THE MODULE OF PSEUDO-RATIONAL RELATIONS OF A QUOTIENT DIVISIBLE GROUP

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ABSTRACT. It is proved that the module of pseudo-rational relations determines a quotient divisible group up to quasi-isomorphism.

INTRODUCTION

In the most general case, an Abelian group G may be called *quotient divisible* if it contains a free subgroup F such that G/F is a divisible periodic group. In the present paper, we consider only quotient divisible torsion-free groups of finite rank that do not contain divisible periodic subgroups. Fomin and Wickless introduced such groups in [1]; moreover, they constructed the category of quotient divisible groups with quasihomomorphisms as morphisms and showed that this category is dual to the well-known category of torsion-free groups of finite rank with quasihomomorphisms as morphisms.

Even earlier, in [2], Beaumont and Pierce considered quotient divisible torsion-free groups of finite rank (the term itself “quotient divisible group” first appeared in that paper), for which they constructed a system of invariants that determine such a group up to quasi-isomorphism. It was an initial example of a description of Abelian groups up to quasi-isomorphism rather than up to isomorphism, which was quite timely in view of the papers [3] by Jónsson and [4] by Yakovlev.

In the present paper, we develop an idea of Fomin from [5] about the invocation of modules over the ring R of pseudo-rational numbers for the study of quotient divisible groups. The point is that any quotient divisible group can be embedded in a certain finitely generated R -module, the structure of which is simpler than that of the group itself but still carries much information about the group.

The main results of the paper are Theorems 6 and 7, in which we prove that the module of pseudo-rational relations determines a quotient divisible group up to quasi-isomorphism. In a sense, the results obtained here may be regarded as a generalization of the results of Beaumont and Pierce on a description of quotient divisible groups up to quasi-isomorphism.

Throughout the paper, by a “group” we mean an Abelian group written additively; by $\mathbb{Z}$, $\mathbb{Q}$, and $\widehat{\mathbb{Z}}_p$ we denote the ring of integers, the ring of rational numbers, and the ring of p -adic integers, respectively, or their additive groups; $\mathbb{Z}(m)$ is the ring (or additive group) of residue classes modulo m , P is the set of all prime numbers, and $\mathbb{N}$ is the set of all natural numbers. If S is a subset of a K -module M , then by $\langle S \rangle$ and $\langle S \rangle_K$ we denote, respectively, the subgroup and the submodule generated by S ; $\langle S \rangle_*$ denotes the pure subgroup generated by S , which consists of all $r \in M$ such that $nr \in \langle S \rangle$ for some natural n .

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The other notions and notation used in the paper are standard and fully correspond to [6].

§1. MODULES OVER THE RING OF PSEUDO-RATIONAL NUMBERS

Let $\chi = (m_p)$ be an arbitrary characteristic, and let $K_p = \mathbb{Z}/p^{m_p}\mathbb{Z}$ or $K_p = \widehat{\mathbb{Z}}_p$ for $m_p < \infty$ and $m_p = \infty$, respectively. If χ contains infinitely many nonzero elements, then we consider the ring $\mathbb{Z}_\chi = \prod_{p \in P} K_p$, in which we construct the ring R_χ that is purely generated by the unity of the ring and by the ideal $\bigoplus_{p \in P} K_p$,

$$R_\chi = \left\langle 1, \bigoplus_{p \in P} K_p \right\rangle_* \subset \prod_{p \in P} K_p.$$

If all p -components of χ except for $p_1, \dots, p_n$ are equal to zero, then we construct the rings $K_\chi = K_{p_1} \oplus \dots \oplus K_{p_n}$ and $R_\chi = \mathbb{Q} \oplus K_\chi$.

Definition 1. The ring R_ρ , where $\rho = (\infty, \infty, \dots)$ is the characteristic of the group $\mathbb{Q}$, is called the ring of *pseudo-rational numbers* and is denoted simply by R .

Consider some properties of the ring of pseudo-rational numbers.

1. An element $r = (\alpha_p) \in \prod_{p \in P} \widehat{\mathbb{Z}}_p$ belongs to R if and only if there exists a rational number $|r| = \frac{m}{n}$ such that $n\alpha_p = m$ for almost all primes p .
2. Denote by ε_p the element in R the p -component of which is equal to 1 and the other components are zero. Then ε_p is an idempotent of the ring R , and consequently, $R = \varepsilon_p R \oplus (1 - \varepsilon_p)R$.
3. The elements of the form

$$(a) \quad \varepsilon = \varepsilon_{p_1} + \dots + \varepsilon_{p_n},$$

where $p_1, \dots, p_n$ are distinct prime numbers, and the elements of the form

$$(b) \quad 1 - \varepsilon,$$

as well as 1 and 0, form the set of all idempotents of the ring R .

4. Any element $r \in R$ can be expressed as $r = \varepsilon r + (1 - \varepsilon)|r|$, where ε is an idempotent of the form (a).
5. A ring K is an epimorphic image of the ring R if and only if $K \cong R_\chi$ or $K \cong K_\chi$.
6. The set $T = \bigoplus_{p \in P} \widehat{\mathbb{Z}}_p$ is a maximal ideal of the ring R , and $R/T \cong \mathbb{Q}$.
7. The ring of pseudo-rational numbers is hereditary.

Let M be an arbitrary module over the ring of pseudo-rational numbers. Since $R/T \cong \mathbb{Q}$, it follows that the quotient module M/TM is a vector space over $\mathbb{Q}$.

Definition 2. The quantity $\dim_{\mathbb{Q}} M/TM$ is called the *pseudo-rational rank* of the R -module M and is denoted by $r^*(M)$.

This invariant was introduced by Fomin. In the theory of R -modules, it is as important as the usual rank in the theory of Abelian groups.

8. If $M = \langle x_1, \dots, x_n \rangle_R$, then $r^*(M) \leq n$.
9. For a quotient module M/L , we have

$$r^*(M) = r^*(M/L) + r^*(L).$$

An R -module M is said to be *divisible* if the structure of the R -module on M coincides with that of a $\mathbb{Q}$ -space; i.e., the additive group of the R -module M is a torsion-free divisible group and $r \cdot m = |r| \cdot m$ for any $r \in R$ and $m \in M$. An R -module M is said to be *reduced* if it contains no divisible submodules.

Let M be an arbitrary finitely generated R -module with a system of generators $\{x_1, \dots, x_n\}$. Then, obviously, the $\widehat{\mathbb{Z}}_p$ -module $M_p = \varepsilon_p M$ is generated by the elements $\{\varepsilon_p x_1, \dots, \varepsilon_p x_n\}$. The finitely generated p -adic module M_p is representable as a direct sum of cyclic $\widehat{\mathbb{Z}}_p$ -modules,

$$M_p = \langle a_1 \rangle_{\widehat{\mathbb{Z}}_p} \oplus \dots \oplus \langle a_n \rangle_{\widehat{\mathbb{Z}}_p},$$

where some summands may be zero.

A cyclic $\widehat{\mathbb{Z}}_p$ -module is isomorphic either to $\mathbb{Z}/p^{k_{ip}}\mathbb{Z}$, where k_{ip} is a nonnegative integer, or to $\widehat{\mathbb{Z}}_p$. Consequently, the isomorphism

$$M_p \cong \mathbb{Z}(p^{k_{p1}}) \oplus \dots \oplus \mathbb{Z}(p^{k_{pt}}) \oplus \bigoplus_s \widehat{\mathbb{Z}}_p, \quad t + s = n,$$

determines the following ordered sequence of nonnegative integers and symbols ∞ :

$$(1) \quad 0 \leq k_{p1} \leq \dots \leq k_{pn} \leq \infty,$$

where the last s terms are the symbols ∞ ($0 \leq s \leq n$). The sequence (1) over all primes p determines a sequence of characteristics (among which zero characteristics may occur):

$$(2) \quad \chi_1 \leq \dots \leq \chi_n.$$

The sequence (2) is called the *generalized cocharacteristic* of a finitely generated R -module M .

§2. QUOTIENT DIVISIBLE GROUPS

Definition 3. A group G is said to be *quotient divisible* if it contains no periodic divisible subgroups, but it contains a free subgroup F of finite rank such that G/F is a periodic divisible group.

The free subgroup F in Definition 3 is called a *fundamental subgroup* of G , and any base of it is called a *fundamental system* of the group.

Let G be a reduced quotient divisible group. Consider its $\mathbb{Z}$ -adic completion $\widehat{G}$. The canonical homomorphism $\alpha : G \rightarrow \widehat{G}$ is a monomorphism, because $\ker \alpha = \bigcap_{n \in \mathbb{N}} nG = G^1 = 0$. The group $\widehat{G}$ is a $\widehat{\mathbb{Z}}$ -module and, thus, a module over the ring of pseudo-rational numbers.

Definition 4. The R -module $\mathcal{R}(G) = \operatorname{div} G \oplus \langle \alpha(G) \rangle_R$ is called the *pseudo-rational envelope* (or the *pseudo-rational type*) of a quotient divisible group G .

The *pseudo-rational rank* of a quotient divisible group is defined to be the pseudo-rational rank of its pseudo-rational envelope and is denoted by $r^*(G)$,

$$r^*(G) = r^*(\mathcal{R}(G)).$$

By the generalized cocharacteristic of a quotient divisible group we mean the generalized cocharacteristic of its pseudo-rational envelope. Obviously, an embedding $\varphi : G \rightarrow \mathcal{R}(G)$ exists; for this reason, throughout the paper we shall identify the group G and its image $\varphi(G)$.

Theorem 1 ([9]). *A quotient divisible group is the additive group of an R -module if and only if its p -adic completion is a finite group for any prime p .*

Consider several statements proved in [5] and [7].

Theorem 2 ([5]). *If $\{x_1, \dots, x_n\}$ is a fundamental system of a quotient divisible group G , then $\mathcal{R}(G) = \langle x_1, \dots, x_n \rangle_R$.*

Theorem 3 ([5]). *If $\{x_1, \dots, x_n\}$ is an arbitrary linearly independent system of generators of an R -module M , then $G = \langle x_1, \dots, x_n \rangle_* \subseteq M$ is a quotient divisible group, and $\{x_1, \dots, x_n\}$ is its fundamental system.*

Lemma 1 ([7]). *Let G and H be some quotient divisible mixed groups; moreover, assume that either H is a reduced group or G is divisible. Let $\bigoplus_{i=1}^n \mathbb{Z}x_i$ be a fundamental subgroup of G , and let $\varphi : G \rightarrow H$ be an arbitrary homomorphism. If*

$$g = r_1x_1 + \dots + r_nx_n \in G, \quad r_1, \dots, r_n \in R,$$

then $\varphi(g) = r_1\varphi(x_1) + \dots + r_n\varphi(x_n)$.

Theorem 4. *Let G and H be quotient divisible groups, let $X = \{x_1, \dots, x_n\}$ be a fundamental system of the group G , and let $\alpha, \beta \in \text{Hom}(G, H)$. Then*

$$\alpha = \beta \Leftrightarrow \alpha(x_i) = \beta(x_i)$$

for any $i \in \{1, \dots, n\}$.

Proof. The implication “ $\Rightarrow$ ” is obvious. We prove “ $\Leftarrow$ ”.

The first case. The group H is reduced. By Theorem 2, any element $g \in G$ can be represented in the form

$$g = r_1x_1 + \dots + r_nx_n, \quad r_1, \dots, r_n \in R.$$

By Lemma 1,

$$\alpha(g) = r_1\alpha(x_1) + \dots + r_n\alpha(x_n) = r_1\beta(x_1) + \dots + r_n\beta(x_n) = \beta(g),$$

whence $\alpha = \beta$.

The second case. $H = \text{div } H \oplus H_1$, where $\text{div } H$ is divisible and H_1 is a reduced part of the group H . Then

$$\text{Hom}(G, H) = \text{Hom}(G, \text{div } H) \oplus \text{Hom}(G, H_1),$$

and $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$, where $\alpha_1, \beta_1 \in \text{Hom}(G, \text{div } H)$ and $\alpha_2, \beta_2 \in \text{Hom}(G, H_1)$. Since $\alpha(X) = \beta(X)$, we have $\alpha_1(X) = \beta_1(X)$ and $\alpha_2(X) = \beta_2(X)$. Then the first case implies that $\alpha_2 = \beta_2$.

Let g be an arbitrary element of the group G . Then the definition of a quotient divisible group shows that there exists a natural number m such that $mg = m_1x_1 + \dots + m_nx_n$, where $m_1, \dots, m_n \in \mathbb{Z}$. Since $\alpha_1(X) = \beta_1(X)$, we have

$$\alpha_1(mg) = m_1\alpha_1(x_1) + \dots + m_n\alpha_1(x_n) = m_1\beta_1(x_1) + \dots + m_n\beta_1(x_n) = \beta_1(mg),$$

whence $m(\alpha_1(g) - \beta_1(g)) = 0$. But $\alpha_1(g) - \beta_1(g) \in \text{div } H$, and $\text{div } H$ contains no nonzero elements of finite order. Consequently, $\alpha_1(g) - \beta_1(g) = 0$ and $\alpha_1(g) = \beta_1(g)$. Therefore, $\alpha_1 = \beta_1$ and, thus, $\alpha = \alpha_1 + \alpha_2 = \beta_1 + \beta_2 = \beta$. $\square$

§3. THE MODULE OF PSEUDO-RATIONAL RELATIONS

Let M be a finitely generated R -module, and $X = \{x_1, \dots, x_n\}$ an arbitrary system of elements in M . Consider the set

$$\Delta M_X = \{(r_1, \dots, r_n) \in R^n \mid r_1x_1 + \dots + r_nx_n = 0\},$$

which is obviously an R -module. If X is a system of generators in M , then ΔM_X will be called the *module of pseudo-rational relations* of M .

Proposition 1. *If $X = \{x_1, \dots, x_n\}$ is a system of generators of an R -module M , then $M \cong R^n / \Delta M_X$.*

Proof. Consider the mapping $\varphi : R^n \rightarrow M$ given by the rule

$$\varphi(r_1, \dots, r_n) = r_1x_1 + \dots + r_nx_n.$$

It is easily seen that φ is a homomorphism. Since X is a system of generators of the R -module M , φ is an epimorphism. Note that

$$(r_1, \dots, r_n) \in \ker \varphi \Leftrightarrow r_1x_1 + \dots + r_nx_n = 0,$$

i.e., $\ker \varphi = \Delta M_X$. Thus, $M \cong R^n / \Delta M_X$. $\square$

Corollary 1. *If $X = \{x_1, \dots, x_n\}$ is a system of generators of an R -module M , then $r^*(M) + r^*(\Delta M_X) = n$.*

Let G be a quotient divisible group, and let $X = \{x_1, \dots, x_n\}$ be a system of its elements. Then G gives rise to the R -module

$$\Delta G_X = \{(r_1, \dots, r_n) \in R^n \mid r_1x_1 + \dots + r_nx_n \in \operatorname{div} G\}.$$

If X is a fundamental system of the group G , then the R -module ΔG_X is called the *module of pseudo-rational relations* of the quotient divisible group G .

If $X = \{x_1, \dots, x_n\}$ is a fundamental system of a quotient divisible group G , then $\mathcal{R}(G) = \langle x_1, \dots, x_n \rangle_R$. So,

$$\mathcal{R}(G) \cong \operatorname{div} G \oplus (R^{r(G)} / \Delta G_X) \quad \text{and} \quad r^*(G) = r(\operatorname{div} G) + r(G) - r^*(\Delta G_X).$$

Theorem 5 ([7]). *Let G and H be arbitrary quotient divisible groups, $X = \{x_1, \dots, x_n\}$ a fundamental system of G , and $Y = \{y_1, \dots, y_n\}$ an arbitrary system of elements of H . A homomorphism $f : G \rightarrow H$ such that $f(x_i) = y_i$ ($1 \leq i \leq n$) exists if and only if $\Delta G_X \subseteq \Delta H_Y$.*

If ΔG_X is the module of pseudo-rational relations of a quotient divisible group G of rank n and A is an $(n \times n)$ -matrix with integral entries, then we define the product $\Delta G_X \cdot A$ in the following way:

$$\Delta G_X \cdot A = \{(r_1, \dots, r_n) \cdot A \mid (r_1, \dots, r_n) \in \Delta G_X\}.$$

Lemma 2. *If $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ are arbitrary maximal linearly independent systems of a quotient divisible group G , then there exists a matrix $A \in GL(n, \mathbb{Q})$ with integral entries such that $\Delta G_X \cdot A \subseteq \Delta G_Y$.*

Proof. Since X and Y are maximal systems, it follows that there are numbers $m, k \in \mathbb{Z}$ and matrices A and B with integral entries such that

$$(1) \quad m \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad \text{and} \quad k \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = B \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Let $(r_1, \dots, r_n) \in \Delta G_X$. Then $(r_1, \dots, r_n)[X] \in \operatorname{div} G$, where $[X]$ is the column of the elements $x_1, \dots, x_n$, so that $m(r_1, \dots, r_n)[X] \in \operatorname{div} G$. It follows that $(r_1, \dots, r_n)A[Y] \in \operatorname{div} G$. Then

$$(2) \quad \Delta G_X \cdot A \subseteq \Delta G_Y.$$

Similarly,

$$(3) \quad \Delta G_Y \cdot B \subseteq \Delta G_X.$$

Relation (1) and the maximality of X and Y imply that

$$AB = BA = kmE.$$

Next, multiplying (3) by A from the right, we get

$$(4) \quad km\Delta G_Y \subseteq \Delta G_X \cdot A.$$

From (2) and (4) it follows that $\Delta G_X \cdot A \doteq \Delta G_Y$. $\square$

Theorem 6. *Quotient divisible groups G and H are quasi-isomorphic if and only if for arbitrary fundamental systems X and Y in G and H , respectively, there exists a matrix $A \in GL(n, \mathbb{Q})$ with integral entries such that $\Delta G_X \cdot A \doteq \Delta H_Y$.*

Proof. Let $\frac{1}{m} \otimes \alpha : G \rightarrow H$ and $\frac{1}{k} \otimes \beta : H \rightarrow G$ be mutually inverse quasi-isomorphisms, and let

$$\alpha([X]) = [U] = A[Y], \quad \beta([Y]) = [T] = B[X].$$

We assume that the homomorphisms α and β are chosen in such a way that the matrices A and B have integral entries (this assumption can be imposed because $\frac{1}{m} \otimes \alpha = \frac{1}{sm} \otimes s\alpha$ and $s\alpha([X]) = s[U] = sA[Y]$). Then

$$\begin{aligned} \alpha(\beta([Y])) &= \alpha(B[X]) = B\alpha([X]) = BA[Y], \\ \beta(\alpha([X])) &= \beta(A[Y]) = A\beta([Y]) = AB[X]. \end{aligned}$$

On the other hand, since $\frac{1}{m} \otimes \alpha$ and $\frac{1}{k} \otimes \beta$ are mutually inverse, we have

$$\alpha\beta = mk \cdot id_H \quad \text{and} \quad \beta\alpha = mk \cdot id_G.$$

Thus, $BA = AB = mkE$.

By Theorem 5, $\Delta G_X \subseteq \Delta H_U$ and $\Delta H_Y \subseteq \Delta G_T$. Since $[U] = A[Y]$ and $[T] = B[X]$, we have $\Delta H_U \cdot A \subseteq \Delta H_Y$ and $\Delta G_T \cdot B \subseteq \Delta G_X$. This implies that

$$\Delta G_X \cdot A \subseteq \Delta H_Y \quad \text{and} \quad \Delta H_Y \cdot B \subseteq \Delta G_X.$$

Multiplying the second inclusion by A from the right, we obtain $km(\Delta H_Y) \subseteq \Delta G_X \cdot A$, which implies the quasirelation $\Delta G_X \cdot A \doteq \Delta H_Y$.

Conversely, assume that for quotient divisible groups G and H we have the quasirelation $\Delta G_X \cdot A \doteq \Delta H_Y$, where $A \in GL(n, \mathbb{Q})$ is a matrix with integral entries. Then $m(\Delta G_X \cdot A) \subseteq \Delta H_Y$ and $k(\Delta H_Y) \subseteq \Delta G_X \cdot A$ for some $m, k \in \mathbb{N}$. Let $[U] = mA[Y]$. If $(r_1, \dots, r_n) \in \Delta G_X$, then

$$(r_1, \dots, r_n)mA[Y] = (r_1, \dots, r_n)[U] \in \text{div } H,$$

so that $\Delta G_X \subseteq \Delta H_U$. By Theorem 5, there exists a homomorphism $\alpha : G \rightarrow H$ such that $\alpha([X]) = [U] = mA[Y]$.

Since the matrix A lies in $GL(n, \mathbb{Q})$, there exists a matrix $B \in GL(n, \mathbb{Q})$ with integral entries such that $AB = BA = nE$, where n is a nonzero integer. We multiply the quasirelation $\Delta G_X \cdot A \doteq \Delta H_Y$ by the matrix B from the right obtaining

$$\Delta H_Y \cdot B \doteq n\Delta G_X \doteq \Delta G_X.$$

In the same way as above, we prove the existence of a homomorphism $\beta : H \rightarrow G$ such that $\beta([Y]) = [T] = kB[X]$, where k is a nonzero integer. We find $\alpha\beta([Y])$:

$$\alpha(\beta([Y])) = \alpha(kB[X]) = kB\alpha([X]) = kBmA[Y] = kmn[Y].$$

Similarly, we check that $\beta\alpha([X]) = kmn[X]$. Then, by Theorem 4,

$$\alpha\beta = kmn \cdot id_H \quad \text{and} \quad \beta\alpha = kmn \cdot id_G.$$

Thus, the quasihomomorphisms $\frac{1}{mn} \otimes \alpha$ and $\frac{1}{k} \otimes \beta$ are mutually inverse; i.e., the groups G and H are quasi-isomorphic. $\square$

Lemma 2 and Theorem 6 imply the following statement.

Corollary 2. *Let $X = \{x_1, \dots, x_n\}$ be a maximal linearly independent system of elements of a quotient divisible group G . Then the subgroup*

$$H = \langle x_1, \dots, x_n \rangle_* \subseteq \langle x_1, \dots, x_n \rangle_R$$

of G is a quotient divisible group quasi-isomorphic to G .

In conclusion of the present section, we touch on the problem concerning quasi-equality of R -modules in general and of modules of pseudo-rational relations in particular.

Proposition 2. *R -modules M and L (not necessarily finitely generated) are quasi-equal if and only if $(1 - \varepsilon)M = (1 - \varepsilon)L$ and εM is quasi-equal to εL , where $(1 - \varepsilon)$ is an idempotent of the form (b) of the ring R .*

Proof. If M and L are quasi-equal R -modules, then $M/(M \cap L)$ and $L/(M \cap L)$ are finite R -modules; therefore,

$$(1 - \varepsilon)M = (1 - \varepsilon)(M \cap L) = (1 - \varepsilon)L$$

for some idempotent $(1 - \varepsilon) \in R$ of the form (b).

Since the R -modules M and L are quasi-equal, the R -modules εM and εL are also quasi-equal.

The converse statement is obvious. $\square$

Let G and H be quasi-isomorphic quotient divisible groups. Then their modules of pseudo-rational relations are connected by the quasi-equality $\Delta G \cdot A \doteq \Delta H$, where A is a nondegenerate matrix with integral entries (the form of which depends on the choice of fundamental systems in G and H). Proposition 2 implies that

$$(1 - \varepsilon)\Delta G \cdot A = (1 - \varepsilon)\Delta H \quad \text{and} \quad \varepsilon(\Delta G \cdot A) \doteq \varepsilon\Delta H.$$

If $\varepsilon = \varepsilon_{p_1} + \dots + \varepsilon_{p_n}$, then $\varepsilon\Delta G = \varepsilon_{p_1}\Delta G \oplus \dots \oplus \varepsilon_{p_n}\Delta G$; consequently,

$$\varepsilon(\Delta G \cdot A) \doteq \varepsilon\Delta H \Leftrightarrow \varepsilon_p(\Delta G \cdot A) \doteq \varepsilon_p\Delta H \quad \text{for every } p \in \{p_1, \dots, p_n\},$$

and $\varepsilon_p(\Delta G \cdot A)$ and $\varepsilon_p\Delta H$ are finitely generated free $\widehat{\mathbb{Z}}_p$ -modules for any prime $p \in \{p_1, p_2, \dots, p_n\}$.

§4. THE CATEGORY $\mathfrak{T}$

In this section, we translate the main results of the preceding section into the language of categories.

Let M be a submodule of the free R -module R^n such that R^n/M is a reduced R -module. The set M consists of some elements of the form $(r_1, \dots, r_n) \in R^n$ and is closed under addition and multiplication by pseudo-rational numbers.

In the quotient module R^n/M , we consider the system of generators

$$x_1 = (1, 0, \dots, 0) + M, \quad x_2 = (0, 1, \dots, 0) + M, \quad \dots, \quad x_n = (0, 0, \dots, 1) + M.$$

In a torsion-free divisible group D of finite rank $n - r$, where r is the number of elements in a maximal subsystem of $\{x_1, \dots, x_n\}$ that is independent over $\mathbb{Z}$, we choose elements $d_1, \dots, d_n$ in such a way that the elements

$$x'_1 = d_1 + x_1, \quad x'_2 = d_2 + x_2, \quad \dots, \quad x'_n = d_n + x_n$$

are linearly independent over $\mathbb{Z}$ in $D \oplus R^n/M$. Consider the group

$$(*) \quad G = \langle x'_1, \dots, x'_n \rangle_* \subseteq D \oplus R^n/M.$$

Since $\{x'_1, \dots, x'_n\}$ is a linearly independent (over $\mathbb{Z}$) system of generators of the R -module $D \oplus R^n/M$, the group G is quotient divisible.

Proposition 3. *If G is a quotient divisible group defined by the condition $(*)$, then $D \oplus R^n/M$ is its pseudo-rational envelope and M is its module of pseudo-rational relations.*

Proof. Indeed, since $G \subseteq D \oplus R^n/M$ by construction, we have $\mathcal{R}(G) \subseteq D \oplus R^n/M$. Since the elements $x'_1, \dots, x'_n$ generate the module $D \oplus R^n/M$ and are contained in G , it follows that $D \oplus R^n/M \subseteq \mathcal{R}(G)$. Thus, $D \oplus R^n/M$ is the pseudo-rational envelope of the group G .

Since the module ΔG_X is the kernel of the natural homomorphism from R^n to $\mathcal{R}(G)/\text{div } G = R^n/M$, we obtain $\Delta G_X = M$. $\square$

Let $\mathfrak{T}_n$ be the set of all submodules $M \subseteq R^n$ such that R^n/M is a reduced R -module. We construct a category $\mathfrak{T}$, the objects of which are elements of the set $\bigcup_{n \in \mathbb{N}} \mathfrak{T}_n$. If $M_1 \in \mathfrak{T}_n$ and $M_2 \in \mathfrak{T}_k$ are arbitrary objects of $\mathfrak{T}$, then by a morphism of M_1 to M_2 we mean any $(n \times k)$ -matrix A with integral entries such that

$$mM_1 \cdot A \subseteq M_2$$

for some natural m (i.e., $M_1 \cdot A$ is quasi-embedded in M_2). The product of morphisms $A : M_1 \rightarrow M_2$ and $B : M_2 \rightarrow M_3$ is the morphism $BA : M_1 \rightarrow M_3$. Obviously, the identity matrix of size n is the identity morphism of an object $M \in \mathfrak{T}_n$. It is easily seen that objects M_1 and M_2 of the category $\mathfrak{T}$ are isomorphic if and only if $M_1, M_2 \in \mathfrak{T}_n$ for some $n \in \mathbb{N}$ and there exists a matrix A in $GL(n, \mathbb{Q})$ with integral entries such that $M_1 \cdot A \subseteq M_2$.

We recall that $\mathcal{QD}$ is the category of quotient divisible groups with quasihomomorphisms as morphisms.

Theorem 7. *The categories $\mathfrak{T}$ and $\mathcal{QD}$ are equivalent.*

Proof. We construct covariant functors $c : \mathfrak{T} \rightarrow \mathcal{QD}$ and $d : \mathcal{QD} \rightarrow \mathfrak{T}$ such that

$$(1) \quad d \cdot c \cong \text{id}_{\mathfrak{T}} \quad \text{and} \quad c \cdot d \cong \text{id}_{\mathcal{QD}},$$

where id is the identity functor.

Let M be an arbitrary object of the category $\mathfrak{T}$. Then we set $c(M) = G$, where G is a quotient divisible group defined by the condition $(*)$.

Consider an arbitrary morphism A of objects M and L in the category $\mathfrak{T}$. Let G and H be quotient divisible groups that correspond to the modules M and L . Since $M = \Delta G_X$ and $L = \Delta H_Y$, we have $m\Delta G_X \cdot A \subseteq \Delta H_Y$ for some $m \in \mathbb{N}$. Then if $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_k\}$, then

$$(r_1, \dots, r_n)mA[Y] \in \text{div } H$$

for any $(r_1, \dots, r_n) \in \Delta G_X$. Let $U = \{u_1, \dots, u_k\}$ be a system of elements in H such that $[U] = mA \cdot [Y]$; then $\Delta G_X \subseteq \Delta H_U$, and thus, by Theorem 5, there exists a homomorphism $\alpha : G \rightarrow H$ such that $\alpha(x_i) = u_i$ for every $i \in \{1, 2, \dots, n\}$. We set $c(A) = \frac{1}{m} \otimes \alpha$ and verify that the quasihomomorphism $c(A)$ does not depend on the choice of the number m . Let $m_1 \in \mathbb{N}$ be such that $m_1\Delta G_X \cdot A \subseteq \Delta H_Y$. Then the pair (m_1, A) determines a homomorphism $\alpha_1 : G \rightarrow H$ such that $\alpha_1([X]) = m_1A[Y]$. Using the relation $\alpha([X]) = mA[Y]$ and Theorem 4, we see that $m_1\alpha = m\alpha_1$, whence

$$\frac{1}{mm_1} \otimes m_1\alpha = \frac{1}{mm_1} \otimes m\alpha_1,$$

i.e., $\frac{1}{m} \otimes \alpha = \frac{1}{m_1} \otimes \alpha_1$.

Let G and H be arbitrary groups in the category $\mathcal{QD}$, and let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_k\}$ be fundamental systems in G and H , respectively. Then we define

$$d(G) = \Delta G_X \quad \text{and} \quad d(H) = \Delta H_Y.$$

If $\frac{1}{m} \otimes \alpha : G \rightarrow H$ is a quasihomomorphism and $\alpha(X) = U$, then $\Delta G_X \subseteq \Delta H_U$ by Theorem 5. Since Y is a fundamental system of the group H and $U \subset H$, there exists an $(n \times k)$ -matrix A with integral entries and $s \in \mathbb{N}$ such that $s[U] = A[Y]$. Since $s\alpha([X]) = s[U] = A[Y]$, it follows that

$$(2) \quad \Delta G_X \subseteq \Delta H_{sU} = \Delta H_{AY}.$$

Suppose $(r_1, \dots, r_n) \in \Delta H_{AY}$. Then $(r_1, \dots, r_n)AY \in \text{div } H$, so that

$$(3) \quad \Delta H_{AY} \cdot A \subseteq \Delta H_Y.$$

Taken together, (2) and (3) imply $\Delta G_X \cdot A \subseteq \Delta H_Y$. Next, we define $d(\frac{1}{m} \otimes \alpha) = A$.

Theorem 6 shows that the functors constructed satisfy condition (1); i.e., the categories $\mathfrak{T}$ and $\mathcal{QD}$ are equivalent. $\square$

§5. COMPLETELY DECOMPOSABLE QUOTIENT DIVISIBLE GROUPS

By analogy with torsion-free groups, a quotient divisible group is said to be *completely decomposable* if it decomposes into a direct sum of quotient divisible groups of rank 1, and it is *almost completely decomposable* if it contains a completely decomposable quotient divisible subgroup of finite index.

Theorem 8. *A quotient divisible group is completely decomposable if and only if it has a fundamental system independent over R .*

Proof. Let G be a completely decomposable quotient divisible group, and let $G = A_1 \oplus \dots \oplus A_n$ be its decomposition into a direct sum of groups of rank 1. For every term A_i ($i \in \{1, \dots, n\}$), we consider a fundamental element x_i . By Theorem 2, we have

$$\mathcal{R}(G) = \mathcal{R}(A_1) \oplus \dots \oplus \mathcal{R}(A_n) = \langle x_1 \rangle_R \oplus \dots \oplus \langle x_n \rangle_R.$$

On the other hand, $\{x_1, \dots, x_n\}$ is a fundamental system of the group G and, thus, $\mathcal{R}(G) = \langle x_1, \dots, x_n \rangle_R$. Consequently,

$$\langle x_1, \dots, x_n \rangle_R = \langle x_1 \rangle_R \oplus \dots \oplus \langle x_n \rangle_R;$$

i.e., the system $\{x_1, \dots, x_n\}$ is independent over the ring R .

Conversely, let $\{x_1, \dots, x_n\}$ be a fundamental system of the group G independent over R , and let $A_i = \langle x_i \rangle_*$ be a rank 1 quotient divisible subgroup of G for every $i \in \{1, \dots, n\}$. We show that $G = A_1 \oplus \dots \oplus A_n$.

Any element $g \in G$ can be represented in the form $g = r_1x_1 + \dots + r_nx_n$, where $r_1, \dots, r_n \in R$. On the other hand, $mg = m_1x_1 + \dots + m_nx_n$ for some $m_1, \dots, m_n \in \mathbb{Z}$ and $m \in \mathbb{N}$. Since $x_1, \dots, x_n$ are independent over R , we have $mr_ix_i = m_ix_i \in \langle x_i \rangle$ for every $i \in \{1, \dots, n\}$, whence $r_ix_i \in A_i$. It follows that $G = A_1 + \dots + A_n$.

If $g = a_1 + \dots + a_n$, where $a_i \in A_i$, then $a_i = s_ix_i$ ($s_i \in R$) for every $i \in \{1, \dots, n\}$. Then $g = s_1x_1 + \dots + s_nx_n = r_1x_1 + \dots + r_nx_n$. But the elements $x_1, \dots, x_n$ are independent over R , and consequently, $a_i = s_ix_i = r_ix_i$ for every $i \in \{1, \dots, n\}$, i.e., $G = A_1 \oplus \dots \oplus A_n$. $\square$

Corollary 3. *A quotient divisible group is almost completely decomposable if and only if it has an independent over R maximal system linearly independent over $\mathbb{Z}$.*

Example. Consider the R -module $M = R \oplus \widehat{\mathbb{Z}}_p$, in which we take the elements $x_1 = (1, 0)$ and $x_2 = (\varepsilon_p, \varepsilon_p)$. We use them to construct the quotient divisible group $G = \langle x_1, x_2 \rangle_* \subset M$. Since x_1 and x_2 are independent over R , it follows that the group G decomposes into a direct sum of quotient divisible groups, $G = H \oplus K$. Moreover, $H = \langle x_1 \rangle_* \subset M$ and $K = \langle x_2 \rangle_* \subset M$, i.e., $H \cong \mathbb{Z}$ and $K \cong \mathbb{Q}_p$.

§6. GROUPS WITH FINITELY GENERATED MODULES OF PSEUDO-RATIONAL RELATIONS

Theorem 6 states that two quotient divisible groups G and H are quasi-isomorphic if and only if there exists a nondegenerate matrix A with integral entries such that

$$\Delta G_X \cdot A \doteq \Delta H_Y,$$

where X and Y are arbitrary fundamental systems in G and H , respectively. Thus, we may say that a quotient divisible group is determined (up to quasi-isomorphism) by its module of pseudo-rational relations. In this connection, it is natural to consider the case where the module of pseudo-rational relations has a rather simple structure, namely, the case where it is finitely generated.

Let G be a quotient divisible group with a finitely generated module ΔG_X of pseudo-rational relations, and let $r(G) = n$. Then its pseudo-rational envelope has the form $\mathcal{R}(G) \cong \operatorname{div} G \oplus R^n / \Delta G_X$, and the R -module $R^n / \Delta G_X$ is finitely representable. In [8], it was shown that a finitely representable R -module decomposes into the direct sum of a projective and a finite module. Also in [8], it was proved that a finitely generated projective R -module M has the form

$$M \cong (1 - \varepsilon)R^k \oplus K_{\varphi_1} \oplus \cdots \oplus K_{\varphi_s},$$

where $\varphi_1, \dots, \varphi_s$ are almost zero idempotent characteristics.

Thus, the module $\mathcal{R}(G)$ has the form

$$\mathcal{R}(G) \cong \operatorname{div} G \oplus (1 - \varepsilon)R^k \oplus K_{\chi_1} \oplus \cdots \oplus K_{\chi_m},$$

where $\chi_1, \dots, \chi_m$ are some (not necessarily idempotent) almost zero characteristics. Consequently, the generalized cocharacteristic of the R -module $\mathcal{R}(G)$ and, thus, of the group G has the form

$$\rho_1, \dots, \rho_k, \chi_1, \dots, \chi_m, 0, \dots, 0,$$

where $\rho_1 = \dots = \rho_k = (1 - \varepsilon)\rho$.

Theorem 9. *A reduced quotient divisible group G has a finitely generated module of pseudo-rational relations if and only if its generalized cocharacteristic has the form $\rho_1, \dots, \rho_k, \chi_1, \dots, \chi_m$, where the characteristics $\rho_1, \dots, \rho_k$ differ from the characteristic $\rho = (\infty)$ by a finite number of ρ -components, the characteristics $\chi_1, \dots, \chi_m$ are almost zero, and $k = r^*(G)$.*

Proof. The “only if” statement follows from all we have said above. We prove the converse statement. Let a group G have a generalized cocharacteristic $\rho_1, \dots, \rho_k, \chi_1, \dots, \chi_m$ satisfying the assumptions of the theorem. Consider the pseudo-rational envelope $\mathcal{R}(G)$ of the group G . Since the characteristics $\chi_1, \dots, \chi_m$ are almost zero, there exists an idempotent $(1 - \varepsilon)$ of the form (b) such that the R -module $(1 - \varepsilon)\mathcal{R}(G)$ has the generalized cocharacteristic $(1 - \varepsilon)\rho_1, \dots, (1 - \varepsilon)\rho_k$. Note that

$$r^*((1 - \varepsilon)\mathcal{R}(G)) = r^*(\mathcal{R}(G)) = k.$$

By [9, Corollary 4.1], we have

$$(1) \quad (1 - \varepsilon)\mathcal{R}(G) \cong R_{(1-\varepsilon)\rho_1} \oplus \cdots \oplus R_{(1-\varepsilon)\rho_k}.$$

The finitely generated R -module $\varepsilon\mathcal{R}(G)$ has pseudo-rational rank 0; applying [9, Corollary 3.1], we obtain

$$(2) \quad \varepsilon\mathcal{R}(G) \cong K_{\varphi_1} \oplus \cdots \oplus K_{\varphi_n},$$

where $\varphi_1, \dots, \varphi_n$ are almost zero characteristics. From (1) and (2) it follows that the R -module $\mathcal{R}(G)$ can be represented in the form $P \oplus K$, where P is a projective R -module and K is a finite R -module. Then, by [8, Corollary 3], $\mathcal{R}(G)$ is a finitely representable

R -module. Since $\mathcal{R}(G) \cong R^{r(G)}/\Delta G_X$, it follows that ΔG_X is a finitely generated R -module. $\square$

Corollary 6. *If a quotient divisible group G has a finitely generated module of pseudo-rational relations, then it can be represented in the form $G = G_1 \oplus K$, where G_1 is a quotient divisible torsion-free group and K is a finite group.*

All the main results obtained above for quotient divisible groups can be carried over to torsion-free groups of finite rank. This follows from the duality constructed in [1] by Fomin and Wickless.

We recall that $\mathcal{QD}$ is the category of quotient divisible groups with quasihomomorphisms as morphisms and $\mathcal{QTF}$ is the category of torsion-free groups of finite rank with quasihomomorphisms as morphisms.

Theorem 10 ([1]). *The categories $\mathcal{QTF}$ and $\mathcal{QD}$ are dual.*

Corollary 7. *The categories $\mathcal{QTF}$ and $\mathcal{T}$ are dual.*

If f is the duality functor from $\mathcal{QTF}$ to $\mathcal{QD}$, then two torsion-free groups of finite rank G and H are quasi-isomorphic if and only if the quotient divisible groups $f(G)$ and $f(H)$ are quasi-isomorphic.

In [5], the pseudo-rational type $\mathcal{R}(G)$ was defined for a torsion-free group G of finite rank, and it was shown there that the R -modules $\mathcal{R}(G)$ and $\mathcal{R}(f(G))$ are quasi-isomorphic. If $\chi_1, \dots, \chi_k$ is the generalized cocharacteristic of the R -module $\mathcal{R}(G)$ (and thus, of the quotient divisible group $f(G)$), then the construction of the functor f implies that the sequence of types $[\chi_1], \dots, [\chi_k]$ is the Richman type of the group G .

Let G be a torsion-free group of finite rank, $X = \{x_1, \dots, x_n\}$ its maximal linearly independent system, $G^* = f(G)$ the quotient divisible group dual to G , and $X^* = \{x_1^*, \dots, x_n^*\}$ the fundamental system in G^* dual to X . Then the R -module $\Delta G_{X^*}^* = \Delta \mathcal{R}(G)_{X^*}$ is called the *module of pseudo-rational relations of a torsion-free group G of finite rank* and is denoted by ΔG_X . In [7], it was proved that

$$\Delta G_X = \{(\varphi(x_1), \dots, \varphi(x_n)) \mid \varphi \in \text{Hom}(G, R)\}.$$

Thus, we obtain the following theorem.

Theorem 11. *Torsion-free groups G and H of finite rank are quasi-isomorphic if and only if there exists a matrix $A \in GL(n, \mathbb{Q})$ with integral entries such that*

$$\Delta G_X \cdot A \doteq \Delta H_Y,$$

where X and Y are some maximal linearly independent systems in the groups G and H , respectively.

The pseudo-rational rank of a torsion-free group G of finite rank is defined as the pseudo-rational rank of its pseudo-rational type

$$r^*(G) = r^*(\mathcal{R}(G)) = r^*(G^*).$$

Theorem 12. *A torsion-free group G of finite rank has a finitely generated module of pseudo-rational relations if and only if its Richman type has the form $\tau_1, \dots, \tau_k, \sigma_1, \dots, \sigma_m$, where the types $\tau_1, \dots, \tau_k$ consist of characteristics that differ from the characteristic $\rho = (\infty)$ by a finite number of p -components, the types $\sigma_1, \dots, \sigma_m$ are almost zero, and $k = r^*(G)$.*

Theorem 12 implies that all groups of the form $A \oplus M$, where A is an almost divisible group and M is a minimax group, have finitely generated modules of pseudo-rational relations. We recall that a group A is said to be *almost divisible* if $pA = A$ for almost all

prime numbers p , and a torsion-free group A of finite rank is *minimax* if it is an extension of a free group of finite rank with the help of a divisible periodic group of finite rank. In other words, A is an almost divisible group if and only if its Richman type consists of types the characteristics of which differ from the characteristic $\rho = (\infty)$ only at a finite number of places, and this number is equal to $r^*(A)$ (it is easily seen that for an almost divisible group A we have $r^*(A) = r(A)$). A torsion-free group M of finite rank is minimax if and only if its Richman type consists of almost zero types.

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