|\tilde{Q}_0\|_{L_\infty}. \end{aligned}$$

Theorem 12.1 is applicable; this yields the following statement.

Proposition 13.6. *Let $\tilde{\mathcal{B}}_\varepsilon$ be the operator (13.35) whose coefficients $\tilde{g}^\varepsilon, \mathbf{A}^\varepsilon, \tilde{v}^\varepsilon, \tilde{v}^\varepsilon$, and $\tilde{\mathcal{V}}^\varepsilon$ satisfy the conditions of Subsection 13.2. Let $\omega(\mathbf{x})$ be the positive Γ -periodic solution of equation (13.27) satisfying (13.28). Let $\mathcal{B}_\varepsilon$ be the operator (13.6) with the coefficients $g^\varepsilon = \tilde{g}^\varepsilon(\omega^\varepsilon)^2, \mathbf{A}^\varepsilon, v^\varepsilon = \tilde{v}^\varepsilon(\omega^\varepsilon)^2$, and $\mathcal{V}^\varepsilon = \tilde{\mathcal{V}}^\varepsilon(\omega^\varepsilon)^2$. Let $\mathcal{B}^0$ be the effective operator for $\mathcal{B}_\varepsilon$ defined by (13.22). Let $\tilde{Q}_0(\mathbf{x})$ be a Γ -periodic, bounded, and positive definite function. Suppose that the parameter λ obeys (12.6), where $f = \omega^{-1}$, and c_0, c_4 correspond to the operator $\mathcal{B}_\varepsilon$. Let $Q_0(\mathbf{x}) = \tilde{Q}_0(\mathbf{x})\omega^2(\mathbf{x})$, and let $\bar{Q}_0$ be the mean value of $Q_0(\mathbf{x})$ over Ω . Then for $0 < \varepsilon \leq 1$ we have*

$$\|(\tilde{\mathcal{B}}_\varepsilon + \lambda \tilde{Q}_0^\varepsilon)^{-1} - \omega^\varepsilon(\mathcal{B}^0 + \lambda \bar{Q}_0)^{-1}\omega^\varepsilon\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon,$$

where the constant C depends only on the initial data (13.38) and the parameters of the lattice Γ .

Applying Proposition 12.4, we get the following.

Proposition 13.7. *Under the assumptions of Proposition 13.6, let $\psi_j(\mathbf{x})$, $j = 1, \dots, d$, be the Γ -periodic solutions of problems (13.16), and let $\tilde{\Lambda}(\mathbf{x})$ be the Γ -periodic function defined by (13.17)–(13.19). Then for $0 < \varepsilon \leq 1$ we have*

$$\left\| (\omega^\varepsilon)^{-1}(\tilde{\mathcal{B}}_\varepsilon + \lambda \tilde{Q}_0^\varepsilon)^{-1} - \left(I + \varepsilon \sum_{j=1}^d \psi_j^\varepsilon \partial_j + \varepsilon \tilde{\Lambda}^\varepsilon \right) (\mathcal{B}^0 + \lambda \bar{Q}_0)^{-1} \omega^\varepsilon \right\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C\varepsilon.$$

The constant C depends only on the initial data (13.38) and the parameters of the lattice Γ .

Proposition 12.8 about approximation of the flux is also applicable, implying the following statement.

Proposition 13.8. *Under the assumptions of Proposition 13.7, $\tilde{u}_\varepsilon = (\tilde{\mathcal{B}}_\varepsilon + \lambda \tilde{Q}_0^\varepsilon)^{-1} F$, where $F \in L_2(\mathbb{R}^d)$. Let $u_\varepsilon^0 = (\mathcal{B}^0 + \lambda \bar{Q}_0)^{-1}(\omega^\varepsilon F)$. Then for $0 < \varepsilon \leq 1$ we have*

$$\|g^\varepsilon \nabla((\omega^\varepsilon)^{-1} \tilde{u}_\varepsilon) - \tilde{g}^\varepsilon \nabla u_\varepsilon^0 - g^\varepsilon (\nabla \tilde{\Lambda})^\varepsilon u_\varepsilon^0\|_{L_2(\mathbb{R}^d)} \leq C\varepsilon \|F\|_{L_2(\mathbb{R}^d)},$$

where the constant C depends only on the initial data (13.38) and the parameters of the lattice Γ .

The expressions for $\mathbf{A}^0$ and W simplify if the magnetic potential $\mathbf{A}(\mathbf{x})$ is subject to the gauge condition $\operatorname{div}(\tilde{g}\omega^2 \mathbf{A}) = 0$ (see Remark 13.4).

§14. APPLICATIONS OF GENERAL RESULTS. THE TWO-DIMENSIONAL PAULI OPERATOR

14.1. Model examples. We start with the model examples that will be useful for the study of the two-dimensional Pauli operator.

Let $d = 2$, and let $\omega_\pm(\mathbf{x})$ be Γ -periodic functions in $\mathbb{R}^2$ such that

$$(14.1\pm) \quad \omega_\pm(\mathbf{x}) > 0, \quad \omega_\pm, \omega_\pm^{-1} \in L_\infty.$$

In $L_2(\mathbb{R}^2)$, we consider the pair of operators

$$(14.2\pm) \quad \mathcal{A}_+ = \partial_+ \omega_+^2 \partial_-, \quad \mathcal{A}_- = \partial_- \omega_-^2 \partial_+,$$

where $\partial_\pm = D_1 \pm iD_2$. The operators $\mathcal{A}_+$ and $\mathcal{A}_-$ are of the same type; it is convenient to study them in parallel. The operator $\mathcal{A}_\pm$ is of the form (5.5) with $m = n = 1$, $b(\mathbf{D}) = D_1 \mp iD_2$, and $g = g_\pm = \omega_\pm^2$. Obviously, $\alpha_0 = \alpha_1 = 1$ (see (5.2)).

Consider the operators

$$(14.3\pm) \quad \mathcal{A}_{+,\varepsilon} = \partial_+ (\omega_+^\varepsilon)^2 \partial_-, \quad \mathcal{A}_{-,\varepsilon} = \partial_- (\omega_-^\varepsilon)^2 \partial_+.$$

Since $m = n$, the effective constant $g_\pm^0$ for the operator $\mathcal{A}_\pm$ coincides with $\underline{g}_\pm$, i.e.,

$$(14.4\pm) \quad g_\pm^0 = \left(|\Omega|^{-1} \int_\Omega (\omega_\pm(\mathbf{x}))^{-2} d\mathbf{x} \right)^{-1}.$$

The corresponding effective operator (see (8.1)) is

$$(14.5\pm) \quad \mathcal{A}_\pm^0 = -g_\pm^0 \Delta.$$

Next, let $\boldsymbol{\eta}_\pm(\mathbf{x}) = \text{col}\{\eta_{1,\pm}(\mathbf{x}), \eta_{2,\pm}(\mathbf{x})\}$, where $\eta_{1,\pm}(\mathbf{x})$ and $\eta_{2,\pm}(\mathbf{x})$ are real-valued Γ -periodic functions in $\mathbb{R}^2$ such that

$$(14.6\pm) \quad \eta_{j,\pm} \in L_\rho(\Omega), \quad \rho > 2, \quad j = 1, 2.$$

Let $v_\pm(\mathbf{x})$ be a real-valued Γ -periodic function in $\mathbb{R}^2$ such that

$$v_\pm \in L_s(\Omega), \quad s > 1; \quad \int_\Omega v_\pm(\mathbf{x}) d\mathbf{x} = 0.$$

Finally, let $Q_\pm(\mathbf{x})$ be a real-valued Γ -periodic function in $\mathbb{R}^2$ such that

$$(14.7\pm) \quad Q_\pm \in L_s(\Omega), \quad s > 1.$$

We consider the operator $\mathcal{B}_{\pm,\varepsilon}$ given formally by the expression

$$(14.8\pm) \quad \mathcal{B}_{\pm,\varepsilon} u = \mathcal{A}_{\pm,\varepsilon} u + i \sum_{j=1}^2 (\eta_{j,\pm}^\varepsilon(\mathbf{x}) \partial_j u + \partial_j (\eta_{j,\pm}^\varepsilon(\mathbf{x}) u)) + \varepsilon^{-1} v_\pm^\varepsilon(\mathbf{x}) u + Q_\pm^\varepsilon(\mathbf{x}) u.$$

More precisely, the operator $\mathcal{B}_{\pm,\varepsilon}$ is defined in terms of the quadratic form

$$\mathbf{b}_{\pm,\varepsilon}[u, u] = \int_{\mathbb{R}^2} \left(|\omega_\pm^\varepsilon \partial_\mp u|^2 - 2 \operatorname{Re} \sum_{j=1}^2 \eta_{j,\pm}^\varepsilon (D_j u) u^* + (\varepsilon^{-1} v_\pm^\varepsilon + Q_\pm^\varepsilon) |u|^2 \right) d\mathbf{x}, \quad u \in H^1(\mathbb{R}^2).$$

Next, let $\Phi_\pm(\mathbf{x})$ be a Γ -periodic solution of the problem

$$\Delta \Phi_\pm(\mathbf{x}) = v_\pm(\mathbf{x}), \quad \int_\Omega \Phi_\pm(\mathbf{x}) d\mathbf{x} = 0,$$

and let $\zeta_{j,\pm}(\mathbf{x}) = -\partial_j \Phi_\pm(\mathbf{x})$ (cf. Subsection 11.1). Then

$$(14.9\pm) \quad v_\pm(\mathbf{x}) = - \sum_{j=1}^d \partial_j \zeta_{j,\pm}(\mathbf{x}).$$

Here, the functions $\Phi_\pm(\mathbf{x})$ and $\zeta_{j,\pm}(\mathbf{x})$ are real-valued, and $\Phi_\pm \in \widetilde{W}_s^2(\Omega)$, $\zeta_{j,\pm} \in \widetilde{W}_s^1(\Omega) \subset L_\varkappa(\Omega)$, where $\varkappa = 2s(2-s)^{-1}$ for $s < 2$, $2 < \varkappa < \infty$ for $s = 2$, $\varkappa = \infty$ for $s > 2$. Since $s > 1$, we have $\varkappa > 2$. Put

$$(14.10\pm) \quad a_{j,\pm}(\mathbf{x}) = -\eta_{j,\pm}(\mathbf{x}) + i\zeta_{j,\pm}(\mathbf{x}).$$

By (14.6 $\pm$), we have

$$(14.11\pm) \quad a_{j,\pm} \in L_{\rho'}(\Omega), \quad \rho' = \min\{\rho, \varkappa\} > 2, \quad j = 1, 2.$$

Relations (14.9 $\pm$) and (14.10 $\pm$) imply that

$$(14.12\pm) \quad \begin{aligned} & \sum_{j=1}^2 (a_{j,\pm}^\varepsilon(\mathbf{x}) D_j u + D_j (a_{j,\pm}^\varepsilon(\mathbf{x}))^* u) \\ &= i \sum_{j=1}^2 (\eta_{j,\pm}^\varepsilon(\mathbf{x}) \partial_j u + \partial_j (\eta_{j,\pm}^\varepsilon(\mathbf{x}) u)) + \varepsilon^{-1} v_\pm^\varepsilon(\mathbf{x}) u. \end{aligned}$$

Comparing (14.8 $\pm$) and (14.12 $\pm$), we write the operator $\mathcal{B}_{\pm,\varepsilon}$ as

$$(14.13\pm) \quad \mathcal{B}_{\pm,\varepsilon} = \mathcal{A}_{\pm,\varepsilon} + \sum_{j=1}^2 (a_{j,\pm}^\varepsilon D_j + D_j (a_{j,\pm}^\varepsilon)^*) + Q_\pm^\varepsilon.$$

Thus, the operator $\mathcal{B}_{\pm,\varepsilon}$ is represented in the form (9.8). Its coefficients satisfy all the conditions of Subsections 5.1–5.3. A condition of the form (5.10) is satisfied due to (14.11 $\pm$). The form generated by the potential $Q_\pm$ corresponds to Example 5.3. Condition (5.31) corresponds to (14.7 $\pm$). In the description of Example 5.3 it was mentioned that Condition 5.2 is satisfied with $\tilde{c} = 2^{-1} \|\omega_\pm^{-1}\|_{L_\infty}^{-2}$, $\tilde{c}_2 = 1$; the constant c_3 is controlled in terms of s , $\|Q_\pm\|_{L_s(\Omega)}$, and the parameters of the lattice Γ , and c_0 is controlled in terms of the same parameters and $\|\omega_\pm^{-1}\|_{L_\infty}$.

Suppose that $Q_{0,\pm}(\mathbf{x})$ is a Γ -periodic, bounded, and positive definite function on $\mathbb{R}^2$. We consider the generalized resolvent $(\mathcal{B}_{\pm,\varepsilon} + \lambda Q_{0,\pm}^\varepsilon)^{-1}$, assuming that the parameter λ is subject to the restriction of the form (5.27) with $Q_0 = Q_{0,\pm}$ and c_0, c_4 corresponding to the operator $\mathcal{B}_{\pm,\varepsilon}$. Note that the lower bound for λ in (5.27) is defined in terms of $s, \rho', \|\omega_\pm^{-1}\|_{L_\infty}, \|a_{j,\pm}\|_{L_{\rho'}(\Omega)}, j = 1, 2, \|Q_\pm\|_{L_s(\Omega)}, \|Q_{0,\pm}\|_{L_\infty}, \|Q_{0,\pm}^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ . The initial data is reduced to the following set:

$$(14.14\pm) \quad \rho, s; \|\omega_\pm\|_{L_\infty}, \|\omega_\pm^{-1}\|_{L_\infty}, \|\boldsymbol{\eta}_\pm\|_{L_\rho(\Omega)}, \|v_\pm\|_{L_s(\Omega)}, \|Q_\pm\|_{L_s(\Omega)}, |\lambda|, \|Q_{0,\pm}\|_{L_\infty}.$$

Now, we calculate the effective operator. The role of $\Lambda(\mathbf{x})$ (see problem (6.10)) is played by the function $\Lambda_\pm(\mathbf{x})$ that is a Γ -periodic solution of the problem

$$(14.15\pm) \quad \partial_\pm \omega_\pm^2(\mathbf{x}) (\partial_\mp \Lambda_\pm(\mathbf{x}) + 1) = 0, \quad \int_\Omega \Lambda_\pm(\mathbf{x}) d\mathbf{x} = 0.$$

Then $\Lambda_\pm(\mathbf{x})$ also solves the problem

$$\partial_\mp \Lambda_\pm(\mathbf{x}) = g_\pm^0(\omega_\pm(\mathbf{x}))^{-2} - 1, \quad \int_\Omega \Lambda_\pm(\mathbf{x}) d\mathbf{x} = 0;$$

see [BSu4, §12]. Hence,

$$(14.16\pm) \quad -\Delta \Lambda_\pm(\mathbf{x}) = \partial_\pm g_\pm^0(\omega_\pm(\mathbf{x}))^{-2}, \quad \int_\Omega \Lambda_\pm(\mathbf{x}) d\mathbf{x} = 0.$$

By Theorem 13.1 of [LaU, Chapter III], the periodic solution of the problem (14.16 $\pm$) is bounded: $\Lambda_\pm \in L_\infty$, and the norm $\|\Lambda_\pm\|_{L_\infty}$ is dominated by a constant depending only on $\|\omega_\pm\|_{L_\infty}, \|\omega_\pm^{-1}\|_{L_\infty}$, and Ω .

By (14.9 $\pm$) and (14.10 $\pm$), problem (6.12) for the Γ -periodic function $\tilde{\Lambda}_\pm(\mathbf{x})$ takes the form

$$(14.17\pm) \quad \partial_\pm \omega_\pm^2(\mathbf{x}) \partial_\mp \tilde{\Lambda}_\pm(\mathbf{x}) + v_\pm(\mathbf{x}) + i \operatorname{div} \boldsymbol{\eta}_\pm(\mathbf{x}) = 0, \quad \int_\Omega \tilde{\Lambda}_\pm(\mathbf{x}) d\mathbf{x} = 0.$$

Since $d = 2$, Proposition 8.11 and Remark 8.12 show that $\tilde{\Lambda}_\pm \in L_p(\Omega)$ with $p > 2$, and the norm $\|\tilde{\Lambda}_\pm\|_{L_p(\Omega)}$ (for a fixed $p > 2$) is dominated by a constant depending only on $\|\omega_\pm^{-1}\|_{L_\infty}, \|\boldsymbol{\eta}_\pm\|_{L_\rho(\Omega)}, \|v_\pm\|_{L_s(\Omega)}$, and the parameters of the lattice Γ .

The role of V (see (7.2)) is played by the number $V_\pm$ given by

$$(14.18\pm) \quad V_\pm = |\Omega|^{-1} \int_{\Omega} (\partial_{\mp} \Lambda_{\pm}(\mathbf{x}))^* \omega_{\pm}^2(\mathbf{x}) (\partial_{\mp} \tilde{\Lambda}_{\pm}(\mathbf{x})) d\mathbf{x}.$$

According to (7.3), we have

$$(14.19\pm) \quad W_\pm = |\Omega|^{-1} \int_{\Omega} \omega_{\pm}^2(\mathbf{x}) |\partial_{\mp} \tilde{\Lambda}_{\pm}(\mathbf{x})|^2 d\mathbf{x}.$$

Obviously, $W_\pm \geq 0$, and $W_\pm = 0$ only in the case where $\tilde{\Lambda}_{\pm}(\mathbf{x}) = 0$.

We calculate the operator $\mathcal{Y}_{\pm}^0 + (\mathcal{Y}_{\pm}^0)^*$ (see (8.2)):

$$(14.20\pm) \quad \begin{aligned} \mathcal{Y}_{\pm}^0 u + (\mathcal{Y}_{\pm}^0)^* u &= -\partial_{\pm} V_{\pm} u - V_{\pm}^* \partial_{\mp} u + \sum_{j=1}^2 (\overline{a_{j,\pm}} + a_{j,\pm}^*) D_j u \\ &= -2(\operatorname{Re} V_{\pm}) D_1 u \pm 2(\operatorname{Im} V_{\pm}) D_2 u - 2 \sum_{j=1}^2 \overline{\eta_{j,\pm}} D_j u \\ &= 2i(\eta_{1,\pm}^0 \partial_1 u + \eta_{2,\pm}^0 \partial_2 u), \end{aligned}$$

where we introduce the notation

$$(14.21\pm) \quad \eta_{1,\pm}^0 = \operatorname{Re} V_{\pm} + \overline{\eta_{1,\pm}}, \quad \eta_{2,\pm}^0 = \mp \operatorname{Im} V_{\pm} + \overline{\eta_{2,\pm}}.$$

As a result, by (14.5 $\pm$) and (14.20 $\pm$), the effective operator $\mathcal{B}_{\pm}^0$ (see (9.10)) takes the form

$$(14.22\pm) \quad \begin{aligned} \mathcal{B}_{\pm}^0 u &= \mathcal{A}_{\pm}^0 u + \mathcal{Y}_{\pm}^0 u + (\mathcal{Y}_{\pm}^0)^* u + (\overline{Q_{\pm}} - W_{\pm}) u \\ &= -g_{\pm}^0 \Delta u + 2i(\eta_{1,\pm}^0 \partial_1 u + \eta_{2,\pm}^0 \partial_2 u) + (\overline{Q_{\pm}} - W_{\pm}) u. \end{aligned}$$

Theorem 9.2 is applicable; this yields the following statement.

Proposition 14.1($\pm$). *Let $\mathcal{B}_{\pm,\varepsilon}$ be the operator (14.8 $\pm$) with coefficients satisfying the conditions of Subsection 14.1. Let $\mathcal{B}_{\pm}^0$ be the operator (14.22 $\pm$) whose coefficients are defined in accordance with (14.4 $\pm$), (14.18 $\pm$), (14.19 $\pm$), and (14.21 $\pm$). Let $Q_{0,\pm}$ be a Γ -periodic, bounded, and positive definite function on $\mathbb{R}^2$. Suppose that the parameter λ is subject to the restriction of the form (5.27) with $Q_0 = Q_{0,\pm}$ and c_0, c_4 corresponding to the operator $\mathcal{B}_{\pm,\varepsilon}$. Then for $0 < \varepsilon \leq 1$ we have*

$$\|(\mathcal{B}_{\pm,\varepsilon} + \lambda Q_{0,\pm}^{\varepsilon})^{-1} - (\mathcal{B}_{\pm}^0 + \lambda \overline{Q_{0,\pm}})^{-1}\|_{L_2(\mathbb{R}^2) \rightarrow L_2(\mathbb{R}^2)} \leq C_{\pm} \varepsilon,$$

where the constant $C_{\pm}$ depends only on the initial data (14.14 $\pm$) and the parameters of the lattice Γ .

Since $d = 2$, then Conditions 8.6 and 8.10 are satisfied. Therefore, statement 3° of Theorem 9.8 is applicable. As was mentioned above, the norms $\|\Lambda_{\pm}\|_{L_{\infty}}$ and $\|\tilde{\Lambda}_{\pm}\|_{L_p(\Omega)}$ are controlled in terms of the data (14.14 $\pm$) and the parameters of the lattice Γ . We arrive at the following result.

Proposition 14.2($\pm$). *Under the assumptions of Proposition 14.1($\pm$), let $\Lambda_{\pm}(\mathbf{x})$ be the Γ -periodic solution of the problem (14.15 $\pm$), and let $\tilde{\Lambda}_{\pm}(\mathbf{x})$ be the Γ -periodic solution of the problem (14.17 $\pm$). Then for $0 < \varepsilon \leq 1$ we have*

$$\|(\mathcal{B}_{\pm,\varepsilon} + \lambda Q_{0,\pm}^{\varepsilon})^{-1} - (I + \varepsilon \Lambda_{\pm}^{\varepsilon} \partial_{\mp} + \varepsilon \tilde{\Lambda}_{\pm}^{\varepsilon})(\mathcal{B}_{\pm}^0 + \lambda \overline{Q_{0,\pm}})^{-1}\|_{L_2(\mathbb{R}^2) \rightarrow H^1(\mathbb{R}^2)} \leq C'_{\pm} \varepsilon,$$

where the constant $C'_{\pm}$ depends only on the initial data (14.14 $\pm$) and the parameters of the lattice Γ .

Now, we approximate the flux. We denote

$$(14.23\pm) \quad u_{\pm,\varepsilon} = (\mathcal{B}_{\pm,\varepsilon} + \lambda Q_{0,\pm}^\varepsilon)^{-1} F, \quad u_\pm^0 = (\mathcal{B}_\pm^0 + \lambda \overline{Q_{0,\pm}})^{-1} F, \quad F \in L_2(\mathbb{R}^2),$$

$$(14.24\pm) \quad p_{\pm,\varepsilon} = (\omega_\pm^\varepsilon)^2 \partial_\mp u_{\pm,\varepsilon}, \quad p_\pm^0 = g_\pm^0 \partial_\mp u_\pm^0.$$

Since $g_\pm^0 = \underline{g}_\pm$, we have $\tilde{g}_\pm = g_\pm^0$. Theorem 10.4 implies the following result.

Proposition 14.3($\pm$). *Under the assumptions of Proposition 14.2*($\pm$), *let $u_{\pm,\varepsilon}$, $p_{\pm,\varepsilon}$ and $u_\pm^0$, $p_\pm^0$ be defined by (14.23* $\pm$), (14.24 $\pm$). *Then for $0 < \varepsilon \leq 1$ we have*

$$\|p_{\pm,\varepsilon} - p_\pm^0 - (\omega_\pm^\varepsilon)^2 (\partial_\mp \tilde{\Lambda}_\pm)^\varepsilon u_\pm^0\|_{L_2(\mathbb{R}^2)} \leq \hat{C}_\pm \varepsilon \|F\|_{L_2(\mathbb{R}^2)}, \quad F \in L_2(\mathbb{R}^2).$$

The constant $\hat{C}_\pm$ depends only on the initial data (14.14 $\pm$) and the parameters of the lattice Γ .

Corollary 14.4($\pm$). *Under the assumptions of Proposition 14.3*($\pm$), *suppose $v_\pm(\mathbf{x}) = 0$ and $\operatorname{div} \tilde{\eta}_\pm(\mathbf{x}) = 0$. Then $\tilde{\Lambda}_\pm = 0$ and*

$$\|p_{\pm,\varepsilon} - p_\pm^0\|_{L_2(\mathbb{R}^2)} \leq \hat{C}_\pm \varepsilon \|F\|_{L_2(\mathbb{R}^2)}, \quad F \in L_2(\mathbb{R}^2), \quad 0 < \varepsilon \leq 1.$$

14.2. The matrix model operator. Let $\omega_+(\mathbf{x})$ and $\omega_-(\mathbf{x})$ be a pair of functions satisfying conditions (14.1 $\pm$). Let $\mathcal{A}_\pm$ be the operators defined by (14.2 $\pm$). In $L_2(\mathbb{R}^2; \mathbb{C}^2)$, we consider the matrix operator with the blocks $\mathcal{A}_-, \mathcal{A}_+$:

$$\mathcal{A}_\times = \operatorname{diag} \{\mathcal{A}_-, \mathcal{A}_+\}.$$

Then the operator $\mathcal{A}_\times$ can be written in a factorized form:

$$(14.25) \quad \mathcal{A}_\times = b_\times(\mathbf{D}) g_\times(\mathbf{x}) b_\times(\mathbf{D}),$$

where

$$(14.26) \quad b_\times(\mathbf{D}) = b_\times(\mathbf{D})^* = \begin{pmatrix} 0 & \partial_- \\ \partial_+ & 0 \end{pmatrix},$$

$$h_\times = \operatorname{diag} \{\omega_+, \omega_-\}, \quad g_\times = h_\times^2 = \operatorname{diag} \{\omega_+^2, \omega_-^2\}.$$

The operator (14.25) is of the form (5.5) with $m = n = 2$, $b(\mathbf{D}) = b_\times(\mathbf{D})$, and $g = g_\times$.

We consider the operator

$$(14.27) \quad \mathcal{A}_{\times,\varepsilon} = b_\times(\mathbf{D}) g_\times^\varepsilon(\mathbf{x}) b_\times(\mathbf{D}) = \operatorname{diag} \{\mathcal{A}_{-,\varepsilon}, \mathcal{A}_{+,\varepsilon}\}.$$

Since $m = n$, the effective matrix $g_\times^0$ for the operator (14.25) is given by

$$(14.28) \quad g_\times^0 = \underline{g}_\times = \operatorname{diag} \{g_+^0, g_-^0\}.$$

The corresponding effective operator (see (8.1)) is given by

$$(14.29) \quad \mathcal{A}_\times^0 = b_\times(\mathbf{D}) g_\times^0 b_\times(\mathbf{D}) = \operatorname{diag} \{\mathcal{A}_-^0, \mathcal{A}_+^0\}, \quad \mathcal{A}_\pm^0 = -g_\pm^0 \Delta.$$

Next, let $\eta_{j,\times}(\mathbf{x})$, $j = 1, 2$, be Γ -periodic Hermitian (2×2) -matrix-valued functions in $\mathbb{R}^2$ such that

$$(14.30) \quad \eta_{j,\times} \in L_\rho(\Omega), \quad \rho > 2.$$

Let $v_\times(\mathbf{x})$ and $Q_\times(\mathbf{x})$ be Γ -periodic Hermitian (2×2) -matrix-valued functions in $\mathbb{R}^2$ such that

$$(14.31) \quad v_\times \in L_s(\Omega), \quad s > 1, \quad \int_\Omega v_\times(\mathbf{x}) d\mathbf{x} = 0,$$

$$Q_\times \in L_s(\Omega), \quad s > 1.$$

We consider the operator $\mathcal{B}_{\times,\varepsilon}$ given formally by the expression

$$(14.32) \quad \mathcal{B}_{\times,\varepsilon} \mathbf{u} = \mathcal{A}_{\times,\varepsilon} \mathbf{u} + i \sum_{j=1}^2 (\eta_{j,\times}^\varepsilon(\mathbf{x}) \partial_j \mathbf{u} + \partial_j (\eta_{j,\times}^\varepsilon(\mathbf{x}) \mathbf{u})) + \varepsilon^{-1} v_\times^\varepsilon(\mathbf{x}) \mathbf{u} + Q_\times^\varepsilon(\mathbf{x}) \mathbf{u}.$$

More precisely, $\mathcal{B}_{\times,\varepsilon}$ is the operator corresponding to the quadratic form

$$\mathbf{b}_{\times,\varepsilon}[\mathbf{u}, \mathbf{u}] = \int_{\mathbb{R}^2} \left(|h_\times^\varepsilon b_\times(\mathbf{D}) \mathbf{u}|^2 - 2 \operatorname{Re} \sum_{j=1}^2 \langle \eta_{j,\times}^\varepsilon D_j \mathbf{u}, \mathbf{u} \rangle + \varepsilon^{-1} \langle v_\times^\varepsilon \mathbf{u}, \mathbf{u} \rangle + \langle Q_\times^\varepsilon \mathbf{u}, \mathbf{u} \rangle \right) d\mathbf{x},$$

$$\mathbf{u} \in H^1(\mathbb{R}^2; \mathbb{C}^2).$$

Next, let $\Phi_\times(\mathbf{x})$ be a Γ -periodic solution of the problem

$$\Delta \Phi_\times(\mathbf{x}) = v_\times(\mathbf{x}), \quad \int_{\Omega} \Phi_\times(\mathbf{x}) d\mathbf{x} = 0,$$

and let $\zeta_{j,\times}(\mathbf{x}) = -\partial_j \Phi_\times(\mathbf{x})$. Then

$$(14.33) \quad v_\times(\mathbf{x}) = - \sum_{j=1}^2 \partial_j \zeta_{j,\times}(\mathbf{x}).$$

Here, $\Phi_\times(\mathbf{x})$ and $\zeta_{j,\times}(\mathbf{x})$ are Hermitian matrix-valued functions such that $\Phi_\times \in \widetilde{W}_s^2(\Omega)$, $\zeta_{j,\times} \in \widetilde{W}_s^1(\Omega) \subset L_\varkappa(\Omega)$, where $\varkappa = 2s(2-s)^{-1}$ for $s < 2$, $2 < \varkappa < \infty$ for $s = 2$, and $\varkappa = \infty$ for $s > 2$. Since, by (14.31), $s > 1$, we have $\varkappa > 2$. Put

$$(14.34) \quad a_{j,\times}(\mathbf{x}) = -\eta_{j,\times}(\mathbf{x}) + i \zeta_{j,\times}(\mathbf{x}), \quad j = 1, 2.$$

Taking (14.30) into account, we see that

$$(14.35) \quad a_{j,\times} \in L_{\rho'}(\Omega), \quad \rho' = \min\{\rho, \varkappa\} > 2, \quad j = 1, 2.$$

As in (14.12 $\pm$) and (14.13 $\pm$), it is easily seen that the operator $\mathcal{B}_{\times,\varepsilon}$ is represented as

$$\mathcal{B}_{\times,\varepsilon} = \mathcal{A}_{\times,\varepsilon} + \sum_{j=1}^2 (a_{j,\times}^\varepsilon D_j + D_j (a_{j,\times}^\varepsilon)^*) + Q_\times^\varepsilon.$$

Thus, the operator $\mathcal{B}_{\times,\varepsilon}$ takes the form (9.8). Its coefficients satisfy all the conditions of Subsections 5.1–5.3. A condition of the form (5.10) is satisfied due to (14.35). The form generated by the potential $Q_\times$ corresponds to Example 5.3. Condition 5.2 is valid with $\tilde{c} = 2^{-1} \|g_\times^{-1}\|_{L_\infty}^{-1}$, $\tilde{c}_2 = 1$; the constant c_3 is controlled in terms of s , $\|Q_\times\|_{L_s(\Omega)}$, and the parameters of the lattice, and c_0 is controlled in terms of the same parameters and $\|g_\times^{-1}\|_{L_\infty}$.

Let $Q_{0,\times}(\mathbf{x})$ be a Γ -periodic, bounded, and positive definite (2×2) -matrix-valued function in $\mathbb{R}^2$. We consider the generalized resolvent $(\mathcal{B}_{\times,\varepsilon} + \lambda Q_{0,\times}^\varepsilon)^{-1}$. We assume that the parameter λ is subject to the restriction (5.27) with $Q_0 = Q_{0,\times}$ and c_0, c_4 corresponding to the operator (14.32). Note that the lower bound for λ in (5.27) is defined in terms of $s, \rho', \|g_\times^{-1}\|_{L_\infty}, \|a_{j,\times}\|_{L_{\rho'}(\Omega)}, j = 1, 2, \|Q_\times\|_{L_s(\Omega)}, \|Q_{0,\times}\|_{L_\infty}, \|Q_{0,\times}^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ . The initial data reduces to the following set:

$$(14.36) \quad \rho, s; \|\omega_+\|_{L_\infty}, \|\omega_+^{-1}\|_{L_\infty}, \|\omega_-\|_{L_\infty}, \|\omega_-^{-1}\|_{L_\infty},$$

$$\|\eta_{j,\times}\|_{L_\rho(\Omega)}, j = 1, 2; \|v_\times\|_{L_s(\Omega)}, \|Q_\times\|_{L_s(\Omega)}, |\lambda|, \|Q_{0,\times}\|_{L_\infty}.$$

Now, we calculate the effective operator. The role of $\Lambda(\mathbf{x})$ (see problem (6.10)) is played by the function $\Lambda_\times(\mathbf{x})$ that is a Γ -periodic solution of the problem

$$(14.37) \quad b_\times(\mathbf{D}) g_\times(\mathbf{x}) (b_\times(\mathbf{D}) \Lambda_\times(\mathbf{x}) + \mathbf{1}) = 0, \quad \int_{\Omega} \Lambda_\times(\mathbf{x}) d\mathbf{x} = 0.$$

Then $\Lambda_\times(\mathbf{x})$ is represented as

$$\Lambda_\times(\mathbf{x}) = \begin{pmatrix} 0 & \Lambda_-(\mathbf{x}) \\ \Lambda_+(\mathbf{x}) & 0 \end{pmatrix},$$

where the $\Lambda_\pm(\mathbf{x})$ are the periodic solutions of problems (14.15 $\pm$). The properties of $\Lambda_\pm$ described above show that $\Lambda_\times \in L_\infty$, and the norm $\|\Lambda_\times\|_{L_\infty}$ is dominated by a constant depending only on $\|\omega_+\|_{L_\infty}$, $\|\omega_+^{-1}\|_{L_\infty}$, $\|\omega_-\|_{L_\infty}$, $\|\omega_-^{-1}\|_{L_\infty}$, and Ω .

By (14.33) and (14.34), problem (6.12) for the Γ -periodic matrix $\tilde{\Lambda}_\times(\mathbf{x})$ takes the form

$$(14.38) \quad b_\times(\mathbf{D})g_\times(\mathbf{x})b_\times(\mathbf{D})\tilde{\Lambda}_\times(\mathbf{x}) + v_\times(\mathbf{x}) + i \sum_{j=1}^2 \partial_j \eta_{j,\times}(\mathbf{x}) = 0, \quad \int_\Omega \tilde{\Lambda}_\times(\mathbf{x}) d\mathbf{x} = 0.$$

By Proposition 8.11 and Remark 8.12, we have $\tilde{\Lambda}_\times \in L_p(\Omega)$, where $p > 2$. Here, the norm $\|\tilde{\Lambda}_\times\|_{L_p(\Omega)}$ (for a fixed $p > 2$) is estimated in terms of $\|\omega_+^{-1}\|_{L_\infty}$, $\|\omega_-^{-1}\|_{L_\infty}$, $\|\eta_{j,\times}\|_{L_\rho(\Omega)}$, $j = 1, 2$, $\|v_\times\|_{L_s(\Omega)}$, and the parameters of the lattice Γ .

The role of V (see (7.2)) is played by the (2×2) -matrix $V_\times$ given by

$$(14.39) \quad V_\times = |\Omega|^{-1} \int_\Omega (b_\times(\mathbf{D})\Lambda_\times(\mathbf{x}))^* g_\times(\mathbf{x}) (b_\times(\mathbf{D})\tilde{\Lambda}_\times(\mathbf{x})) d\mathbf{x}.$$

By (7.3), we have

$$(14.40) \quad W_\times = |\Omega|^{-1} \int_\Omega (b_\times(\mathbf{D})\tilde{\Lambda}_\times(\mathbf{x}))^* g_\times(\mathbf{x}) (b_\times(\mathbf{D})\tilde{\Lambda}_\times(\mathbf{x})) d\mathbf{x}.$$

We calculate the operator $\mathcal{Y}_\times^0 + (\mathcal{Y}_\times^0)^*$ (see (8.2)):

$$(14.41) \quad \begin{aligned} \mathcal{Y}_\times^0 \mathbf{u} + (\mathcal{Y}_\times^0)^* \mathbf{u} &= -b_\times(\mathbf{D})V_\times \mathbf{u} - V_\times^* b_\times(\mathbf{D}) \mathbf{u} - 2 \sum_{j=1}^2 \overline{\eta_{j,\times}} D_j u \\ &= 2i(\eta_{1,\times}^0 \partial_1 + \eta_{2,\times}^0 \partial_2), \end{aligned}$$

where

$$(14.42) \quad \begin{aligned} \eta_{1,\times}^0 &= \overline{\eta_{1,\times}} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V_\times + V_\times^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \eta_{2,\times}^0 &= \overline{\eta_{2,\times}} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} V_\times + V_\times^* \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \end{aligned}$$

Finally, by (14.29) and (14.41), the effective operator $\mathcal{B}_\times^0$ (see (9.10)) takes the form

$$(14.43) \quad \begin{aligned} \mathcal{B}_\times^0 \mathbf{u} &= \mathcal{A}_\times^0 \mathbf{u} + \mathcal{Y}_\times^0 \mathbf{u} + (\mathcal{Y}_\times^0)^* \mathbf{u} + (\overline{Q_\times} - W_\times) \mathbf{u} \\ &= -g_\times^0 \Delta \mathbf{u} + 2i(\eta_{1,\times}^0 \partial_1 \mathbf{u} + \eta_{2,\times}^0 \partial_2 \mathbf{u}) + (\overline{Q_\times} - W_\times) \mathbf{u}. \end{aligned}$$

Theorem 9.2 is applicable and yields the following statement.

Proposition 14.5. *Let $\mathcal{B}_{\times,\varepsilon}$ be the operator (14.32) with coefficients satisfying the conditions of Subsection 14.2. Let $\mathcal{B}_\times^0$ be the operator (14.43) with coefficients defined as in (14.28), (14.39), (14.40), and (14.42). Let $Q_{0,\times}$ be a Γ -periodic, bounded, and positive definite (2×2) -matrix-valued function in $\mathbb{R}^2$. Suppose that the parameter λ obeys the restriction (5.27) with $Q_0 = Q_{0,\times}$ and c_0, c_4 corresponding to the operator (14.32). Then for $0 < \varepsilon \leq 1$ we have*

$$\|(\mathcal{B}_{\times,\varepsilon} + \lambda Q_{0,\times}^\varepsilon)^{-1} - (\mathcal{B}_\times^0 + \lambda \overline{Q_{0,\times}})^{-1}\|_{L_2(\mathbb{R}^2) \rightarrow L_2(\mathbb{R}^2)} \leq C_\times \varepsilon,$$

where the constant $C_\times$ depends only on the initial data (14.36) and the parameters of the lattice Γ .

Since $d = 2$, Conditions 8.6 and 8.10 are satisfied. Therefore, statement 3° of Theorem 9.8 applies. Now, the corrector is represented as

$$(\Lambda_{\times}^{\varepsilon} b_{\times}(\mathbf{D}) + \tilde{\Lambda}_{\times}^{\varepsilon})(\mathcal{B}_{\times}^0 + \lambda \overline{Q_{0,\times}})^{-1} = \left(\text{diag} \{ \Lambda_{-}^{\varepsilon} \partial_{+}, \Lambda_{+}^{\varepsilon} \partial_{-} \} + \tilde{\Lambda}_{\times}^{\varepsilon} \right) (\mathcal{B}_{\times}^0 + \lambda \overline{Q_{0,\times}})^{-1}.$$

As was mentioned above, the norms $\|\Lambda_{\times}\|_{L_{\infty}}$ and $\|\tilde{\Lambda}_{\times}\|_{L_p(\Omega)}$ are controlled in terms of the initial data (14.36) and the parameters of the lattice Γ . We obtain the following result.

Proposition 14.6. *Under the assumptions of Proposition 14.5, let $\Lambda_{\times}(\mathbf{x})$ be the Γ -periodic solution of problem (14.37), and let $\tilde{\Lambda}_{\times}(\mathbf{x})$ be the Γ -periodic solution of problem (14.38). Then for $0 < \varepsilon \leq 1$ we have*

$$\|(\mathcal{B}_{\times,\varepsilon} + \lambda Q_{0,\times}^{\varepsilon})^{-1} - (I + \varepsilon \Lambda_{\times}^{\varepsilon} b_{\times}(\mathbf{D}) + \varepsilon \tilde{\Lambda}_{\times}^{\varepsilon})(\mathcal{B}_{\times}^0 + \lambda \overline{Q_{0,\times}})^{-1}\|_{L_2(\mathbb{R}^2) \rightarrow H^1(\mathbb{R}^2)} \leq C'_{\times} \varepsilon,$$

where the constant $C'_{\times}$ depends only on the initial data (14.36) and the parameters of the lattice Γ .

We pass to approximation of the flux. We denote

$$(14.44) \quad \mathbf{u}_{\times,\varepsilon} = (\mathcal{B}_{\times,\varepsilon} + \lambda Q_{0,\times}^{\varepsilon})^{-1} \mathbf{F}, \quad \mathbf{u}_{\times}^0 = (\mathcal{B}_{\times}^0 + \lambda \overline{Q_{0,\times}})^{-1} \mathbf{F}, \quad \mathbf{F} \in L_2(\mathbb{R}^2; \mathbb{C}^2),$$

$$(14.45) \quad \mathbf{p}_{\times,\varepsilon} = g_{\times}^{\varepsilon} b_{\times}(\mathbf{D}) \mathbf{u}_{\times,\varepsilon}, \quad \mathbf{p}_{\times}^0 = g_{\times}^0 b_{\times}(\mathbf{D}) \mathbf{u}_{\times}^0.$$

Since $g_{\times}^0 = \underline{g}_{\times}$, now we have $\tilde{g}_{\times} = g_{\times}^0$. Then Theorem 10.4 implies the following result.

Proposition 14.7. *Under the assumptions of Proposition 14.6, let $\mathbf{u}_{\times,\varepsilon}$, $\mathbf{p}_{\times,\varepsilon}$ and $\mathbf{u}_{\times}^0$, $\mathbf{p}_{\times}^0$ be defined by (14.44), (14.45). Then for $0 < \varepsilon \leq 1$ we have*

$$\|\mathbf{p}_{\times,\varepsilon} - \mathbf{p}_{\times}^0 - g_{\times}^{\varepsilon} (b_{\times}(\mathbf{D}) \tilde{\Lambda}_{\times})^{\varepsilon} \mathbf{u}_{\times}^0\|_{L_2(\mathbb{R}^2)} \leq \hat{C}_{\times} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^2)}, \quad \mathbf{F} \in L_2(\mathbb{R}^2; \mathbb{C}^2).$$

The constant $\hat{C}_{\times}$ depends only on the initial data (14.36) and the parameters of the lattice Γ .

Corollary 14.8. *Under the assumptions of Proposition 14.7, suppose also that*

$$v_{\times}(\mathbf{x}) = 0, \quad \sum_{j=1}^2 \partial_j \eta_{j,\times}(\mathbf{x}) = 0.$$

Then $\tilde{\Lambda}_{\times} = 0$ and

$$\|\mathbf{p}_{\times,\varepsilon} - \mathbf{p}_{\times}^0\|_{L_2(\mathbb{R}^2)} \leq \hat{C}_{\times} \varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^2)}, \quad \mathbf{F} \in L_2(\mathbb{R}^2; \mathbb{C}^2), \quad 0 < \varepsilon \leq 1.$$

14.3. Definition and factorization of the Pauli operator. (See, e.g., [BSu2, Chapter 6, §2].) Suppose that the magnetic potential $\check{\mathbf{A}}(\mathbf{x}) = \{\check{A}_1(\mathbf{x}), \check{A}_2(\mathbf{x})\}$ is given, where the $\check{A}_j(\mathbf{x})$, $j = 1, 2$, are Γ -periodic real-valued functions in $\mathbb{R}^2$ such that

$$(14.46) \quad \check{\mathbf{A}} \in L_{\rho}(\Omega; \mathbb{C}^2), \quad \rho > 2.$$

We recall the standard notation for the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In $L_2(\mathbb{R}^2; \mathbb{C}^2)$, we consider the operator

$$\mathcal{D}\mathbf{u} = (D_1 - \check{A}_1(\mathbf{x}))\sigma_1\mathbf{u} + (D_2 - \check{A}_2(\mathbf{x}))\sigma_2\mathbf{u}, \quad \mathbf{u} \in H^1(\mathbb{R}^2; \mathbb{C}^2).$$

The operator $\mathcal{D}$ is closed and selfadjoint. By definition, the Pauli operator $\mathcal{P}$ is the square of $\mathcal{D}$:

$$(14.47) \quad \mathcal{P} = \mathcal{D}^2 = \begin{pmatrix} P_{-} & 0 \\ 0 & P_{+} \end{pmatrix}.$$

The precise definition of $\mathcal{P}$ is given in terms of the quadratic form $\|\mathcal{D}\mathbf{u}\|_{L_2(\mathbb{R}^2)}^2$, $\mathbf{u} \in H^1(\mathbb{R}^2; \mathbb{C}^2)$. If the potential $\check{\mathbf{A}}(\mathbf{x})$ is sufficiently smooth, then the blocks $P_{\pm}$ of the operator (14.47) are given by

$$P_{\pm} = (\mathbf{D} - \check{\mathbf{A}}(\mathbf{x}))^2 \pm \check{B}(\mathbf{x}), \quad \check{B}(\mathbf{x}) := \partial_1 \check{A}_2(\mathbf{x}) - \partial_2 \check{A}_1(\mathbf{x}).$$

Here $\check{B}(\mathbf{x})$ has the meaning of the magnetic field.

We use a known factorization for the Pauli operator. A gauge transformation allows us to assume that the potential $\check{\mathbf{A}}$ is subject to the conditions

$$(14.48) \quad \operatorname{div} \check{\mathbf{A}}(\mathbf{x}) = 0, \quad \int_{\Omega} \check{\mathbf{A}}(\mathbf{x}) d\mathbf{x} = 0$$

and still satisfies (14.46). Under the conditions (14.46) and (14.48), there exists a (unique) Γ -periodic real-valued function $\varphi(\mathbf{x})$ such that

$$\nabla \varphi(\mathbf{x}) = \operatorname{col} \{ \check{A}_2(\mathbf{x}), -\check{A}_1(\mathbf{x}) \}, \quad \int_{\Omega} \varphi(\mathbf{x}) d\mathbf{x} = 0.$$

Herewith, we have $\varphi \in \widetilde{W}_{\rho}^1(\Omega) \subset C^{\alpha}$, $\alpha = 1 - 2\rho^{-1}$.

We denote

$$\omega_{\pm}(\mathbf{x}) = e^{\pm \varphi(\mathbf{x})}.$$

Then $\omega_{\pm} \in \widetilde{W}_{\rho}^1(\Omega)$, and

$$\omega_+(\mathbf{x})\omega_-(\mathbf{x}) = 1, \quad \mathbf{x} \in \mathbb{R}^2.$$

Moreover, the functions $\omega_{\pm}$ satisfy the conditions (14.1 $\pm$) automatically and are multipliers both in $H^1(\mathbb{R}^2)$ and in $\widetilde{H}^1(\Omega)$. The norms $\|\omega_+\|_{L_{\infty}}$ and $\|\omega_-\|_{L_{\infty}}$ are controlled in terms of the norm $\|\check{\mathbf{A}}\|_{L_{\rho}(\Omega)}$ and the parameters of the lattice Γ . Let $h_{\times}(\mathbf{x})$ and $g_{\times}(\mathbf{x})$ be the matrices defined by (14.26). The operators $\mathcal{D}$ and $\mathcal{P}$ admit the following factorization:

$$(14.49) \quad \begin{aligned} \mathcal{D} &= h_{\times}(\mathbf{x})b_{\times}(\mathbf{D})h_{\times}(\mathbf{x}), \\ \mathcal{P} &= h_{\times}(\mathbf{x})b_{\times}(\mathbf{D})g_{\times}(\mathbf{x})b_{\times}(\mathbf{D})h_{\times}(\mathbf{x}). \end{aligned}$$

The blocks $P_{\pm}$ of the operator $\mathcal{P}$ can be represented as

$$P_+ = \omega_- \partial_+ \omega_+^2 \partial_- \omega_-, \quad P_- = \omega_+ \partial_- \omega_-^2 \partial_+ \omega_+.$$

14.4. The operators $\tilde{\mathcal{B}}_{\pm, \varepsilon}$. Now we consider the operators

$$(14.50\pm) \quad P_{\pm, \varepsilon} = \omega_{\mp}^{\varepsilon} \partial_{\pm} (\omega_{\pm}^{\varepsilon})^2 \partial_{\mp} \omega_{\mp}^{\varepsilon}.$$

They are of the form (12.2) with $m = n = 1$, $b(\mathbf{D}) = \partial_{\mp}$, $g = \omega_{\pm}^2$, and $f = \omega_{\mp}$. If the magnetic potential is sufficiently smooth, then

$$P_{\pm, \varepsilon} = (\mathbf{D} - \varepsilon^{-1} \check{\mathbf{A}}^{\varepsilon})^2 \pm \varepsilon^{-2} \check{B}^{\varepsilon}.$$

Next, let $\mathbf{A}(\mathbf{x}) = \operatorname{col} \{A_1(\mathbf{x}), A_2(\mathbf{x})\}$ be a vector-valued potential, where $A_1(\mathbf{x})$ and $A_2(\mathbf{x})$ are Γ -periodic real-valued functions such that $A_j \in L_{\rho}(\Omega)$, $\rho > 2$. Let $\check{v}_{\pm}(\mathbf{x})$ be a real-valued Γ -periodic function on $\mathbb{R}^2$ such that

$$\check{v}_{\pm} \in L_s(\Omega), \quad s > 1; \quad \int_{\Omega} \omega_{\pm}^2(\mathbf{x}) \check{v}_{\pm}(\mathbf{x}) d\mathbf{x} = 0.$$

Finally, let $\check{Q}(\mathbf{x})$ be a Γ -periodic real-valued function in $\mathbb{R}^2$ such that $\check{Q} \in L_s(\Omega)$, $s > 1$.

We consider the operator $\tilde{\mathcal{B}}_{\pm, \varepsilon}$ given formally by the expression

$$(14.51\pm) \quad \tilde{\mathcal{B}}_{\pm, \varepsilon} u = P_{\pm, \varepsilon} u + i \sum_{j=1}^2 (A_j^{\varepsilon}(\mathbf{x}) \partial_j u + \partial_j (A_j^{\varepsilon}(\mathbf{x}) u)) + \varepsilon^{-1} \check{v}_{\pm}^{\varepsilon}(\mathbf{x}) u + \check{Q}^{\varepsilon}(\mathbf{x}) u.$$

More precisely, the operator $\tilde{\mathcal{B}}_{\pm,\varepsilon}$ is defined in terms of the quadratic form

$$\tilde{\mathfrak{b}}_{\pm,\varepsilon}[u, u] = \int_{\mathbb{R}^2} (|\omega_{\pm}^{\varepsilon} \partial_{\mp}(\omega_{\mp}^{\varepsilon} u)|^2 - 2 \operatorname{Re} \langle \mathbf{A}^{\varepsilon} \mathbf{D} u, u \rangle + (\varepsilon^{-1} \tilde{v}_{\pm}^{\varepsilon} + \tilde{Q}^{\varepsilon}) |u|^2) d\mathbf{x},$$

$u \in H^1(\mathbb{R}^2).$

Remark 14.9. Assuming that the potentials $\mathbf{A}(\mathbf{x})$ and $\tilde{\mathbf{A}}(\mathbf{x})$ are Lipschitz functions, we consider the operator

$$(14.52\pm) \quad \tilde{\mathcal{B}}_{\pm,\varepsilon} u = (\mathbf{D} - \varepsilon^{-1} \tilde{\mathbf{A}}^{\varepsilon} - \mathbf{A}^{\varepsilon})^2 u \pm (\varepsilon^{-2} \tilde{B}^{\varepsilon} + \varepsilon^{-1} B^{\varepsilon}) u + \varepsilon^{-1} \tilde{v}^{\varepsilon} u + \mathcal{V}^{\varepsilon} u,$$

where $B(\mathbf{x}) = \partial_1 A_2(\mathbf{x}) - \partial_2 A_1(\mathbf{x})$, and $\tilde{v}, \mathcal{V}$ are Γ -periodic real-valued functions such that $\tilde{v}, \mathcal{V} \in L_s(\Omega)$, $s > 1$. Assume also that

$$\int_{\Omega} \omega_{\pm}^2(\mathbf{x}) (\tilde{v}(\mathbf{x}) + 2 \langle \tilde{\mathbf{A}}(\mathbf{x}), \mathbf{A}(\mathbf{x}) \rangle \pm B(\mathbf{x})) d\mathbf{x} = 0.$$

Then the operator (14.52 $\pm$) can be written in the form (14.51 $\pm$) with

$$\tilde{v}_{\pm}(\mathbf{x}) = \tilde{v}(\mathbf{x}) + 2 \langle \tilde{\mathbf{A}}(\mathbf{x}), \mathbf{A}(\mathbf{x}) \rangle \pm B(\mathbf{x}), \quad \tilde{Q}(\mathbf{x}) = \mathcal{V}(\mathbf{x}) + |\mathbf{A}(\mathbf{x})|^2.$$

The operators (14.52 $\pm$) are the blocks of the Pauli operator with the magnetic potential $\varepsilon^{-1} \tilde{\mathbf{A}}^{\varepsilon} + \mathbf{A}^{\varepsilon}$ (which contains the singular term $\varepsilon^{-1} \tilde{\mathbf{A}}^{\varepsilon}$), perturbed by the scalar electric potential $\varepsilon^{-1} \tilde{v}^{\varepsilon} + \mathcal{V}^{\varepsilon}$ (which also contains a singular term).

Now, taking (14.50 $\pm$) and (14.3 $\pm$) into account, we see that the operators (14.51 $\pm$) and (14.8 $\pm$) satisfy the relations

$$\tilde{\mathcal{B}}_{+,\varepsilon} = \omega_{-}^{\varepsilon} \mathcal{B}_{+,\varepsilon} \omega_{-}^{\varepsilon}, \quad \tilde{\mathcal{B}}_{-,\varepsilon} = \omega_{+}^{\varepsilon} \mathcal{B}_{-,\varepsilon} \omega_{+}^{\varepsilon}.$$

Note that in (14.8 $\pm$) we put

$$(14.53\pm) \quad \eta_{j,\pm} = \omega_{\pm}^2 A_j, \quad j = 1, 2; \quad v_{\pm} = \omega_{\pm}^2 \tilde{v}_{\pm}, \quad Q_{\pm} = \omega_{\pm}^2 \tilde{Q}.$$

Let $\tilde{Q}_0(\mathbf{x})$ be a Γ -periodic, bounded, and positive definite real-valued function. We consider the generalized resolvent $(\tilde{\mathcal{B}}_{\pm,\varepsilon} + \lambda \tilde{Q}_0^{\varepsilon})^{-1}$. Assume that the parameter λ is subject to the restriction (12.6) with $f = \omega_{\mp}$ and with c_0, c_4 corresponding to the operator $\mathcal{B}_{\pm,\varepsilon}$. The lower bound for λ is controlled in terms of $s, \rho, \|\omega_{+}\|_{L_{\infty}}, \|\omega_{-}\|_{L_{\infty}}, \|\mathbf{A}\|_{L_{\rho}(\Omega)}, \|\tilde{v}_{\pm}\|_{L_s(\Omega)}, \|\tilde{Q}\|_{L_s(\Omega)}, \|\tilde{Q}_0\|_{L_{\infty}}, \|\tilde{Q}_0^{-1}\|_{L_{\infty}}$, and the parameters of the lattice Γ .

As was mentioned in Subsection 14.3, the norms $\|\omega_{+}\|_{L_{\infty}}$ and $\|\omega_{-}\|_{L_{\infty}}$ are controlled in terms of $\|\tilde{\mathbf{A}}\|_{L_{\rho}(\Omega)}$ and the parameters of the lattice Γ . Hence, the initial data reduces to the following set:

$$(14.54\pm) \quad \rho, s; \quad \|\tilde{\mathbf{A}}\|_{L_{\rho}(\Omega)}, \quad \|\mathbf{A}\|_{L_{\rho}(\Omega)}, \quad \|\tilde{v}_{\pm}\|_{L_s(\Omega)}, \quad \|\tilde{Q}\|_{L_s(\Omega)}, \quad |\lambda|, \quad \|\tilde{Q}_0\|_{L_{\infty}}.$$

Applying Theorem 12.1, we get the following statement.

Proposition 14.10($\pm$). *Let $\tilde{\mathcal{B}}_{\pm,\varepsilon}$ be the operator (14.51 $\pm$) with coefficients satisfying the conditions formulated above in Subsections 14.3, 14.4. Let $\mathcal{B}_{\pm,\varepsilon}$ be the operator (14.8 $\pm$), whose coefficients are related to those of the operator $\tilde{\mathcal{B}}_{\pm,\varepsilon}$ by relations (14.53 $\pm$). Let $\mathcal{B}_{\pm,\varepsilon}^0$ be the effective operator for $\mathcal{B}_{\pm,\varepsilon}$, defined by (14.22 $\pm$). Let $\tilde{Q}_0(\mathbf{x})$ be a Γ -periodic, bounded, and positive definite function in $\mathbb{R}^2$. Suppose that the parameter λ is subject to the restriction (12.6) with $f = \omega_{\mp}$ and with c_0, c_4 corresponding to the operator $\mathcal{B}_{\pm,\varepsilon}$. Let $Q_{0,\pm} = \tilde{Q}_0 \omega_{\pm}^2$. Then for $0 < \varepsilon \leq 1$ we have*

$$\|(\tilde{\mathcal{B}}_{\pm,\varepsilon} + \lambda \tilde{Q}_0^{\varepsilon})^{-1} - \omega_{\pm}^{\varepsilon} (\mathcal{B}_{\pm}^0 + \lambda \overline{Q_{0,\pm}})^{-1} \omega_{\pm}^{\varepsilon}\|_{L_2(\mathbb{R}^2) \rightarrow L_2(\mathbb{R}^2)} \leq \tilde{C}_{\pm} \varepsilon,$$

where the constant $\tilde{C}_{\pm}$ depends only on the initial data (14.54 $\pm$) and the parameters of the lattice Γ .

Since $d = 2$, Conditions 8.6 and 8.10 are satisfied. We use the properties of the functions $\Lambda_{\pm}$ and $\tilde{\Lambda}_{\pm}$ mentioned in Subsection 14.1, and also the fact that the norms $\|\omega_{+}\|_{L_{\infty}}$ and $\|\omega_{-}\|_{L_{\infty}}$ are controlled in terms of $\|\tilde{\mathbf{A}}\|_{L_{\rho}(\Omega)}$ and the parameters of the lattice. Then $\|\Lambda_{\pm}\|_{L_{\infty}}$ and $\|\tilde{\Lambda}_{\pm}\|_{L_p(\Omega)}$ (for a fixed $p > 2$) are controlled in terms of the data (14.54 $\pm$) and the parameters of the lattice Γ . Applying statement 3 $^{\circ}$ of Theorem 12.3 to the operator $\tilde{\mathcal{B}}_{\pm,\varepsilon}$, we arrive at the following proposition.

Proposition 14.11($\pm$). *Under the assumptions of Proposition 14.10($\pm$), let $\Lambda_{\pm}(\mathbf{x})$ be the Γ -periodic solution of problem (14.16 $\pm$), and let $\tilde{\Lambda}_{\pm}(\mathbf{x})$ be the Γ -periodic solution of problem (14.17 $\pm$). Then for $0 < \varepsilon \leq 1$ we have*

$$\|\omega_{\mp}^{\varepsilon}(\tilde{\mathcal{B}}_{\pm,\varepsilon} + \lambda\tilde{Q}_0^{\varepsilon})^{-1} - (I + \varepsilon\Lambda_{\pm}^{\varepsilon}\partial_{\mp} + \varepsilon\tilde{\Lambda}_{\pm}^{\varepsilon})(\mathcal{B}_{\pm}^0 + \lambda\overline{Q_{0,\pm}})^{-1}\omega_{\pm}^{\varepsilon}\|_{L_2(\mathbb{R}^2) \rightarrow H^1(\mathbb{R}^2)} \leq C_{\pm}\varepsilon,$$

where the constant $C_{\pm}$ depends only on the initial data (14.54 $\pm$) and the parameters of the lattice Γ .

Now, we approximate the flux. We denote

$$(14.55\pm) \quad \tilde{u}_{\pm,\varepsilon} = (\tilde{\mathcal{B}}_{\pm,\varepsilon} + \lambda\tilde{Q}_0^{\varepsilon})^{-1}F, \quad u_{\pm,\varepsilon}^0 = (\mathcal{B}_{\pm}^0 + \lambda\overline{Q_{0,\pm}})^{-1}\omega_{\pm}^{\varepsilon}F, \quad F \in L_2(\mathbb{R}^2),$$

$$(14.56\pm) \quad \tilde{p}_{\pm,\varepsilon} = (\omega_{\pm}^{\varepsilon})^2\partial_{\mp}\omega_{\mp}^{\varepsilon}\tilde{u}_{\pm,\varepsilon}, \quad p_{\pm,\varepsilon}^0 = g_{\pm}^0\partial_{\mp}u_{\pm,\varepsilon}^0.$$

Since $g_{\pm}^0 = \underline{g}_{\pm}$, now we have $\tilde{g}_{\pm} = g_{\pm}^0$. Then statement 3 $^{\circ}$ of Theorem 12.7 implies the following result.

Proposition 14.12($\pm$). *Under the assumptions of Proposition 14.11($\pm$), let $\tilde{u}_{\pm,\varepsilon}$, $\tilde{p}_{\pm,\varepsilon}$ and $u_{\pm,\varepsilon}^0$, $p_{\pm,\varepsilon}^0$ be defined by (14.55 $\pm$) and (14.56 $\pm$). Then for $0 < \varepsilon \leq 1$ we have*

$$\|\tilde{p}_{\pm,\varepsilon} - p_{\pm,\varepsilon}^0 - (\omega_{\pm}^{\varepsilon})^2(\partial_{\mp}\tilde{\Lambda}_{\pm})^{\varepsilon}u_{\pm,\varepsilon}^0\|_{L_2(\mathbb{R}^2)} \leq \hat{C}_{\pm}\varepsilon\|F\|_{L_2(\mathbb{R}^2)}, \quad F \in L_2(\mathbb{R}^2).$$

The constant $\hat{C}_{\pm}$ depends only on the initial data (14.54 $\pm$) and the parameters of the lattice Γ .

Corollary 14.13($\pm$). *Under the assumptions of Proposition 14.12($\pm$), suppose that $\tilde{v}_{\pm}(\mathbf{x}) = 0$ and $\operatorname{div} \omega_{\pm}^2(\mathbf{x})\mathbf{A}(\mathbf{x}) = 0$. Then $\tilde{\Lambda}_{\pm} = 0$ and*

$$\|\tilde{p}_{\pm,\varepsilon} - p_{\pm,\varepsilon}^0\|_{L_2(\mathbb{R}^2)} \leq \hat{C}_{\pm}\varepsilon\|F\|_{L_2(\mathbb{R}^2)}, \quad F \in L_2(\mathbb{R}^2), \quad 0 < \varepsilon \leq 1.$$

14.5. The operator $\tilde{\mathcal{B}}_{\times,\varepsilon}$. We return to the matrix Pauli operator (14.49) and consider the corresponding operator

$$(14.57) \quad \mathcal{P}_{\varepsilon} = h_{\times}^{\varepsilon}b_{\times}(\mathbf{D})g_{\times}^{\varepsilon}b_{\times}(\mathbf{D})h_{\times}^{\varepsilon}.$$

Then

$$\mathcal{P}_{\varepsilon} = \begin{pmatrix} P_{-,\varepsilon} & 0 \\ 0 & P_{+,\varepsilon} \end{pmatrix},$$

where the blocks $P_{\pm,\varepsilon}$ are as in (14.50 $\pm$). The operator (14.57) is of the form (12.2) with $m = n = 2$, $b(\mathbf{D}) = b_{\times}(\mathbf{D})$, $g = g_{\times}$, and $f = h_{\times}$.

Next, let $\tilde{\eta}_{j,\times}(\mathbf{x})$, $j = 1, 2$, be Γ -periodic *diagonal* (2×2) -matrix-valued functions in $\mathbb{R}^2$ with real entries and such that $\tilde{\eta}_{j,\times} \in L_{\rho}(\Omega)$, $\rho > 2$. Let $\tilde{v}_{\times}(\mathbf{x})$ and $\tilde{Q}_{\times}(\mathbf{x})$ be Γ -periodic Hermitian (2×2) -matrix-valued functions in $\mathbb{R}^2$ such that

$$\begin{aligned} \tilde{v}_{\times} &\in L_s(\Omega), \quad s > 1; \quad \int_{\Omega} (h_{\times}(\mathbf{x}))^{-1}\tilde{v}_{\times}(\mathbf{x})(h_{\times}(\mathbf{x}))^{-1}d\mathbf{x} = 0, \\ \tilde{Q}_{\times} &\in L_s(\Omega), \quad s > 1. \end{aligned}$$

Consider the operator $\tilde{\mathcal{B}}_{\times,\varepsilon}$ given formally by the expression

$$(14.58) \quad \tilde{\mathcal{B}}_{\times,\varepsilon} \mathbf{u} = \mathcal{P}_\varepsilon \mathbf{u} + i \sum_{j=1}^2 (\tilde{\eta}_{j,\times}^\varepsilon(\mathbf{x}) \partial_j \mathbf{u} + \partial_j (\tilde{\eta}_{j,\times}^\varepsilon(\mathbf{x}) \mathbf{u})) + \varepsilon^{-1} \tilde{v}_\times^\varepsilon(\mathbf{x}) \mathbf{u} + \tilde{Q}_\times^\varepsilon(\mathbf{x}) \mathbf{u}.$$

More precisely, the operator $\tilde{\mathcal{B}}_{\times,\varepsilon}$ is defined in terms of the quadratic form

$$\tilde{\mathfrak{b}}_{\times,\varepsilon}[\mathbf{u}, \mathbf{u}] = \int_{\mathbb{R}^2} \left(|h_\times^\varepsilon b_\times(\mathbf{D}) h_\times^\varepsilon \mathbf{u}|^2 - 2 \operatorname{Re} \sum_{j=1}^2 \langle \tilde{\eta}_{j,\times}^\varepsilon D_j \mathbf{u}, \mathbf{u} \rangle + \varepsilon^{-1} \langle \tilde{v}_\times^\varepsilon \mathbf{u}, \mathbf{u} \rangle + \langle \tilde{Q}_\times^\varepsilon \mathbf{u}, \mathbf{u} \rangle \right) d\mathbf{x},$$

$\mathbf{u} \in H^1(\mathbb{R}^2; \mathbb{C}^2).$

Remark 14.14. Let $\mathbf{A}(\mathbf{x}) = \operatorname{col}\{A_1(\mathbf{x}), A_2(\mathbf{x})\}$, where the $A_j(\mathbf{x})$, $j = 1, 2$, are Γ -periodic real-valued functions. Assume that the vector-valued potentials $\tilde{\mathbf{A}}(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ are Lipschitz functions. Let $\tilde{v}_\times(\mathbf{x})$ and $\mathcal{V}_\times(\mathbf{x})$ be Γ -periodic Hermitian matrix-valued functions such that $\tilde{v}_\times, \mathcal{V}_\times \in L_s(\Omega)$, $s > 1$. We consider the operator

$$(14.59) \quad \tilde{\mathcal{B}}_{\times,\varepsilon} = \operatorname{diag} \{ \mathfrak{P}_{-,\varepsilon}, \mathfrak{P}_{+,\varepsilon} \} + \varepsilon^{-1} \tilde{v}_\times^\varepsilon + \mathcal{V}_\times^\varepsilon,$$

where

$$\mathfrak{P}_{\pm,\varepsilon} = (\mathbf{D} - \varepsilon^{-1} \tilde{\mathbf{A}}^\varepsilon - \mathbf{A}^\varepsilon)^2 \pm \varepsilon^{-2} \tilde{B}^\varepsilon \pm \varepsilon^{-1} B^\varepsilon.$$

Moreover, we assume that

$$\int_{\Omega} (h_\times(\mathbf{x}))^{-1} (\tilde{v}_\times(\mathbf{x}) + 2 \langle \tilde{\mathbf{A}}(\mathbf{x}), \mathbf{A}(\mathbf{x}) \rangle \mathbf{1} + B(\mathbf{x}) \operatorname{diag} \{-1, 1\}) (h_\times(\mathbf{x}))^{-1} d\mathbf{x} = 0.$$

Then the operator (14.59) can be represented in the form (14.58) with

$$\begin{aligned} \tilde{\eta}_{j,\times}(\mathbf{x}) &= A_j(\mathbf{x}) \mathbf{1}, \quad j = 1, 2; \\ \tilde{v}_\times(\mathbf{x}) &= \tilde{v}_\times(\mathbf{x}) + 2 \langle \tilde{\mathbf{A}}(\mathbf{x}), \mathbf{A}(\mathbf{x}) \rangle \mathbf{1} + B(\mathbf{x}) \operatorname{diag} \{-1, 1\}; \\ \tilde{Q}_\times(\mathbf{x}) &= \mathcal{V}_\times(\mathbf{x}) + |\mathbf{A}(\mathbf{x})|^2 \mathbf{1}. \end{aligned}$$

The operator (14.59) is the Pauli operator with the magnetic potential $\varepsilon^{-1} \tilde{\mathbf{A}}^\varepsilon + \mathbf{A}^\varepsilon$ (containing the singular term $\varepsilon^{-1} \tilde{\mathbf{A}}^\varepsilon$) perturbed by the matrix electric potential $\varepsilon^{-1} \tilde{v}_\times^\varepsilon + \mathcal{V}_\times^\varepsilon$ (also containing a singular term).

Now, taking (14.27) and (14.57) into account, we see that the operator (14.58) is related to the operator (14.32) by the formula

$$(14.60) \quad \tilde{\mathcal{B}}_{\times,\varepsilon} = h_\times^\varepsilon \mathcal{B}_{\times,\varepsilon} h_\times^\varepsilon.$$

Note that in (14.32) we put

$$(14.61) \quad \eta_{j,\times} = h_\times^{-1} \tilde{\eta}_{j,\times} h_\times^{-1}, \quad j = 1, 2; \quad v_\times = h_\times^{-1} \tilde{v}_\times h_\times^{-1}; \quad Q_\times = h_\times^{-1} \tilde{Q}_\times h_\times^{-1}.$$

In order to check identity (14.60), we use the fact that the matrices $\tilde{\eta}_{j,\times}$ and $h_\times$ are diagonal, and therefore, they commute.

Let $\tilde{Q}_{0,\times}(\mathbf{x})$ be a Γ -periodic, positive definite, and bounded (2×2) -matrix-valued function on $\mathbb{R}^2$. We consider the generalized resolvent $(\tilde{\mathcal{B}}_{\times,\varepsilon} + \lambda \tilde{Q}_{0,\times}^\varepsilon)^{-1}$, assuming that the parameter λ obeys (12.6) with $f = h_\times$, $\tilde{Q}_0 = \tilde{Q}_{0,\times}$, and with c_0, c_4 corresponding to the operator $\mathcal{B}_{\times,\varepsilon}$. The lower bound for λ is controlled in terms of $s, \rho, \|\omega_+\|_{L_\infty}, \|\omega_-\|_{L_\infty}, \|\tilde{\eta}_{j,\times}\|_{L_\rho(\Omega)}, j = 1, 2, \|\tilde{v}_\times\|_{L_s(\Omega)}, \|\tilde{Q}_\times\|_{L_s(\Omega)}, \|\tilde{Q}_{0,\times}\|_{L_\infty}, \|\tilde{Q}_{0,\times}^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ . The initial data reduces to the following set:

$$(14.62) \quad \rho, s; \|\tilde{\mathbf{A}}\|_{L_\rho(\Omega)}, \|\tilde{\eta}_{j,\times}\|_{L_\rho(\Omega)}, j = 1, 2, \|\tilde{v}_\times\|_{L_s(\Omega)}, \|\tilde{Q}_\times\|_{L_s(\Omega)}, |\lambda|, \|\tilde{Q}_{0,\times}\|_{L_\infty}.$$

Theorem 12.1 is applicable; this yields the following result.

Proposition 14.15. *Let $\tilde{\mathcal{B}}_{\times,\varepsilon}$ be the operator (14.58) whose coefficients satisfy the conditions formulated above. Let $\mathcal{B}_{\times,\varepsilon}$ be the operator (14.32) whose coefficients are related to those of the operator $\tilde{\mathcal{B}}_{\times,\varepsilon}$ by (14.61). Let $\mathcal{B}_{\times}^0$ be the effective operator for $\mathcal{B}_{\times,\varepsilon}$ defined by (14.43). Let $\tilde{Q}_{0,\times}$ be a Γ -periodic, positive definite, and bounded (2×2) -matrix-valued function. Suppose λ satisfies (12.6) with $f = h_{\times}$, $\tilde{Q}_0 = \tilde{Q}_{0,\times}$ and with c_0, c_4 corresponding to the operator $\mathcal{B}_{\times,\varepsilon}$. Let $Q_{0,\times} = (h_{\times})^{-1}\tilde{Q}_{0,\times}(h_{\times})^{-1}$. Then for $0 < \varepsilon \leq 1$ we have*

$$\|(\tilde{\mathcal{B}}_{\times,\varepsilon} + \lambda\tilde{Q}_{0,\times}^{\varepsilon})^{-1} - (h_{\times}^{\varepsilon})^{-1}(\mathcal{B}_{\times}^0 + \lambda\overline{Q_{0,\times}})^{-1}(h_{\times}^{\varepsilon})^{-1}\|_{L_2(\mathbb{R}^2) \rightarrow L_2(\mathbb{R}^2)} \leq \tilde{C}_{\times}\varepsilon,$$

where the constant $\tilde{C}_{\times}$ depends only on the initial data (14.62) and the parameters of the lattice Γ .

Since $d = 2$, Conditions 8.6 and 8.10 are satisfied. We use the properties of the functions $\Lambda_{\times}$ and $\tilde{\Lambda}_{\times}$ mentioned above in Subsection 14.2 and the fact that the norms $\|\omega_{+}\|_{L_{\infty}}$, $\|\omega_{-}\|_{L_{\infty}}$ are dominated by a constant depending only on $\|\tilde{\mathbf{A}}\|_{L_p(\Omega)}$ and the lattice. Then the norms $\|\Lambda_{\times}\|_{L_{\infty}}$ and $\|\tilde{\Lambda}_{\times}\|_{L_p(\Omega)}$ (for fixed $p > 2$) are controlled in terms of the data (14.62) and the parameters of the lattice Γ . Statement 3° of Theorem 12.3 applies to the operator $\tilde{\mathcal{B}}_{\times,\varepsilon}$. This leads to the following statement.

Proposition 14.16. *Under the assumptions of Proposition 14.15, let $\Lambda_{\times}(\mathbf{x})$ be the Γ -periodic solution of problem (14.37), and let $\tilde{\Lambda}_{\times}(\mathbf{x})$ be the Γ -periodic solution of problem (14.38). Then for $0 < \varepsilon \leq 1$ we have*

$$\|h_{\times}^{\varepsilon}(\tilde{\mathcal{B}}_{\times,\varepsilon} + \lambda\tilde{Q}_{0,\times}^{\varepsilon})^{-1} - (I + \varepsilon\Lambda_{\times}^{\varepsilon}b_{\times}(\mathbf{D}) + \varepsilon\tilde{\Lambda}_{\times}^{\varepsilon})(\mathcal{B}_{\times}^0 + \lambda\overline{Q_{0,\times}})^{-1}(h_{\times}^{\varepsilon})^{-1}\|_{L_2(\mathbb{R}^2) \rightarrow H^1(\mathbb{R}^2)} \leq \tilde{C}'_{\times}\varepsilon,$$

where the constant $\tilde{C}'_{\times}$ depends only on the initial data (14.62) and the parameters of the lattice Γ .

We proceed to an approximation of the flux. We denote

$$(14.63) \quad \tilde{\mathbf{u}}_{\times,\varepsilon} = (\tilde{\mathcal{B}}_{\times,\varepsilon} + \lambda\tilde{Q}_{0,\times}^{\varepsilon})^{-1}\mathbf{F}, \quad \mathbf{u}_{\times,\varepsilon}^0 = (\mathcal{B}_{\times}^0 + \lambda\overline{Q_{0,\times}})^{-1}(h_{\times}^{\varepsilon})^{-1}\mathbf{F}, \quad \mathbf{F} \in L_2(\mathbb{R}^2; \mathbb{C}^2),$$

$$(14.64) \quad \tilde{\mathbf{p}}_{\times,\varepsilon} = g_{\times}^{\varepsilon}b_{\times}(\mathbf{D})h_{\times}^{\varepsilon}\tilde{\mathbf{u}}_{\times,\varepsilon}, \quad \mathbf{p}_{\times,\varepsilon}^0 = g_{\times}^0b_{\times}(\mathbf{D})\mathbf{u}_{\times,\varepsilon}^0.$$

Since $g_{\times}^0 = \underline{g}_{\times}$, now we have $\tilde{g}_{\times} = g_{\times}^0$. Then statement 3° of Theorem 12.7 implies the following result.

Proposition 14.17. *Under the assumptions of Proposition 14.16, let $\tilde{\mathbf{u}}_{\times,\varepsilon}$, $\tilde{\mathbf{p}}_{\times,\varepsilon}$ and $\mathbf{u}_{\times,\varepsilon}^0$, $\mathbf{p}_{\times,\varepsilon}^0$ be defined by (14.63) and (14.64). Then for $0 < \varepsilon \leq 1$ we have*

$$\|\tilde{\mathbf{p}}_{\times,\varepsilon} - \mathbf{p}_{\times,\varepsilon}^0 - g_{\times}^{\varepsilon}(b_{\times}(\mathbf{D})\tilde{\Lambda}_{\times})^{\varepsilon}\mathbf{u}_{\times,\varepsilon}^0\|_{L_2(\mathbb{R}^2)} \leq \mathcal{C}_{\times}\varepsilon\|\mathbf{F}\|_{L_2(\mathbb{R}^2)}, \quad \mathbf{F} \in L_2(\mathbb{R}^2; \mathbb{C}^2).$$

The constant $\mathcal{C}_{\times}$ depends only on the initial data (14.62) and the parameters of the lattice Γ .

Corollary 14.18. *Under the assumptions of Proposition 14.17, suppose also that*

$$\check{v}_{\times}(\mathbf{x}) = 0, \quad \sum_{j=1}^2 \partial_j(h_{\times}(\mathbf{x}))^{-1}\check{\eta}_{j,\times}(\mathbf{x})(h_{\times}(\mathbf{x}))^{-1} = 0.$$

Then $\tilde{\Lambda}_{\times} = 0$, and we have

$$\|\tilde{\mathbf{p}}_{\times,\varepsilon} - \mathbf{p}_{\times,\varepsilon}^0\|_{L_2(\mathbb{R}^2)} \leq \mathcal{C}_{\times}\varepsilon\|\mathbf{F}\|_{L_2(\mathbb{R}^2)}, \quad \mathbf{F} \in L_2(\mathbb{R}^2; \mathbb{C}^2), \quad 0 < \varepsilon \leq 1.$$

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