CLUSTER $\mathcal{X}$ -VARIETIES FOR DUAL POISSON–LIE GROUPS. I

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ABSTRACT. We associate a family of cluster $\mathcal{X}$ -varieties with the dual Poisson–Lie group G^* of a complex semisimple Lie group G of adjoint type given with the standard Poisson structure. This family is described by the W -permutohedron associated with the Lie algebra $\mathfrak{g}$ of G , vertices being labeled by cluster $\mathcal{X}$ -varieties and edges by new Poisson birational isomorphisms on appropriate seed $\mathcal{X}$ -tori, called *saltation*. The underlying combinatorics is based on a factorization of the Fomin–Zelevinsky twist maps into mutations and other new Poisson birational isomorphisms on seed $\mathcal{X}$ -tori, called *tropical mutations* (because they are obtained by a tropicalization of the mutation formula), associated with an enrichment of the combinatorics on double words of the Weyl group W of G .

§1. INTRODUCTION

The rising of the cluster combinatorics goes back to two sources: Berenstein, Fomin and Zelevinsky on the one hand [FZ02, FZ03a, BFZ05, FZ07], in their study of total positivity, and Fock and Goncharov on the other hand [FG07a, FG06b, FG07b], in their higher Teichmüller theory. These structures quickly spread to diverse mathematical areas such as: quiver representations, Poisson geometry, integrable systems, convex polytopes, tropical geometry, and so on. In this paper and its sequel [B], we use them to sharpen the geometry of dual Poisson–Lie groups. Therefore, according to the quantum duality principle [STS93], these two papers can be seen as the semiclassical starting point towards a cluster combinatorics describing the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ associated with a complex semisimple Lie algebra $\mathfrak{g}$.

Let us recall that a Lie group G given with a Poisson structure is called a *Poisson–Lie group* if the multiplication $m : G \times G \rightarrow G$ is a Poisson map, when the set $G \times G$ is endowed with the Poisson product structure. Let $(\mathfrak{g}, \mathfrak{g}^*)$ be the tangent Lie bialgebra of G . According to the standard theory, $(\mathfrak{g}^*, \mathfrak{g})$ is also a Lie bialgebra, and hence the Lie group associated with $\mathfrak{g}^*$ is again a Poisson–Lie group, called the *dual Poisson–Lie group* of G and denoted by G^* . If the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is factorizable, the Poisson–Lie group G^* can be embedded as a dense subset G_0 of G , when the latter is given an appropriate Poisson structure. The symplectic leaves of G are then the G^* -orbits on G via the dressing transformations and the symplectic leaves of G^* are the conjugacy classes in G [STS85]. We denote this appropriate Poisson structure by π_* when G is a complex semisimple Lie group given with the standard Poisson structure π_G , that is, a Sklyanin bracket associated with the standard r -matrix of the Belavin–Drinfeld classification. In that case, the dual of the Poisson–Lie group (G, π_G) may be identified with a subgroup in the direct product of two opposite Borel subgroups B and B_- of G , and we denote it by (G^*, π_{G^*}) .

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When G is a real split semisimple Lie group with trivial center, the geometry of (G, π_G) was described by Fock and Goncharov via the combinatorics involved in the cluster $\mathcal{X}$ -variety. Recall that a cluster $\mathcal{X}$ -variety is a Poisson variety obtained by gluing a set of tori along some specific birational isomorphisms called $(\mathcal{X})$ -mutations. Each torus is given a log-canonical Poisson structure, that is, a set of coordinates x_i and a skew-symmetric matrix $\hat{e}$, with generic integer values, such that $\{x_i, x_j\} = \hat{e}_{ij}x_ix_j$. Since the mutations are Poisson maps relative to these log-canonical Poisson structures, the cluster $\mathcal{X}$ -varieties are naturally given a kind of Darboux coordinates. In [FG06b], using a coarser Poisson stratification of (G, π_G) into *double Bruhat cells* $G^{u,v}$ defined as the intersection of the cells BuB and B_-vB_- , where u, v belong to the Weyl group W of G , Fock and Goncharov constructed canonical Poisson birational maps, called evaluation maps, of cluster $\mathcal{X}$ -varieties into $(G^{u,v}, \pi_G)$ (one map $\text{ev}_{\mathbf{i}}$ for each seed $\mathcal{X}$ -torus $\mathcal{X}_{\mathbf{i}}$ associated with a double reduced word $\mathbf{i}$ associated with the pair $u, v \in W$); this construction provides a natural set of rational canonical coordinates. The canonical maps $\mu_{\mathbf{i} \rightarrow \mathbf{j}}$ associated with different double reduced words $\mathbf{i}, \mathbf{j}$ are given by composition of mutations simply related to the composition of generalized d -moves linking the double reduced words $\mathbf{i}$ and $\mathbf{j}$.

In the present paper, using a key result of Evens and Lu [EL07], we adapt the construction of Fock and Goncharov to study the dual Poisson–Lie group (G^*, π_{G^*}) of (G, π_G) in the case where G is a complex semisimple group of adjoint type. It turns out that the description of (G^*, π_{G^*}) requires not one but a family of cluster $\mathcal{X}$ -varieties, indexed by the Weyl group W of G . This family is in fact described by the W -permutohedron associated with the Lie algebra $\mathfrak{g}$ of G , vertices being labeled by cluster $\mathcal{X}$ -varieties and edges by new Poisson birational isomorphisms, on appropriate seed $\mathcal{X}$ -tori, called *saltation*. Roughly speaking, with every cluster variety of this family we associate a twisted evaluation, i.e., a composition of an evaluation map as above with a new map, called a twisted map, which generalizes the birational isomorphisms constructed by Evens and Lu in their study of Grothendieck resolutions [EL07], and use saltations to relate them. The combinatorics underlying this result are based on two new moves on double words, added to the previous generalized d -moves; these moves are called τ -moves and *dual moves*. The maps on seed $\mathcal{X}$ -tori associated with dual moves are saltations, whereas the maps on seed $\mathcal{X}$ -tori associated with τ -moves are obtained by tropicalization of the mutation formulas, and therefore, are called *tropical mutations*. In fact, one of the key technical results here is an explicit factorization of the Fomin–Zelevinsky twist maps given in [FZ99] in terms of mutations and tropical mutations. In the sequel [B] of this paper, we shall see how to use tropical mutations to include the DeConcini–Kac–Procesi Poisson automorphisms on (G_0, π_*) in the story.

Here is the organization of the paper. In §2 we fix notation, give backgrounds on semisimple Lie algebras and dual Poisson–Lie groups, and recall the basic definitions leading to the notion of a cluster $\mathcal{X}$ -variety. In §3 we show, in a way useful for our purposes, how to naturally attach a cluster $\mathcal{X}$ -variety $\mathcal{X}_{|\mathbf{i}|}$ to every double Bruhat cell $(G^{u,v}, \pi_G)$ via evaluation maps associated with any double (reduced) word $\mathbf{i}$ (this section sums up the results of [FG06b]). In §4, we introduce new evaluation maps and new seeds related to double reduced words to state an analog of the previous construction of Fock and Goncharov for the dual Poisson–Lie group (G_0, π_*) ; although the results of this section are strongly generalized in §8, this is the occasion to give a flavor of our construction without using the machinery of generalized cluster transformations and saltations, developed later. In §5, we enlarge the combinatorics on double words and on their related seed $\mathcal{X}$ -tori by introducing (respectively) new moves, called τ -moves, and related birational Poisson isomorphisms on seed $\mathcal{X}$ -tori, called tropical mutations; this

enables us to describe the Fomin–Zelevinsky twist maps and their variations in terms of mutations and tropical mutations. In §6, we use the W -permutohedron associated with the Lie algebra $\mathfrak{g}$ to study the combinatorics on double reduced words generated by generalized d -moves and enriched with tropical moves, as well as the related combinatorics on cluster $\mathcal{X}$ -varieties; the idea is to prepare the ground for the cluster combinatorics related to twisted evaluations and dual Poisson–Lie groups, developed in §8. In §7, we generalized the results of §4: we introduce twisted evaluation and adapt the combinatorics of the previous section to get a family of cluster $\mathcal{X}$ -varieties $\mathcal{X}_w$ associated with each element w of W parameterizing (BB_-, π_*) . (The results of §4 are rediscovered by taking the unity of W for w .) In §8, we relate the previous twisted evaluations by cluster transformations and new birational Poisson isomorphisms called saltations; as a consequence, we get a parameterization of the dual Poisson–Lie group (BB_-, π_*) by a family of cluster $\mathcal{X}$ -varieties; moreover, the cluster $\mathcal{X}$ -varieties of this family are related by saltations described by the 1-skeleton of the W -permutohedron P_W . In §9, we start by giving an alternative way to describe twist maps with mutations and tropical mutations and provide evaluations for (G^*, π_{G^*}) in the spirit of the Kirillov–Reshetikhin multiplicative formula for the quantum R -matrix associated with $\mathcal{U}_q(\mathfrak{g})$; moreover, the birational Poisson isomorphisms used to pass from the positive part to the negative part of (G^*, π_{G^*}) (and *vice versa*) are easily encoded by paths on the 1-skeleton of the W -permutohedron relating the identity and the longest element w_0 of W . Finally, we apply all our constructions to the very special case of $G = \mathrm{SL}(2, \mathbb{C})$ in §10, and, as a conclusion, we give the quantization of this elementary construction by considering the cluster combinatorics associated with the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

§2. PRELIMINARIES

We fix notation, give some background on semisimple Lie algebras and dual Poisson–Lie groups, and recall the basic definitions leading to the notion of a cluster $\mathcal{X}$ -variety.

2.1. Background on semisimple Lie algebras. Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank l , A its Cartan matrix, and G its Lie group of adjoint type. Fix a Borel subgroup $B \subset G$, and let B_- be the opposite Borel subgroup, $H = B \cap B_-$ the associated Cartan subgroup and N (respectively, N_-) the unipotent radical of B (respectively, B_-). Let $\mathfrak{h}, \mathfrak{n}, \mathfrak{n}_- \subset \mathfrak{g}$ be the Cartan and nilpotent subalgebras of $\mathfrak{g}$ corresponding (respectively) to H, N , and N_- . In the following, we denote $[1, l] := \{1, \dots, l\}$.

Let $\alpha_1, \dots, \alpha_l$ be the simple roots of $\mathfrak{g}$, and let $\omega_1, \omega_2, \dots, \omega_l \in \mathfrak{h}^*$ be the corresponding fundamental weights. For every $i \in [1, l]$, let (e_i, f_i, h_i) be the Chevalley generators of $\mathfrak{g}$; they generate a Lie subalgebra $\mathfrak{g}_{\alpha_i}$ of $\mathfrak{g}$. In particular, we have $\omega_j(h_k) = \delta_{jk}$ for every $j, k \in [1, l]$. We recall that the *weight lattice* P is the set of all weights $\gamma \in \mathfrak{h}^*$ such that $\gamma(h_i) \in \mathbb{Z}$ for all i . So, the group P has a $\mathbb{Z}$ -basis formed by the fundamental weights. Every weight $\gamma \in P$ gives rise to a multiplicative character $a \mapsto a^\gamma$ of the maximal torus H ; this character is given by $\exp(h)^\gamma = e^{\gamma(h)}$, with $h \in \mathfrak{h}$.

The Lie algebra $\mathfrak{g}$ being semisimple, its Cartan matrix A is invertible, and we can introduce a new basis $\{h^i, 1 \leq i \leq l\}$ on $\mathfrak{h}$ by putting

$$(2.1) \quad h^i := \sum (A^{-1})_{ij} h_j.$$

Let $D = \mathrm{diag}(d_1, \dots, d_l)$ be the diagonal matrix associated with the set of Cartan symmetrizers; we put $\hat{a}_{ij} = d_i a_{ij} = a_{ij} d_j$. For every $x \in \mathbb{C}$ and $i \in [1, l]$, we define the

group elements

$$\begin{aligned} E^i &= \exp(e_i), \quad F^i = \exp(f_i), \\ H_i(x) &= \exp(\log(x)h_i), \quad H^i(x) = \exp(\log(x)h^i) \end{aligned}$$

related (respectively) to the generators e_i , f_i , h_i , and h^i of $\mathfrak{g}$. In particular, the canonical inclusions $\varphi_i : \mathrm{SL}(2, \mathbb{C}) \hookrightarrow G$ give the following identities for every nonzero complex number x :

$$(2.2) \quad \varphi_i \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = H^i(x)E^iH^i(x^{-1}), \quad \varphi_i \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = H^i(x^{-1})F^iH^i(x),$$

$$(2.3) \quad \text{and } \varphi_i \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = H_i(x) = \prod_{j \in [1, l]} H^j(x)^{a_{ij}}.$$

We denote by W the Weyl group of G . As an abstract group, W is a finite Coxeter group of rank l generated by the set of *simple reflections* $S = \{s_1, \dots, s_l\}$; it acts on $\mathfrak{h}^*$, $\mathfrak{h}$ and the Cartan subgroup H by

$$(2.4) \quad s_i(\gamma) = \gamma - \gamma(\alpha_i^\vee)\alpha_i, \quad s_i(h) = h - \alpha_i(h)\alpha_i^\vee, \quad \text{and} \quad a^{w(\gamma)} = (\hat{w}^{-1}a\hat{w})^\gamma$$

for every $\gamma \in \mathfrak{h}^*$, $h \in \mathfrak{h}$, $w \in W$, and $a \in H$. Recall now that a *reduced word* for $w \in W$ is an expression for w in the generators belonging to S that is minimal in length among all such expressions for w . Let us denote by $\ell(w)$ this minimal length and by $R(w)$ the set of reduced words associated with w . As usual, the notation w_0 will refer to the longest word of W .

We denote by Π the set of positive roots of the Lie algebra $\mathfrak{g}$. It is well known that if $i_1 \dots i_{\ell(w_0)}$ is a reduced expression for w_0 , then

$$\Pi = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \dots, s_{i_1} \dots s_{i_{\ell(w_0)-1}}(\alpha_{i_{\ell(w_0)}})\},$$

each positive root occurring exactly once on the right-hand side. There are automorphisms $T_1, \dots, T_l$ of $\mathfrak{g}$ such that

$$(2.5) \quad \begin{aligned} T_i(e_i) &= -f_i, \quad T_i(f_i) = -e_i, \quad T_i(h_j) = h_j - a_{ji}h_i, \\ T_i(e_j) &= (-a_{ij})!^{-1}(\mathrm{ad}_{e_i})^{-a_{ij}}(e_j) \quad \text{if } i \neq j, \\ T_i(f_j) &= (-a_{ij})!^{-1}(\mathrm{ad}_{f_i})^{-a_{ij}}(f_j) \quad \text{if } i \neq j, \end{aligned}$$

where $\mathrm{ad}_a(b) = [a, b]$ for every $a, b \in \mathfrak{g}$. With any positive root $\beta = s_{i_1}s_{i_2} \dots s_{i_{k-1}}(\alpha_{i_k}) \in \Pi$, where $i_1 \dots i_{\ell(w_0)}$ is a reduced expression of the longest word w_0 of W , we associate the *positive* and *negative root vectors*

$$(2.6) \quad e_\beta = T_{i_1}T_{i_2} \dots T_{i_{k-1}}(e_{i_k}) \quad \text{and} \quad f_\beta = T_{i_1} \dots T_{i_{k-1}}(f_{i_k}).$$

We recall that W can also be seen as the subgroup $\mathrm{Norm}_G(H)/H$ of G . Thus, with every simple reflection $s_i \in S$ we associate the group element

$$\hat{s}_i = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We can choose representatives in G for every element of W by setting $\widehat{w_1 w_2} = \widehat{w_1} \widehat{w_2}$ for every $w_1, w_2 \in W$ as long as $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$. Finally, we have the *Bruhat decompositions* associated with G :

$$(2.7) \quad G = \bigcup_{u \in W} B \hat{u} B = \bigcup_{v \in W} B_- \hat{v} B_- = \bigcup_{w \in W} B \hat{w} B_-.$$

2.2. Background on dual Poisson–Lie groups. A Lie group G given with a Poisson structure is called a *Poisson–Lie group* if the multiplication $m : G \times G \longrightarrow G$ is a Poisson map, when the set $G \times G$ is given the Poisson product structure. In the following, we shall focus on the following Poisson–Lie groups.

2.2.1. The Poisson–Lie group (G, π_G) . According to the Belavin–Drinfel’d classification, the so-called *standard* classical r -matrix is given by the formula

$$r = \sum e_\alpha \wedge f_\alpha \in \mathfrak{g} \wedge \mathfrak{g},$$

where the summation is taken over all positive roots. Let $\langle \cdot, \cdot \rangle$ be the Killing form on $\mathfrak{g}$. For every $x \in \mathfrak{g}$, $X \in G$ and every function $f \in \mathcal{F}(G)$, the left and right gradients are defined, respectively, by

$$\langle \nabla f(X), x \rangle = \left. \frac{d}{dt} \right|_{t=0} f(e^{tx} X) \quad \text{and} \quad \langle \nabla' f(X), x \rangle = \left. \frac{d}{dt} \right|_{t=0} f(X e^{tx}).$$

If $f, g \in \mathcal{F}(G)$, let π_G be the following Poisson structure on G , given by the *Sklyanin bracket*, which transforms G into a Poisson–Lie group,

$$(2.8) \quad \{f, g\}_G = \frac{1}{2}(\langle \nabla f \otimes \nabla g, r \rangle - \langle \nabla' f \otimes \nabla' g, r \rangle).$$

A Poisson stratification of (G, π_G) is obtained by using the Bruhat decompositions given by (2.7). As in [FZ99, §1.2], with any $u, v \in W$ we associate the *double Bruhat cell* $G^{u,v} \subset G$ defined by

$$G^{u,v} = B \hat{u} B \cap B_- \hat{v} B_-.$$

Each double Bruhat cell $G^{u,v}$ has dimension $\ell(u) + \ell(v) + l$ by [FZ99, Theorem 1.1]. The following result leads to the Poisson stratification

$$(2.9) \quad G = \bigcup_{u,v \in W} G^{u,v}.$$

Proposition 2.1 ([HKKR00, KZ02, R03]). *For every $u, v \in W$, the double Bruhat cells $G^{u,v}$ are the H -orbits, by the right-multiplication action, of the symplectic leaves of (G, π_G) .*

2.2.2. The Poisson–Lie group (G^*, π_{G^*}) . We recall that any multiplicative Poisson bracket on G vanishes identically at its unit element $e \in G$; its linearization at e gives rise to the structure of a Lie algebra on the dual space $\mathfrak{g}^*$; then multiplicativity implies that the dual of the commutator map $[\cdot, \cdot] : \mathfrak{g}^* \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a 1-cocycle on $\mathfrak{g}$. A pair $(\mathfrak{g}, \mathfrak{g}^*)$ with these properties is called a *Lie bialgebra*. Because of an equivalence between the category of Poisson–Lie groups (whose morphisms are Lie group homomorphisms that are also Poisson mappings) and the category of Lie bialgebras (whose morphisms are homomorphisms of Lie algebras such that their duals are homomorphisms of the dual algebras), finite-dimensional Poisson–Lie groups always come by pairs. The Poisson–Lie group associated with the Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$ is called *the Poisson–Lie group dual to G* .

Let $\mathfrak{b}_\pm \subset \mathfrak{g}$ be the opposite Borel subalgebras associated with $B_\pm$. The dual Lie algebra $\mathfrak{g}^*$ associated with the standard r -matrix can be identified with the following subalgebra of $\mathfrak{b}_+ \oplus \mathfrak{b}_-$:

$$\mathfrak{g}^* = \{(X_+, X_-) \in \mathfrak{b}_+ \oplus \mathfrak{b}_- \mid \text{diag } X_+ + \text{diag } X_- = 0\}.$$

Here, $\text{diag} : \mathfrak{g} \rightarrow \mathfrak{h}$ denotes the projection of $\mathfrak{g}$ to its Cartan subalgebra. It can be lifted to give a projection from the Lie group G to its Cartan subgroup H , also denoted diag . The Lie group G^* associated with $\mathfrak{g}^*$ can be identified with the following subgroup in $B_+ \times B_-$:

$$(2.10) \quad G^* = \{(b_+, b_-) \in B_+ \times B_- \mid \text{diag } b_+ \cdot \text{diag } b_- = I\}.$$

It carries a natural Poisson bracket, which makes it a Poisson–Lie group; this is the dual Poisson–Lie group (G^*, π_{G^*}) of G . Let $t \in \mathfrak{g} \otimes \mathfrak{g}$ be the Casimir element of $\mathfrak{g}$, given by the following formula, and put $r_{\pm} = r \pm t$:

$$t = \frac{1}{2} \left(\sum_{i,j \in [1,l]} (\hat{A}^{-1})_{ij} h_i \otimes h_j + \sum_{i \in [1,l]} (e_i \otimes f_i + f_i \otimes e_i) \right).$$

Proposition 2.2 ([STS85]). *Let us equip G with the Poisson structure π_* given by*

$$\{f, g\}_* = \frac{1}{2} (\langle \nabla f \otimes \nabla g, r \rangle + \langle \nabla' f \otimes \nabla' g, r \rangle) - \langle \nabla f \otimes \nabla' g, r_+ \rangle - \langle \nabla' f \otimes \nabla g, r_- \rangle.$$

The map $\phi : (G^, \pi_{G^*}) \rightarrow (BB_-, \pi_*) : (b_+, b_-) \mapsto b_+ b_-^{-1}$ is a Poisson covering of degree 2^l of Poisson manifolds.*

Proposition 2.3 ([STS85]). *The following conjugation action is a Poisson action:*

$$\begin{aligned} (G, \pi_G) \times (G, \pi_0) &\longrightarrow (G, \pi_0), \\ (g, h) &\longmapsto ghg^{-1}. \end{aligned}$$

Now we give a Poisson stratification for (G, π_*) . Following [EL07], we recall that a *regular class function* on G is a regular function on G that is invariant under conjugation. Two elements $g_1, g_2 \in G$ are said to be in the same *Steinberg fiber* if $f(g_1) = f(g_2)$ for every regular class function f on G . For $t \in H$, let F_t be the Steinberg fiber containing t . By the Jordan decomposition of elements in G , every Steinberg fiber is of the form F_t for some $t \in H$. Moreover, we have $F_{t'} = F_t$ if and only if there exists $w \in W$ such that $t' = w(t)$, where W acts on H by formula (2.4). Therefore, the group G has the decompositions

$$(2.11) \quad G = \bigcup_{t \in H, w \in W} F_{t,w} = \bigsqcup_{t \in H \setminus W, w \in W} F_{t,w}, \quad \text{where } F_{t,w} := B\hat{w}B_- \cap F_t.$$

Proposition 2.4 ([EL07, Proposition 3.3]). *For every $t \in H$ and $v \in W$,*

- $F_{t,w}$ is a nonempty irreducible subvariety of G with dimension equal to $\dim(G) - l - \ell(w)$;
- $F_{t,v}$ is a finite union of H -orbits, for the conjugation action, of the symplectic leaves of (G, π_*) .

2.3. Background on cluster $\mathcal{X}$ -varieties. We recall the definitions, introduced by Fock and Goncharov, that underlie the cluster $\mathcal{X}$ -varieties. We add the notion of the *erasing map*, which was already used in [FG06b] without being named.

Definition 2.5 ([FG07a, Definition 1.4]). A *seed* $\mathbf{I}$ is a quadruple (I, I_0, ε, d) , where

- I is a finite set;
- $I_0 \subset I$;
- ε is a matrix ε_{ij} , $i, j \in I$, such that $\varepsilon_{ij} \in \mathbb{Z}$ unless $i, j \in I_0$;
- $d = \{d_i\}$, $i \in I$, is a subset of positive integers such that the matrix $\hat{\varepsilon}_{ij} = \varepsilon_{ij} d_j$ is skew-symmetric.

Elements of the set I_0 are sometimes called *frozen vertices*. Here, we shall not use this terminology, for the simple reason that, in what follows, we shall allow some birational Poisson isomorphisms in the direction of these frozen vertices, called tropical mutations. For every real number $x \in \mathbb{R}$, we denote $[x]_+ = \max(x, 0)$ and

$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Definition 2.6 ([FG07a, Section 1.2],[FZ02, Definition 4.2]). Let $\mathbf{I} = (I, I_0, \varepsilon, d)$, $\mathbf{I}' = (I', I'_0, \varepsilon', d')$ be two seeds, and let $k \in I \setminus I_0$. A *mutation in the direction k* is a map $\mu_k : I \longrightarrow I'$ satisfying the following conditions:

- $\mu_k(I_0) = I'_0$;
- $d'_{\mu_k(i)} = d_i$;
- $\varepsilon'_{\mu_k(i)\mu_k(j)} = \begin{cases} -\varepsilon_{ij} & \text{if } i = k \text{ or } j = k, \\ \varepsilon_{ij} + \text{sgn}(\varepsilon_{ik})[\varepsilon_{ik}\varepsilon_{kj}]_+ & \text{if } i, j \neq k. \end{cases}$

Definition 2.7 ([FG07a, Section 1.2]). A *symmetry* on a seed $\mathbf{I} = (I, I_0, \varepsilon, d)$ is an automorphism σ of the set I preserving the subset I_0 , the matrix ε , and the numbers d_i . That is to say:

- $\sigma(I_0) = I_0$;
- $d_{\sigma(i)} = d_i$;
- $\varepsilon_{\sigma(i)\sigma(j)} = \varepsilon_{ij}$.

Let $|I|$ be the cardinal of every finite set I and $\mathbb{C}_{\neq 0}$ be the set of nonzero complex numbers.

Definition 2.8 ([FG07a, Section 1.2]). Let $\mathbf{I}$ be a seed. The related *seed $\mathcal{X}$ -torus* $\mathcal{X}_{\mathbf{I}}$ is the torus $(\mathbb{C}_{\neq 0})^{|I|}$ with the Poisson bracket

$$\{x_i, x_j\} = \hat{\varepsilon}_{ij} x_i x_j,$$

where $\{x_i | i \in I\}$ are the standard coordinates on the factors. The *exchange part* of the seed $\mathcal{X}$ -torus $\mathcal{X}_{\mathbf{I}}$ is the subtorus obtained by keeping only the x_j with $j \in I \setminus I_0$.

Symmetries and mutations on seeds induce involutive maps between the corresponding seed $\mathcal{X}$ -tori, denoted by the same symbols μ_k and σ , and given by

- $x_{\sigma(i)} = x_i$;
- $x_{\mu_k(i)} = \begin{cases} x_k^{-1} & \text{if } i = k, \\ x_i x_k^{[\varepsilon_{ik}]_+} (1 + x_k)^{-\varepsilon_{ik}} & \text{if } i \neq k. \end{cases}$

Definition 2.9 ([FG07a, Section 1.2]). A *cluster transformation* linking two seeds (and two seed $\mathcal{X}$ -tori) is a composition of symmetries and mutations. Let $\mathbf{I}$ be a seed. A *cluster $\mathcal{X}$ -variety* $\mathcal{X}_{|\mathbf{I}|}$ is obtained by taking every seed $\mathcal{X}$ -tori obtained from $\mathcal{X}_{\mathbf{I}}$ by cluster transformations, and gluing them via the previous birational isomorphisms.

The correspondence between these definitions and the setup given by Fomin and Zelevinsky in [FZ07] is done easily. In particular, we have the following properties.

- Each seed $\mathcal{X}$ -torus $\mathcal{X}_{\mathbf{I}}$ associated with a seed $\mathbf{I} = (I, I_0, \varepsilon, d)$ leads to a labeled Y -seed $(\mathbf{y}, B)$ in the semifield $\mathbb{C}_{\neq 0}$ [FZ07, Definition 2.3] by setting

$$\mathbf{y} = \{x_i \mid i \in I \setminus I_0\} \text{ and } B = (\varepsilon_{ij})_{i,j \in I \setminus I_0}.$$

- The Y -seed mutations [FZ07, Definition 2.4] associated with the labeled Y -seed $(\mathbf{y}, B)$ are given by restriction to the subset $I \setminus I_0$ of the mutations, relative to the seed $\mathcal{X}$ -torus $\mathcal{X}_{\mathbf{I}}$, given by Definition 2.8.
- The exchange graph [FZ07, Definition 4.2] of the Y -pattern [FZ07, Definition 2.9] associated with $(\mathbf{y}, B)$ is the graph whose vertices are the seed $\mathcal{X}$ -tori constituting the cluster $\mathcal{X}$ -variety $\mathcal{X}_{|\mathbf{I}|}$, modulo the symmetries given in Definition 2.7, and whose edges are the restriction of mutations to the subset $I \setminus I_0$. Thus, the paths in this associated exchange graph are given by restriction of cluster transformations to this subset $I \setminus I_0$.

Definition 2.10 ([FG06b]). Let $\mathbf{I} = (I, I_0, \eta, c)$ and $\mathbf{J} = (J, J_0, \varepsilon, d)$ be two seeds, and let L be a set embedded into both I_0 and J_0 in such a way that for any $i, j \in L$ we have $c(i) = d(j)$. Then the amalgamation of $\mathbf{J}$ and $\mathbf{I}$ is a seed $\mathbf{K} = (K, K_0, \zeta, b)$ such that $K = I \cup_L J$, $K_0 = I_0 \cup_L J_0$ and

$$\zeta_{ij} = \begin{cases} 0 & \text{if } i \in I \setminus L \text{ and } j \in J \setminus L, \\ 0 & \text{if } i \in J \setminus L \text{ and } j \in I \setminus L, \\ \eta_{ij} & \text{if } i \in I \setminus L \text{ and } j \in I \setminus L, \\ \varepsilon_{ij} & \text{if } i \in J \setminus L \text{ and } j \in J \setminus L, \\ \eta_{ij} + \varepsilon_{ij} & \text{if } i, j \in L. \end{cases}$$

This operation induces a homomorphism $\mathcal{X}_{\mathbf{J}} \times \mathcal{X}_{\mathbf{I}} \rightarrow \mathcal{X}_{\mathbf{K}}$ between the corresponding seed $\mathcal{X}$ -tori, given by the rule

$$z_i = \begin{cases} x_i & \text{if } i \in I \setminus L, \\ y_i & \text{if } i \in J \setminus L, \\ x_i y_i & \text{if } i \in L. \end{cases}$$

It is easy to check that this homomorphism respects the Poisson structure and commutes with cluster transformations; thus, it is defined for the cluster $\mathcal{X}$ -varieties, and not only for seeds.

Definition 2.11. Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ be a seed and $k \in I$. A k -erasing map on a seed $\mathbf{I} = (I, I_0, \varepsilon, d)$ is a morphism ς_k on $\mathbf{I}$ such that

- $\varsigma_k(I_0) = I_0$ and $\varsigma_k(I) = I \setminus \{k\}$;
- $d_{\varsigma_k(i)} = d_i$;
- $\varepsilon_{\varsigma_k(i)\varsigma_k(j)} = \varepsilon_{ij}$.

The erasing maps on seeds induce maps between the corresponding seed $\mathcal{X}$ -tori, which are denoted by the same symbols ς_k , and are given by $x_{\varsigma_k(i)} = x_i$.

§3. CLUSTER $\mathcal{X}$ -VARIETIES RELATED TO (G, π_G)

We show, in a way useful for our purposes, how to naturally attach a cluster $\mathcal{X}$ -variety $\mathcal{X}_{|\mathbf{i}|}$ to every double Bruhat cell $(G^{u,v}, \pi_G)$ via evaluation maps associated with any double (reduced) word $\mathbf{i}$. This section sums up the results of [FG06b].

3.1. Combinatorics on double words of W . We start by recalling the combinatorics on double words of W , which is derived from a well-known result of Tits. Following [FZ99], a (reduced) word of $W \times W$ is called a *double (reduced) word*. In order to avoid confusion, we denote by $\bar{1}, \dots, \bar{l}$ the indices of the reflections associated with the first copy of W , and by $1, \dots, l$ the indices of the reflections associated with the second copy. A double (reduced) word of (u, v) is none other than a shuffle of a (reduced) word of u , written in the alphabet $[\bar{1}, \bar{l}]$, and of a (reduced) word of v , written in the alphabet $[1, l]$. We denote by $D(u, v)$ and $R(u, v)$ the set of double words and double reduced words of (u, v) , respectively. (Therefore, we have the inclusion $R(u, v) \subset D(u, v)$.) In particular, let $\mathbf{1} \in R(1, 1)$ be the double word associated with the unity of $W \times W$.

Let $w \in W$ and $\mathbf{i}$ be a word of w . Following [BZ01, §7], we call a *d-move* (also named “braid-move” in [BB05]) a transformation of $\mathbf{i}$ that replaces d consecutive entries $i, j, i, j, \dots$ by $j, i, j, i, \dots$ for some i and j such that d is the order of $s_i s_j$; that is, if $a_{ij} a_{ji} = 0$ (respectively, $1, 2, 3$), then $d = 2$ (respectively, $3, 4, 6$). We call a *nil-move* a transformation of $\mathbf{i}$ that replaces the string $i i$ by the elementary string i for some $i \in [1, l]$. The following result, called *the Tits theorem*, is standard and can be found, for example, in [BB05, Theorem 3.3.1].

Theorem 3.1. *Let (W, S) be a Coxeter group and $w \in W$.*

- *Any expression for w can be transformed into a reduced expression for w by a sequence of nil-moves and d -moves.*
- *Every two reduced words for w can be connected by a sequence of d -moves.*

We say that a letter i of $\mathbf{i}$ is *positive* if $i \in [1, l]$ and *negative* if $i \in [\bar{1}, \bar{l}]$; a double word $\mathbf{i}$ is said to be *positive* (respectively, *negative*) if all its letters are positive (respectively, negative). Considering the group $W \times W$, we conclude that every two double reduced words $\mathbf{i}, \mathbf{j} \in R(u, v)$ can be obtained from each other by a sequence of *generalized d -moves*, listed in Figure 1.

They contain

- *positive d -moves* for the alphabet $[1, l]$;
- *negative d -moves* for the alphabet $[\bar{1}, \bar{l}]$;
- *mixed 2-moves* that interchange two consecutive indices of opposite signs.

In the same way, any double word for the couple $(u, v) \in W \times W$ can be transformed to give any double reduced word of $R(u, v)$ by a sequence of *generalized dn -moves*, including

- *positive nil-moves* for the alphabet $[1, l]$;
- *negative nil-moves* for the alphabet $[\bar{1}, \bar{l}]$;
- *generalized d -moves*.

$$\begin{array}{llll}
\text{or} & \begin{array}{ccc} \dots & i & \bar{j} & \dots \\ & \dots & \bar{i} & j & \dots \end{array} & \rightsquigarrow & \begin{array}{ccc} \dots & \bar{j} & i & \dots \\ & \dots & j & \bar{i} & \dots \end{array} & \text{for every } i, j \in [1, l] \\
\text{or} & \begin{array}{ccc} \dots & i & j & \dots \\ & \dots & \bar{i} & \bar{j} & \dots \end{array} & \rightsquigarrow & \begin{array}{ccc} \dots & j & i & \dots \\ & \dots & \bar{j} & \bar{i} & \dots \end{array} & \text{when } a_{ij}a_{ji} = 0 \\
\text{or} & \begin{array}{ccc} \dots & i & j & i & \dots \\ & \dots & \bar{i} & \bar{j} & \bar{i} & \dots \end{array} & \rightsquigarrow & \begin{array}{ccc} \dots & j & i & j & \dots \\ & \dots & \bar{j} & \bar{i} & \bar{j} & \dots \end{array} & \text{when } a_{ij}a_{ji} = 1 \\
\text{or} & \begin{array}{ccc} \dots & i & j & i & j & \dots \\ & \dots & \bar{i} & \bar{j} & \bar{i} & \bar{j} & \dots \end{array} & \rightsquigarrow & \begin{array}{ccc} \dots & j & i & j & i & \dots \\ & \dots & \bar{j} & \bar{i} & \bar{j} & \bar{i} & \dots \end{array} & \text{when } a_{ij}a_{ji} = 2 \\
\text{or} & \begin{array}{ccc} \dots & i & j & i & j & i & j & \dots \\ & \dots & \bar{i} & \bar{j} & \bar{i} & \bar{j} & \bar{i} & \bar{j} & \dots \end{array} & \rightsquigarrow & \begin{array}{ccc} \dots & j & i & j & i & j & i & \dots \\ & \dots & \bar{j} & \bar{i} & \bar{j} & \bar{i} & \bar{j} & \bar{i} & \dots \end{array} & \text{when } a_{ij}a_{ji} = 3
\end{array}$$

FIGURE 1. The generalized d -moves.

$$\text{and } \begin{array}{ccc} \dots & i & i & \dots \\ & \dots & \bar{i} & \bar{i} & \dots \end{array} \rightsquigarrow \begin{array}{ccc} \dots & i & \dots \\ & \dots & \bar{i} & \dots \end{array} \text{ for every } i \in [1, l]$$

FIGURE 2. The positive and negative *nil*-moves.

3.2. Quivers and seeds associated with a double word. Now we reformulate the procedure to attach a seed to a double word, given in [FG06b], via gluing on quivers in the spirit of [FST, §13].

3.2.1. Dynkin quivers. Let $\Gamma_{\mathfrak{g}}$ be the Dynkin diagram of $\mathfrak{g}$; we denote its vertex by $\binom{1}{0} \dots \binom{l}{0}$ and choose some $i \in [1, l]$. The *elementary Dynkin quiver* $\Gamma_{\mathfrak{g}}(i)$ is the directed graph obtained from Γ by the following procedure.

- Create a new vertex $\binom{i}{1}$ and call *i-vertices* the vertices $\binom{i}{0}, \binom{i}{1}$, and *j-vertex* the vertex $\binom{j}{0}$ for any $j \neq i$. The vertex $\binom{i}{0}$ and the *j-vertices* are then called *left outlets*, whereas $\binom{i}{1}$ and these *j-vertices* are called *right outlets*.
- Erase all the edges of the Dynkin diagram except these involving the vertex $\binom{i}{0}$.
- Connect the vertices $\binom{i}{1}$ to the vertex $\binom{i}{0}$ by an arrow such that $\binom{i}{1}$ is the tail of the arrow and $\binom{i}{0}$ its head.
- Create as many arrows between $\binom{i}{1}$ and each remaining vertex as there are between $\binom{i}{0}$ and this remaining vertex. The heads and the tails of the arrows are directed in such a way that the triangle(s) thus created is/are oriented.

The elementary Dynkin quiver $\Gamma_{\mathfrak{g}}(\bar{i})$ is then obtained from $\Gamma_{\mathfrak{g}}(i)$ by reversing the orientation of all the arrows. To these elementary Dynkin quivers, we add the *trivial Dynkin quiver* $\Gamma_{\mathfrak{g}}(\mathbf{1})$ obtained from the Dynkin diagram by removing all the edges; as above, the vertices of this quiver are labeled by $\binom{i}{0}$ with $i \in [1, l]$. (For the trivial Dynkin quiver $\Gamma_{\mathfrak{g}}(\mathbf{1})$, the set of left outlets is, by definition, the set of right outlets.) Figure 3 and Figure 4 describe (respectively) the elementary Dynkin quivers of the cases $\mathfrak{g} = A_3$ and $\mathfrak{g} = B_2$. Let us stress that, in all our examples, outlets will be marked by unfilled circles. Other conventions in our drawing will be the following: the type of vertices is given by a kind of height function where vertices of type 1 are at the top of the quiver and vertices of type l at the bottom; moreover, left outlets will always be drawn at the left of right outlets if both are k -vertices but different.

FIGURE 3. Elementary Dynkin quivers for $\mathfrak{g} = A_3$.

FIGURE 4. Elementary Dynkin quivers for $\mathfrak{g} = B_2$.

A quiver Γ is called a *Dynkin quiver* if it can be obtained from a collection of disjoint elementary Dynkin quivers, coming from the same Dynkin diagram, by the following procedure, called *amalgamation*. Let $\mathbf{i} = i_1 \dots i_n$ be a double word and $\Gamma_{\mathfrak{g}}(i_1), \dots, \Gamma_{\mathfrak{g}}(i_n)$ be the associated elementary Dynkin quivers. For every $k \in [1, l]$, we put a total order, called the *k-order* on the set of the *k-vertices* of all the elementary Dynkin quivers, in such a way that $\binom{i_j}{0} < \binom{i_l}{1}$ for every j , and that all *k-vertices* of $\Gamma_{\mathfrak{g}}(i_l)$ are lower than the *k-vertices* of $\Gamma_{\mathfrak{g}}(i_j)$ when $l < j$.

- For every $j \in [1, n - 1]$, glue every right outlet of $\Gamma_{\mathfrak{g}}(i_j)$ to every left outlet of $\Gamma_{\mathfrak{g}}(i_{j+1})$ in such a way that *k-vertices* are glued together.

- Relabel each k -vertex following the k -order; that is, $\binom{k}{i}$ is the i th k -vertex, with an increasing numbering, of the set of all k -vertices given by the k -order.
- Redefine the set of outlets: the left outlets (respectively, right outlets) are vertices $\binom{k}{i}$ such that $\binom{k}{i} \leq \binom{k}{j}$ (respectively, $\binom{k}{j} \leq \binom{k}{i}$) for every j .
- If Γ contains a pair of edges connecting the same pair of vertices but going in opposite directions, then remove each such pair of edges (as a result, all arrows linking two vertices now point in the same direction).
- For every $k_1 \neq k_2$, erase arrows linking k_1 -vertices to k_2 -vertices until the only arrows linking these vertices in the resulting graph are those linking the k_1 th vertex and the k_2 th vertex in the corresponding Dynkin diagram.

The amalgamation process is certainly easier to figure out from examples. Figures 5, 6, and 7 describe some amalgamations in the case where $\mathfrak{g} = A_3$. (The obvious labeling is left to the reader.)

FIGURE 5. The amalgamation $(\Gamma_{\mathfrak{g}}(1), \Gamma_{\mathfrak{g}}(\bar{2})) \mapsto \Gamma_{\mathfrak{g}}(1\bar{2})$ for $\mathfrak{g} = A_3$.

FIGURE 6. The amalgamation $(\Gamma_{\mathfrak{g}}(2), \Gamma_{\mathfrak{g}}(2)) \mapsto \Gamma_{\mathfrak{g}}(22)$ for $\mathfrak{g} = A_3$.

FIGURE 7. The amalgamation $(\Gamma_{\mathfrak{g}}(2), \Gamma_{\mathfrak{g}}(\bar{2})) \mapsto \Gamma_{\mathfrak{g}}(2\bar{2})$ for $\mathfrak{g} = A_3$.

The resulting graph is called the *Dynkin quiver* $\Gamma_{\mathfrak{g}}(\mathbf{i})$. Therefore, a Dynkin quiver is associated with every double word $\mathbf{i}$. In particular, every elementary Dynkin quiver is a Dynkin quiver, and the elementary Dynkin quiver $\Gamma_{\mathfrak{g}}(\mathbf{1})$ does not affect gluing.

It is easily seen that amalgamation is associative. In particular, the Dynkin quivers $\Gamma_{\mathfrak{g}}(\mathbf{i})$ and $\Gamma_{\mathfrak{g}}(\mathbf{j})$ can be amalgamated to obtain the Dynkin quiver $\Gamma_{\mathfrak{g}}(\mathbf{ij})$. We also remark that for every vertex i of a Dynkin quiver Γ , there exists $k \in [1, l]$ such that i is a k -vertex. Such a k is called the *vertex-type* of i and is denoted by $k(i)$. It is clear that the vertex-type remains unchanged by amalgamation. Moreover, for every $j \in [1, l]$, we denote by $N^j(\mathbf{i})$ the number of vertices in $\Gamma_{\mathfrak{g}}(\mathbf{i})$ whose vertex-type is j . Stated otherwise, $N^j(\mathbf{i})$ is the number of times the letter j or $\bar{j}$ appears in the double word $\mathbf{i}$.

FIGURE 8. The quiver $\Gamma_{\mathfrak{g}}(123121)$ for $\mathfrak{g} = A_3$.

3.2.2. Seeds associated with double words. We derive the seeds constructed by Fock and Goncharov from the previous Dynkin quivers in the following way. Let $\mathbf{i}$ be a double word, and let Γ be the associated Dynkin quiver. Let $\bar{B}(\mathbf{i}) = (\bar{b}_{ij})$ denote the skew-symmetric matrix whose rows and columns are labeled by the vertices of Γ , and whose entry $\bar{b}_{ij}$ is equal to the number of edges going from i to j minus the number of edges going from j to i . The skew-symmetrizable matrix $B(\Gamma) = (b_{ij})$ is obtained from $\bar{B}(\mathbf{i})$ by the following skew-symmetrizing formula, involving the vertex-types $k(i)$ and $k(j)$ of i and j :

$$d_{k(i)}b_{ij} = \bar{b}_{ij} = -\bar{b}_{ji} = -b_{ji}d_{k(j)},$$

where $d_1, \dots, d_l$ are nonzero natural numbers that symmetrize the Cartan matrix. Let $\mathbf{i}$ be a double word and $\Gamma_{\mathfrak{g}}(\mathbf{i})$ its associated Dynkin quiver. The seed

$$\mathbf{I}(\mathbf{i}) = (I(\mathbf{i}), I_0(\mathbf{i}), \varepsilon(\mathbf{i}), d(\mathbf{i}))$$

is defined in the following way.

- The set $I(\mathbf{i})$ is the set of vertices of $\Gamma_{\mathfrak{g}}(\mathbf{i})$, with a partial order induced by the k -orders on the set of k -vertices, the sets $I_0^{\mathfrak{R}}(\mathbf{i}), I_0^{\mathfrak{L}}(\mathbf{i})$ are (respectively) the set of right outlets and left outlets of $\Gamma_{\mathfrak{g}}(\mathbf{i})$, and $I_0(\mathbf{i})$ is the set of outlets. Stated otherwise, the set $I(\mathbf{i})$ (respectively, $I_0^{\mathfrak{L}}(\mathbf{i}), I_0^{\mathfrak{R}}(\mathbf{i})$ and $I_0(\mathbf{i})$) is the set of all ordered pairs $\binom{j}{k}$ such that $j \in [1, l]$ and $0 \leq k \leq N^j(\mathbf{i})$ (respectively, $k = 0, k = N^j(\mathbf{i})$, and $k \in \{0, N^j(\mathbf{i})\}$), where $N^j(\mathbf{i})$ is the number of times the letter j or $\bar{j}$ appears in $\mathbf{i}$.
- The matrix $\varepsilon(\mathbf{i})$ is given by a normalization of $B(\mathbf{i})$:

$$(3.1) \quad \varepsilon(\mathbf{i})_{kl} = \begin{cases} \frac{B(\mathbf{i})_{kl}}{2} & \text{if } k \in I_0(\mathbf{i}) \text{ and } l \in I_0(\mathbf{i}), \\ B(\mathbf{i})_{kl} & \text{otherwise.} \end{cases}$$

- The multiplier $d(\mathbf{i})$ is given by associating, with every vertex j , the Cartan symmetrizer of its type-vertex, i.e.,

$$d(\mathbf{i})_j = d_{k(j)}.$$

It is easy to translate the amalgamation procedure at the level of seeds: for all double words $\mathbf{i}, \mathbf{j}$, the *amalgamated seed* $(I(\mathbf{ij}), I_0(\mathbf{ij}), \varepsilon(\mathbf{ij}), d(\mathbf{ij}))$ is defined in the following way [FG06b]. The elements of the set $d(\mathbf{ij})$ are equal to the corresponding Cartan symmetrizer as above, and the matrix $\varepsilon(\mathbf{ij})$ is given by

$$(3.2) \quad \varepsilon(\mathbf{ij})_{\binom{i}{k}\binom{j}{l}} = \begin{cases} \varepsilon(\mathbf{i})_{\binom{i}{k}\binom{j}{l}} & \text{if } k < N^i(\mathbf{i}) \text{ and } l < N^j(\mathbf{i}), \\ \varepsilon(\mathbf{i})_{\binom{i}{k}\binom{j}{l}} + \varepsilon(\mathbf{j})_{\binom{i}{0}\binom{j}{0}} & \text{if } k = N^i(\mathbf{i}) \text{ and } l = N^j(\mathbf{i}), \\ \varepsilon(\mathbf{j})_{\binom{i}{k-N^i(\mathbf{i})}\binom{j}{l-N^j(\mathbf{i})}} & \text{if } k > N^i(\mathbf{i}) \text{ and } l > N^j(\mathbf{i}), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for every $i \in [1, l]$ and $\mathbf{i} \in \{i, \bar{i}\}$, the matrices $\varepsilon(\bar{i})$ and $\varepsilon(i)$ have their entries labeled by the elements of $I(i) = I(\bar{i})$ and are given by the following relations and zero

otherwise:

$$(3.3) \quad \varepsilon(i)_{(i)(j)} = \frac{a_{ij}}{2} = -\varepsilon(\bar{i})_{(i)(j)}, \quad \varepsilon(\bar{i})_{(i)(j)} = -\frac{a_{ij}}{2} = -\varepsilon(i)_{(i)(j)}.$$

3.3. Cluster $\mathcal{X}$ -varieties related to (G, π_G) . We attach a seed $\mathcal{X}$ -torus to each of the previous seeds. We use them to describe the combinatorics underlying the Poisson geometry of double Bruhat cells, as in [FG06b]. Let us recall the basis of $\mathfrak{h}$ given by (2.1) and the related group elements $H^i(x)$ given by (2.3). For every $\mathbf{i} \in \{\mathbf{1}, i, \bar{i}\}$, we denote by $\mathcal{X}_{\mathbf{i}}$ the seed $\mathcal{X}$ -torus associated with the elementary seed $(I(\mathbf{i}), I_0(\mathbf{i}), \varepsilon(\mathbf{i}), d(\mathbf{i}))$ and by $\text{ev}_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}} \rightarrow G$ the related evaluation map given by

$$\begin{aligned} \text{ev}_{\mathbf{1}} : \mathbb{C}_{\neq 0}^l &\longrightarrow G : \left(x_{(1)}^{(1)}, \dots, x_{(i)}^{(i)}, \dots, x_{(l)}^{(l)}\right) \longmapsto \prod_j H^j \left(x_{(j)}^{(j)}\right), \\ \text{ev}_i : \mathbb{C}_{\neq 0}^{l+1} &\longrightarrow G : \left(x_{(1)}^{(1)}, \dots, x_{(i)}^{(i)}, x_{(1)}^{(i)}, x_{(0)}^{(i+1)}, \dots, x_{(l)}^{(l)}\right) \\ &\longmapsto \prod_j H^j \left(x_{(j)}^{(j)}\right) E^i H^i \left(x_{(1)}^{(i)}\right), \\ \text{ev}_{\bar{i}} : \mathbb{C}_{\neq 0}^{l+1} &\longrightarrow G : \left(x_{(1)}^{(1)}, \dots, x_{(i)}^{(i)}, x_{(1)}^{(i)}, x_{(0)}^{(i+1)}, \dots, x_{(l)}^{(l)}\right) \\ &\longmapsto \prod_j H^j \left(x_{(j)}^{(j)}\right) F^i H^i \left(x_{(1)}^{(i)}\right). \end{aligned}$$

Proposition 3.2 ([FG06b, Proposition 3.11]). *For every $i \in [1, l]$, $\mathbf{i} \in \{\mathbf{1}, i, \bar{i}\}$ and $(u, v) \in W \times W$ such that $\mathbf{i} \in R(u, v)$, the evaluation map $\text{ev}_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}} \longrightarrow (G^{u,v}, \pi_G)$ is a Poisson birational isomorphism on a Zariski open set of $G^{u,v}$.*

Lemma 3.3 ([FG06b]). *The amalgamation procedure induces a Poisson homomorphism $\mathbf{m} : \mathcal{X}_{\mathbf{i}} \times \mathcal{X}_{\mathbf{j}} \rightarrow \mathcal{X}_{\mathbf{ij}}$ between the corresponding seed $\mathcal{X}$ -tori given by*

$$(3.4) \quad z_{(k)}^{(i)} = \begin{cases} x_{(k)}^{(i)} & \text{if } 0 \leq k < N^i(\mathbf{i}), \\ x_{(k)}^{(i)} y_{(0)}^{(i)} & \text{if } k = N^i(\mathbf{i}), \\ y_{(k - N^i(\mathbf{i}))}^{(i)} & \text{if } N^i(\mathbf{i}) < k \leq N^i(\mathbf{i}) + N^i(\mathbf{j}), \end{cases}$$

where the x_i , y_j and z_k are the associated variables.

Notice that this amalgamated product is associative. Now, let $\mathbf{i} = i_1 \dots i_k$ be a double word, $\mathcal{X}_{\mathbf{i}}$ the seed $\mathcal{X}$ -torus given by the associated amalgamation $\mathbf{m} : \mathcal{X}_{i_1} \times \dots \times \mathcal{X}_{i_k} \longrightarrow \mathcal{X}_{\mathbf{i}}$, and $\mathbf{z}$ the amalgamated variable $\mathbf{m}(\mathbf{x}_1, \dots, \mathbf{x}_k)$. We define the *evaluation map*:

$$(3.5) \quad \text{ev}_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}} \rightarrow G : \mathbf{z} \mapsto \text{ev}_{i_1}(\mathbf{x}_1) \dots \text{ev}_{i_k}(\mathbf{x}_k), \quad \text{where } \mathbf{z} = \mathbf{m}(\mathbf{x}_1, \dots, \mathbf{x}_k).$$

Using the multiplicative property of the Poisson-Lie group (G, π_G) , we see that this evaluation is also a Poisson map. So, Proposition 3.2 leads to the Poisson statement of the following result.

Theorem 3.4 ([FG06b]). *For any $u, v \in W$ and $\mathbf{i} \in R(u, v)$, the map $\text{ev}_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}} \rightarrow (G^{u,v}, \pi_G)$ is a Poisson birational isomorphism onto a Zariski open set of the double Bruhat cell $G^{u,v}$.*

Now we introduce cluster transformations in the framework. We say that a double reduced word of length d is *d-minimal* if we can perform a generalized d -move on it. The double words i and $i \bar{i}$ for every $i \in [1, l] \cup [\bar{1}, \bar{l}]$ are said to be *n-minimal*. Finally, a double word is said to be *dn-minimal* if it is d -minimal or n -minimal. With any two dn -minimal

double words $\mathbf{i}$ and $\mathbf{i}'$ related by a generalized dn -move $\delta : \mathbf{i} \mapsto \mathbf{i}'$, we associate a cluster transformation $\mu_{\mathbf{i} \rightarrow \mathbf{i}'} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{i}'}$ in the following way:

$$(3.6) \quad \mu_{\mathbf{i} \rightarrow \mathbf{i}'} = \begin{cases} \varsigma_{(1)}^{(i)} \circ \mu_{(1)}^{(i)} & \text{if } \delta \text{ is a nil-move,} \\ \mu_{(1)}^{(i)} & \text{if } \delta \text{ is a move } i \bar{i} \leftrightarrow \bar{i} i \text{ or a 3-move,} \\ \mu_{(1)}^{(i)} \mu_{(1)}^{(j)} \mu_{(1)}^{(i)} & \text{if } \delta \text{ is a 4-move,} \\ \mu_{(2)}^{(j)} \mu_{(1)}^{(i)} \mu_{(1)}^{(j)} \mu_{(2)}^{(i)} \mu_{(2)}^{(j)} \mu_{(1)}^{(i)} \mu_{(2)}^{(j)} \mu_{(1)}^{(i)} \mu_{(2)}^{(j)} & \text{if } \delta \text{ is a 6-move,} \\ \text{the identity map} & \text{otherwise,} \end{cases}$$

where, as in [FG06b], we have denoted $\mu_{\mu_i(j)} \mu_i$ by $\mu_j \mu_k$, $\mu_{\mu_{\mu_i(j)} \mu_i(k)} \mu_{\mu_i(j)} \mu_i$ by $\mu_k \mu_j \mu_i$, and so on.

Since mutations commute with amalgamation, we can extend these definitions to any two double words $\mathbf{i}, \mathbf{i}' \in D(u, v)$ related by a generalized dn -move. Finally, if $\mathbf{i}, \mathbf{j}$ are double words linked by a sequence $\delta_{\mathbf{i} \rightarrow \mathbf{j}}$ of generalized dn -moves and $\mathbf{i} \rightarrow \mathbf{i}_1 \rightarrow \cdots \rightarrow \mathbf{i}_{n-1} \rightarrow \mathbf{j}$ is the associated chain of elements, we define the cluster transformation $\mu_{\mathbf{i} \rightarrow \mathbf{j}}$ as the composition $\mu_{\mathbf{i}_{n-1} \rightarrow \mathbf{j}} \circ \cdots \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_1}$. The following result is derived easily from Theorem 3.1.

Lemma 3.5. *A double reduced word $\mathbf{j} \in R(u, v)$ can be obtained from a double reduced word $\mathbf{i} \in R(u', v')$ by a sequence of generalized d -moves $\delta_{\mathbf{i} \rightarrow \mathbf{j}}$ if and only if $u' = u$ and $v' = v$.*

Let $u, v \in W$ and $\mathbf{i}, \mathbf{j} \in R(u, v)$. Since the birational Poisson isomorphism $\mu_{\mathbf{i} \rightarrow \mathbf{j}} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{j}}$, associated with a sequence $\delta_{\mathbf{i} \rightarrow \mathbf{j}}$ of generalized d -moves is a cluster transformation, we shall denote by $\mathcal{X}^{u, v}$ the cluster $\mathcal{X}$ -variety associated with the set $R(u, v)$. Stated otherwise, a local chart $(\mathcal{X}_{\mathbf{i}}, \varphi_{\mathbf{i}})$ in the cluster $\mathcal{X}$ -variety $\mathcal{X}^{u, v}$ corresponds to any double reduced word $\mathbf{i} \in R(u, v)$, and the cluster transformation $\mu_{\mathbf{i} \rightarrow \mathbf{j}}$ is the transition map between the local charts $(\mathcal{X}_{\mathbf{i}}, \varphi_{\mathbf{i}})$ and $(\mathcal{X}_{\mathbf{j}}, \varphi_{\mathbf{j}})$ for any $\mathbf{j} \in R(u, v)$. We also denote by $\iota_{\mathbf{i}} : R(u, v) \rightarrow \mathbf{i}$ every time we choose an element $\mathbf{i} \in R(u, v)$. Therefore, the diagrams in Figure 9 are commutative.

FIGURE 9. The set $R(u, v)$ and the cluster $\mathcal{X}$ -variety $\mathcal{X}^{u, v}$.

Finally we show the way to attach the cluster $\mathcal{X}$ -variety $\mathcal{X}^{u, v}$ to the double Bruhat cell $(G^{u, v}, \pi_G)$, for every $u, v \in W$.

Lemma 3.6 ([FG06b]). *For any $u, v \in W$ and $\mathbf{i}, \mathbf{j} \in D(u, v)$ such that $\mathbf{j}$ is obtained from $\mathbf{i}$ by a sequence of generalized dn -moves, we have $\text{ev}_{\mathbf{i}} = \text{ev}_{\mathbf{j}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}$.*

Theorem 3.7 ([FG06b]). *For any $u, v \in W$ and $\mathbf{i}, \mathbf{j} \in R(u, v)$, we have $\text{ev}_{\mathbf{i}} = \text{ev}_{\mathbf{j}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}$.*

The cluster $\mathcal{X}$ -variety $\mathcal{X}^{u, v}$ is therefore attached to the double Bruhat cell $(G^{u, v}, \pi_G)$ for every $u, v \in W$. Let $\mathcal{X}$ denote the map that takes any double word $\mathbf{i}$ to the corresponding seed $\mathcal{X}$ -torus $\mathcal{X}_{\mathbf{i}}$. We then summarize Theorem 3.4 and Theorem 3.7 by abusively saying that there exists a Poisson birational isomorphism

$$\text{ev}^{u, v} : \mathcal{X}^{u, v} \rightarrow (G^{u, v}, \pi_G).$$

The commutative diagrams in Figure 9 can finally be related and completed to get the commutative diagram given in Figure 10.

FIGURE 10. The cluster $\mathcal{X}$ -variety $\mathcal{X}^{u,v}$ associated with $(G^{u,v}, \pi_G)$.§4. TRUNCATION MAPS AND CLUSTER $\mathcal{X}$ -VARIETIES RELATED TO (G, π_*)

We introduce new evaluation maps and new seeds related to double reduced words to state analogs of Theorem 3.4 and Theorem 3.7 for the dual Poisson–Lie group $G^* \subset (G, \pi_*)$. Although the result in this section will be strongly generalized in §8, this is an occasion to introduce truncation maps and to give a flavor of what will be later called twisted evaluations and its related combinatorics, without using the machinery of generalized cluster transformations and saltations, which we start to develop in §5.

4.1. Double reduced Bruhat cells and reduced evaluations. According to [BZ01, Section 4.3], let $L^{u,v}$ be the *reduced double Bruhat cell* associated with every $u, v \in W$, that is, the quotient of a double Bruhat cell $G^{u,v}$ by the H -right multiplication,

$$(4.1) \quad L^{u,v} = G^{u,v}/H.$$

We are going to slightly modify the cluster $\mathcal{X}$ -varieties previously constructed in order to evaluate these double reduced Bruhat cells. For every double word $\mathbf{i}$, let $\varsigma_{\mathfrak{R}, \mathbf{i}}$ be the erasing map associated with the set $I_0^{\mathfrak{R}}(\mathbf{i})$ of right outlets, defined as the product over the set $I_0^{\mathfrak{R}}(\mathbf{i})$ of the erasing maps ς_j given by Definition 2.11,

$$(4.2) \quad \varsigma_{\mathfrak{R}, \mathbf{i}} = \prod_{j \in I_0^{\mathfrak{R}}(\mathbf{i})} \varsigma_j.$$

We denote by $\mathbf{i}^{\text{red}}$ the image of the seed $\mathbf{I}(\mathbf{i})$ under $\varsigma_{\mathfrak{R}, \mathbf{i}}$, and by $\mathcal{X}_{\mathbf{i}}^{\text{red}}$ the seed $\mathcal{X}$ -torus associated with the seed $\mathbf{i}^{\text{red}}$. Therefore, we have $\varsigma_{\mathfrak{R}, \mathbf{i}} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{i}}^{\text{red}}$. Every cluster transformation $\mu_{\mathbf{i} \rightarrow \mathbf{j}} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{j}}$ canonically leads to a cluster transformation between the associated seed $\mathcal{X}$ -tori $\mu_{\mathbf{i} \rightarrow \mathbf{j}}^{\text{red}} : \mathcal{X}_{\mathbf{i}}^{\text{red}} \rightarrow \mathcal{X}_{\mathbf{j}}^{\text{red}}$:

$$\mu_{\mathbf{i} \rightarrow \mathbf{j}}^{\text{red}} \circ \varsigma_{\mathfrak{R}, \mathbf{i}} = \varsigma_{\mathfrak{R}, \mathbf{j}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}.$$

Definition 4.1. Let $\mathbf{i}$ be a double reduced word. It is clear that the map defined on $\mathcal{X}_{\mathbf{i}}$ and given by

$$\mathbf{x} \mapsto \text{ev}_{\mathbf{i}}(\mathbf{x}) \prod_{j \in [1, l]} H^j(x_{\binom{j}{N^j(\mathbf{i})}}^{-1})$$

does not depend on x_k for $k \in I_0^{\mathfrak{R}}(\mathbf{i})$, so that we get an evaluation $\text{ev}_{\mathbf{i}}^{\text{red}} : \mathcal{X}_{\mathbf{i}}^{\text{red}} \rightarrow G$, called a *reduced evaluation*. Stated in a rougher way, the reduced evaluation associated with the double word $\mathbf{i}$ is obtained from the evaluation map $\text{ev}_{\mathbf{i}}$ by setting the cluster variables x_j to 1 (or forgetting the related Cartan element $H^i(x_j)$) for every $j \in I_0^{\mathfrak{R}}(\mathbf{i})$.

When the double word $\mathbf{i}$ is a double reduced word, Theorem 3.4 and Theorem 3.7 are easily adapted to reduced double Bruhat cells by using reduced evaluations. We get the following result.

Corollary 4.2. *For any $u, v \in W$ and $\mathbf{i} \in R(u, v)$, the map $\text{ev}_{\mathbf{i}}^{\text{red}} : \mathcal{X}_{\mathbf{i}}^{\text{red}} \rightarrow (L^{u,v}, \pi_G)$ is a Poisson birational isomorphism onto a Zariski open set of the double Bruhat cell $L^{u,v}$, and for any $u, v \in W$ and $\mathbf{i}, \mathbf{j} \in R(u, v)$, we have $\text{ev}_{\mathbf{i}}^{\text{red}} = \text{ev}_{\mathbf{j}}^{\text{red}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}^{\text{red}}$.*

Remark 4.3. These corollaries remain valid even if the Lie group G is not of adjoint type but simply connected.

4.2. Truncation maps on cluster $\mathcal{X}$ -varieties. The cluster $\mathcal{X}$ -variety we are going to associate with the dual Poisson–Lie group (BB_-, π_*) in the next subsection can be obtained easily from the cluster $\mathcal{X}$ -variety $\mathcal{X}^{w_0, w_0}$ of §3 with the help of the notions of truncation map and truncated cluster $\mathcal{X}$ -varieties we are going to introduce now. The underlying idea is to force the apparition of the Casimir of (BB_-, π_*) . We start by giving the general setting.

Definition 4.4. Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ be a seed, and let $J \subset I$. The *truncation map associated with J* is a map $\mathbf{t}_J : \mathbf{I} \rightarrow \mathbf{I}_J$ such that the seed $\mathbf{I}_J = (I, I_0, \varepsilon', d)$ is given by

$$(4.3) \quad \varepsilon'_{ij} = \begin{cases} \varepsilon_{ij} & \text{if } i, j \in I \setminus J, \\ 0 & \text{otherwise.} \end{cases}$$

For any finite set J , we denote by $\mathcal{X}_J^0$ the seed $\mathcal{X}$ -torus associated with a seed $(J, J, 0, d)$. Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ be a seed such that $J \subset I$, and let $\mathbf{I}' = (I, I_0, \varepsilon', d)$ be the image of $\mathbf{I}$ by $\mathbf{t}_J$. With every $\mathbf{t} \in \mathcal{X}_J$, we associate the subtorus $\mathcal{X}_{\mathbf{I}'}(t) \subset \mathcal{X}_{\mathbf{I}'}$ formed by the elements $\mathbf{x} \in \mathcal{X}_{\mathbf{I}'}$ such that $x_i = t_i$ for every $i \in J$. It is a Poisson subtorus, because of formula (4.3). The map $\mathbf{t}_J$ induces a homomorphism $\mathbf{t}_{J(t)} : \mathcal{X}_{\mathbf{I}} \rightarrow \mathcal{X}_{\mathbf{I}'}(t)$ associated with any $t \in \mathcal{X}_J^0$ and given by

$$x_{\mathbf{t}_{J(t)}(i)} = \begin{cases} x_i & \text{if } i \in I \setminus J, \\ t_i & \text{if } i \in J. \end{cases}$$

In particular, the *trivial truncated map*, associated with the empty set $J = \emptyset$, is the identity map. Now, let $\mathbf{I} = (I, I_0, \varepsilon, d)$ be a seed and J a subset of I . With any cluster transformation $\phi_{\mathbf{I} \rightarrow \mathbf{J}} : \mathcal{X}_{\mathbf{I}} \rightarrow \mathcal{X}_{\mathbf{J}}$, we associate the following Poisson birational isomorphism $\phi_{\mathbf{I}_J \rightarrow \mathbf{J}_J} : \mathcal{X}_{\mathbf{I}_J} \rightarrow \mathcal{X}_{\mathbf{J}_J}$, called the *truncated cluster transformation*:

$$x_{\phi_{\mathbf{I}_J \rightarrow \mathbf{J}_J}(i)} = \begin{cases} x_{\phi_{\mathbf{I} \rightarrow \mathbf{J}}(i)} & \text{if } i \in I \setminus J, \\ x_i & \text{if } i \in J. \end{cases}$$

It is clear that for every $t \in \mathcal{X}_J^0$ this map admits a restriction $\phi_{\mathbf{I}_J \rightarrow \mathbf{J}_J} : \mathcal{X}_{\mathbf{I}_J}(t) \rightarrow \mathcal{X}_{\mathbf{J}_J}(t)$, which is also a birational Poisson isomorphism.

We would like to define truncation maps at the level of cluster $\mathcal{X}$ -varieties. A sufficient condition is given by the following immediate result.

Lemma 4.5. *Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ be a seed, and let $J \subset I_0$. The following relation is satisfied for every cluster transformation $\phi_{\mathbf{I} \rightarrow \mathbf{J}} : \mathcal{X}_{\mathbf{I}} \rightarrow \mathcal{X}_{\mathbf{J}}$ and every $t \in \mathcal{X}_J^0$:*

$$(4.4) \quad \phi_{\mathbf{I}_J \rightarrow \mathbf{J}_J} \circ \mathbf{t}_{J(t)} = \mathbf{t}_{\phi_{\mathbf{I} \rightarrow \mathbf{J}}(J)(t)} \circ \phi_{\mathbf{I} \rightarrow \mathbf{J}}.$$

Remark 4.6. Formula (4.4) is not necessarily true if the condition $J \subset I_0$ is omitted. Indeed, consider the seed $\mathbf{I}(i)$ for any $i \in [1, l]$, and the set $J = \{\binom{i}{1}\}$, with the cluster transformation $\mu_{\binom{i}{1}}$. In fact, a generalization of formula (4.4) will be the starting point for the definition of the Poisson birational isomorphisms, called saltations; see Subsection 8.2.

Definition 4.7. Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ be a seed, J a subset of I , and $\mathbf{I}'$ the truncated seed associated with the seed $\mathbf{I}$ and the set J . The *truncated cluster $\mathcal{X}$ -variety* $\mathcal{X}_{|\mathbf{I}'|}$ of the cluster $\mathcal{X}$ -variety $\mathcal{X}_{|\mathbf{I}|}$ is obtained by taking every seed $\mathcal{X}$ -tori obtained from $\mathcal{X}_{\mathbf{I}'}$ by

cluster transformations, and gluing them as usual. Moreover, if we denote by $\mathcal{X}_{|\mathbf{I}'|}(t)$ the cluster $\mathcal{X}$ -variety obtained from every seed $\mathcal{X}$ -torus $\mathcal{X}_{\mathbf{I}'}(t) \subset \mathcal{X}_{\mathbf{I}'}$, then we have the following Poisson stratification; see (4.3):

$$\mathcal{X}_{|\mathbf{I}'|} = \bigcup_{t \in \mathcal{X}_J^0} \mathcal{X}_{|\mathbf{I}'|}(t).$$

The truncated map $\mathbf{t}_{J(t)}$ is therefore well defined at the level of cluster $\mathcal{X}$ -varieties for every fixed $t \in \mathcal{X}_J^0$ if J is a subset of I_0 . It is denoted by $\mathbf{t}_{J(t)} : \mathcal{X}_{|\mathbf{I}|} \rightarrow \mathcal{X}_{|\mathbf{t}_{J(t)}(\mathbf{I})|}(t)$.

Remark 4.8. The exchange graph [FZ07, Definition 4.2] associated to an initial seed always covers the exchange graph associated with any truncated seed, in the sense of [FZ07, Definition 4.5].

Now we focus on particular truncations on cluster $\mathcal{X}$ -varieties associated with double words. For every double word $\mathbf{i}$, let $[\mathbf{i}]_{\mathfrak{R}}$ be the image of the seed $\mathbf{I}(\mathbf{i})$ under the *right truncation map* $\mathbf{t}_{\mathfrak{R}}$ associated with the set of right outlets $I_0^{\mathfrak{R}}(\mathbf{i})$ of $\mathbf{I}(\mathbf{i})$. Stated otherwise, for every double word $\mathbf{i}$, the seed $[\mathbf{i}]_{\mathfrak{R}} = (I(\mathbf{i}), I_0(\mathbf{i}), \eta(\mathbf{i}), d(\mathbf{i}))$ is the seed defined by

$$(4.5) \quad \eta(\mathbf{i})_{ij} = \begin{cases} \varepsilon(\mathbf{i})_{ij} & \text{if } i, j \in I(\mathbf{i}) \setminus I_0^{\mathfrak{R}}(\mathbf{i}), \\ 0 & \text{otherwise.} \end{cases}$$

An example of a right truncation map is given in Figure 11. The *right truncated seed*

FIGURE 11. The right truncation map $\mathbf{t}_{\mathfrak{R}} : \Gamma_{A_3}(\mathbf{i}) \rightarrow \Gamma_{A_3}([\mathbf{i}]_{\mathfrak{R}})$ for $\mathbf{i} = 123121$.

$\mathcal{X}$ -torus $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}(t)$ associated with any $t \in H$ is the subset of $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ obtained by fixing the following cluster variables $\mathbf{x}(\mathfrak{R})$ associated with right outlets:

$$(4.6) \quad \mathbf{x}(\mathfrak{R}) = \{x_j \mid j \in I_0^{\mathfrak{R}}(\mathbf{i})\}$$

via the relation

$$(4.7) \quad \text{ev}_{\mathbf{1}} \left(x_{(N^1(\mathbf{i}))}, \dots, x_{(N^l(\mathbf{i}))} \right) = t.$$

Equation (4.5) implies that $\mathcal{X}_{\mathbf{i}}(t)$ is a Poisson submanifold of $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$, and, according to Definition 4.4, the right truncation map $\mathbf{t}_{\mathfrak{R}}$ on seeds induces *right truncation maps* on seed $\mathcal{X}$ -tori, $\mathbf{t}_{\mathfrak{R}}(t) : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}(t)$. The following result is clear.

Lemma 4.9. *Let $\mathbf{i}$ be a double word. The right truncated seed $\mathcal{X}$ -torus $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ is Poisson isomorphic to the direct product $\mathcal{X}_{\mathbf{i}}^{\text{red}} \times \mathcal{X}_{\mathbf{1}}$ of seed $\mathcal{X}$ -tori. Moreover, the following relation is satisfied for any double word $\mathbf{j}$ obtained from $\mathbf{i}$ by a sequence of generalized d -moves:*

$$(4.8) \quad \begin{aligned} &\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{j}]_{\mathfrak{R}}} : \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}} \rightarrow \mathcal{X}_{[\mathbf{j}]_{\mathfrak{R}}}, \\ &x_{\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{j}]_{\mathfrak{R}}}(i)} = \begin{cases} x_{\mu_{\mathbf{i} \rightarrow \mathbf{j}}(i)} & \text{if } i \in I \setminus J, \\ x_i & \text{if } i \in J. \end{cases} \end{aligned}$$

In particular, this implies that the cluster transformation $\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{j}]_{\mathfrak{R}}}$ sends $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}(t)$ to $\mathcal{X}_{[\mathbf{j}]_{\mathfrak{R}}}(t)$ for every $\mathbf{j}$ linked to $\mathbf{i}$ by a composition of generalized d -moves. Finally, the cluster $\mathcal{X}$ -variety $\mathcal{X}_{|[\mathbf{i}]_{\mathfrak{R}}|}$ constructed from the seed $\mathcal{X}$ -torus $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ associated with a double word $\mathbf{i}$ is called a *right truncated cluster $\mathcal{X}$ -variety*. By Lemma 4.5, there exists a *truncation*

map $\mathfrak{t}_{\mathfrak{R}}$ that takes every cluster $\mathcal{X}$ -variety $\mathcal{X}_{|\mathfrak{i}|}$ to its right truncated cluster $\mathcal{X}$ -variety $\mathcal{X}_{[|\mathfrak{i}|]_{\mathfrak{R}}}$:

$$\mathfrak{t}_{\mathfrak{R}} : \mathcal{X}_{|\mathfrak{i}|} \mapsto \mathcal{X}_{[|\mathfrak{i}|]_{\mathfrak{R}}}.$$

4.3. Dual evaluations and first cluster $\mathcal{X}$ -varieties related to (G, π_*) . We use the previous truncated cluster $\mathcal{X}$ -varieties to start the geometrical combinatorics of the Poisson manifold (G, π_*) . For every element $x \in B_-B$, let $x = [x]_-[x]_0[x]_+$ be its Gauss decomposition, that is, $[x]_{\pm}$ belongs to the unipotent parts $N_{\pm}$ of the respective Borel subgroups $B_{\pm}$ and $[x]_0$ to the Cartan part H of G . We shall also use the following notation:

$$(4.9) \quad [x]_-[x]_{\geq 0} = [x]_-[x]_0[x]_+ = [x]_{\leq 0}[x]_+.$$

Definition 4.10. Let $v \in W$. For every $\mathfrak{i} \in R(v, w_0)$, we define the *dual evaluation map* $\text{ev}_{\mathfrak{i}}^{\text{dual}} : \mathcal{X}_{[|\mathfrak{i}|]_{\mathfrak{R}}} \rightarrow G$ by the formula

$$(4.10) \quad \text{ev}_{\mathfrak{i}}^{\text{dual}}(\mathbf{x}) = \text{ev}_{\mathfrak{i}}(\mathbf{x}) \widehat{w_0} [\text{ev}_{\mathfrak{i}}^{\text{red}}(\mathbf{x}) \widehat{w_0}]_{\leq 0}^{-1}.$$

The dual evaluations will be generalized into twisted evaluations in Subsection 7.1. (We refer to Remark 7.4 for more details.) For the moment, let us remember the Poisson stratification (2.7) of (G, π_*) .

Theorem 4.11. For every $v \in W$, $t \in H$, and $\mathfrak{i} \in R(v, w_0)$, the map $\text{ev}_{\mathfrak{i}}^{\text{dual}} : \mathcal{X}_{[|\mathfrak{i}|]_{\mathfrak{R}}}(t) \rightarrow (F_{t, w_0 v^{-1}}, \pi_*)$ is a Poisson birational isomorphism onto a Zariski open set $F_{t, w_0 v^{-1}}^0$ of $F_{t, w_0 v^{-1}}$.

Theorem 4.11 will be deduced from Theorem 8.12. The synthesis diagram, given by Figure 10 and relating, for every $u, v \in W$, the cluster $\mathcal{X}$ -variety $\mathcal{X}^{u, v}$ to the double Bruhat cell $G^{u, v}$ can therefore be adapted to get a cluster $\mathcal{X}$ -variety $\mathcal{X}_{v \leq v}(t)$ associated with $(F_{t, w_0 v^{-1}}, \pi_*)$, for every $v \in W$ and every $t \in H$. It is illustrated in Figure 12. (The weird terminology for “ $\mathcal{X}_{v \leq v}(t)$ ” and “ $\text{ev}_{v \leq v}$ ” will be explained in Subsection 7.3.) Finally, for every double word $\mathfrak{i}$, let $\mathcal{X}_{\mathfrak{i}}^{\text{dual}} \subset \mathcal{X}_{[|\mathfrak{i}|]_{\mathfrak{R}}}$ be such that the elements of the set of

FIGURE 12. The cluster $\mathcal{X}$ -variety $\mathcal{X}_{v \leq v}(t)$ associated to $(F_{t, w_0 v^{-1}}, \pi_*)$.

variables $\mathbf{x}(\mathfrak{R})$ are pairwise disjoint. It is a Poisson submanifold of $\mathcal{X}_{[|\mathfrak{i}|]_{\mathfrak{R}}}$ because of (4.5). Thus, the following corollaries are deduced, respectively, from the second decomposition of (2.11) with Theorem 4.11, and Theorem 3.7 with Lemma 4.9 and equation (4.10).

Corollary 4.12. For every $\mathfrak{i} \in R(w_0, w_0)$, the map $\text{ev}_{\mathfrak{i}}^{\text{dual}} : \mathcal{X}_{\mathfrak{i}}^{\text{dual}} \rightarrow (BB_-, \pi_*)$ is a Poisson birational isomorphism on a Zariski open set of BB_- .

Corollary 4.13. For any $v \in W$ and $\mathfrak{i}, \mathfrak{j} \in R(v, w_0)$, we have $\text{ev}_{\mathfrak{i}}^{\text{dual}} = \text{ev}_{\mathfrak{j}}^{\text{dual}} \circ \mu_{[\mathfrak{i}]_{\mathfrak{R}} \rightarrow [\mathfrak{j}]_{\mathfrak{R}}}$.

Remark 4.14. For every $\mathfrak{i} \in R(w_0, w_0)$, the set $\mathbf{x}(\mathfrak{R})$ is the set of Casimir functions on the seed $\mathcal{X}$ -tori $\mathcal{X}_{[|\mathfrak{i}|]_{\mathfrak{R}}}$ and, hence, on the associated right truncated cluster $\mathcal{X}$ -variety.

§5. τ -MOVES, TROPICAL MUTATIONS, AND TWIST MAPS

We enlarge the combinatorics on double words and on their related seed $\mathcal{X}$ -tori by introducing new moves based on the involution $i \mapsto \bar{i}$ and new birational Poisson isomorphisms on seed $\mathcal{X}$ -tori obtained by tropicalization of the mutation formula. This enables us to describe the Fomin–Zelevinsky twist maps and their variations in terms on cluster transformations and tropical mutations.

5.1. τ -moves and tropical mutations. We introduce τ -moves and *tropical mutations*. From now on, we suppose that for every seed $\mathbf{I} = (I, I_0, \varepsilon, d)$, the related matrix ε is such that ε_{ij} is a rational number, for every $i, j \in I$.

5.1.1. The right and left τ -moves. We now consider the map $i \mapsto \bar{i}$ as an involution on $[1, l] \cup [\bar{1}, \bar{l}]$. First, we enrich the combinatorics on double words.

Definition 5.1. For every double word $\mathbf{i} = i_1 \dots i_n$, let $\mathfrak{L}(\mathbf{i})$ (respectively, $\mathfrak{R}(\mathbf{i})$) be the double word obtained by changing the first letter (respectively, last letter) i of $\mathbf{i}$ into $\bar{i}$:

$$\mathfrak{L}_{i_1}(\mathbf{i}) = \bar{i}_1 i_2 \dots i_n \quad \text{and} \quad \mathfrak{R}_{i_n}(\mathbf{i}) = i_1 \dots i_{n-1} \bar{i}_n.$$

The map $\mathbf{i} \mapsto \mathfrak{L}_{i_1}(\mathbf{i})$ (respectively, $\mathbf{i} \mapsto \mathfrak{R}_{i_n}(\mathbf{i})$) is called a *left* (respectively, *right*) τ -move on $\mathbf{i}$. (We will simply write $\mathbf{i} \mapsto \mathfrak{L}(\mathbf{i})$ and $\mathbf{i} \mapsto \mathfrak{R}(\mathbf{i})$ when no confusion may occur.)

Remark 5.2. The set $R(u, v)$ of double reduced words associated to the elements $u, v \in W$ is generally stable by neither left nor right τ -moves. Even the set of all the double reduced words associated with W is stable under neither left nor right τ -moves. However, this set contains subsets that are stable. For example, consider the following set $R^\tau(w)$ associated with any $w \in W$:

$$R^\tau(w) = \bigcup_{w' \leq w \in W} R(w'^{-1}, ww'^{-1}).$$

It is stable under right τ -moves and will be used to describe the combinatorics associated with the dual Poisson–Lie group (G^*, π_{G^*}) .

5.1.2. Tropical mutations. The τ -moves on double words lead to a new type of mutations on the associated seed $\mathcal{X}$ -tori, called tropical mutations and defined in the following way. We first identify $\mathbb{Q}$ with the Cartesian product $\mathbb{Z} \times \mathbb{N} \setminus \{0\}$, by decomposing every element into its numerator p and its positive denominator q with $(p, q) = 1$, and associate with every value ε_{ij} of ε its numerator b_{ij} . So, we have $b_{ij} = \varepsilon_{ij}$ unless $i, j \in I_0$; from now on, we suppose that the denominator q is one and the same for every $i, j \in I_0$. In particular, we recall that for every double word $\mathbf{i}$ we have

$$(5.1) \quad b(\mathbf{i})_{kl} = \begin{cases} 2\varepsilon(\mathbf{i})_{kl} & \text{if } k, l \in I_0, \\ \varepsilon(\mathbf{i})_{kl} & \text{otherwise.} \end{cases}$$

Definition 5.3. Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ and $\mathbf{I}' = (I', I'_0, \varepsilon', d')$ be two seeds, and let $k \in I_0$. A *tropical mutation in the direction k* is an involution $\mu_k : \mathbf{I} \longrightarrow \mathbf{I}'$ satisfying the following conditions:

- (i) $\mu_k(I_0) = I'_0$;
- (ii) $d'_{\mu_k(i)} = d_i$;
- (iii)

$$\varepsilon'_{\mu_k(i)\mu_k(j)} = \begin{cases} -\varepsilon_{ij} & \text{if } i = k \text{ or } j = k, \\ \varepsilon_{ij} & \text{if } i, j \in I_0 \setminus \{k\}, \\ \varepsilon_{ij} - \varepsilon_{ik}b_{kj} & \text{otherwise.} \end{cases}$$

Tropical mutations induce involutive maps between the corresponding seed $\mathcal{X}$ -tori, which are denoted by the same symbols μ_k and are given by

$$(5.2) \quad x_{\mu_k(i)} = \begin{cases} x_k^{-1} & \text{if } i = k, \\ x_i x_k^{b_{ki}} & \text{if } i \in I_0 \setminus \{k\}, \\ x_i & \text{otherwise.} \end{cases}$$

A *generalized cluster transformation* linking two seeds (and two seed $\mathcal{X}$ -tori) is a composition of symmetries, mutations, and tropical mutations.

Let us stress that a tropical mutation is not a cluster transformation. Indeed, the cluster variable x_k associated with an outlet k is mapped onto its inverse x_k^{-1} by the tropical mutation in the direction k . Now, any symmetry is a permutation of the set of outlets, and the mutations do not mix the cluster variables associated with the set of outlets, by the formula of Definition 2.8, so that there is no way to obtain the inverse variable x_k^{-1} from x_k , as given by the first line of (5.2), by a cluster transformation.

Remark 5.4. Here is the reason underlying the terminology for *tropical mutation*. As in [FZ02, Example 5.6], let $\text{Trop}(x_i : i \in I)$ be the Abelian group (written multiplicatively) freely generated by the cluster variables x_i ($i \in I$). We define the addition $\boxplus$ in $\text{Trop}(x_i : i \in I)$ by

$$\prod_i x_i^{a_i} \boxplus \prod_i x_i^{b_i} = \prod_i x_i^{\min(a_i, b_i)}.$$

This tropical addition $\boxplus$ should not be confused with the tropical addition $\oplus$ in the usual tropical setting

$$(5.3) \quad a \odot b = a + b, \quad a \oplus b = \min(a, b).$$

But it is easily seen that they are strongly related:

$$\prod_i x_i^{a_i} \boxplus \prod_i x_i^{b_i} = \prod_i x_i^{a_i \oplus b_i}.$$

It turns out that the left and right tropical mutations, associated respectively to left and right τ -moves $\mathfrak{L}_i$ and $\mathfrak{R}_j$ with $i, j \in [1, l]$ and described in the next subsection, can alternatively be defined by the following formula, obtained by tropicalizing the mutation formula in Definition 2.8:

$$x_{\mu_k(i)} = \begin{cases} x_k^{-1} & \text{if } i = k, \\ x_i x_k^{(b_{ik})+} (1 \boxplus x_k)^{-b_{ik}} & \text{otherwise.} \end{cases}$$

This last formula has relatives in the cluster algebra literature:

- The formula covers the mutation combinatorics attached to a labeled Y -seed $(\mathbf{y}, B)$, in the tropical semifield given above, defined by

$$\mathbf{y} = \{x_i \mid i \in I_0\} \quad \text{and} \quad B = (b_{ij})_{i,j \in I_0}.$$

In fact, the combinatorics on seed $\mathcal{X}$ -tori induced by mutations and tropical mutations is very near to that induced by mutations on the special cases of $(M, \mathbb{P}, \hat{\mathbb{P}})$ -seeds [FZ07, Remark 6.5].

- Up to the rescaling $\varepsilon \rightarrow b$ given by (5.1) and the choice of direction, this formula is equal to the monomial part μ'_k of mutations respecting the decomposition given in [FG07b, Section 2.3].

Remark 5.5. Like the amalgamation product, the tropical mutations act on the cluster variables x_j associated with outlets; therefore, commutation between them may fail. In fact, a tropical mutation acting nontrivially on a set of cluster variables $\{x_j\}_{j \in J}$ associated with a set of outlets J commutes with the amalgamation product if and only

if we have $L \cap J = \emptyset$, where the set L denotes the set of amalgamation, as given in Definition 2.10.

5.1.3. Left and right tropical mutations. We are now ready to describe the way to relate left and τ -moves to particular tropical mutations, called left and right tropical mutations, respectively.

Proposition 5.6. *The following tropical mutations are Poisson birational isomorphisms for every double word $\mathbf{i} = i_1 \dots i_n$:*

$$(5.4) \quad \mu_{\begin{pmatrix} i_1 \\ 0 \end{pmatrix}} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathfrak{L}(\mathbf{i})} \quad \text{and} \quad \mu_{\begin{pmatrix} i_n \\ N^{i_n}(\mathbf{i}) \end{pmatrix}} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathfrak{R}(\mathbf{i})} .$$

Proof. The birational part is clear, so we focus on the Poisson part. We notice that, since the right (respectively, left) tropical mutations commute with an amalgamated product done on the left side (respectively, right side), as detailed in Remark 5.8, it suffices to show that the proposition is true for $\mathbf{i} \in \{i, \bar{i}\}$. Consider the case where $\mathbf{i} = i$. Then we have $\mathfrak{L}(\mathbf{i}) = \mathfrak{R}(\mathbf{i}) = \bar{i}$, and $b(\mathbf{i}) = 2\varepsilon(\mathbf{i})$. Now it is straightforward to check that the matrices $\varepsilon(\mathfrak{L}(\mathbf{i}))$ and $\varepsilon(\mathfrak{R}(\mathbf{i}))$ are equal to $\varepsilon(\bar{i})$. The case where $\mathbf{i} = \bar{i}$ is treated in the same way. $\square$

Definition 5.7. Let $\mathbf{i} = i_1 \dots i_n$ be a double word. The tropical mutations given by (5.4) are called *left* and *right tropical mutations*, respectively, and we denote the associated directions by $\diamond_{i_1}^{\mathfrak{L}}$ and $\diamond_{i_n}^{\mathfrak{R}}$:

$$\diamond_{i_1}^{\mathfrak{L}} = \begin{pmatrix} i_1 \\ 0 \end{pmatrix} \quad \text{and} \quad \diamond_{i_n}^{\mathfrak{R}} = \begin{pmatrix} i_n \\ N^{i_n}(\mathbf{i}) \end{pmatrix}.$$

Remark 5.8. Remark 5.5 can be refined in the following way: the tropical mutation in a left (respectively, right) outlet direction applied to a seed $\mathcal{X}$ -torus $\mathcal{X}_{\mathbf{I}}$ (respectively, $\mathcal{X}_{\mathbf{J}}$) is equal to the tropical mutation in the same direction applied to the amalgamated seed $\mathcal{X}$ -torus $\mathcal{X}_{\mathbf{m}(\mathbf{I}, \mathbf{J})}$.

Left and right tropical mutations are easily described on Dynkin quivers. We take a Dynkin quiver Γ and let $\mathbf{i} = i_1 \dots i_n$ be the associated double word, so that $\Gamma = \Gamma_{\mathbf{g}}(\mathbf{i})$. In particular, Γ is obtained by amalgamation of the elementary Dynkin quivers $\Gamma_{\mathbf{g}}(i_1), \dots, \Gamma_{\mathbf{g}}(i_n)$. Let us remember the relationship between tropical mutations and the amalgamated product given by Remark 5.8, and change the orientation of the arrows of $\Gamma_{\mathbf{g}}(i_1)$ (respectively, $\Gamma_{\mathbf{g}}(i_m)$) if we have a left (respectively, right) tropical mutation. We get the elementary Dynkin quiver $\Gamma_{\mathbf{g}}(\bar{i}_1)$ (respectively, $\Gamma_{\mathbf{g}}(\bar{i}_m)$). Now, we perform the amalgamation

$$\begin{aligned} \mathbf{m} : \Gamma_{\mathbf{g}}(\bar{i}_1) \times \dots \times \Gamma_{\mathbf{g}}(i_n) &\rightarrow \Gamma_{\mathbf{g}}(\mathfrak{L}(\mathbf{i})) \\ (\text{resp. } \mathbf{m} : \Gamma_{\mathbf{g}}(i_1) \times \dots \times \Gamma_{\mathbf{g}}(\bar{i}_n) &\rightarrow \Gamma_{\mathbf{g}}(\mathfrak{R}(\mathbf{i}))). \end{aligned}$$

The resulting quiver Γ' is the image of Γ by the left (respectively, right) tropical mutation. This procedure is illustrated in Figure 13.

Remark 5.9. The left and right tropical mutations acting on a double word $\mathbf{i} = i_1 \dots i_n$ such that the letter i_1 is negative and the letter i_n is positive leave invariant the cluster variables x_i with $i \in I \setminus I_0$.

5.1.4. From left and right tropical mutations to (simple) reflections of W .

We are going to describe left and right tropical mutations as geometrical lifts of simple reflections of W , according to the tropicalization formula (5.3). We recall that for every double word $\mathbf{i}$ and every cluster $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$, the cluster variables $\mathbf{x}(\mathfrak{R})$ are the variables associated with right outlets and given by (4.6). In the same way, we define the following cluster variables $\mathbf{x}(\mathfrak{L})$ associated with left outlets:

$$\mathbf{x}(\mathfrak{L}) = \{x_j \mid j \in I_0^{\mathfrak{L}}(\mathbf{i})\}.$$

FIGURE 13. The tropical mutation $\mu_{\binom{1}{3}} : \Gamma_{A_3}(123121) \mapsto \Gamma_{A_3}(12312\bar{1})$.

We recall that Π denotes the set of roots of the Lie algebra $\mathfrak{g}$. It is easy to see that for every $u, v \in W$ and all double words $\mathbf{i}, \mathbf{j} \in D(u, v)$, there exist maps $a_{\mathbf{i}}^{\mathcal{L}} : \mathbf{x}(\mathcal{L}) \rightarrow \Pi$ and $a_{\mathbf{j}}^{\mathcal{R}} : \mathbf{x}(\mathcal{R}) \rightarrow \Pi$, which are bijections on the set of simple roots and are given by

$$a_{\mathbf{i}}^{\mathcal{L}}(x_{\binom{i}{0}}) = \alpha_i \quad \text{and} \quad a_{\mathbf{j}}^{\mathcal{R}}(x_{\binom{j}{N^i(\mathbf{j})}}) = \alpha_i.$$

Proposition 5.10. *Suppose $u, v \in W$, $i, j \in [1, l]$, and $\mathbf{i} \in R(u, 1)$ and $\mathbf{j} \in R(1, v)$ are double reduced words such that the first letter of $\mathbf{i}$ is $\bar{i}$ and the last letter of $\mathbf{j}$ is j . Then*

$$[\mu_{\binom{i}{0}}]_{\text{trop}} = s_i \circ a_{\mathbf{i}}^{\mathcal{L}} \quad \text{and} \quad [\mu_{\binom{j}{N^j(\mathbf{j})}}]_{\text{trop}} = s_j \circ a_{\mathbf{j}}^{\mathcal{R}}.$$

Proof. The proof follows that of Proposition 5.6 and employs the first identity of (2.4), the definition (5.3) of the tropicalization, and the definition (5.3) of the tropical mutations. $\square$

Example 5.11. Let $\mathfrak{g} = A_2$, and let $\mathbf{i} \in R(1, w_0)$, $\mathbf{j} \in R(1, w_0)$ be (respectively) the double reduced words 121 and $\bar{2}1\bar{2}$. Using the definition of the tropical mutations and the values of the matrices $\varepsilon(\mathbf{i})$ and $\varepsilon(\mathbf{j})$, we get the following formulas for every $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$ and $\mathbf{y} \in \mathcal{X}_{\mathbf{j}}$:

$$\begin{aligned} \mu_{\binom{1}{0}} \left(x_{\binom{1}{0}}, x_{\binom{2}{0}}, x_{\binom{1}{1}}, x_{\binom{1}{2}}, x_{\binom{2}{1}} \right) &= \left(x_{\binom{1}{0}}^{-1}, x_{\binom{1}{0}} x_{\binom{2}{0}}, x_{\binom{1}{1}}, x_{\binom{1}{2}}, x_{\binom{2}{1}} \right), \\ \mu_{\binom{2}{2}} \left(y_{\binom{1}{0}}, y_{\binom{2}{0}}, y_{\binom{2}{1}}, y_{\binom{1}{1}}, y_{\binom{2}{2}} \right) &= \left(y_{\binom{1}{0}}, y_{\binom{2}{0}}, y_{\binom{2}{1}}, y_{\binom{1}{1}} y_{\binom{2}{2}}, y_{\binom{2}{2}}^{-1} \right). \end{aligned}$$

The formulas involving the cluster variables associated with the sets $I_0^{\mathcal{L}}(\mathbf{i})$ and $I_0^{\mathcal{L}}(\mathcal{L}(\mathbf{i}))$ on the one hand, and with the sets $I_0^{\mathcal{R}}(\mathbf{j})$ and $I_0^{\mathcal{R}}(\mathcal{R}(\mathbf{j}))$ on the other hand, are then easily tropicalized to get formulas underlying the first identity in (2.4):

$$s_1(\alpha_1, \alpha_2) = (-\alpha_1, \alpha_1 + \alpha_2) \quad \text{and} \quad s_2(\alpha_1, \alpha_2) = (\alpha_1 + \alpha_2, -\alpha_2).$$

Therefore, Proposition 5.10 gives a clue (but not a reason: geometrical lifts associated with a map are generally not unique) why the W -permutohedron associated with the Lie algebra $\mathfrak{g}$ appears in the combinatorics of §6 and §8.

5.2. Symmetries on seed $\mathcal{X}$ -tori. In this subsection and in the next one, we define involutions on double words, and the related symmetries on seed $\mathcal{X}$ -tori. They will be used to describe various automorphisms and anti-automorphisms on the group G . We recall that the *fundamental weights* $\omega_i \in \mathfrak{h}^*$, given by $\omega_i(h_j) = \delta_{ij}$ for every $i, j \in [1, l]$,

are permuted by the transformation $(-w_0)$. Let $i \mapsto i^*$ be the induced permutation of the indices of these weights, i.e., $\omega_{i^*} = -w_0(\omega_i)$. This permutation is nontrivial only for root systems of the types A_n, D_n, E_6 , where it acts as follows:

$$(5.5) \quad \begin{cases} A_n : & i^* = n + 1 - i, \quad i = 1, \dots, n; \\ D_n : & i^* = \begin{cases} n & \text{if } i = n - 1, \\ n - 1 & \text{if } i = n, \\ i & \text{otherwise;} \end{cases} \\ E_6 : & i^* = \begin{cases} 6 & \text{if } i = 1, \\ 1 & \text{if } i = 6, \\ i & \text{otherwise.} \end{cases} \end{cases}$$

This automorphism on the Dynkin diagram leads to symmetries on seed $\mathcal{X}$ -tori described by the use of Dynkin quivers: for every $u, v \in W$ and $\mathbf{i} \in R(u, v)$ we define the involutions

$$\begin{array}{ccc} \star : R(u, v) & \rightarrow & R(v^*, u^*) \\ \mathbf{i} & \mapsto & \mathbf{i}^* \end{array} \quad \text{op} : \begin{array}{ccc} R(u, v) & \rightarrow & R(u^{-1}, v^{-1}) \\ \mathbf{i} & \mapsto & \mathbf{i}^{\text{op}} \end{array}$$

in the following way: the double reduced word $\mathbf{i}^* \in R(v^*, u^*)$ is obtained by transforming each letter i of $[1, l] \cup [\bar{1}, \bar{l}]$ into $\bar{i}^*$, so that if $\mathbf{i} = i_1 \dots i_n$, then $\mathbf{i}^* = \bar{i}_1^* \dots \bar{i}_n^*$, and the double reduced word $\mathbf{i}^{\text{op}} \in R(u^{-1}, v^{-1})$ is obtained by reading $\mathbf{i}$ backwards: $\mathbf{i}^{\text{op}} = i_n \dots i_1$. Now, let

$$\circlearrowleft : R(u, v) \rightarrow R(v^{\star-1}, u^{\star-1})$$

be the map such that the double reduced word $\mathbf{i}^{\circlearrowleft} \in R(v^{\star-1}, u^{\star-1})$ is obtained by converting each i into $\bar{i}^*$ and then reading the result backwards. Stated otherwise, this transformation is defined by $\circlearrowleft = \text{op} \circ \star$. It induces the following symmetry on seed $\mathcal{X}$ -tori.

Proposition 5.12. *For every $u, v \in W$ and every double reduced word $\mathbf{i} \in R(u, v)$, the involution $\circlearrowleft$ induces the following isomorphism on the related seed $\mathcal{X}$ -tori:*

$$(5.6) \quad \begin{array}{ccc} \circlearrowleft : \mathcal{X}_{\mathbf{i}} & \longrightarrow & \mathcal{X}_{\mathbf{i}^{\circlearrowleft}} \\ x_{\binom{i}{j}} & \longmapsto & x_{\binom{i^*}{N^{\mathbf{i}^*}(\mathbf{i}^{\circlearrowleft})-j}}. \end{array}$$

Proof. This follows from the fact that a rotation by 180 degrees performed on the quiver $\Gamma_{\mathfrak{g}}(\mathbf{i})$, as shown in Figure 15, gives the quiver $\Gamma_{\mathfrak{g}}(\mathbf{i}^{\circlearrowleft})$. $\square$

Remark 5.13. The involutions $\star$ and op were already defined on reduced words $\mathbf{i} \in R(w_0)$ in [BZ01, Equation (3.1)]. Moreover, the notation $\circlearrowleft$ for the last involution on double reduced words symbolizes the rotation by 180 degrees performed on the quiver $\Gamma_{\mathfrak{g}}(\mathbf{i})$, as illustrated in Figure 15.

FIGURE 14. The $\star$ involution on Γ_{A_3} .

In the same way, with any double word $\mathbf{i}$ we associate the double word $\mathbf{i}^{\square}$ obtained by reading $\mathbf{i}$ backwards and applying the involution $i \mapsto \bar{i}$ to every letter of the result. In particular, for every $u, v \in W$ we get an involution

$$\square : R(u, v) \rightarrow R(v^{-1}, u^{-1}).$$

FIGURE 15. The involution $\circ : \Gamma_{A_3}(123121) \mapsto \Gamma_{A_3}(\overline{323123})$.

This involution induces a symmetry on the related seed: it corresponds to a rotation along the vertical axis passing by the center of the Dynkin quiver $\Gamma_{\mathfrak{g}}(\mathbf{i})$, as illustrated in Figure 16. It also induces a symmetry on seed $\mathcal{X}$ -tori, also denoted by $\square$ and given by

$$\begin{aligned} \square : \mathcal{X}_{\mathbf{i}} &\longrightarrow \mathcal{X}_{\mathbf{i}\square}, \\ x_{(j)}^{(i)} &\longmapsto x_{(N^i(\mathbf{i})-j)}^{(i)} \end{aligned}$$

FIGURE 16. The involution $\square : \Gamma_{A_3}(123121) \mapsto \Gamma_{A_3}(\overline{121321})$.

Now, here are some related isomorphisms on (G, π_G) . Starting with an elementary double word $\mathbf{i} \in \{\mathbf{1}, i, \bar{i}\}$, where $i \in [1, l]$, and then applying the properties of the amalgamation product, we easily prove the following results.

Lemma 5.14. *Let $u, v \in W$ and $\mathbf{i} \in R(u, v)$. For every cluster $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$, let $\mathbf{x}^* \in \mathcal{X}_{\mathbf{i}^*}$ be the cluster such that*

$$\widehat{w_0} \text{ev}_{\mathbf{i}}(\mathbf{x}) \widehat{w_0}^{-1} = \text{ev}_{\mathbf{i}^*}(\mathbf{x}^*).$$

Then

$$(5.7) \quad x_{(j)}^{*(i)} = \begin{cases} -x_{(i^*)}^{-1} & \text{if } 0 = j \neq N^{i^*}(\mathbf{i}) \text{ or } 0 \neq j = N^{i^*}(\mathbf{i}), \\ x_{(j)}^{-1} & \text{otherwise.} \end{cases}$$

Lemma 5.15. *Let $u, v \in W$ and $\mathbf{i} \in R(u, v)$. For every cluster $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$, let $\mathbf{x}^{\text{op}} \in \mathcal{X}_{\mathbf{i}^{\text{op}}}$ be the cluster such that*

$$\text{ev}_{\mathbf{i}}(\mathbf{x})^{-1} = \text{ev}_{\mathbf{i}^{\text{op}}}(\mathbf{x}^{\text{op}}).$$

Then

$$x_{(j)}^{\text{op}(i)} = \begin{cases} -x_{(N^i(\mathbf{i})-j)}^{-1} & \text{if } 0 = j \neq N^i(\mathbf{i}) \text{ or } 0 \neq j = N^i(\mathbf{i}), \\ x_{(N^i(\mathbf{i})-j)}^{-1} & \text{otherwise.} \end{cases}$$

Let $\kappa : G \times G \rightarrow G$ be the composition of the conjugacy map associated with the first argument $g \in G$ and the inverse map $x \mapsto x^{-1}$ for the second argument $x \in G$, that is,

$$(5.8) \quad \kappa : (g, x) \mapsto gx^{-1}g^{-1}.$$

Proposition 5.16. *For every $u, v \in W$, $\mathbf{i} \in R(u, v)$, and every $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$, we have*

$$\kappa(\widehat{w_0}, \text{ev}_{\mathbf{i}}(\mathbf{x})) = \text{ev}_{\mathbf{i}\circ}(\mathbf{x}^{\circ}).$$

Proof. The involutions defined by Lemma 5.14, Lemma 5.15, and equation (5.6) on seed $\mathcal{X}$ -tori clearly imply that the relation $\circ = \text{op} \circ \star$ remains valid on seed $\mathcal{X}$ -tori. Therefore, Lemma 5.14 and Lemma 5.15 lead to the required identity. $\square$

5.3. Chiral dual and other involutions. We introduce other involutions on double words. These will be directly useful to describe the combinatorics associated with the twist maps.

Definition 5.17 ([FG07b]). The *chiral dual* of a seed $\mathbf{I} = (I, I_0, \varepsilon, d)$ is the seed $\mathbf{I}^\circ = (I, I_0, -\varepsilon, d)$. The chiral dual induces an involutive map between the corresponding seed $\mathcal{X}$ -tori, denoted in the same way and given by $x_{\circ(i)} = x_i^{-1}$.

Remark 5.18. It can be checked that the chiral dual commutes with cluster transformations but not with tropical mutations.

The chiral dual leads to the following involution on double reduced words, also denoted by $\circ$. For every $u, v \in W$ and every double word $\mathbf{i} \in D(u, v)$, let $\mathbf{i}^\circ \in D(v, u)$ be the double word obtained by applying the map $i \mapsto \bar{i}$ as a homomorphism to the double word $\mathbf{i}$. It is clear that the double word $\mathbf{i}^\circ$ is reduced if and only if $\mathbf{i}$ is reduced. The following result is immediate.

Lemma 5.19. *For every double word $\mathbf{i}$, the chiral dual $\mathbf{I}(\mathbf{i})^\circ$ of the seed $\mathbf{I}(\mathbf{i})$ associated with the double word $\mathbf{i}$ is the seed $\mathbf{I}(\mathbf{i}^\circ)$ associated with the double word $\mathbf{i}^\circ$.*

Now we give a link with the group G . We recall that the *involutive Cartan group automorphism* $\theta : G \rightarrow G : x \mapsto x^\theta$ is the map given by

$$(5.9) \quad a^\theta = a^{-1} \in H, \quad E^{i^\theta} = F^i, \quad F^{i^\theta} = E^i.$$

Proposition 5.20. *For every $u, v \in W$ and every double reduced word $\mathbf{i} \in R(u, v)$, the following diagram commutes. Its vertical edges are labeled by birational Poisson isomorphisms, whereas its horizontal edges are labeled by birational anti-Poisson isomorphisms:*

$$\begin{array}{ccc} & \circ & \\ \mathcal{X}_{\mathbf{i}} & \xrightarrow{\quad} & \mathcal{X}_{\mathbf{i}^\circ} \\ \text{ev}_{\mathbf{i}} \downarrow & & \downarrow \text{ev}_{\mathbf{i}^\circ} \\ (G^{u,v}, \pi_G) & \xrightarrow{\quad \theta \quad} & (G^{v,u}, \pi_G) \end{array}$$

Proof. Since the involution $\circ$ commutes with the amalgamated product, which intertwines the product on G via the evaluation map by equation (3.5), and since θ is a group automorphism for this product on G , it suffices to focus on the case where $\mathbf{i} \in \{i, \bar{i}\}$. The result is then derived easily from the definition of the involutions $\circ$ and θ , and formula (3.3). $\square$

5.4. Generalized cluster transformations and twist maps on (G, π_G) . In this subsection, we use generalized cluster transformations to give the cluster combinatorics underlying the Fomin–Zelevinsky twist maps.

5.4.1. Tropical mutations and twist maps. Let $w, w' \in W$. We write $w \rightarrow w'$ if and only if we can find a letter $i \in [1, l]$ such that $w = s_i w'$ and $\ell(w) = \ell(w') + 1$, and we denote by $\leq$ the *right weak order* on W , i.e. $w' \leq w$ if there exists a chain $w \rightarrow \cdots \rightarrow w'$. Recall that for every $w' \leq w$, a reduced word $\mathbf{i} = i_1 \dots i_{\ell(w)} \in R(w)$ is said to be *adapted* to w' if $s_{i_1} \dots s_{i_{\ell(w')}} = w'$.

With $i \in [1, l] \cup [\bar{1}, \bar{l}]$, we associate the positive letter $|i| \in [1, l]$ given by the formula

$$(5.10) \quad |i| = \begin{cases} i & \text{if } i \in [1, l], \\ \bar{i} & \text{otherwise.} \end{cases}$$

We extend this notation to every double word $\mathbf{i} = i_1 \dots i_n$ by setting $|\mathbf{i}| := |i_1| \dots |i_n|$.

Definition 5.21. Suppose $e < w_1 \leq u, e < w_2 \leq v \in W$. A double reduced word $\mathbf{i} = i_1 \dots i_n \in R(u, v)$ is said to be $\mathfrak{L}$ -adapted to w_1 (respectively, $\mathfrak{R}$ -adapted to w_2) if the reduced word

$$|i_1 \dots i_{\ell(w_1)}| \in R(w_1) \quad (\text{respectively, } |i_{\ell(u)+\ell(w_2^{-1}v)+1} \dots i_{\ell(u)+\ell(v)}| \in R(w_2))$$

is adapted to w_1 (respectively, adapted to w_2), and the double word $\mathbf{i}$ is (w_1, w_2) -adapted if it is $\mathfrak{L}$ -adapted to w_1 and $\mathfrak{R}$ -adapted to w_2 . In particular, a (w_1, w_2) -adapted double reduced word is (w'_1, w'_2) -adapted for every $w'_1 \leq w_1, w'_2 \leq w_2$.

For example, the double reduced word $\mathbf{i} = \bar{2}\bar{1}2$ is $\mathfrak{L}$ -adapted for the elements $s_2, s_2s_1 \in W$, $\mathfrak{R}$ -adapted for $s_2, (s_2s_1, s_2)$ -adapted, and (s_2, s_2) -adapted, whereas the double reduced word $\mathbf{j} = 2\bar{2}\bar{1}$ is neither $\mathfrak{L}$ -adapted, nor $\mathfrak{R}$ -adapted, hence nor (w_1, w_2) -adapted, for any $w_1, w_2 \in W$.

Remark 5.22. For every $u, v \in W$, if the double reduced word $\mathbf{i} \in R(u, v)$ is (u, v) -adapted, then its first $\ell(u)$ letters (respectively, $\ell(v)$ last letters) are negative (respectively, positive) and give a reduced expression for u (respectively, v). Moreover, the following assertions are equivalent for every double reduced word $\mathbf{i} \in R(u, v)$:

- the double reduced word $\mathbf{i}$ is (u, v) -adapted;
- the double reduced word $\mathbf{i}$ is $\mathfrak{L}$ -adapted to u ;
- the double reduced word $\mathbf{i}$ is $\mathfrak{R}$ -adapted to v .

Proposition 5.23. The following relations are satisfied for every $u, v \in W$, every (u, v) -adapted double word $\mathbf{i} = i_1 \dots i_n \in R(u, v)$, and every $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$:

$$\begin{aligned} [\widehat{\text{ev}_{\mathbf{i}}(\mathbf{x})v^{-1}}]_{\leq 0} &= [\widehat{\text{ev}_{\mathfrak{R}(\mathbf{i})} \circ \mu_{\binom{i_n}{N^{i_n}(\mathbf{i})}}(\mathbf{x})s_{i_n}v^{-1}}]_{\leq 0}, \\ [\widehat{u^{-1} \text{ev}_{\mathbf{i}}(\mathbf{x})}]_{\geq 0} &= [\widehat{s_{i_1}u^{-1} \text{ev}_{\mathfrak{L}(\mathbf{i})} \circ \mu_{\binom{i_1}{0}}(\mathbf{x})}]_{\geq 0}. \end{aligned}$$

Proof. Let us remember the map $\varphi_j : \text{SL}(2, \mathbb{C}) \hookrightarrow G$ defined in Subsection 2.1. For any nonzero $t \in \mathbb{C}$ and any $i \in [1, l]$, we denote

$$(5.11) \quad x_i(t) = \varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad x_{\bar{i}}(t) = \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

For every $j \in [1, l]$, the following identity, easily checked on $\text{SL}(2, \mathbb{C})$ by elementary matrix calculus, extends to G via the map φ_j of Subsection 2.1:

$$(5.12) \quad \widehat{s_j^{-1} x_{\bar{j}}(t)} = x_{\bar{j}}(-t^{-1}) t^{h_j} x_j(t^{-1}).$$

Using the definition of tropical mutation and the fact that tropical mutations commute with the amalgamated product according to Remark 5.8, we obtain

$$(5.13) \quad \widehat{s_{i_1}^{-1} \text{ev}_{\mathbf{i}}(\mathbf{x})} = x_{i_1}(-x_{i_1}^{-1}) \widehat{\text{ev}_{\mathfrak{L}(\mathbf{i})} \circ \mu_{\binom{i_1}{0}}(\mathbf{x})}.$$

Moreover, we have the inequality $\ell(s_{i_1}u) < \ell(u)$, because the double word $\mathbf{i}$ is $\mathfrak{L}$ -adapted to u . This inequality implies that $\widehat{s_{i_1}u^{-1} x_{\bar{j}}(t) \widehat{s_{i_1}u}} \in N_-$ for every $t \in \mathbb{C}$. Therefore, the second equation is proved. The first equation is proved in the same way, using the following identity instead of (5.12):

$$(5.14) \quad x_i(t) \widehat{s_i} = x_{\bar{i}}(t^{-1}) t^{h_i} x_i(-t^{-1}). \quad \square$$

Definition 5.24 ([FZ99, Definition 1.5]). Let $u, v \in W$. The *twist map* $\zeta_\theta^{u,v} : x \mapsto x'$ is the map defined by

$$(5.15) \quad x' = ([\widehat{u}^{-1}x]_-^{-1} \widehat{u}^{-1} x \widehat{v}^{-1} [x \widehat{v}^{-1}]_+^{-1})^\theta.$$

By [FZ99, Theorem 1.6], the right-hand side of (5.15) is well defined for every $x \in G^{u,v}$, and the twist map $\zeta_\theta^{u,v}$ establishes a biregular isomorphism between $G^{u,v}$ and $G^{u^{-1},v^{-1}}$. We define the related map

$$(5.16) \quad \begin{aligned} \zeta^{u,v} : G^{u,v} &\longrightarrow G^{v^{-1},u^{-1}}, \\ x &\longmapsto [\widehat{u}^{-1}x]_-^{-1} \widehat{u}^{-1} x \widehat{v}^{-1} [x \widehat{v}^{-1}]_+^{-1}. \end{aligned}$$

Then $\zeta_\theta^{u,v} = \theta \circ \zeta^{u,v}$. Observe that for every $x \in G^{u,1}$ and $y \in G^{1,v}$ we have the relations

$$\zeta^{u,1}(x) = [\widehat{u}^{-1}x]_{\geq 0} \quad \text{and} \quad \zeta^{1,v}(y) = [y \widehat{v}]_{\leq 0}.$$

Since the map (5.16) will appear repeatedly in what follows, it will be useful to name it also the “twist map”. When we shall need to avoid confusion, we will refer to the map (5.15) as the “Fomin–Zelevinsky twist map”. We are going to describe twist maps at the level of seed $\mathcal{X}$ -tori.

5.4.2. Generalized cluster transformations and twist maps on $(B_\pm, \pi_G)$. Now we start to associate a generalized cluster transformation with any twist map on $(B_\pm, \pi_G)$. For that, we first need to sharpen the preceding involution $\square$ on double words. For every positive reduced word $\mathbf{i} = i_1 \dots i_n$, every negative reduced word $\mathbf{j} = j_1 \dots j_n$, and every $k \in [1, n+1]$, we introduce the double words

$$(5.17) \quad \begin{aligned} \mathbf{i}(k) &= \mathbf{i}(k)_- \mathbf{i}(k)_+, \quad \text{where} \quad \mathbf{i}(k)_+ = i_1 \dots i_{k-1} \quad \text{and} \quad \mathbf{i}(k)_- = \overline{i_n} \dots \overline{i_k}, \\ \mathbf{j}(k) &= \mathbf{j}(k)_- \mathbf{j}(k)_+, \quad \text{where} \quad \mathbf{j}(k)_+ = j_{k-1} \dots j_1 \quad \text{and} \quad \mathbf{j}(k)_- = \overline{j_k} \dots \overline{j_n}. \end{aligned}$$

The following insight on these double reduced words will be developed in the next section, by considering the W -permutohedron. It is derived from an easy induction on the number $k \in [1, \ell(w)]$ that appears in the statement.

Lemma 5.25. *Let $w \in W$, let $\mathbf{i} = i_1 \dots i_{\ell(w)} \in R(1, w)$ be a positive reduced word, and let $w_{\geq k} = s_{i_k} \dots s_{i_{\ell(w)}}$ be the element of w associated with any $k \in [1, k]$. The double reduced word $\mathbf{i}(k)$ belongs to the set $R(w_{\geq k}^{-1}, w w_{\geq k}^{-1})$.*

The involution $\square$ on positive or negative words is rediscovered from equation (5.17), because

$$\mathbf{i} = \mathbf{i}(n+1), \quad \mathbf{j} = \mathbf{j}(1), \quad \text{and} \quad \mathbf{i}^\square = \mathbf{i}(1), \quad \mathbf{j}^\square = \mathbf{j}(n).$$

Example 5.26. Let us choose the positive reduced word $\mathbf{i} = 121 \in R(1, w_0)$, for $\mathfrak{g} = A_2$. Then we get the following double reduced words:

$$\mathbf{i}(4) = 121, \quad \mathbf{i}(3) = \overline{1}12, \quad \mathbf{i}(2) = \overline{1}2\overline{1}, \quad \text{and} \quad \mathbf{i}(1) = \overline{1}\overline{2}\overline{1}.$$

In the same way, if we consider the negative double word $\mathbf{j} = \overline{1}\overline{2}\overline{1} \in R(w_0, 1)$, we get the double reduced words

$$\mathbf{j}(1) = \overline{1}\overline{2}\overline{1}, \quad \mathbf{j}(2) = \overline{1}2\overline{1}, \quad \mathbf{j}(3) = \overline{1}12, \quad \text{and} \quad \mathbf{j}(4) = 121.$$

For every positive reduced word $\mathbf{i} = i_1 \dots i_n$, every negative reduced word $\mathbf{j} = j_1 \dots j_n$, and every $k \in [1, n]$, we define the generalized cluster transformations $\zeta_{\mathbf{i}(k)} : \mathcal{X}_{\mathbf{i}(k)} \rightarrow \mathcal{X}_{\mathbf{i}(k-1)}$ and $\zeta_{\mathbf{j}(k)} : \mathcal{X}_{\mathbf{j}(k)} \rightarrow \mathcal{X}_{\mathbf{j}(k-1)}$ by the following formulas:

$$(5.18) \quad \begin{aligned} \zeta_{\mathbf{i}(k)} &= \mu_{\binom{i_k}{N^{i_k}(\mathbf{i}(k)_-)}} \circ \mu_{\binom{i_k}{N^{i_k}(\mathbf{i}(k)_-)+1}} \circ \dots \circ \mu_{\binom{i_k}{N^{i_k}(\mathbf{i})}} \quad \text{and} \\ \zeta_{\mathbf{j}(k)} &= \mu_{\binom{j_k}{j_0}} \circ \mu_{\binom{j_k}{j_1}} \circ \dots \circ \mu_{\binom{j_k}{N^{j_k}(\mathbf{j}(k)_-)-1}}. \end{aligned}$$

Example 5.27. Here are the generalized cluster transformations related to the double reduced words of the previous example:

$$\begin{aligned}\zeta_{\mathbf{i}(4)} &= \mu_{(1)}^{(1)} \circ \mu_{(2)}^{(1)}, & \zeta_{\mathbf{i}(3)} &= \mu_{(1)}^{(2)}, & \zeta_{\mathbf{i}(2)} &= \mu_{(2)}^{(1)}, & \text{and} \\ \zeta_{\mathbf{j}(1)} &= \mu_{(1)}^{(1)} \circ \mu_{(0)}^{(1)}, & \zeta_{\mathbf{j}(2)} &= \mu_{(0)}^{(2)}, & \zeta_{\mathbf{j}(3)} &= \mu_{(0)}^{(1)}.\end{aligned}$$

Corollary 5.28. *For every $u, v \in W$, $\mathbf{i} \in R(1, v)$, and $\mathbf{j} \in R(u, 1)$, the following maps are Poisson birational isomorphisms:*

$$(5.19) \quad \begin{aligned} \zeta_{\mathbf{i}(\geq k)} : \mathcal{X}_{\mathbf{i}} &\longrightarrow \mathcal{X}_{\mathbf{i}(k-1)} & \zeta_{\mathbf{j}(\leq k)} : \mathcal{X}_{\mathbf{j}} &\longrightarrow \mathcal{X}_{\mathbf{j}(k-1)} \\ \mathbf{x} &\longmapsto \zeta_{\mathbf{i}(k)} \circ \cdots \circ \zeta_{\mathbf{i}(n)}(\mathbf{x}) & \mathbf{x} &\longmapsto \zeta_{\mathbf{j}(k)} \circ \cdots \circ \zeta_{\mathbf{j}(1)}(\mathbf{x}). \end{aligned}$$

Proof. We use Proposition 5.6, equation (5.18), and the fact that the mutations are Poisson birational isomorphisms. $\square$

Remark 5.29. The generalized cluster transformations (5.19) will be used in §9 to describe the unipotent parts of the dual Poisson–Lie group (G^*, π_{G^*}) .

Example 5.30. Let us keep the positive reduced word $\mathbf{i} = 121 \in R(1, w_0)$, for $\mathfrak{g} = A_2$. Then the generalized cluster transformations associated with the double reduced words of the example above are ($\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$ denotes the cluster $(x_{(0)}^{(1)}, x_{(1)}^{(1)}, x_{(2)}^{(1)}, x_{(0)}^{(2)}, x_{(1)}^{(2)})$)

$$\begin{aligned}\zeta_{\mathbf{i}(\geq 3)}(\mathbf{x}) &= \zeta_{\mathbf{i}(3)}(\mathbf{x}) = \begin{pmatrix} x_{(0)}^{(1)}(1 + x_{(1)}^{(1)}), & x_{(1)}^{-1}, & x_{(2)}^{-1}(1 + x_{(1)}^{(1)}), \\ x_{(0)}^{(2)}(1 + x_{(1)}^{-1})^{-1}, & x_{(1)}^{(2)}x_{(2)}^{(1)} \end{pmatrix}; \\ \zeta_{\mathbf{i}(\geq 2)}(\mathbf{x}) &= \zeta_{\mathbf{i}(2)} \circ \zeta_{\mathbf{i}(3)}(\mathbf{x}) = \begin{pmatrix} x_{(0)}^{(1)}(1 + x_{(1)}^{(1)}), & x_{(1)}^{-1}, & x_{(2)}^{(1)}(1 + x_{(1)}^{(1)}), \\ x_{(0)}^{(2)}(1 + x_{(1)}^{-1})^{-1}, & x_{(1)}^{-1}x_{(2)}^{-1} \end{pmatrix}; \\ \zeta_{\mathbf{i}(\geq 1)}(\mathbf{x}) &= \zeta_{\mathbf{i}(1)} \circ \zeta_{\mathbf{i}(2)} \circ \zeta_{\mathbf{i}(3)}(\mathbf{x}) = \begin{pmatrix} x_{(0)}^{(1)}(1 + x_{(1)}^{(1)}), & x_{(1)}^{-1}, & x_{(2)}^{-1}(1 + x_{(1)}^{(1)})^{-1}, \\ x_{(0)}^{(2)}(1 + x_{(1)}^{-1})^{-1}, & x_{(2)}^{-1}(1 + x_{(1)}^{(1)}) \end{pmatrix}.\end{aligned}$$

Special cases of the generalized cluster transformations (5.19) are given by the following birational Poisson isomorphisms. These are the ones that we are going to associate with twist maps on (G, π_G) :

$$(5.20) \quad \zeta_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{i}\square} : \mathbf{x} \mapsto \zeta_{\mathbf{i}(\geq 1)}(\mathbf{x}) \quad \text{and} \quad \zeta_{\mathbf{j}} : \mathcal{X}_{\mathbf{j}} \rightarrow \mathcal{X}_{\mathbf{j}\square} : \mathbf{x} \mapsto \zeta_{\mathbf{j}(\leq \ell(u))}(\mathbf{x}).$$

Proposition 5.31. *Let $w' \leq u, w \leq v \in W$, and let $\mathbf{i} \in R(w'^{-1}u, w'^{-1})$ and $\mathbf{j} \in R(w^{-1}, vw^{-1})$ be double reduced words $(w'^{-1}u, w'^{-1})$ -adapted and (w^{-1}, vw^{-1}) -adapted, respectively. The following identities are true for every cluster $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$ and $\mathbf{y} \in \mathcal{X}_{\mathbf{j}}$:*

$$[\widehat{w'^{-1}u}^{-1} \text{ev}_{\mathbf{i}}(\mathbf{x})]_{\geq 0} = \text{ev}_{\mathbf{i}\square_+} \circ \zeta_{\mathbf{i}_-}(\mathbf{x}) \quad \text{and} \quad [\text{ev}_{\mathbf{j}}(\mathbf{y})\widehat{wv}^{-1}]_{\leq 0} = \text{ev}_{\mathbf{j}\square_-} \circ \zeta_{\mathbf{j}_+}(\mathbf{y}).$$

Proof. Let $w \in W$ and $j \in [1, l]$ be such that $w < ws_j \leq v$ for the right weak order. Then, for every $g \in G^{v,1}$, we have

$$[\widehat{s_j}^{-1}[\widehat{w}^{-1}g]_{\geq 0}]_{\geq 0} = [\widehat{s_j}^{-1}\widehat{w}^{-1}g]_{\geq 0}.$$

Induction on the length of v , involving at each step equation (5.13), Theorem 3.7, and the definition (5.20) leads to the second identity. The first is proved in the same way. $\square$

Corollary 5.32. *For every $u, v \in W$, every (double) reduced words $\mathbf{i} \in R(1, u)$ and $\mathbf{j} \in R(v, 1)$, and every cluster $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$ and $\mathbf{y} \in \mathcal{X}_{\mathbf{j}}$, we have*

$$(5.21) \quad [\widehat{u}^{-1} \text{ev}_{\mathbf{i}}(\mathbf{x})]_{\geq 0} = \text{ev}_{\mathbf{i}\square} \circ \zeta_{\mathbf{i}}(\mathbf{x}) \quad \text{and} \quad [\text{ev}_{\mathbf{j}}(\mathbf{y}) \widehat{v^{-1}}]_{\leq 0} = \text{ev}_{\mathbf{j}\square} \circ \zeta_{\mathbf{j}}(\mathbf{y}).$$

Proof. We apply Proposition 5.31 with $w = w' = e$. $\square$

Corollary 5.33. *The following relation is valid for every $v \in W$ and all reduced words $\mathbf{i}, \mathbf{j} \in R(1, v)$, or $\mathbf{i}, \mathbf{j} \in R(v, 1)$:*

$$\mu_{\mathbf{i}\square \rightarrow \mathbf{j}\square} \circ \zeta_{\mathbf{i}} = \zeta_{\mathbf{j}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}.$$

Proof. Suppose that $\mathbf{i}, \mathbf{j} \in R(1, v)$. We recall that the involution $\square$ maps double reduced words to double reduced words, and that the evaluation map $\text{ev}_{\mathbf{j}}$ associated with any double reduced word $\mathbf{j}$ is birational by Theorem 3.4. Therefore, the equality $\mathbf{y} = \mathbf{z}$ between cluster variables on $\mathcal{X}_{\mathbf{i}\square}$ is satisfied if and only if the equality $\text{ev}_{\mathbf{i}\square}(\mathbf{y}) = \text{ev}_{\mathbf{i}\square}(\mathbf{z})$ is satisfied on G . Now, it suffices to apply Theorem 3.7 and the second equation of (5.21) to obtain the following equality for every $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$. The case $\mathbf{i}, \mathbf{j} \in R(v, 1)$ is proved in the same way:

$$\begin{aligned} \text{ev}_{\mathbf{i}\square} \circ \zeta_{\mathbf{i}}(\mathbf{x}) &= [\text{ev}_{\mathbf{j}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}(\mathbf{x}) \widehat{v^{-1}}]_{\leq 0} \\ &= \text{ev}_{\mathbf{j}\square} \circ \zeta_{\mathbf{j}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}(\mathbf{x}) = \text{ev}_{\mathbf{i}\square} \circ \mu_{\mathbf{i}\square \rightarrow \mathbf{j}\square} \circ \zeta_{\mathbf{j}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}(\mathbf{x}). \end{aligned} \quad \square$$

5.4.3. From twist maps on $(B_{\pm}, \pi_G)$ to twist maps on (G, π_G) . Now we are ready to give the generalized cluster transformations associated with any twist maps on (G, π_G) . For every double word $\mathbf{i}$, let $\mathbf{i}_+$ and $\mathbf{i}_-$ be the words obtained by erasing all the negative and positive letters of $\mathbf{i}$, respectively, without changing the order of the remaining letters. The word $\mathbf{i}_+$ (respectively, $\mathbf{i}_-$) is called the *positive part* (respectively, *negative part*) of $\mathbf{i}$. In particular, the double word $\mathbf{i}$ is linked by compositions of mixed 2-moves to the double words $\mathbf{i}_+\mathbf{i}_-$ and $\mathbf{i}_-\mathbf{i}_+$. (This definition is compatible with the notation used in equation (5.17).) Using Corollary 5.33, for every $\mathbf{i} \in R(u, v)$ we introduce the maps $\zeta_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{i}\square}$ and $\zeta_{\mathbf{i}\square} : \mathcal{X}_{\mathbf{i}\square} \rightarrow \mathcal{X}_{\mathbf{i}\square\text{op}}$ by the following formulas and state the main result of this subsection:

$$(5.22) \quad \zeta_{\mathbf{i}} = \mu_{\mathbf{i}\square \rightarrow \mathbf{i}_+\mathbf{i}_-} \circ \zeta_{\mathbf{i}_-} \circ \zeta_{\mathbf{i}_+} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_-\mathbf{i}_+} \quad \text{and} \quad \zeta_{\mathbf{i}\square} = \square \circ \zeta_{\mathbf{i}}.$$

Theorem 5.34. *For every $u, v \in W$ and every double reduced word $\mathbf{i} \in R(u, v)$, the following diagrams are commutative. All edges of the left diagram are Poisson birational isomorphisms, whereas the vertical edges of the right diagram are Poisson birational isomorphisms and the horizontal edges are anti-Poisson birational isomorphisms:*

Proof. First, we notice that Proposition 5.20 and the definitions of $\zeta^{u,v}$ and $\zeta_{\theta}^{u,v}$ show that the commutativity of the right diagram can be derived from the commutativity of the left one. So, we focus on the left one. The cases where $(u, v) = (u, 1)$ and $(u, v) = (1, v)$ are proved by Proposition 5.31. Moreover, among all the remaining cases, it suffices to prove the case where $\mathbf{i} = \mathbf{i}_-\mathbf{i}_+$ (with $\mathbf{i}_- \in R(u, 1)$ and $\mathbf{i}_+ \in R(1, v)$), by the definition (5.22) of $\zeta_{\mathbf{i}}$ and Theorem 3.7. The demonstration relies on identity (5.23)

below, borrowed from [GSV03, Theorem 3.1]. For every $x \in G^{u,v}$, the definition (5.16) leads to the following:

$$\begin{aligned}
 \zeta^{u,v}(x) &= [\widehat{u}^{-1}x]_-^{-1} \widehat{u}^{-1} x v^{-1} [\widehat{xv^{-1}}]_+^{-1} \\
 &= [\widehat{u}^{-1}x]_0 [\widehat{u}^{-1}x]_+ v^{-1} [\widehat{xv^{-1}}]_+^{-1} \\
 &= [\widehat{u}^{-1}[x]_- [x]_0 [x]_+]_0 [\widehat{u}^{-1}[x]_- [x]_0 [x]_+]_+ (x^{-1}x) \widehat{v^{-1}} [\widehat{xv^{-1}}]_+^{-1} \\
 &= [\widehat{u}^{-1}[x]_{\leq 0}]_0 [\widehat{u}^{-1}[x]_-]_+ [x]_+ x^{-1} [\widehat{xv^{-1}}]_{\leq 0} \\
 &= [\widehat{u}^{-1}[x]_{\leq 0}]_{\geq 0} [x]_+ x^{-1} [x]_- [[x]_{\geq 0} \widehat{v^{-1}}]_{\leq 0} \\
 &= \zeta^{u,1}([x]_{\leq 0}) [x]_0^{-1} \zeta^{1,v}([x]_{\geq 0}).
 \end{aligned}
 \tag{5.23}$$

Let $\mathbf{i} = \mathbf{i}_- \mathbf{i}_+ \in R(u, v)$, and let $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$, $\mathbf{x}_- \in \mathcal{X}_{\mathbf{i}_-}$, $\mathbf{x}_+ \in \mathcal{X}_{\mathbf{i}_+}$ be the cluster variables such that $x = \text{ev}_{\mathbf{i}}(\mathbf{x})$ and $\mathbf{m}(\mathbf{x}_-, \mathbf{x}_+) = \mathbf{x}$. We start with introducing the following maps $\pi_{\mathbf{j}} : \mathcal{X}_{\mathbf{j}} \rightarrow \mathcal{X}_{\mathbf{1}}$, for every double word $\mathbf{j}$:

$$\begin{aligned}
 \pi_{\mathbf{j}} : \quad \mathcal{X}_{\mathbf{j}} &\rightarrow \mathcal{X}_{\mathbf{1}} \\
 x_{\pi_{\mathbf{i}}((i)_j)} &= x_{(i)_0} x_{(i)_1} \cdots x_{(N^i(i))}.
 \end{aligned}
 \tag{5.24}$$

We use these maps to define the elements $\mathbf{x}_{\leq \mathbf{0}} \in \mathcal{X}_{\mathbf{i}_-}$, $\mathbf{x}_{\mathbf{0}} \in \mathcal{X}_{\mathbf{1}}$, and $\mathbf{x}_{\geq \mathbf{0}} \in \mathcal{X}_{\mathbf{i}_+}$ related to $\mathbf{x}$ in the following way (the associated identities are proved easily):

$$\begin{aligned}
 \mathbf{x}_{\leq \mathbf{0}} &= \mathbf{m}(\mathbf{x}_-, \pi_{\mathbf{i}_+}(\mathbf{x}_+)), \quad \mathbf{x}_{\mathbf{0}} = \pi_{\mathbf{i}}(\mathbf{x}), \quad \text{and} \quad \mathbf{x}_{\geq \mathbf{0}} = \mathbf{m}(\pi_{\mathbf{i}_-}(\mathbf{x}_-), \mathbf{x}_+), \\
 [x]_{\leq 0} &= \text{ev}_{\mathbf{i}_-}(\mathbf{x}_{\leq \mathbf{0}}), \quad [x]_0^{-1} = \text{ev}_{\mathbf{1}}(\mathbf{x}_{\mathbf{0}}^{\circ}), \quad \text{and} \quad [x]_{\geq 0} = \text{ev}_{\mathbf{i}_+}(\mathbf{x}_{\geq \mathbf{0}}).
 \end{aligned}
 \tag{5.25}$$

Now, Remark 5.8 implies the following relations:

$$\begin{aligned}
 \zeta_{\mathbf{i}_-}(\mathbf{x}_{\leq \mathbf{0}}) &= \mathbf{m}(\zeta_{\mathbf{i}_-}(\mathbf{x}_-), \pi_{\mathbf{i}_+}(\mathbf{x}_+)) \quad \text{and} \quad \zeta_{\mathbf{i}_+}(\mathbf{x}_{\geq \mathbf{0}}) = \mathbf{m}(\pi_{\mathbf{i}_-}(\mathbf{x}_-), \zeta_{\mathbf{i}_+}(\mathbf{x}_+)); \\
 \zeta_{\mathbf{i}}(\mathbf{x}) &= \zeta_{\mathbf{i}_-} \circ \zeta_{\mathbf{i}_+}(\mathbf{x}) = \mathbf{m}(\zeta_{\mathbf{i}_-}(\mathbf{x}_{\leq \mathbf{0}}), \mathbf{x}_{\mathbf{0}}^{\circ}, \zeta_{\mathbf{i}_+}(\mathbf{x}_{\geq \mathbf{0}})) = \zeta_{\mathbf{i}_+} \circ \zeta_{\mathbf{i}_-}(\mathbf{x}).
 \end{aligned}$$

Then Proposition 5.31 and identities (5.23), (5.25) lead to $\zeta^{u,v}(\text{ev}_{\mathbf{i}}(\mathbf{x})) = \text{ev}_{\mathbf{i} \square}(\zeta_{\mathbf{i}}(\mathbf{x}))$. Finally, the Poisson and birational statements are clear from Theorem 3.7, Proposition 5.32, and Proposition 5.20. $\square$

§6. τ -COMBINATORICS, W -PERMUTOHEDRON, AND EVALUATIONS ON (G, π_G)

We will refer to τ -combinatorics as the combinatorics on double reduced words generated by generalized d -moves and enriched with right tropical moves. The idea is to prepare the ground for the cluster combinatorics related to twisted evaluations and dual Poisson-Lie groups, developed in §8. Here, we associate a family of cluster $\mathcal{X}$ -varieties with every double Bruhat cell $G^{w,1}$ by linking cluster $\mathcal{X}$ -varieties with tropical mutations via the W -permutohedron associated with the Lie algebra $\mathfrak{g}$.

6.1. The W -permutohedron, $\uparrow$ -paths and $\downarrow$ -paths. We recall here that any reduced expression of any element of W can be described as a monotone paths on a particular polytope: the W -permutohedron (or moment polytope, or weight polytope [P05]). Recall that Λ denotes the integer weight lattice associated with $\mathfrak{g}$; we denote by $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$ the weight space. The roots in Π span the root lattice $L \subseteq \Lambda$. The associated Weyl group W acts on the weight space $\Lambda_{\mathbb{R}}$. For $x \in \Lambda_{\mathbb{R}}$, we can define the W -permutohedron $P_W(x)$ as the convex hull of a Weyl group orbit:

$$P_W(x) := \text{ConvexHull}(w(x) \mid w \in W) \subset \Lambda_{\mathbb{R}}.$$

For the Lie type A_n , the W -permutohedron $P_W(x)$ is the permutohedron $P_{n+1}(x)$ defined as the convex hull of all vectors obtained from $(x_1, \dots, x_{n+1})$ by permutations of the coordinates:

$$P_{n+1}(x_1, \dots, x_{n+1}) := \text{ConvexHull}((x_{w(1)}, \dots, x_{w(n+1)}) \mid w \in S_{n+1}).$$

From now on, we fix a generic $x \in \Lambda_{\mathbb{R}}$ such that the associated W -permutohedron $P_W(x)$ has maximal dimension. This W -permutohedron will be (abusively) denoted by P_W . It is well known that we can label vertices and edges of P_W (respectively) by the set of elements of W and the set of elementary reflections $s_i \in S$ that generate W , in such a way that

- every vertex has a different label;
- all labeled vertices w_1 and w_2 of P_W are connected by a labeled edge s_i if and only if $w_2 = w_1 s_i$.

In particular, the number of vertices of P_W is given by the cardinal of W . When we draw a picture of the W -permutohedron P_W , the bottom vertex can be associated with the identity element $1 \in W$, so that the top vertex is the longest element $w_0 \in W$. As remarked in [FR07], a reduced word for w then corresponds to a path along edges from 1 to w that moves up in a monotone fashion. We call a $\uparrow$ -path a path along edges of P_W that moves up monotonically on P_W . In the same way, a path that moves down monotonically will be called a $\downarrow$ -path. A $\uparrow$ -path relating the vertex w to the vertex w' is called a $w \nearrow w'$ -path and the corresponding $\downarrow$ -path is called a $w' \searrow w$ -path. In particular, a $\uparrow$ -path along edges from 1 to w is called a $\uparrow_w$ -path. The following result is clear.

Proposition 6.1. *Let $u \leq v \in W$. The $u \nearrow v$ -paths (respectively, $v \searrow u$ -paths) are in bijection with reduced expressions of the element $vu^{-1} \in W$ (respectively, $uv^{-1} \in W$). In particular, for a given $w \in W$, the number of $\uparrow_w$ -paths is equal to the number of reduced expressions of w .*

A $\uparrow$ -path and a $\downarrow$ -path on the permutohedron P_3 are given in Figure 17. Observe that the $\uparrow$ -path at the left of Figure 17 is a $\uparrow_{s_1 s_2}$ -path and that it is the only one. In fact, the only $w \in W$ such that the related $\uparrow_w$ -path is not unique is $w = w_0$ and then there are two $\uparrow_{w_0}$ -paths.

FIGURE 17. A $\uparrow$ -path and a $\downarrow$ -path on the W -permutohedron P_3 .

6.2. The set $R^\tau(w)$. It turns out, as seen in Remark 5.2, that we can define stable subsets $R^\tau(w)$ of double reduced words. In fact, their combinatorics, involving right τ -moves and generalized d -moves (or left τ -moves and generalized d -moves) is given by the W -permutohedron associated with $\mathbf{g}$. In this subsection, we choose to focus on the right τ -moves, but the same combinatorics could be developed by considering left τ -moves.

Definition 6.2. Let $\mathbf{i}$ be a double word. A *right d^τ -move* (or simply a *d^τ -move*) on $\mathbf{i}$ is given by one of these transformations:

- a generalized d -move;
- a right τ -move.

For every $w \in W$, let $R^\tau(w)$ be the set of all the double words obtained from a word $\mathbf{i} \in R(1, w)$ by composition of d^τ -moves. (The choice of the double word $\mathbf{i}$ does not matter, by Theorem 3.1.)

It is easy to check that this definition coincides with that given in Remark 5.2: the set $R^\tau(w)$ is the union of the disjoint sets $R(w'^{-1}, ww'^{-1})$ for every $w' \leq w \in W$, i.e.,

$$(6.1) \quad R^\tau(w) = \bigcup_{w' \leq w \in W} R(w'^{-1}, ww'^{-1}).$$

(In particular, for every $\mathbf{i} \in R^\tau(w)$ there exists $w' \in W$ such that $\mathbf{i} \in R(w'^{-1}, ww'^{-1})$.)

In the same way, Lemma 3.5 relates generalized d -moves to the set $R(u, v)$ of double reduced words associated with the elements $u, v \in W$; the link between d^τ -moves and the set $R^\tau(w)$ associated with $w \in W$ is the following statement, whose proof is immediate.

Lemma 6.3. A double reduced word $\mathbf{j} \in R^\tau(w)$ can be obtained from a double reduced word $\mathbf{i} \in R^\tau(w')$ by a sequence of d^τ -moves $\delta_{\mathbf{i} \rightarrow \mathbf{j}}^\tau$ if and only if $w' = w$ on W .

Using Proposition 5.10, the link between double reduced words and the set $R^\tau(w)$ given by (6.1) can be strengthened by relating $R^\tau(w)$ to the vertices of the W -permutohedron in the following way. Recall that with $i \in [1, l] \cup [\bar{1}, \bar{l}]$ we can associate the positive letter $|i| \in [1, l]$ given by formula (5.10). We denote, for every double word $\mathbf{i} = i_1 \dots i_n$, by $\mathfrak{R}_{|i_n|}(\mathbf{i})$ (respectively, $\mathfrak{L}_{|i_1|}(\mathbf{i})$) the result of a right (respectively, left) τ -move on $\mathbf{i}$.

Lemma 6.4. We have the following statements for every $w \in W$.

- The set $R^\tau(w)$ is the disjoint union of the labels $R(w'^{-1}, ww'^{-1})$ associated with the vertices w' of the W -permutohedron P_W that are crossed or reached by a $\uparrow_w$ -path.
- Two labeled vertices $R(u, v) \subset R^\tau(w)$ and $R(u', v') \subset R^\tau(w)$ of P_W are related by an edge s_j if and only if there exist double reduced words $\mathbf{i} \in R(u, v)$ and $\mathbf{j} \in R(u', v')$ such that $\mathbf{j} = \mathfrak{R}_j(\mathbf{i})$.

Proof. Let $W_w \subset W$ be the set of elements w' such that $w' \leq w$. Since the map $W_w \rightarrow R^\tau(w) : w' \mapsto R(w'^{-1}, ww'^{-1})$ is a bijection for every $w \in W$, the first statement is just a translation of equation (6.1). So, consider the second statement. It is easily seen that if double reduced words $\mathbf{i} \in R(u, v)$ and $\mathbf{j} \in R(u', v')$ are related by a right τ -move $\mathfrak{R}_j$, then there exist w_1 and w_2 such that $\mathbf{i} \in R(w_1^{-1}, w_0 w_1^{-1})$ and $\mathbf{j} \in R(w_2^{-1}, w_0 w_2^{-1})$, and $w_2 = w_1 s_j$ by Lemma 5.25. Now, if the vertices w'_1 and w'_2 are linked by the edge s_j , we have the equality $w'_2 = w'_1 s_j$. The associated sets of double reduced words are therefore $R(w'_1{}^{-1}, w_0 w'_1{}^{-1})$ and $R(s_j w'_1{}^{-1}, w_0 s_j w'_1{}^{-1})$, and we still use Lemma 5.25 to get some double reduced words $\mathbf{i}$ and $\mathbf{j}$ such that $\mathbf{j} = \mathfrak{R}_j(\mathbf{i})$. $\square$

Example 6.5. Let $\mathbf{g} = A_2$. Figure 18 gives examples of sets $R^\tau(w)$ and their link with the permutohedron P_3 .

FIGURE 18. The sets $R^\tau(s_2 s_1)$ and $R^\tau(w_0)$ when $\mathfrak{g} = A_2$.

6.3. The family $\mathcal{X}^{\tau(w)}$ of cluster $\mathcal{X}$ -varieties related to $G^{1,w}$. The same ideas can be applied at the level of cluster $\mathcal{X}$ -varieties. We recall the notation of Definition 5.7. With any double reduced words $\mathbf{i}, \mathbf{i}' \in R^\tau(w)$ such that there exists a d^τ -move $\delta : \mathbf{i} \rightarrow \mathbf{i}'$, we associate the generalized cluster transformation $\mu_{\mathbf{i} \rightarrow \mathbf{i}'}^\tau : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{i}'}$ given by

- the cluster transformation $\mu_{\mathbf{i} \rightarrow \mathbf{i}'}^\tau$ if δ is a generalized d -move;
- the tropical mutation $\mu_{\diamond_{\mathfrak{R}_i}}^\tau$ if δ is the τ -move $\mathfrak{R}_i$.

We extend this definition to every $\mathbf{i}, \mathbf{j} \in R^\tau(w)$ in the following way. If $\mathbf{i}, \mathbf{j}$ are double words linked by a sequence $\delta_{\mathbf{i} \rightarrow \mathbf{j}}^\tau$ of d^τ -moves, and $\mathbf{i} \rightarrow \mathbf{i}_1 \rightarrow \cdots \rightarrow \mathbf{i}_{n-1} \rightarrow \mathbf{j}$ is the associated chain of elements, we define the map $\mu_{\mathbf{i} \rightarrow \mathbf{j}}^\tau$ as the composition $\mu_{\mathbf{i}_{n-1} \rightarrow \mathbf{j}}^\tau \circ \cdots \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_1}^\tau$.

Since the birational Poisson isomorphism $\mu_{\mathbf{i} \rightarrow \mathbf{j}}^\tau : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{j}}$ associated with such a sequence $\delta_{\mathbf{i} \rightarrow \mathbf{j}}^\tau$ is a generalized cluster transformation for every $\mathbf{i}, \mathbf{j} \in R^\tau(w)$, we get a family of cluster $\mathcal{X}$ -varieties $\mathcal{X}_{[\mathbf{i}]}$ associated with the set $R^\tau(w)$ and related by tropical mutations, which we denote by $\mathcal{X}_w^\tau$. The combinatorics is in fact encoded by the W -permutohedron P_W , as stated by the following result, deduced straightforwardly from Proposition 5.10 and Lemma 6.4.

Lemma 6.6. *Let $w \in W$. Replace each label $w' \in W$ of a vertex of the W -permutohedron P_W by the cluster $\mathcal{X}$ -variety $\mathcal{X}^{w'^{-1}, ww'^{-1}}$. Then we have the following properties.*

- The family $\mathcal{X}_w^\tau$ of cluster $\mathcal{X}$ -varieties contains the cluster $\mathcal{X}$ -variety $\mathcal{X}^{w'^{-1}, ww'^{-1}}$ associated with any $w' \in W$ of the W -permutohedron that can be crossed or reached by a $\uparrow_w$ path.
- For every $i \in [1, l]$, if two vertices related to the labels $\mathcal{X}^{u,v}, \mathcal{X}^{u',v'} \subset \mathcal{X}_w^\tau$ of P_W , respectively, are linked by an edge $s_i \in W$, then there exist two double reduced words $\mathbf{i}, \mathbf{j} \in R^\tau(w_0)$ such that the associated seed $\mathcal{X}$ -tori $\mathcal{X}_{\mathbf{i}}$ and $\mathcal{X}_{\mathbf{j}}$ are related by the right tropical mutation $\mu_{\diamond_{\mathfrak{R}_i}}^\tau$ associated with i .

Finally, we define new evaluation maps to associate the family $\mathcal{X}_w^\tau$ with every double Bruhat cell $G^{w,1}$. With any $w \in W$ and any double word $\mathbf{i} \in R^\tau(w)$, we associate the evaluation map $\text{ev}_{\mathbf{i}}^\tau : \mathcal{X}_{\mathbf{i}} \rightarrow (G^{w,1}, \pi_G)$ by the formula

$$(6.2) \quad \text{ev}_{\mathbf{i}}^\tau(\mathbf{x}) = [\text{ev}_{\mathbf{i}}(\mathbf{x}) \widehat{w'w^{-1}}]_{\leq 0}, \quad \text{for every } \mathbf{x} \in \mathcal{X}_{\mathbf{i}}.$$

Lemma 6.7. *For every $w \in W$ and every $\mathbf{i} \in R^\tau(w)$, the map $\text{ev}_\mathbf{i}^\tau$ is a birational Poisson isomorphism on a Zariski open set of $G^{w,1}$.*

Proof. From [FZ99, Theorem 1.6] we know that the map $g \mapsto [g\hat{w}]_{\leq 0}$ is biregular on $G^{1,w}$. Therefore, the statement is implied by Theorem 3.4 and Proposition 5.6. $\square$

Lemma 6.8. *For every $w \in W$ and every $\mathbf{i}, \mathbf{j} \in R^\tau(w)$, we have $\text{ev}_\mathbf{i}^\tau = \text{ev}_\mathbf{j}^\tau \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}^\tau$.*

Proof. This is derived easily from Theorem 3.7 and Proposition 5.23, involving cluster transformations and tropical mutations, respectively. $\square$

For $\mathfrak{g} = A_2$, the synthesis of this section is provided by Figure 19.

FIGURE 19. The set $R^\tau(w_0)$ and related cluster $\mathcal{X}$ -varieties in the case where $\mathfrak{g} = A_2$.

§7. TWISTED EVALUATIONS, $(w_1, w_2)_v$ -MAPS, AND CLUSTER VARIETIES RELATED TO (G, π_*)

We continue the evaluation for dual Poisson–Lie groups, started in §4. For every $v \in W$, we introduce a new set $D(v)$ of double words, containing the set $R(v, w_0)$. With each double word $\mathbf{i}$ in this set, we associate a *twisted evaluation* $\hat{\text{ev}}_\mathbf{i} : \mathcal{X}_{[\mathbf{i}]} \rightarrow G$, which then generalizes the dual evaluation of §4. These twisted evaluations are obtained by composing the Fock–Goncharov evaluation maps of §3 with new maps called the $(w_1, w_2)_v$ -maps. When $w_1 = w_2 = e$, we rediscover the Evens–Lu morphisms [EL07, Section 5]. In particular, if $v = w_0$, we get the birational Poisson isomorphisms $\hat{\text{ev}}_\mathbf{i} : \mathcal{X}_{[\mathbf{i}]} \rightarrow (BB_-, \pi_*)$ providing the dual Poisson–Lie group with log-canonical coordinates. Then we use the τ -combinatorics of §6 with the truncation maps of §4 to get a family of cluster $\mathcal{X}$ -varieties $\mathcal{X}_w$ associated with each element w of W parameterizing (BB_-, π_*) . In particular, setting $w = e$, we rediscover the result of §4. (The way to relate these cluster $\mathcal{X}$ -varieties will be given in §8 by introducing new birational Poisson isomorphisms on seed $\mathcal{X}$ -tori called saltations.)

7.1. The $(w_1, w_2)_v$ -maps. The following $(w_1, w_2)_v$ -maps generalize the Poisson birational isomorphisms studied in [EL07, §5], which link direct products of double Bruhat cells and double reduced Bruhat cells to Steinberg fibers. We recall the involution $i \mapsto i^*$ on double words given by formula (5.5). Now, let $w \in W$ and $s_{i_1} \dots s_{i_n}$ be a reduced decomposition of w . Then $w^* \in W$ is the element given by $w^* = s_{i_1^*} \dots s_{i_n^*}$. (Using the

(Tits theorem, it is easy to show that the result does not depend on the choice of the decomposition of w into simple reflections.) Recall the notation (4.1) for double reduced Bruhat cells. We denote by $\pi_{G \times G}$ the Poisson product structure on the manifold $G \times G$ induced by the Poisson manifold (G, π_G) .

For every $w_1 \leq v$ and $w_2 \in W$, let $((G, L)^{(w_1, w_2)_v}, \pi_{(w_1, w_2)_v})$ be the quotient of the direct product $(G^{w_1^{*-1}, v w_1^{-1}} \times G^{w_2^{-1}, w_0 w_2^{-1}}, \pi_{G \times G})$ by the $(H \times H)$ -right action given by

$$(7.1) \quad \begin{aligned} (g_1, g_2) \cdot (h_1, h_2) &= (g_1 h_1, h_1^{-1} g_2 h_2) \quad \text{with} \\ g_1 &\in G^{w_1^{*-1}, v w_1^{-1}}, \quad g_2 \in G^{w_2^{-1}, w_0 w_2^{-1}}, \quad \text{and} \quad h_1, h_2 \in H. \end{aligned}$$

In particular, if $w_1 = v$ and $w_2 = e$, then the quotient set $(G, L)^{(w_1, w_2)_v}$ is the set L^{v, w_0} .

Definition 7.1. Suppose $w_1 \leq v$ and $w_2 \in W$. The $(w_1, w_2)_v$ -right map $\varrho_{(w_1, w_2)_v}$ and the $(w_1, w_2)_v$ -left map $\lambda_{(w_1, w_2)_v}$ are defined by the formulas

$$\begin{aligned} \varrho_{(w_1, w_2)_v} : (G, L)^{(w_1, w_2)_v} &\rightarrow L^{w_0, v w_1^{-1}} : (b_1, bH) \mapsto b_1 [b \widehat{w_2 w_0}]_{\leq 0} H, \\ \lambda_{(w_1, w_2)_v} : (G, L)^{(w_1, w_2)_v} &\rightarrow L^{w_1^{*-1}, w_0} : (b_1, bH) \mapsto b_1 [[b \widehat{w_2 w_0}]_{\leq 0}^{\theta} \widehat{w_0}]_{\leq 0}^{\theta} H, \end{aligned}$$

where θ denotes the Cartan involution on G given by (5.9). For every $t \in H$, the $(w_1, w_2)_v$ -maps (or simply (w_1, w_2) -maps if no confusion can occur) are the maps given by

$$\rho_{t, (w_1, w_2)_v} : (G, L)^{(w_1, w_2)_v} \rightarrow G : (b_1, bH) \mapsto \lambda_{(w_1, w_2)_v}(b_1, bH) \widehat{w_0} t \varrho_{(w_1, w_2)_v}(b_1, bH)^{-1}.$$

Example 7.2. The $(w_1, w_2)_v$ -maps associated with $\mathfrak{g} = A_2$ are obtained in the following way. First we describe, for every $v \in W$, the set $W_{\leq v}$ of elements $w_1 \in W$ such that $w_1 \leq v$. They are the following: $W_{\leq e} = \{e\}$, $W_{\leq s_i} = \{e, s_i\}$, $W_{\leq s_i s_j} = \{e, s_i, s_i s_j\}$, and $W_{\leq s_i s_j s_i} = W$, $i, j \in [1, 2]$ being different numbers. Then, suppose $g_1 \in G^{w_1^{*-1}, v w_1^{-1}}$, $b \in G^{1, w_0}$, $g \in G^{s_i, s_i s_j}$, $g' \in G^{s_i s_j, s_i}$, and $c \in G^{w_0, 1}$; the different $(w_1, w_2)_v$ -maps are given by

$$\begin{aligned} \rho_{t, (w_1, e)_v}(g_1, b) &= g_1 b t \widehat{w_0} (g_1 [b \widehat{w_0}]_{\leq 0})^{-1}; \\ \rho_{t, (w_1, s_i)_v}(g_1, g) &= g_1 [[g \widehat{s_j s_i}]_{\leq 0}^{\theta} \widehat{w_0}]_{\leq 0}^{\theta} t \widehat{w_0} (g_1 [g \widehat{s_j s_i}]_{\leq 0})^{-1}; \\ \rho_{t, (w_1, s_i s_j)_v}(g_1, g') &= g_1 [[g' \widehat{s_i}]_{\leq 0}^{\theta} \widehat{w_0}]_{\leq 0}^{\theta} t \widehat{w_0} (g_1 [g' \widehat{s_i}]_{\leq 0})^{-1}; \\ \rho_{t, (w_1, w_0)_v}(g_1, c) &= g_1 [c^{\theta} \widehat{w_0}]_{\leq 0}^{\theta} t \widehat{w_0} (g_1 c)^{-1}. \end{aligned}$$

The way to relate the geometries of (G, π_G) and (G, π_*) is given by the following result, which is deduced directly from Theorem 3.4, Lemma 7.8, and the forthcoming Theorem 7.9.

Proposition 7.3. For every $t \in H$ and $w_1 \leq v, w_2 \in W$, the $(w_1, w_2)_v$ -map $\rho_{t, (w_1, w_2)_v}$ is a birational Poisson isomorphism of $((G, L)^{(w_1, w_2)_v}, \pi_{(w_1, w_2)_v})$ on a Zariski open set of $(F_{t, w_0 v^{-1}}, \pi_*)$.

Remark 7.4. For every $v \in W$, the $(e, e)_v$ -maps are the maps denoted by $\mu\rho$ in [EL07, §5], and Proposition 7.3 below then rephrases [EL07, Corollary 5.11]. Another interesting case is given by $w_1 = v$ and $w_2 = e$; in this case, the related $(w_1, w_2)_v$ -maps are the following maps, strongly related to the dual evaluations of §4:

$$\begin{aligned} \rho_{t, (v, e)_v} : (L^{v, w_0}, \pi_{G \setminus H}) &\rightarrow (F_{t, w_0 v^{-1}}, \pi_*), \\ gH &\mapsto g \widehat{w_0} t [g \widehat{w_0}]_{\leq 0}^{-1}. \end{aligned}$$

Indeed, the relation $\text{ev}_{\mathbf{i}}^{\text{dual}} = \rho_{t, (v, e)_v} \circ \text{ev}_{\mathbf{i}}^{\text{red}}$ is clearly satisfied on $\mathcal{X}_{[\mathbf{i}]\mathfrak{N}}(t)$ for every $t \in H$ and every $\mathbf{i} \in R(v, w_0)$.

7.2. Twisted evaluations on (G, π_*) . As will be stated by Lemma 7.8, the composition of evaluations and reduced evaluations with $(w_1, w_2)_v$ -maps leads to a generalization of dual evaluations, called twisted evaluations and given by formula (7.4). But first, we introduce new sets of double words, denoted $W(w_1, w_2)_v$, and a new operation on seed $\mathcal{X}$ -tori called $\mathcal{X}$ -split, as preliminaries to the definition of twisted evaluations.

Definition 7.5. Suppose $w_1 \leq v$ and $w_2 \in W$. A $(w_1, w_2)_v$ -word $\mathbf{i}$ is a double word linked to a product $\mathbf{i}_1 \mathbf{i}_2$ with $\mathbf{i}_1 \in R(w_1^{-1}, vw_1^{-1})$ and $\mathbf{i}_2 \in R(w_2^{-1}, w_0 w_2^{-1})$ by a sequence of mixed 2-moves. The product $\mathbf{i}_1 \mathbf{i}_2$ is called a *trivial* $(w_1, w_2)_v$ -word and the decomposition $(\mathbf{i}_1, \mathbf{i}_2)$ associated with $\mathbf{i}$ is called the $(w_1, w_2)_v$ -decomposition of $\mathbf{i}$. (For example, every $(e, e)_v$ -word is a trivial $(e, e)_v$ -word.) The set of $(w_1, w_2)_v$ -words will be denoted by $W(w_1, w_2)_v$.

In particular, the set $R(v, w_0)$ is the set of $(v, e)_v$ -words. Let $D(v)$ be the (disjoint) union over $w_1 \leq v, w_2 \in W$ of all the $(w_1, w_2)_v$ -words:

$$D(v) = \bigsqcup_{w_1 \leq v, w_2 \in W} W(w_1, w_2)_v.$$

Therefore, $R(v, w_0) \subset D(v)$ for every $v \in W$.

A complete description of the sets $D(v)$ will be given in Subsection 8.1 via the W -permutohedron. Here is, for the moment, an example with $\mathfrak{g} = A_2$ and $v = s_1$. The associated sets $R(w_1^{-1}, vw_1^{-1})$ and $R(w_2^{-1}, w_0 w_2^{-1})$ are given by the following list:

$$\begin{aligned} R(1, s_1), R(s_2, 1) \text{ and } R(1, w_0), R(s_1, s_1 s_2), R(s_1 s_2, s_1), \\ R(s_2, s_2 s_1), R(s_2 s_1, s_2), R(w_0, 1). \end{aligned}$$

The trivial $(w_1, w_2)_v$ -words are therefore products $\mathbf{i}_1 \mathbf{i}_2$ of some double words:

$$\begin{aligned} \mathbf{i}_1 \in \{1, \bar{2}\} \text{ and } \mathbf{i}_2 \in \{121, 212, \bar{1}1\bar{2}, 1\bar{1}\bar{2}, 12\bar{1}, \bar{1}2\bar{1}, \bar{1}\bar{1}\bar{2}, 1\bar{1}\bar{2}, \bar{2}21, \bar{2}\bar{2}1, 21\bar{2}, \bar{2}\bar{1}\bar{2}, \\ \bar{2}\bar{2}\bar{1}, \bar{2}\bar{2}\bar{1}, \bar{1}\bar{2}\bar{1}, \bar{2}\bar{1}\bar{2}\}. \end{aligned}$$

Now we introduce a new operation on seed $\mathcal{X}$ -tori, which will be useful to describe the oncoming combinatorics related to the following twisted evaluations.

Definition 7.6. A *split* of a seed $\mathbf{I}$ is a pair of seeds $(\mathbf{I}_1, \mathbf{I}_2)$ such that $\mathbf{I}$ is their amalgamated product, i.e., $\mathbf{I} = \mathbf{m}(\mathbf{I}_1, \mathbf{I}_2)$. An associated $\mathcal{X}$ -split is a section of the amalgamation map $\mathbf{m} : \mathcal{X}_{\mathbf{I}_1} \times \mathcal{X}_{\mathbf{I}_2} \rightarrow \mathcal{X}_{\mathbf{I}}$, i.e., a map $\mathfrak{s} : \mathcal{X}_{\mathbf{I}} \rightarrow \mathcal{X}_{\mathbf{I}_1} \times \mathcal{X}_{\mathbf{I}_2}$ such that the product $\mathbf{m} \circ \mathfrak{s}$ gives the identity map on $\mathcal{X}_{\mathbf{I}}$. For every $\mathcal{X}$ -split $\mathfrak{s}$ associated with the decomposition $\mathbf{I} \rightarrow (\mathbf{I}_1, \mathbf{I}_2)$, with any $\mathbf{x} \in \mathcal{X}_{\mathbf{I}}$ we associate the elements $\mathbf{x}_{(1)} \in \mathcal{X}_{\mathbf{I}_1}$ and $\mathbf{x}_{(2)} \in \mathcal{X}_{\mathbf{I}_2}$ given by

$$\mathfrak{s}(\mathbf{x}) = (\mathbf{x}_{(1)}, \mathbf{x}_{(2)}).$$

Figure 20 describes the split associated with the decomposition $2\bar{2} \rightarrow (2, \bar{2})$ in the case where $\mathfrak{g} = A_3$. We stress that, since the amalgamated product is not an isomorphism

FIGURE 20. The split $\mathfrak{s} : \Gamma_{\mathfrak{g}}(2\bar{2}) \mapsto (\Gamma_{\mathfrak{g}}(2), \Gamma_{\mathfrak{g}}(\bar{2}))$ for $\mathfrak{g} = A_3$.

of seed $\mathcal{X}$ -tori, different $\mathcal{X}$ -splits can be associated with a given split of seed. Indeed, let $\mathbf{i}, \mathbf{i}_1, \mathbf{i}_2$ be double words such that $\mathbf{i} = \mathbf{i}_1 \mathbf{i}_2$; if $\mathfrak{s}$ is an $\mathcal{X}$ -split associated with the decomposition $\mathbf{i} \rightarrow (\mathbf{i}_1, \mathbf{i}_2)$, then for every $\mathbf{t} \in \mathcal{X}_1$ the following map $\mathfrak{s}_{\mathbf{t}}$ is also an $\mathcal{X}$ -split:

$$\begin{aligned} \mathfrak{s}_{\mathbf{t}}(\mathbf{x}) &= (\mathbf{m}(\mathbf{x}_{(1)}, \mathbf{t}), \mathbf{m}(\mathbf{t}^{-1}, \mathbf{x}_{(2)})), \text{ where } \mathfrak{s}(\mathbf{x}) = (\mathbf{x}_{(1)}, \mathbf{x}_{(2)}) \\ \text{and } \mathbf{t}^{-1} &= (t_1^{-1}, \dots, t_l^{-1}) \text{ if } \mathbf{t} = (t_1, \dots, t_l). \end{aligned}$$

However, in what follows, nearly each time we will need an $\mathcal{X}$ -split, this freedom of choice will not affect the related result. We are now ready to define the twisted evaluations on (G, π_*) .

Definition 7.7. Let $w_1 \leq v, w_2 \in W$, let $\mathbf{i} = \mathbf{i}_1 \mathbf{i}_2$ be a trivial $(w_1, w_2)_v$ -word, and let $\mathfrak{s}$ be an $\mathcal{X}$ -split associated with the decomposition $\mathbf{i} \rightarrow (\mathbf{i}_1, \mathbf{i}_2)$. We define the *left* and *right evaluations* $\text{ev}_{\mathbf{i}}^{\mathfrak{s}}, \text{ev}_{\mathbf{i}}^{\mathfrak{R}} : \mathcal{X}_{\mathbf{i}} \rightarrow G$ by the formulas

$$(7.2) \quad \begin{aligned} \text{ev}_{\mathbf{i}}^{\mathfrak{R}}(\mathbf{x}) &= \text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)})[\text{ev}_{\mathbf{i}_2}^{\text{red}}(\mathbf{x}_{(2)})\widehat{w_2 w_0}]_{\leq 0} \quad \text{and} \\ \text{ev}_{\mathbf{i}}^{\mathfrak{s}}(\mathbf{x}) &= \text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)})[(\text{ev}_{\mathbf{i}_2}^{\mathfrak{R}}(\mathbf{x}_{(2)}))^{\theta} \widehat{w_0}]_{\leq 0}^{\theta}, \end{aligned}$$

where the reduced evaluations ev^{red} are the evaluation maps given in §4. (Note that these left and right evaluations do not depend on the choice of the $\mathcal{X}$ -split $\mathfrak{s}$.) These maps are extended to every $\mathbf{i} \in D(v)$ by setting

$$(7.3) \quad \text{ev}_{\mathbf{i}}^{\mathfrak{s}} = \text{ev}_{\mathbf{i}_1 \mathbf{i}_2}^{\mathfrak{s}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_1 \mathbf{i}_2} \quad \text{and} \quad \text{ev}_{\mathbf{i}}^{\mathfrak{R}} = \text{ev}_{\mathbf{i}_1 \mathbf{i}_2}^{\mathfrak{R}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_1 \mathbf{i}_2}.$$

Finally, for every $v \in W$ and $\mathbf{i} \in D(v)$, we define the *twisted evaluation*

$$(7.4) \quad \begin{aligned} \widehat{\text{ev}}_{\mathbf{i}} : \mathcal{X}_{[\mathbf{i}]\mathfrak{R}} &\rightarrow G : \mathbf{x} \mapsto \text{ev}_{\mathbf{i}}^{\mathfrak{s}}(\mathbf{x}) x_1(\mathbf{x}(\mathfrak{R}))\widehat{w_0} \text{ev}_{\mathbf{i}}^{\mathfrak{R}}(\mathbf{x})^{-1}, \\ \text{where } \text{ev}_1(\mathbf{x}(\mathfrak{R})) &= \prod_{j=1}^l H^j(x_{(N^j(\mathbf{i}))}). \end{aligned}$$

The relationships among twisted evaluations, (w_1, w_2) -maps, and the Fock–Goncharov evaluation maps are given by the following lemma, checked straightforwardly.

Lemma 7.8. Let $w_1 \leq v, w_2 \in W$, let $\mathbf{i}$ be a trivial $(w_1, w_2)_v$ -word, and let $\mathfrak{s}$ be an $\mathcal{X}$ -split associated with the $(w_1, w_2)_v$ -decomposition $\mathbf{i} \rightarrow (\mathbf{i}_1, \mathbf{i}_2)$. Then:

$$\begin{aligned} \text{ev}_{\mathbf{i}}^{\mathfrak{R}}(\mathbf{x}) &= \varrho_{(w_1, w_2)_v}(\text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)}), \text{ev}_{\mathbf{i}_2}^{\text{red}}(\mathbf{x}_{(2)})), \\ \text{ev}_{\mathbf{i}}^{\mathfrak{s}}(\mathbf{x}) &= \lambda_{(w_1, w_2)_v}(\text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)}), \text{ev}_{\mathbf{i}_2}^{\text{red}}(\mathbf{x}_{(2)})), \\ \widehat{\text{ev}}_{\mathbf{i}}(\mathbf{x}) &= \rho_{\text{ev}_1(\mathbf{x}(\mathfrak{R})), (w_1, w_2)_v}(\text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)}), \text{ev}_{\mathbf{i}_2}^{\text{red}}(\mathbf{x}_{(2)})). \end{aligned}$$

Finally we give an analog of Theorem 3.4 for (G, π_*) .

Theorem 7.9. For every $v \in W$, $t \in H$, and $\mathbf{i} \in D(v)$, the restriction $\widehat{\text{ev}}_{\mathbf{i}} : \mathcal{X}_{[\mathbf{i}]\mathfrak{R}}(t) \rightarrow (F_{t, w_0 v^{-1}}, \pi_*)$ is a Poisson birational isomorphism onto a Zariski open set $F_{t, w_0 v^{-1}}^0$ of $F_{t, w_0 v^{-1}}$.

Proof. We are going to use the following three lemmas, the first one being crucial to our construction.

Lemma 7.10 ([EL07, Corollary 5.11]). Let $v \in W$. The $(e, e)_v$ -map associated with every $t \in H$ is a birational Poisson isomorphism of $((G, L)^{(e, e)_v}, \pi_{(e, e)_v})$ onto a Zariski open set of $(F_{t, w_0 v^{-1}}, \pi_*)$.

Lemma 7.11. Suppose $t \in H$, $v \in W$, $\mathbf{i}$ is a $(e, e)_v$ -word, and $\mathfrak{s}$ is an $\mathcal{X}$ -split associated with the $(e, e)_v$ -decomposition $\mathbf{i} \rightarrow (\mathbf{i}_1, \mathbf{i}_2)$. The following evaluation map is a birational Poisson isomorphism onto a Zariski open set $F_{t, w_0 v^{-1}}^0 \subset F_{t, w_0 v^{-1}}$:

$$\widehat{\text{ev}}_{t, \mathbf{i}} : \mathcal{X}_{\mathbf{i}}^{\text{red}} \rightarrow (F_{t, w_0 v^{-1}}, \pi_*) : \mathbf{x} \mapsto \text{ev}_{\mathbf{i}}^{\text{red}}(\mathbf{x})\widehat{w_0} t (\text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)})[\text{ev}_{\mathbf{i}_2}^{\text{red}}(\mathbf{x}_{(2)})\widehat{w_0}]_{\leq 0})^{-1}.$$

Lemma 7.11 is deduced easily from Lemma 7.10 and Theorem 3.7.

Lemma 7.12. *The map $b \mapsto [b\widehat{w}_0]_{\leq 0}^\theta$ is an involution on G^{e,w_0} .*

Lemma 7.12 is well known; to prove it, we need to recall that the map θ is an involution and use successively the facts that $\widehat{w}_0^\theta = \widehat{w}_0^{-1}$, $b^\theta \in G^{w_0,e}$, and $\widehat{w}_0^{-1}[b\widehat{w}_0]_+^\theta \widehat{w}_0 \in N$ to get

$$b^\theta = (b\widehat{w}_0)^\theta \widehat{w}_0 = [(b\widehat{w}_0)^\theta \widehat{w}_0]_{\leq 0} = [[b\widehat{w}_0]_{\leq 0}^\theta \widehat{w}_0]_{\leq 0}.$$

Now we choose $\mathbf{x}(\mathfrak{R}) \in \mathcal{X}_1$ such that $\text{ev}_1(\mathbf{x}(\mathfrak{R})) = t$. Lemma 7.11 and Lemma 7.12 show that if a $(w_1, w_2)_v$ -word $\mathbf{i} \in D(v)$ is such that $w_1 = w_2 = e$, then the evaluation $\widehat{\text{ev}}_{t,\mathbf{i}}$ gives the restriction of the map $\widehat{\text{ev}}_{\mathbf{i}}$ to the set $\mathcal{X}_{[\mathbf{i}]\mathfrak{R}}(t)$. The general case will be deduced from Theorem 8.12 by noticing that the associated map $\widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{j}}$, defined in §8, is a birational Poisson isomorphism for all double words $\mathbf{i}, \mathbf{j} \in D(v)$. $\square$

In particular, Theorem 4.11 is deduced from Theorem 7.9 because $\widehat{\text{ev}}_{\mathbf{i}}$ and $\text{ev}_{\mathbf{i}}^{\text{dual}}$ coincide for every $\mathbf{i} \in R(v, w_0)$, that is, for every $(w_1, w_2)_v$ -word $\mathbf{i}$ such that $w_1 = v$ and $w_2 = e$. Finally, we recall that for every double word $\mathbf{i}$, the set $\mathcal{X}_{\mathbf{i}}^{\text{dual}} \subset \mathcal{X}_{[\mathbf{i}]\mathfrak{R}}$ is such that the elements of the set of variables $\mathbf{x}(\mathfrak{R})$ are pairwise disjoint. Thus, we get the following corollary to the second decomposition of (2.11) and Theorem 7.9.

Corollary 7.13. *For every $\mathbf{i} \in D(w_0)$, the map $\widehat{\text{ev}}_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}}^{\text{dual}} \rightarrow (BB_-, \pi_*)$ is a Poisson birational isomorphism on a Zariski open set of BB_- .*

Remark 7.14. The same construction can be done (with the same results) if G is not of adjoint type, but simply connected. In this case, the twisted evaluations we have to consider are as follows:

$$\begin{aligned} \widehat{\text{ev}}_{\mathbf{i}} : \mathcal{X}_{[\mathbf{i}]\mathfrak{R}} &\rightarrow G : \mathbf{x} \mapsto \text{ev}_{\mathbf{i}}^{\mathcal{S}}(\mathbf{x}) \, x_1(\mathbf{x}(\mathfrak{R})) \widehat{w}_0 \, \text{ev}_{\mathbf{i}}^{\mathfrak{R}}(\mathbf{x})^{-1}, \\ \text{where } x_1(\mathbf{x}(\mathfrak{R})) &= \prod_{j=1}^l H_j(x_{(N^j(\mathbf{i}))}), \end{aligned}$$

the generators $H_j(\cdot)$ being given by (2.3).

7.3. τ -combinatorics and cluster $\mathcal{X}$ -varieties related to (G, π_*) . We now relate twisted evaluations by cluster transformations to get Poisson parameterizations related to (G, π_*) by cluster $\mathcal{X}$ -varieties. This is achieved by mixing the truncation maps of §4 with the τ -combinatorics developed in §6. Then we get a family of $\mathcal{X}$ -varieties $\mathcal{X}_w$ indexed by the Weyl group W of G , evaluating the dual Poisson–Lie group (BB_-, π_*) .

7.3.1. Double reduced words, the set $D_{w_1}(v)$, and the W -permutohedron associated with $\mathfrak{g}$. As was done in §3, we start with the study of combinatorics on double words before considering the related birational Poisson isomorphisms on seed $\mathcal{X}$ -tori. We fix $w_1 \leq v \in W$, and denote by $D_{w_1}(v)$ the set of $(w_1, w_2)_v$ -words for every $w_2 \in W$. Therefore, the set $D(v)$ is the union over all $w_1 \leq v \in W$ of the sets $D_{w_1}(v)$,

$$(7.5) \quad D_{w_1}(v) = \bigcup_{w_2 \in W} W(w_1, w_2)_v \quad \text{and} \quad D(v) = \bigcup_{w_1 \leq v} D_{w_1}(v).$$

The following result is clear from Definition 7.5. It uses amalgamation to relate the sets $D_{w_1}(v)$ to the set $R^\tau(w_0)$ already studied in §6.

Lemma 7.15. *The set $D_{w_1}(v)$ is the set of double words obtained by composition of mixed 2-moves from an amalgamation of a double reduced word $\mathbf{i}_1 \in R(w_1^{\star-1}, vw_1^{-1})$ with any double reduced word $\mathbf{i}_2 \in R^\tau(w_0)$.*

Therefore, we can use the fact that the set $R^\tau(w)$ was described in §6 via the W -permutohedron to relate the sets $W(w_1, w_2)_v$ to $D_{w_1}(v)$, for every $w_1 \leq v \in W$.

Lemma 7.16. *We have the following statements for every $v, w_1 \in W$ such that $w_1 \leq v$.*

- *The set $D_{w_1}(v)$ is the disjoint union of the labels $W(w_1, w'^{-1})_v$ associated with the vertices w' of the W -permutohedron P_W that are crossed or reached by a $\uparrow_v$ -path.*
- *Two labeled vertices $W(w_1, w^{-1})_v, W(w_1, w'^{-1})_v \subset D_{w_1}(v)$ of P_W are linked by an edge s_j if and only if there exist double reduced words $\mathbf{i} \in W(w_1, w^{-1})_v$ and $\mathbf{j} \in W(w_1, w'^{-1})_v$ such that $\mathbf{j} = \mathfrak{R}_j(\mathbf{i})$.*

Proof. It suffices to apply Lemma 6.4 and the fact that the map $R(w'^{-1}, vw'^{-1}) \mapsto W(w_1, w'^{-1})_v$ gives a bijection from $R^\tau(v)$ to $D_{w_1}(v)$. $\square$

Example 7.17. We still take $\mathfrak{g} = A_2$. From Figure 18, we get the W -permutohedron on the right in Figure 19. We stress that this picture is valid for *any* $w_1 \in W$.

FIGURE 21. From the set $R^\tau(w_0)$ to the set $D_{w_1}(w_0)$ when $\mathfrak{g} = A_2$.

7.3.2. The cluster $\mathcal{X}$ -varieties $\mathcal{X}_{w_1 \leq v}(t)$ associated to the set $D_{w_1}(v)$. Now we consider the related cluster transformations and associate a right truncated cluster $\mathcal{X}$ -variety $\mathcal{X}_{w_1 \leq v}$ with each set $D_{w_1}(v)$. Let $w_1 \leq v \in W$. With any d^τ -move δ linking two double words $\mathbf{i}, \mathbf{i}' \in D_{w_1}(v)$, we associate the cluster transformation $\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{i}']_{\mathfrak{R}}} : \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}} \rightarrow \mathcal{X}_{[\mathbf{i}']_{\mathfrak{R}}}$ such that

$$\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{i}']_{\mathfrak{R}}} \circ \mathbf{t}_{\mathfrak{R}}(t) = \mathbf{t}_{\mathfrak{R}}'(t) \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}'}^\tau,$$

where, as in Subsection 6.3, the generalized cluster $\mu_{\mathbf{i} \rightarrow \mathbf{j}}^\tau$ is

- the cluster transformation $\mu_{\mathbf{i} \rightarrow \mathbf{i}'}$ if δ is a generalized d -move;
- the tropical mutation $\mu_{\diamond_{\mathfrak{R}_j}^{i_{\ell(w)}}}$ if δ is the τ -move $\mathfrak{R}_j$.

As usual, we extend this definition to every $\mathbf{i}, \mathbf{j} \in D_{w_1}(v)$: if $\mathbf{i}, \mathbf{j}$ are double words linked by a sequence $\delta_{\mathbf{i} \rightarrow \mathbf{j}}^\tau$ of d^τ -moves, and if $\mathbf{i} \rightarrow \mathbf{i}_1 \rightarrow \dots \rightarrow \mathbf{i}_{n-1} \rightarrow \mathbf{j}$ is the associated chain of elements, we define the map $\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{j}]_{\mathfrak{R}}} : \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}} \rightarrow \mathcal{X}_{[\mathbf{j}]_{\mathfrak{R}}}$ as the composition

$$\mu_{[\mathbf{i}_{n-1}]_{\mathfrak{R}} \rightarrow [\mathbf{j}]_{\mathfrak{R}}} \circ \dots \circ \mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{i}_1]_{\mathfrak{R}}}.$$

We denote by $\mathcal{X}_{w_1 \leq v}$, or simply $\mathcal{X}_w$ if $v = w_0$, the cluster $\mathcal{X}$ -variety associated with the set $D_{w_1}(v)$. Moreover, equation (4.5) shows that the cluster $\mathcal{X}$ -variety $\mathcal{X}_{w_1 \leq v}$ can be Poisson stratified into the disjoint union over H of the cluster $\mathcal{X}$ -varieties $\mathcal{X}_{w_1 \leq v}(t)$:

$$\mathcal{X}_{w_1 \leq v} = \bigsqcup_{t \in H} \mathcal{X}_{w_1 \leq v}(t).$$

We can relate the cluster $\mathcal{X}$ -variety $\mathcal{X}_{w_1 \leq v}(t)$ to the W -permutohedron as follows. As a preliminary, to every set $W(w_1, w_2)_v$ we attach a cluster $\mathcal{X}$ -variety, denoted $\mathcal{X}_{(w_1, w_2)_v}$ and obtained as a result of the following amalgamation:

$$\mathfrak{m} : \mathcal{X}^{w_1^{*-1}, vw_1^{-1}} \times \mathcal{X}^{w_2^{-1}, vw_2^{-1}} \rightarrow \mathcal{X}_{(w_1, w_2)_v}.$$

The first lemma is deduced directly from the definition of the amalgamated product and the definition of the set $W(w_1, w_2)_v$.

Lemma 7.18. *Let $w_1 \leq v, w_2 \in W$. The cluster $\mathcal{X}$ -variety $\mathcal{X}_{(w_1, w_2)_v}$ contains the seed $\mathcal{X}$ -torus $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ associated with any double word $\mathbf{i} \in W(w_1, w_2)_v$.*

Lemma 7.19. *Replace each label $w' \in W$ of a vertex of the W -permutohedron P_W by the cluster $\mathcal{X}$ -variety $\mathcal{X}^{w'^{-1}, w_0 w'^{-1}}$. Then:*

- the cluster $\mathcal{X}$ -variety $\mathcal{X}_w$ indexed by any element $w \in W$ contains the seed $\mathcal{X}$ -torus $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ associated with any double word $\mathbf{i} \in D_w(w_0)$;
- the cluster $\mathcal{X}$ -variety $\mathcal{X}_{w_1 \leq v}$ is the image of the cluster $\mathcal{X}$ -variety $\mathcal{X}_{(w_1, w')_v}$ under the right truncation map $\mathfrak{t}_{\mathfrak{R}}$ for every $w' \in W$;
- for every $i \in [1, l]$, if two vertices labeled by the cluster $\mathcal{X}$ -varieties $\mathcal{X}^{u, v}, \mathcal{X}^{u', v'} \subset \mathcal{X}_{w_0}$ of P_W , respectively, are related by the edge $s_i \in W$, then there exist two double reduced words $\mathbf{i}, \mathbf{j} \in D_{w_1}(v)$ such that the associated seed $\mathcal{X}$ -tori $\mathcal{X}_{\mathbf{i}}$ and $\mathcal{X}_{\mathbf{j}}$ are related by the right tropical mutation associated with i and denoted by $\mu_{\diamond_i}^{\mathfrak{R}}$.

Proof. It is clear that the seed $\mathcal{X}$ -tori obtained by right truncation from the seed $\mathcal{X}$ -tori related by a right tropical mutation are the same. Therefore, the cluster $\mathcal{X}$ -varieties included in the family of cluster $\mathcal{X}$ -varieties and described by the W -permutohedron in §6 are sent to one and the same truncated cluster $\mathcal{X}$ -variety. Therefore, it suffices to apply Lemma 6.6, because right tropical mutations and right truncations commute with left amalgamation; indeed, we change each cluster $\mathcal{X}$ -variety label $\mathcal{X}^{u, v}$ of Lemma 6.6 into the amalgamated cluster $\mathcal{X}$ -variety $\mathcal{X}_{(w_1, w_2)_v}$. $\square$

This result is illustrated by Figure 19 in the case where $\mathfrak{g} = A_2$.

FIGURE 22. Truncation maps and the cluster $\mathcal{X}$ -variety $\mathcal{X}_{w_1}(t)$ when $\mathfrak{g} = A_2$.

7.3.3. Twisted evaluation maps relating $\mathcal{X}_{w_1 \leq v}(t)$ to $(F_{t,v}, \pi_*)$. Finally, we use twisted evaluation maps to get, for every $v \in W$, a family of truncated cluster $\mathcal{X}$ -varieties

$$\{\mathcal{X}_{w_1 \leq v}(t), w_1 \leq v \in W\}$$

parameterizing the Poisson submanifold $(F_{t,w_0 v^{-1}}, \pi_*)$. In particular, the family of truncated cluster $\mathcal{X}$ -varieties $\{\mathcal{X}_{w_1 \leq w_0}, w_1 \in W\}$, also denoted by $\{\mathcal{X}_{w_1}, w_1 \in W\}$, will parameterize the dual Poisson-Lie group (BB_-, π_*) .

Proposition 7.20. *For every $t \in H$ and $w_1 \leq v \in W$ and all double words $\mathbf{i}, \mathbf{i}' \in D_{w_1}(v)$, we have $\widehat{\text{ev}}_{\mathbf{i}} = \widehat{\text{ev}}_{\mathbf{i}'} \circ \mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{i}']_{\mathfrak{R}}}$.*

Proof. We introduce relatives of the left and right evaluations of (7.2) in the following way. Suppose $w_1 \leq v$ and $w_2 \in W$, and let $\mathfrak{s}$ be an $\mathcal{X}$ -split associated with the $(w_1, w_2)_v$ -decomposition $\mathbf{i} \mapsto (\mathbf{i}_1, \mathbf{i}_2)$. We define maps $\overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{L}}, \overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{R}} : \mathcal{X}_{\mathbf{i}} \rightarrow G$ and $\overline{\text{ev}}_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}} \times \mathcal{X}_{\mathbf{i}_1} \rightarrow G$ by the following formulas: we start with a definition on the trivial double words,

$$\begin{aligned} \overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{R}}(\mathbf{x}) &= \text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)})[\text{ev}_{\mathbf{i}_2}(\mathbf{x}_{(2)})\widehat{w_2 w_0}]_{\leq 0}, \\ \overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{L}}(\mathbf{x}) &= \text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)})[(\overline{\text{ev}}_{\mathbf{i}_2}^{\mathfrak{R}}(\mathbf{x}_{(2)}))^{\theta}\widehat{w_0}]_{\leq 0}^{\theta}, \end{aligned}$$

before using the same idea as in (7.3) to extend this definition for every $\mathbf{i} \in D_{w_1}(v)$ by setting $\overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{L}} = \overline{\text{ev}}_{\mathbf{i}_1 \mathbf{i}_2}^{\mathfrak{L}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_1 \mathbf{i}_2}$ and $\overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{R}} = \overline{\text{ev}}_{\mathbf{i}_1 \mathbf{i}_2}^{\mathfrak{R}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_1 \mathbf{i}_2}$; finally, we introduce

$$\overline{\text{ev}}_{\mathbf{i}}(\mathbf{x}, \mathbf{t}) \mapsto \overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{L}}(\mathbf{x}) \text{ev}_{\mathbf{1}}(\mathbf{t}) \widehat{w_0} \overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{R}}(\mathbf{x})^{-1}.$$

Then, using Lemma 6.8 and the extension of the kind (7.3) just described, we see that the following identities are satisfied for all double words $\mathbf{i}, \mathbf{i}' \in D_{w_1}(v)$:

$$\overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{R}} = \overline{\text{ev}}_{\mathbf{i}'}^{\mathfrak{R}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}'}^{\tau}, \quad \overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{L}} = \overline{\text{ev}}_{\mathbf{i}'}^{\mathfrak{L}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}'}^{\tau}, \quad \text{and} \quad \overline{\text{ev}}_{\mathbf{i}} = \overline{\text{ev}}_{\mathbf{i}'}(\mu_{\mathbf{i} \rightarrow \mathbf{i}'}^{\tau}(\mathbf{x}), \mathbf{t}).$$

Moreover, it is easy to show that the element $\overline{\text{ev}}_{\mathbf{i}}(\mathbf{x}, \mathbf{t})$ does not depend on x_j for $j \in I_0^{\mathfrak{R}}(\mathbf{i})$. Now, we recall Remark 5.9: the tropical mutations associated with right τ -moves affect only the variables associated with right outlets. Therefore,

$$\overline{\text{ev}}_{\mathbf{i}}(\mathbf{x}, \mathbf{t}) = \overline{\text{ev}}_{\mathbf{i}'}(\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{i}']_{\mathfrak{R}}}(\mathbf{x}), \mathbf{t}).$$

Finally, it is clear that $\widehat{\text{ev}}_{\mathbf{i}}(\mathbf{x}) = \overline{\text{ev}}_{\mathbf{i}}(\mathbf{x}, \mathbf{x}(\mathfrak{R}))$ for every $\mathbf{i} \in D_{w_1}(v)$. This proves the proposition, because, by equation (4.5), the cluster variables belonging to $\mathbf{x}(\mathfrak{R})$ are invariant under mutations. $\square$

For every $t \in H$ and every $w_1 \leq v \in W$, we have attached the cluster $\mathcal{X}$ -variety $\mathcal{X}_{w_1 \leq v}(t)$ to the Poisson submanifold $(F_{t,v}, \pi_*)$. We denote by $\mathcal{X}_{[\cdot]_{\mathfrak{R}}}(t)$ the map that takes any double word $\mathbf{i}$ to the corresponding seed $\mathcal{X}$ -torus $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}(t)$. Applied to a seed $\mathbf{i}$, $\mathcal{X}_{[\cdot]_{\mathfrak{R}}}(t)$ is therefore the composition of the map $\mathcal{X}$ and the truncation map $\mathbf{t}_{\mathbf{i}_{\mathfrak{R}}}(t)$. Then we summarize Theorem 7.9 and Proposition 7.20 by (abusively) saying that there exists a Poisson birational isomorphism $\widehat{\text{ev}}_{w_1 \leq v} : \mathcal{X}_{w_1 \leq v} \rightarrow (F_{t,v}, \pi_*)$. Thus, we get the following commutative diagram, which generalizes Figure 12. (The link between these truncated cluster varieties $\mathcal{X}_{w_1 \leq v}(t)$, as well as the way to link different twisted evaluations, will be given in §8, via the introduction of saltation maps.)

Remark 7.21. This description can be adapted if G is not adjoint, but simply connected. We consider the evaluation maps $\widehat{\text{ev}}_{\mathbf{i}}$ given in Remark 7.14, which also lead to birational Poisson isomorphisms related to (G, π_*) when G is simply connected. Then we introduce the following Poisson covering on right truncated seed $\mathcal{X}$ -tori, by mimicking formula (2.3):

$$\mathfrak{p}_{\mathcal{X}} : \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}} \rightarrow \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}} : x_{\mathfrak{p}_{\mathcal{X}}(i)} = \begin{cases} \prod_{j=1}^l x_{\binom{a_{ij}}{N^j(\mathbf{i})}} & \text{if } i \in I_0^{\mathfrak{R}}(\mathbf{i}), \\ x_i & \text{otherwise.} \end{cases}$$

FIGURE 23. The cluster $\mathcal{X}$ -variety $\mathcal{X}_{w_1 \leq v}(t)$ associated to $(F_{t, w_0 v^{-1}}, \pi_*)$.

But let us notice that the seed $\mathcal{X}$ -tori $\mathcal{X}_{[i]_{\mathfrak{R}}}(t)$ are generically not preserved by the covering map $\mathfrak{p}_{\mathcal{X}}$. Let $\mathfrak{p}_{\mathcal{X}}(t) \in H$ be the element such that $\mathfrak{p}_{\mathcal{X}} : \mathcal{X}_{[i]_{\mathfrak{R}}}(t) \rightarrow \mathcal{X}_{[i]_{\mathfrak{R}}}(\mathfrak{p}_{\mathcal{X}}(t))$. Let $w_1 \leq v$ and $\mathbf{i}, \mathbf{j} \in D_{w_1}(v)$. An analog of Proposition 7.20 is then given by the commutativity of the following diagram:

§8. SALTATIONS AND CLUSTER $\mathcal{X}$ -VARIETIES FOR (BB_-, π_*)

We relate the twisted evaluations $\widehat{\text{ev}}_i : \mathcal{X}_{[i]_{\mathfrak{R}}} \rightarrow (G, \pi_*)$ of §7 by composition of cluster transformations to new birational Poisson isomorphisms called saltations. As a corollary, we get a parameterization of the dual Poisson–Lie group (BB_-, π_*) by a family of cluster $\mathcal{X}$ -varieties; moreover, the cluster $\mathcal{X}$ -varieties of this family are related by saltations indexed by the 1-skeleton of the W -permutohedron P_W .

8.1. Various moves on the set $D(v)$. We refine the description of the set $D(v)$ for every $v \in W$. For that, we enlarge the combinatorics on double words by introducing *dual-moves* and mix them with the d^τ -moves described in §6. We start by adding a variation of τ -moves, involving the involution $i \mapsto i^*$ on the set $[1, l]$ given by (5.5).

Definition 8.1. Let $\mathbf{i} = i_1 \dots i_n$ be a double word. We denote by $\mathfrak{R}_{|i_n|}^*(\mathbf{i})$, or simply $\mathfrak{R}^*(\mathbf{i})$ if no confusion may occur, the double word obtained by changing the last letter i of $\mathbf{i}$ into $\overline{i}^*$:

$$\mathfrak{R}_{i_n}^*(\mathbf{i}) = i_1 \dots i_{n-1} \overline{i_n}^* \quad \text{if } i_n \in [1, l],$$

$$\mathfrak{R}_{\overline{i_n}}^*(\mathbf{i}) = i_1 \dots i_{n-1} \overline{i_n}^* \quad \text{if } i_n \in [\overline{1}, \overline{l}].$$

The map $\mathbf{i} \mapsto \mathfrak{R}^*(\mathbf{i})$ is called a *right τ^* -move* on $\mathbf{i}$. Because the maps $i \mapsto i^*$ and $i \mapsto \overline{i}$ are involutions, it is clear that every map $\mathfrak{R}_j^*$ is an involution on the set of double words.

A d^{τ^*} -move is then given by one of the following transformations:

- a generalized d -move;
- a τ^* -move.

For every $w \in W$, let $R^{\tau^*}(w)$ be the set of all double words obtained from a double reduced word $\mathbf{i} \in R(1, w)$ by composition of d^{τ^*} -moves. (The choice of the double word $\mathbf{i}$ does not matter, by Theorem 3.1.)

Example 8.2. If $\mathfrak{g} = A_2$, the action of a τ^* -move on the double reduced words $\mathbf{i}_1 = 121$, $\mathbf{i}_2 = \bar{2}12$, and $\mathbf{i}_3 = \bar{2}\bar{1}1$ is as follows:

$$\mathfrak{R}_1^*(\mathbf{i}_1) = 12\bar{2}, \quad \mathfrak{R}_2^*(\mathbf{i}_2) = \bar{2}1\bar{1}, \quad \text{and} \quad \mathfrak{R}_1^*(\mathbf{i}_3) = \bar{2}\bar{1}\bar{2}.$$

As for every set $R^{\tau}(w)$, the set $R^{\tau^*}(w)$, $w \in W$, is described easily via the W -permutohedron. Indeed, we have the following analog of Lemma 6.4 for the set $R^{\tau^*}(w)$. It is proved in the same way.

Lemma 8.3. *The following statements are valid for every $w \in W$.*

- The set $R^{\tau^*}(w)$ is the disjoint union of the labels $R(w'^{\star-1}, ww'^{-1})$ associated with the vertices w' of the W -permutohedron P_W that are crossed or reached by a $\uparrow_w$ -path.
- Two labeled vertices $R(u, v) \subset R^{\tau^*}(w)$ and $R(u', v') \subset R^{\tau^*}(w)$ of P_W are related by an edge s_i if and only if there exist double reduced words $\mathbf{i} \in R(u, v)$ and $\mathbf{j} \in R(u', v')$ such that $\mathbf{j} = \mathfrak{R}^*(\mathbf{i})$.

Here is a variation of Example 6.5 that has described the sets $R^{\tau}(s_2s_1)$ and $R^{\tau}(w_0)$ when $\mathfrak{g} = A_2$: Figure 18 is replaced by Figure 24. Now, we use the previous τ^* -moves and

FIGURE 24. The sets $R^{\tau^*}(s_2s_1)$ and $R^{\tau^*}(w_0)$ in the case where $\mathfrak{g} = A_2$.

the involution $\square$ given in Subsection 5.2 to define new moves on double words, besides the generalized dn -moves and τ -moves; they are called the dual-moves.

Definition 8.4. Let $\mathbf{i} \in R(1, w_0) \cup R(w_0, 1)$ be a positive or negative reduced word associated with w_0 , and let $\mathbf{j}$ be a double word. The *dual-move* Δ_j associated with the last letter of $\mathbf{j}$ transforms the product $\mathbf{j}\mathbf{i}$ into the following double word:

$$(8.1) \quad \Delta_j : \mathbf{j}\mathbf{i} \mapsto \mathfrak{R}_j^*(\mathbf{j}) \mathbf{i}^\square.$$

Since the right τ^* -moves and the map $\square$ are involutions, the dual-moves are in fact involutions on the set of such products $\mathbf{j}\mathbf{i}$. Let $v \in W$, and let $\mathbf{i} \in D(v)$ be a double word.

A $\hat{d}$ -move on $\mathbf{i}$ is one of the following transformations (in particular, every $\hat{d}$ -move is a d^τ -move when $v = 1$):

- a d^τ -move;
- a dual-move Δ_i .

As an example, let us keep the notation of Example 8.2 and consider the double words $\mathbf{j}_1 = \mathbf{i}_1\mathbf{j}$, $\mathbf{j}_2 = \mathbf{i}_2\mathbf{j}$ and $\mathbf{j}_3 = \mathbf{i}_3\mathbf{j}$, with $\mathbf{j} = \overline{121}$. Since $\mathbf{j}^\square = 121$, the action of dual-moves on these double words looks like this:

$$\Delta_2(\mathbf{j}_1) = 12\overline{2} \ 121, \quad \Delta_1(\mathbf{j}_2) = \overline{2}1\overline{1} \ 121, \quad \text{and} \quad \Delta_2(\mathbf{j}_3) = \overline{2}\overline{1}\overline{2} \ 121.$$

Lemma 8.5. *The following statements are valid for every $v \in W$.*

- The set $D(v)$ is the disjoint union of the labels $D_{w_1}(v)$ associated with the vertices w_1 of the W -permutohedron P_W that are crossed or reached by a $\uparrow_v$ -path.
- Two labeled vertices $D_{w_1}(v) \subset D(v)$ and $D_{w'_1}(v) \subset D(v)$ of P_W are related by an edge s_i if and only if there exist trivial double reduced words $\mathbf{i} \in D_{w_1}(v)$ and $\mathbf{j} \in D_{w'_1}(v)$ such that $\mathbf{j} = \Delta_i(\mathbf{i})$.

Proof. The first statement is a reformulation of the second relation in (7.5). The second statement is obtained by applying Lemma 7.16 and Lemma 8.3, because of formula (8.1) describing the dual-moves. $\square$

FIGURE 25. The sets $D(s_2s_1)$ and $D(w_0)$ in the case where $\mathfrak{g} = A_2$.

We continue our running example of the case where $\mathfrak{g} = A_2$. Using Figure 18, we give in Figure 24 the description of the sets $D(s_2s_1)$ and $D(w_0)$ in terms of subsets $D_{w_1}(s_2s_1)$ and $D_{w_1}(w_0)$ related by dual-moves. Now, in Example 7.17 we saw that each subset $D_{w_1}(w_0)$ can be decomposed into sets $W(u, v)$ with appropriate $u, v \in W$, related by right τ^* -moves; this is described by Figure 19. Therefore, mixing Figure 19 and Figure 24, in Figure 26 we get a description of the set $D(w_0)$ as unions of sets $W(u, v)$ related by $\hat{d}$ -moves. Observe the double occurrence of the permutohedron P_3 in this picture.

8.2. Saltations. Now we are ready to introduce the saltations, and use them to describe the cluster combinatorics associated with double words differing from a dual-move. Roughly speaking, saltations generalize the generalized cluster transformations involving truncation maps. When we deal with generalized cluster transformations, the combinatorics giving the formulas is described by the Poisson bivector of the seed $\mathcal{X}$ -torus

FIGURE 26. The set $D(w_0)$ in the case where $\mathfrak{g} = A_2$.

(i.e., the seed matrix usually denoted by ε), which, in its turn, is transformed by these generalized cluster transformations. The idea underlying the definition of the saltations is simple: we allow a little more freedom between the combinatorics on seed $\mathcal{X}$ -tori and their Poisson geometry. We recall the truncated torus $\mathcal{X}_J^0$ associated with the truncation map $\mathfrak{t}_J$, as given by Definition 4.4.

Definition 8.6. Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ and $\mathbf{I}' = (I', I'_0, \varepsilon', d')$ be two seeds, $J \subset I$, $J' \subset I'$ two isomorphic subsets, and $\mathfrak{t}_J(\mathbf{I})$, $\mathfrak{t}_{J'}(\mathbf{I}')$ the related truncation maps. A birational Poisson isomorphism $\Xi : \mathcal{X}_{\mathfrak{t}_J(\mathbf{I})} \rightarrow \mathcal{X}_{\mathfrak{t}_{J'}(\mathbf{I})}$ is said to be a *saltation* (relative to the subsets J, J') if there exists a generalized cluster transformation $\phi_{\mathbf{I} \rightarrow \mathbf{I}'}^T : \mathcal{X}_{\mathbf{I}} \rightarrow \mathcal{X}_{\mathbf{I}'}$ that

makes the following diagram commutative for every $\mathbf{t} \in \mathcal{X}_J^0$:

$$(8.2) \quad \begin{array}{ccc} & \xrightarrow{\phi_{\mathbf{I} \rightarrow \mathbf{I}'}^\tau} & \\ \mathcal{X}_{\mathbf{I}} & & \mathcal{X}_{\mathbf{I}'} \\ \downarrow \mathbf{t}_J(\mathbf{t}) & & \downarrow \mathbf{t}_{J'}(\mathbf{t}) \\ \mathcal{X}_{\mathbf{t}_J(\mathbf{I})}(\mathbf{t}) & \xrightarrow{\Xi} & \mathcal{X}_{\mathbf{t}_{J'}(\mathbf{I}')}(\mathbf{t}) \end{array}$$

Saltations are composed easily: if Ξ_1 is a saltation relative to the sets J, J' and to a generalized cluster transformation ϕ_1 , and if Ξ_2 is a saltation relative to the sets J', J'' and to the generalized cluster transformation ϕ_2 , then the composition $\Xi_2 \circ \Xi_1$ is a saltation relative to the sets J, J'' and to the generalized cluster transformation $\phi_2 \circ \phi_1$.

A few saltations have already been encountered before.

- Every generalized cluster transformation is a saltation: take the sets J and J' equal to the empty set $\emptyset$.
- For every $u, v \in W$ and all double reduced words $\mathbf{i}, \mathbf{j} \in R(u, v)$, the cluster transformation $\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{j}]_{\mathfrak{R}}}$ is the saltation relative to the cluster transformation $\mu_{\mathbf{i} \rightarrow \mathbf{j}}$, J being the set of right outlets relative to the seed $\mathbf{I}(\mathbf{i})$ and J' the set of right outlets relative to the seed $\mathbf{I}(\mathbf{j})$.
- For every $w \in W$ and every $\mathbf{i} \in R^\tau(w)$, the cluster transformation $\zeta_{[\mathbf{i}]_{\mathfrak{R}}}$ is the saltation relative to the generalized cluster transformation $\zeta_{\mathbf{i}}$, where the set J is the set of right outlets relative to the seed $\mathbf{I}(\mathbf{i})$ and J' is the set of right outlets relative to the seed $\mathbf{I}(\mathbf{i}^\square)$.
- More generally, the saltation associated with any generalized cluster transformation ϕ is always a cluster transformation if the set J contains the directions relative to all the tropical mutations that are used to factorize ϕ , and $\phi(J) \subset J'$.

Remark 8.7. Every generalized cluster transformation $\phi_{\mathbf{I} \rightarrow \mathbf{I}'}^\tau$ is a product of symmetries, mutations, and tropical mutations, and every symmetry of a finite set J can be decomposed into a product of transpositions. Therefore, it is tempting to decompose every saltation Ξ as a product of elementary saltations of the following forms:

$$\begin{array}{ccc} \mathcal{X}_{\mathbf{I}} & \xrightarrow{\text{id}} & \mathcal{X}_{\mathbf{I}} \\ \downarrow \mathbf{t}_J & & \downarrow \mathbf{t}_{J'} \\ \mathcal{X}_{\mathbf{I}_J}(t) & \xrightarrow{\Xi_1} & \mathcal{X}_{\mathbf{I}_{J'}}(t) \end{array} \quad \begin{array}{ccc} \mathcal{X}_{\mathbf{I}} & \xrightarrow{\mu_k} & \mathcal{X}_{\mu_k(\mathbf{I})} \\ \downarrow \mathbf{t}_J & & \downarrow \mathbf{t}_J \\ \mathcal{X}_{\mathbf{I}_J}(t) & \xrightarrow{\Xi_2} & \mathcal{X}_{\mu_k(\mathbf{I})_J}(t) \end{array}$$

$$\begin{array}{ccc} \mathcal{X}_{\mathbf{I}} & \xrightarrow{\mu_k^\tau} & \mathcal{X}_{\mu_k^\tau(\mathbf{I})} \\ \downarrow \mathbf{t}_J & & \downarrow \mathbf{t}_J \\ \mathcal{X}_{\mathbf{I}_J}(t) & \xrightarrow{\Xi_3} & \mathcal{X}_{\mu_k^\tau(\mathbf{I})_J}(t) \end{array}$$

The problem is that the first map Ξ_1 is clearly undefined if the symmetry $s : J \mapsto J'$ is not the identity!

Here is the main reason for introducing saltations. For every double reduced word $\mathbf{j}$,

every positive word $\mathbf{i}_+ \in R(1, w_0)$, and every $i \in [1, l]$, we define a map $\Xi_k : \mathcal{X}_{[\mathbf{j}\mathbf{i}_+ \bar{k}]_{\mathfrak{R}}} \rightarrow \mathcal{X}_{[\mathbf{j}\mathbf{i}_+^{\square} k^*]_{\mathfrak{R}}}$ by the rule

$$(8.3) \quad x_{\Xi_k(i)} = \begin{cases} x_{\zeta_{\mathbf{i}_+}(j)} & \text{if } j < N^i(\mathbf{j}\mathbf{i}_+), \\ x_{\zeta_{\mathbf{i}_+}(j)} x_{(N^i(\mathbf{j}\mathbf{i}_+ \bar{k}))}^{-1} & \text{if } j = N^i(\mathbf{j}\mathbf{i}_+) < N^i(\mathbf{j}\mathbf{i}_+ \bar{k}), \\ x_{(j)} & \text{otherwise.} \end{cases}$$

Proposition 8.8. *The map $\Xi_i : \mathcal{X}_{[\mathbf{j}\mathbf{i}_+ \bar{i}]_{\mathfrak{R}}} \rightarrow \mathcal{X}_{[\mathbf{j}\mathbf{i}_+^{\square} i^*]_{\mathfrak{R}}}$ is a saltation, but not a generalized cluster transformation.*

Proposition 8.8 will be proved in Subsection 8.4. For the moment, we focus on the link between saltations and dual-moves, given by the following result.

Corollary 8.9. *Let $i \in [1, l]$, and let $\mathbf{i}$ be a double word such that we can apply the dual-move Δ_i on it. Then the following product is a birational Poisson isomorphism:*

$$(8.4) \quad \begin{aligned} \Xi_{s_i} : \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}} &\longrightarrow \mathcal{X}_{[\Delta_i(\mathbf{i})]_{\mathfrak{R}}} \\ \mathbf{x} &\longmapsto \mu_{[\mathbf{i}^{\square} i^*]_{\mathfrak{R}} \rightarrow [\Delta_i(\mathbf{i})]_{\mathfrak{R}}} \circ \Xi_i \circ \mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{i}_+ \bar{i}]_{\mathfrak{R}}}(\mathbf{x}). \end{aligned}$$

Proof. We use Proposition 8.8 and the fact that, for all double words $\mathbf{j}$ and $\mathbf{k}$, the cluster transformation $\mu_{[\mathbf{j}]_{\mathfrak{R}} \rightarrow [\mathbf{k}]_{\mathfrak{R}}}$, when it exists, is a birational Poisson isomorphism between the seed $\mathcal{X}$ -tori $\mathcal{X}_{[\mathbf{j}]_{\mathfrak{R}}}$ and $\mathcal{X}_{[\mathbf{k}]_{\mathfrak{R}}}$. $\square$

Finally, we prove that the relations among various cluster $\mathcal{X}$ -varieties $\mathcal{X}_w$ associated with different double words $\mathbf{i} \in D(w_0)$ involve saltations and are described by the W -permutohedron P_W .

Lemma 8.10. *Let us replace every label $w' \in W$ of the vertices of the W -permutohedron P_W by the cluster $\mathcal{X}$ -variety $\mathcal{X}_w$ associated with a seed $\mathcal{X}$ -torus $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ related to a double word $\mathbf{i} \in D_w(w_0)$. The following statements are valid.*

- The cluster $\mathcal{X}$ -variety $\mathcal{X}_w$ related to any $w \in W$ contains the seed $\mathcal{X}$ -torus $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ associated with any double word $\mathbf{i} \in D_w(w_0)$.
- For any $i \in [1, l]$, if two vertices of P_W are respectively labeled by $\mathcal{X}_w$ and $\mathcal{X}_{w'}$ and are related by an edge $s_i \in W$, then there exist two trivial double words $\mathbf{i}, \mathbf{j} \in D(w_0)$ such that the seed $\mathcal{X}$ -tori associated with $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ and $\mathcal{X}_{[\mathbf{j}]_{\mathfrak{R}}}$ are related by the saltation Ξ_{s_i} .

Proof. The first statement is given by Lemma 7.19. The second statement comes from Lemma 8.5 and Corollary 8.9. $\square$

8.3. Cluster $\mathcal{X}$ -varieties for (G, π_*) , saltations, and the W -permutohedron. We obtain finally, in Theorem 8.12, the cluster combinatorics relating the twisted evaluations of §7. This cluster combinatorics involves cluster transformations and saltations.

With any trivial double words $\mathbf{i}, \mathbf{i}' \in D(v)$ such that there exists a $\hat{d}$ -move $\delta : \mathbf{i} \rightarrow \mathbf{i}'$, we associate a birational Poisson isomorphism $\hat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}'} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{i}'}$ given by

- the cluster transformation $\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{i}']_{\mathfrak{R}}}$ if δ is a d^{τ} -move;
- the saltation map Ξ_i if δ is the dual-move Δ_i .

By Lemma 8.5, there exists a sequence of $\hat{d}$ -moves relating any two trivial double words $\mathbf{i}, \mathbf{i}' \in D(v)$. Therefore, we can extend this definition to every $\mathbf{i}, \mathbf{j} \in D(v)$ in the usual way: if $\mathbf{i}, \mathbf{j}$ are trivial double words linked by a sequence of $\hat{d}$ -moves and $\mathbf{i} \rightarrow \mathbf{i}_1 \rightarrow \dots \rightarrow \mathbf{i}_{n-1} \rightarrow \mathbf{j}$ is the associated chain of elements, we define the map $\hat{\mu}_{\mathbf{i} \rightarrow \mathbf{j}}$ as the composition $\hat{\mu}_{\mathbf{i}_{n-1} \rightarrow \mathbf{j}} \circ \dots \circ \hat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}_1}$. Finally, since every double word $\mathbf{k} \in D(v)$ is related to a trivial double word of $D(v)$ by a sequence of generalized d -moves, we consider the

cluster transformation $\mu_{\mathbf{k} \rightarrow \mathbf{i}}$ to complete the picture. Finally, we get a birational Poisson isomorphism $\hat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}'} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{i}'}$ associated with any double words $\mathbf{i}, \mathbf{j} \in D(v)$. We can now relate the Poisson birational isomorphism of Corollary 8.9 with the twisted evaluations of §7.

Proposition 8.11. *For every $i \in [\bar{1}, \bar{l}]$ and every double reduced word $\mathbf{i} \in R(s_i, w_0)$ starting with the letter i , we have*

$$\hat{\text{ev}}_{\mathbf{i}} = \hat{\text{ev}}_{\Delta_i(\mathbf{i})} \circ \Xi_{s_i}.$$

Proposition 8.11 is proved in Subsection 8.5. Now, by Lemma 8.5, there exists a composition of $\hat{d}$ -moves δ that satisfy the relation $\delta : \mathbf{i} \rightarrow \mathbf{j}$. Therefore, it suffices to apply Proposition 7.20 and Proposition 8.11 to prove the following result, which was the missing argument in the proof of Theorem 7.9.

Theorem 8.12. *For every $v \in W$ and $\mathbf{i}, \mathbf{j} \in D(v)$, the maps $\hat{\text{ev}}_{\mathbf{i}}$ and $\hat{\text{ev}}_{\mathbf{j}}$ satisfy $\hat{\text{ev}}_{\mathbf{i}} = \hat{\text{ev}}_{\mathbf{j}} \circ \hat{\mu}_{\mathbf{i} \rightarrow \mathbf{j}}$.*

Figure 27 describes, in the case where $\mathfrak{g} = A_2$, the full picture of cluster combinatorics we obtain for (BB_-, π_*) from the cluster $\mathcal{X}$ -varieties related to (G, π_G) and described in §3.

8.4. Proof of Proposition 8.8. We need a few preliminaries to prove Proposition 8.8. Recall the map $\pi_{\mathbf{i}}$ associated with any double word $\mathbf{i}$, given by (5.24). Here is a generalization. Let $\mathfrak{s}$ be a $\mathcal{X}$ -split associated with the decomposition $\mathbf{ij} \rightarrow (\mathbf{i}, \mathbf{j})$: for every $\mathbf{x} \in \mathcal{X}_{\mathbf{ij}}$, we define

$$(8.5) \quad \pi_{\mathbf{ij} \rightarrow \mathbf{i}}(\mathbf{x}) = \mathbf{m}(\mathbf{x}_{(1)}, \pi_{\mathbf{j}}(\mathbf{x}_{(2)})) \quad \text{and} \quad \pi_{\mathbf{ij} \rightarrow \mathbf{j}}(\mathbf{x}) = \mathbf{m}(\pi_{\mathbf{i}}(\mathbf{x}_{(1)}), \mathbf{x}_{(2)}).$$

(It is clear that these identities do not depend on the choice of $\mathfrak{s}$ and that they are also valid for every $\mathbf{x} \in \mathcal{X}_{[\mathbf{ij}]_{\text{or}}}$.) Figure 28 and Figure 29 give examples of these maps in the case where $\mathfrak{g} = A_3$, whereas the following result is straightforward.

Lemma 8.13. *Let $u, v \in W$, and let $\mathbf{i} \in D(u, 1) \cup D(1, v)$ be a positive or negative double word. The following equality is valid for every $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$:*

$$[\text{ev}_{\mathbf{i}}(\mathbf{x})]_0 = \text{ev}_{\mathbf{1}} \circ \pi_{\mathbf{i}}(\mathbf{x}).$$

Lemma 8.14. *Let $w_1 \leq u, w_2 \leq v \in W$, and let $\mathbf{i} \in R(w_2, v)$ and $\mathbf{j} \in R(u, w_1)$ be such that $\mathbf{i} = \mathbf{i}_+ \mathbf{i}_-$ and $\mathbf{j} = \mathbf{j}_+ \mathbf{j}_-$. The following equalities are valid for every $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$ and $\mathbf{y} \in \mathcal{X}_{\mathbf{j}}$:*

$$\begin{aligned} [\widehat{\text{ev}}_{\mathbf{i}}(\mathbf{x})v^{-1}]_{\leq 0} &= [\widehat{\text{ev}}_{\mathbf{i}_+} \circ \pi_{\mathbf{i} \rightarrow \mathbf{i}_+}(\mathbf{x})v^{-1}]_{\leq 0} \quad \text{and} \\ [\widehat{u}^{-1} \widehat{\text{ev}}_{\mathbf{j}}(\mathbf{y})]_{\geq 0} &= [\widehat{u}^{-1} \widehat{\text{ev}}_{\mathbf{j}_-} \circ \pi_{\mathbf{j} \rightarrow \mathbf{j}_-}(\mathbf{y})]_{\geq 0}. \end{aligned}$$

Proof. We prove the first identity. Let $\mathfrak{s}$ be a $\mathcal{X}$ -split associated with the decomposition $\mathbf{i} \rightarrow (\mathbf{i}_+, \mathbf{i}_-)$. Since $\mathbf{i}_- \in R(w_2, 1)$ and $w_2 \leq v$, the conjugation of $\text{ev}_{\mathbf{i}_-}(\mathbf{x}_{(2)})$ by $\widehat{v}$ belongs to the Borel subgroup B . But it is clear that for every $b \in B$ we have $[b]_{\leq 0} = [b]_0$. Therefore, we get the result by applying the definition (8.5) for $\pi_{\mathbf{i} \rightarrow \mathbf{i}_+}$ and Lemma 8.13. The second identity is proved in the same way. $\square$

Lemma 8.15. *Let $w_1 \leq u, w_2 \leq v \in W$, and let $\mathbf{i} \in R(w_2, v)$ and $\mathbf{j} \in R(u, w_1)$ be double reduced words such that $\mathbf{i} = \mathbf{i}_+ \mathbf{i}_-$ and $\mathbf{j} = \mathbf{j}_+ \mathbf{j}_-$. Then*

$$(8.6) \quad \begin{aligned} \pi_{\mathbf{i} \rightarrow \mathbf{i}_+} &= \zeta_{\mathbf{i}_+}^{-1} \circ \mu_{\mathbf{i}_- \mathbf{i}_+ \rightarrow \mathbf{i}_+} \circ \zeta_{\mathbf{i}_+} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_+}, \\ \pi_{\mathbf{j} \rightarrow \mathbf{j}_-} &= \zeta_{\mathbf{j}_-}^{-1} \circ \mu_{\mathbf{j}_+ \mathbf{j}_- \rightarrow \mathbf{j}_-} \circ \zeta_{\mathbf{j}_-} \circ \mu_{\mathbf{j} \rightarrow \mathbf{j}_-}. \end{aligned}$$

FIGURE 27. Cluster $\mathcal{X}$ -varieties evaluating (BB_-, π_*) in the case where $\mathfrak{g} = A_2$.

FIGURE 28. The generalized folding $\pi_{\mathbf{i}} : \Gamma_{A_3}(\mathbf{i}) \rightarrow \Gamma_{A_3}(\mathbf{1})$ for $\mathbf{i} = 123121$.

FIGURE 29. The generalized folding $\pi_{\mathbf{ij} \rightarrow \mathbf{i}} : \Gamma_{A_3}(\mathbf{ij}) \rightarrow \Gamma_{A_3}(\mathbf{i})$ for $\mathbf{i} = 123121$ and $\mathbf{j} = \bar{1}$.

Proof. Let $\mathbf{z} \in \mathcal{X}_{\mathbf{i}_+ \mathbf{i}_+}$ and $b_- := \zeta^{1,v}(\text{ev}_{\mathbf{i}_+ \mathbf{i}_+}(\mathbf{z}))$. Using Proposition 5.31, equation (3.5), Remark 5.8, and Theorem 3.7, we see that the evaluation map $\text{ev}_{\mathbf{i}_+ \mathbf{i}_+}$ sends the element

$$\mu_{\mathbf{i}_+ \mathbf{i}_+ \rightarrow \mathbf{i}_+ \mathbf{i}_+} \circ \zeta_{\mathbf{i}_+}(\mathbf{z})$$

to b_- . But this evaluation also sends the element $\zeta_{\mathbf{i}_+} \circ \pi_{\mathbf{i} \rightarrow \mathbf{i}_+} \circ \mu_{\mathbf{i}_+ \mathbf{i}_+ \rightarrow \mathbf{i}}(\mathbf{z})$ to b_- , by the first identity in Lemma 8.14, Theorem 3.7, and equation (3.5). Now, the double word $\mathbf{i}_+ \mathbf{i}_+$ is reduced, because $\mathbf{i}_+$ is a double reduced word. Therefore, the maps $\text{ev}_{\mathbf{i}_+ \mathbf{i}_+}$ and $\zeta_{\mathbf{i}_+}$ are birational isomorphisms, and the first relation in (8.6) is proved. The second is proved in the same way, by using the second identity of Lemma 8.14. $\square$

Here is the last preliminary. We recall that, since the erasing map occurs in the first line of (3.6), the Poisson map $\mu_{\mathbf{i} \rightarrow \mathbf{j}}$ associated with a nil-move $\delta : \mathbf{i} \mapsto \mathbf{j}$ is not a birational isomorphism. Then we define the related cluster transformation $\bar{\mu}_{\mathbf{i} \rightarrow \mathbf{j}}$ by

$$\bar{\mu}_{\mathbf{i} \rightarrow \mathbf{j}} = \varsigma_{\binom{i}{1}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}.$$

Lemma 8.16. *Let $i \in [1, l]$, and let $\mathbf{i} \in R(s_i, w_0)$ be a double reduced word starting with the letter $\bar{i}$. Then for every $\mathbf{t} \in \mathcal{X}_{\mathbf{1}}$ we have*

$$\mathbf{t}'_{\mathbf{i}\mathfrak{R}}(\mathbf{t}) \circ \bar{\mu}_{\bar{\mathbf{i}}\mathbf{i}_+ \rightarrow \mathbf{i}_+ \mathbf{i}_+} \circ \zeta_{\mathbf{i}_+} \circ \mu_{\mathbf{i} \rightarrow \bar{\mathbf{i}}\mathbf{i}_+} = \Xi_i \circ \mathbf{t}_{\mathbf{i}\mathfrak{R}}(\mathbf{t}),$$

where $\mathbf{t}'_{\mathbf{i}\mathfrak{R}}$ denotes the truncation map associated with the set $I_0^{\mathfrak{R}}(\mathbf{i})'$ given by

$$(8.7) \quad I_0^{\mathfrak{R}}(\mathbf{i})' = I_0^{\mathfrak{R}}(\mathbf{i}_+^{\square} \mathbf{i}^*) \cup \left\{ \binom{i}{1} \right\} \setminus \left\{ \binom{i^*}{N^{i^*}(\mathbf{i}_+^{\square} \mathbf{i}^*)} \right\}.$$

Proof. For every $i \in [1, l]$ and every double word $\mathbf{i}$, we define the erasing map $\varsigma_{\mathfrak{R}^i, \mathbf{i}}$ as the product of j -erasing maps ς_j for every right outlet j that is not an i -vertex:

$$\varsigma_{\mathfrak{R}^i, \mathbf{i}} = \prod_{j \in I_0^{\mathfrak{R}}(\mathbf{i}) \setminus \{ \binom{i}{N^i(\mathbf{i})} \}} \varsigma_j.$$

Note that the map Ξ_i can also be defined as

$$(8.8) \quad \Xi_i : \mathcal{X}_{[\mathbf{i}_+ \bar{i}]_{\mathfrak{R}}} \longrightarrow \mathcal{X}_{[\mathbf{i}_+^{\square} \mathbf{i}^*]_{\mathfrak{R}}} : \mathbf{x} \longmapsto (\zeta_{\mathbf{i}_+} \circ \pi_{\mathbf{i}_+ \bar{i} \rightarrow \mathbf{i}_+}(\mathbf{x}), \mathbf{x}(\mathfrak{R})).$$

Then we use formulas (8.6) and (8.8) to deduce the following identity, valid for every $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}_+ \bar{i}]_{\mathfrak{R}}}$:

$$(\varsigma_{\mathfrak{R}^{i^*}, \mathbf{i}_+^{\square}} \circ \varsigma_{\binom{i}{1}} \circ \bar{\mu}_{\bar{\mathbf{i}}\mathbf{i}_+ \rightarrow \mathbf{i}_+ \mathbf{i}_+} \circ \zeta_{\mathbf{i}_+} \circ \mu_{\mathbf{i} \rightarrow \bar{\mathbf{i}}\mathbf{i}_+}(\mathbf{x}), \mathbf{x}(\mathfrak{R})) = \Xi_i \circ \mathbf{t}_{\mathbf{i}\mathfrak{R}}(\mathbf{x}).$$

Now, it suffices to use the definition of truncation maps to end the proof of the lemma. $\square$

Lemma 8.16 implies that the map Ξ_i is a saltation associated with the generalized cluster transformation $\bar{\mu}_{\bar{\mathbf{i}}\mathbf{i}_+ \rightarrow \mathbf{i}_+ \mathbf{i}_+} \circ \zeta_{\mathbf{i}_+} \circ \mu_{\mathbf{i} \rightarrow \bar{\mathbf{i}}\mathbf{i}_+}$, and this proves the first part of Proposition 8.8.

Lemma 8.17. *Let $\mathbf{I}$ be a seed such that there exist a cluster variable x_i of the seed $\mathcal{X}$ -torus $\mathcal{X}_{\mathbf{I}}$ that is a Casimir function. Then, for every cluster $\mathbf{x}$ of $\mathcal{X}_{\mathbf{I}}$ and every generalized cluster transformation ϕ , we have $x_{\phi(i)} = x_i$ and, for every $j \neq i$, the cluster variable $x_{\phi(j)}$ does not depend on the cluster variable x_i .*

Proof. It suffices to factorize the generalized cluster transformation ϕ into a product of tropical mutations, symmetries, and mutations ϕ_k . The properties are then checked easily for each ϕ_k , with the help of formulas for mutations and tropical mutations. $\square$

To prove that the saltation Ξ_i is not a generalized cluster transformation, it suffices to apply Lemma 8.17, the second line of formula (8.3), and the fact that every cluster variable x_i , $i \in I_0^{\mathfrak{A}}(\mathbf{i})$, is a Casimir function for every right truncated seed $[\mathbf{i}]_{\mathfrak{A}}$.

8.5. Proof of Proposition 8.11. We recall that every element $x \in B_-B$ can be written as $x = [x]_-[x]_0[x]_+$, where $[x]_- \in N_-$, $[x]_0 \in H$, and $[x]_+ \in N$. If $x \in B_-B \cap BB_-$, we can also write $x = [[x]]_+[[x]]_0[[x]]_-$, with $[[x]]_- \in N_-$, $[[x]]_0 \in H$, and $[[x]]_+ \in N$. These two decompositions are related easily: since the map $x \mapsto x^{-1}$ is an involution, we get

$$(8.9) \quad [[x]]_+[[x]]_0[[x]]_- = [x^{-1}]_+^{-1}[x^{-1}]_0^{-1}[x^{-1}]_-^{-1}.$$

Now we recall the map $\kappa : G \times G \rightarrow G$ given by (5.8). Let $t \in H$, $v \in W$; first, we introduce the map $\Xi_t : L^{v,w_0} \rightarrow L^{w_0,v^{\star-1}}$ given by

$$(8.10) \quad \Xi_t(gH) = [[[gH]]_{\geq 0}\widehat{w_0}]_{\leq 0} [\kappa((t\widehat{w_0})^{-1}, [[gH]]_{< 0})]_+ H.$$

Lemma 8.18. *Let $t \in H$ and $v \in W$. For every $g \in G^{v,w_0}$ we have*

$$\rho_{t,(v,e)_v}(gH) = [\Xi_t(gH)^{\theta}\widehat{w_0}]_{\leq 0}^{\theta} t\widehat{w_0} \Xi_t(gH)^{-1}.$$

Proof. Set $b = [[g]]_{\geq 0}$ and $n_- = [[g]]_{< 0}$. We use successively the fact that $[\kappa(\widehat{w_0}^{-1}, n_-)]_0$ and $[\kappa((t\widehat{w_0})^{-1}, n_-)]_0$ are equal (to the unit 1_G of G), the fact that the map $\kappa(g, \cdot)$ commutes with the inverse map $x \mapsto x^{-1}$ for every $g \in G$, and Lemma 7.12, obtaining

$$\begin{aligned} \rho_{t,(v,e)_v}(gH) &= bn_-t\widehat{w_0}[bn_-\widehat{w_0}]_{\leq 0}^{-1} \\ &= bt\widehat{w_0} \kappa((t\widehat{w_0})^{-1}, n_-^{-1}) ([b\widehat{w_0}]_{\leq 0}[\kappa((\widehat{w_0})^{-1}, n_-^{-1})]_0)^{-1} \\ &= bt\widehat{w_0} ([b\widehat{w_0}]_{\leq 0}[\kappa((t\widehat{w_0})^{-1}, n_-)]_+)^{-1} \\ &= [([b\widehat{w_0}]_{\leq 0}[\kappa((t\widehat{w_0})^{-1}, n_-)]_+)^{\theta}\widehat{w_0}]_{\leq 0}^{\theta} t\widehat{w_0} ([b\widehat{w_0}]_{\leq 0}[\kappa((t\widehat{w_0})^{-1}, n_-)]_+)^{-1} \\ &= [\Xi_t(gH)^{\theta}\widehat{w_0}]_{\leq 0}^{\theta} t\widehat{w_0} \Xi_t(gH)^{-1}. \end{aligned} \quad \square$$

As an immediate consequence, for every $t \in H$ and every $v \in W$ we get

$$(8.11) \quad \rho_{t,(e,w_0)_v} = \rho_{t,(v,e)_v} \circ \Lambda_t, \quad \text{where } \Lambda_t(b_1, cH) = \Xi_t(b_1 cH).$$

Now we describe the related cluster combinatorics. We will focus on the case where $v = s_i$, because it is all that we need, but similar statements, although more technical, can in fact be obtained for a general $v \in W$. We recall the maps $\pi_{\mathbf{i} \rightarrow \mathbf{i}}$ given by (8.5).

Lemma 8.19. *Let $i \in [1, l]$ and $t \in H$, and let $\mathbf{i} \in R(s_i, w_0)$ be a double reduced word satisfying $\mathbf{i} = \mathbf{i}_+ \mathbf{i}_-$. Then for every $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}]_{\mathfrak{A}}}(t)$ we have*

$$[\Xi_t \circ \text{ev}_{\mathbf{i}}(\mathbf{x})]_{\leq 0} = \text{ev}_{\mathbf{i}_+} \circ \zeta_{\mathbf{i}_+} \circ \pi_{\mathbf{i} \rightarrow \mathbf{i}_+}(\mathbf{x}).$$

Proof. Since the double reduced word $\mathbf{i}$ is (s_i, w_0) -adapted, it is easy to show that

$$[[\text{ev}_{\mathbf{i}}(\mathbf{x})]]_{\geq 0} = \text{ev}_{\mathbf{i}_+} \circ \pi_{\mathbf{i} \rightarrow \mathbf{i}_+}(\mathbf{x})$$

for every $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}]_{\mathfrak{A}}}(t)$. Therefore, it suffices to apply Theorem 5.34 and (8.10) to prove the lemma. $\square$

Lemma 8.20. *Let $i \in [1, l]$, $g \in G^{s_i, w_0}$, and $t \in H$. Denoting by $t^\star$ the conjugation of t by $\widehat{w_0}$, we obtain $[\Xi_t(gH)]_+ H = t^\star E^{i^\star} H$.*

Proof. Since $g \in G^{s_i, w_0}$, we get $[[gH]]_{<0} H = F^i H$. So, we see that, up to H we have $[\kappa((t\widehat{w_0})^{-1}, [[gH]]_{<0})]_+$ equal to $t^\star E^{i^\star} t^{\star-1}$. Now it suffices to apply (8.10). $\square$

We recall the reduced evaluations of Subsection 4.1. Since the identity $t = \text{ev}_1(\mathbf{x}(\mathfrak{R}))$ is clear for every $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}]\mathfrak{R}}(t)$, the following proposition is directly implied by the definitions of Ξ_i and Ξ_t , with Lemma 8.19 and Lemma 8.20, via the amalgamation procedure.

Proposition 8.21. *Let $v \in W$ and $t \in H$, and let $\mathbf{i} \in R(v, w_0)$ be a double reduced word satisfying $\mathbf{i} = \mathbf{i}_+ \bar{\mathbf{i}}$. Then for every $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}]\mathfrak{R}}(t)$ we have*

$$\Xi_t(\text{ev}_{\mathbf{i}}^{\text{red}}(\mathbf{x}^{\text{red}})) = \text{ev}_{\mathbf{i}_+ \bar{\mathbf{i}}^\star}^{\text{red}}(\Xi_i(\mathbf{x})^{\text{red}}).$$

Finally, Proposition 8.11 is deduced from the definition of (w_1, w_2) -maps given in §7, equation (8.11) and Proposition 8.21, the properties of the amalgamated product, and the definition (8.4) of the birational Poisson isomorphism Ξ_{s_i} .

§9. EVALUATIONS AND CLUSTER $\mathcal{X}$ -VARIETIES FOR $(G^\star, \pi_{G^\star})$

We start by presenting an alternative way to describe twist maps with mutations and tropical mutations. Then we describe the dual Poisson–Lie group $(G^\star, \pi_{G^\star})$ via (w_1, w_2) -maps and provide, in Theorem 9.12, evaluations for $(G^\star, \pi_{G^\star})$ in the spirit of the Kirillov–Reshetikhin multiplicative formula for the quantum R -matrix associated with $\mathcal{U}_q(\mathfrak{g})$. Moreover, the birational Poisson isomorphisms, used for passing from the positive part to the negative part of $(G^\star, \pi_{G^\star})$ (and vice versa), can be read on the W -permutohedron: they are described by the $\uparrow$ -paths linking the cluster $\mathcal{X}$ -varieties corresponding to the identity and the longest element w_0 of W .

9.1. Twist maps and coordinates in Schubert cells. We introduce parameterizations of unipotent subgroups of G that will be used to evaluate the Poisson–Lie group $(G^\star, \pi_{G^\star})$. They involve the generalized cluster transformations of Subsection 5.4. We start by recalling a few facts from [FZ99, §2.4]. For every $w \in W$, the corresponding Schubert cell $(BwB)/B \subset G/B$ is the image of the Bruhat cell BwB under the natural projection of G onto the flag variety G/B . We recall the subgroups $N_+(w) \subset N$ and $N_-(w) \subset N_-$ given by

$$N_+(w) = N \cap \widehat{w} N_- \widehat{w}^{-1}, \quad N_-(w) = N_- \cap \widehat{w}^{-1} N \widehat{w}.$$

The following proposition is, in essence, well known (cf. [FH91, Corollary 23.60]).

Proposition 9.1. *An element $x \in G$ lies in the Bruhat cell BwB if and only if we have $\widehat{w}^{-1}x \in G_0$ and $[\widehat{w}^{-1}x]_- \in N_-(w)$. Furthermore, the correspondence $\tau_+ : x \mapsto y_+$ given by*

$$y_+ = \tau_+(x) = \widehat{w}[\widehat{w}^{-1}x]_- \widehat{w}^{-1} \in N_+(w)$$

induces a biregular isomorphism between the Schubert cell $(BwB)/B$ and $N_+(w)$.

Let $T : G \rightarrow G : x \mapsto x^T$ be the involutive anti-automorphism of G defined in [FZ99] and given, for every $i \in [1, l]$ and every complex number t , by

$$a^T = a \ (a \in H), \quad x_i(t)^T = x_{\bar{i}}(t), \quad x_{\bar{i}}(t)^T = x_i(t).$$

Using the transpose map T , we obtain a counterpart of Proposition 9.1 for the opposite Bruhat cell B_-wB_- .

Proposition 9.2. *An element $x \in G$ lies in B_-wB_- if and only if we have $x\hat{w}^{-1} \in G_0$ and $[x\hat{w}^{-1}]_+ \in N_+(w)$. Furthermore, the correspondence $\tau_- : x \mapsto y_-$ given by*

$$y_- = \tau_-(x) = \hat{w}^{-1}[x\hat{w}^{-1}]_+\hat{w} \in N_-(w)$$

induces a biregular isomorphism between the “opposite Schubert cell” $B_- \setminus (B_-wB_-)$ and $N_-(w)$.

The maps τ_+ and τ_- are in fact easily described by using mutations and tropical mutations. We recall that the group $N_-(w)$ is a unipotent Lie group of dimension $\ell(w)$; hence it is isomorphic to the affine space $\mathbb{C}^{\ell(w)}$ as an algebraic variety. With any negative reduced word $\mathbf{i} = i_1 \dots i_{\ell(w)}$ and every positive reduced word $\mathbf{j} = j_1 \dots j_{\ell(w)}$ we are going to associate the following system of coordinates on $N_{\pm}(w)$, which involves the generalized cluster transformations of equation (5.19). For every $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$ and $\mathbf{y} \in \mathcal{X}_{\mathbf{j}}$, we set

$$(9.1) \quad \begin{aligned} \tau_{\mathbf{i}}(\mathbf{x}) &= \hat{w}^{-1} \cdot \hat{s}_{i_1} y_{i_1} \left(-x_{\binom{i_1}{0}}^{-1} \right) \cdots \hat{s}_{i_{\ell(w)}} y_{i_{\ell(w)}} \left(-x_{\zeta_{\mathbf{i}(\leq \ell(w)-1)} \binom{i_{\ell(w)}}{0}}^{-1} \right), \\ \tau_{\mathbf{j}}(\mathbf{y}) &= x_{j_1} \left(-y_{\zeta_{\mathbf{j}(\geq 2)} \binom{j_1}{N^{j_1}(\mathbf{j})}}^{-1} \right) \hat{s}_{j_1} \cdots x_{j_{\ell(w)}} \left(-y_{\binom{j_{\ell(w)}}{N^{j_{\ell(w)}}(\mathbf{j})}}^{-1} \right) \hat{s}_{j_{\ell(w)}} \cdot \hat{w}^{-1}. \end{aligned}$$

Now, for every w , every reduced word $\mathbf{i} = i_1 \dots i_{\ell(w)} \in R(w)$, and every $k \in [1, \ell(w)]$, we set $w_{\mathbf{i}_{>k}} := s_{i_{k+1}} \dots s_{i_n}$ and $w_{\mathbf{i}_{<k}} := s_{i_1} \dots s_{i_{k-1}}$. For every $u \in W$, $t \in \mathbb{C}$ and $i \in [1, l]$, we denote

$$(9.2) \quad x_{u(i)}(t) := \hat{u}^{-1} x_i(t) \hat{u} \quad \text{and} \quad y_{u(i)}(t) := \hat{u}^{-1} y_i(t) \hat{u}.$$

It is well known, and straightforward to check via induction on the length on W , that the following identities are fulfilled for every w and all complex numbers $t_1, \dots, t_{\ell(w)}$:

$$(9.3) \quad \begin{aligned} \prod_{k=1}^{\ell(w)} y_{w_{\mathbf{i}_{>k}}(i_k)}(t_k) &= \hat{w}^{-1} \cdot \prod_{k=1}^{\ell(w)} \hat{s}_{i_k} y_{i_k}(t_k) \quad \text{and} \\ \prod_{k=1}^{\ell(w)} x_{w_{\mathbf{i}_{<k}}(i_k)}(t_k) &= \prod_{k=1}^{\ell(w)} x_{i_k}(t_k) \hat{s}_{i_k} \cdot \hat{w}^{-1}. \end{aligned}$$

Lemma 9.3. *For every $w \in W$, $\mathbf{i} \in R(w, 1)$, $\mathbf{j} \in R(1, w)$, and $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$, $\mathbf{y} \in \mathcal{X}_{\mathbf{j}}$, we have*

$$[\hat{w}^{-1} \text{ev}_{\mathbf{i}}(\mathbf{x})]_- = \tau_{\mathbf{i}}(\mathbf{x}) \quad \text{and} \quad [\text{ev}_{\mathbf{j}}(\mathbf{y})\hat{w}]_+ = \tau_{\mathbf{j}}(\mathbf{y}).$$

Proof. We focus on the first relation. Using equation (5.13) and the negative projection $[\cdot]_- : B_-B \rightarrow N_- : x \mapsto [x]_-$ to the unipotent subgroup $N_- \subset G$ associated with the Gauss decomposition (4.9), we are led to the formula

$$\begin{aligned} &[\widehat{w_{\mathbf{i}_{>k-1}}}^{-1} \text{ev}_{\mathbf{i}(k-1)} \circ \zeta_{\mathbf{i}(\leq k-1)}(\mathbf{x})]_- \\ &= y_{w_{\mathbf{i}_{>k}}(i_k)} \left(-x_{\zeta_{\mathbf{i}(\leq k-1)} \binom{i_k}{0}}^{-1} \right) [\widehat{w_{>k}}^{-1} \text{ev}_{\mathbf{i}(k)} \circ \zeta_{\mathbf{i}(\leq k)}(\mathbf{x})]_- . \end{aligned}$$

The first relation is then obtained by iteration of this formula, because of the first identity in (9.3). The second relation is proved in the same way, by using the second identity in (9.3). $\square$

Proposition 9.4. *For every $w \in W$, $\mathbf{i} \in R(w, 1)$, $\mathbf{j} \in R(1, w)$, and $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$, $\mathbf{y} \in \mathcal{X}_{\mathbf{j}}$, we have*

$$\tau_-(\text{ev}_{\mathbf{i}}(\mathbf{x})) = \hat{w} \tau_{\mathbf{i}}(\mathbf{x}) \hat{w}^{-1} \quad \text{and} \quad \tau_+(\text{ev}_{\mathbf{j}}(\mathbf{y})) = \hat{w}^{-1} \tau_{\mathbf{j}}(\mathbf{y}) \hat{w}.$$

Proof. This is deduced simply from the preceding lemma and the expressions for τ_- and τ_+ given by Propositions 9.1 and 9.2. $\square$

9.2. From (G_0, π_*) to (G^*, π_{G^*}) via $(w_1, w_2)_{w_0}$ -maps. Here are a few preliminary maps to get the evaluation maps related to (G^*, π_{G^*}) , which are given in the next subsection. We recall the variation of the Gauss decomposition given by formula (8.9).

Lemma 9.5. *For every $w_2 \in W$ and every $t \in H$, we have*

$$\begin{aligned} [[\rho_{t,(e,w_2)_{w_0}}(b_1, gH)]_-]_- &= [b_1 \widehat{w_0}^{-1}]_- \quad \text{and} \\ [[\rho_{t,(w_0,w_2)_{w_0}}(c_1, gH)]_+]_+ &= [(c_1 \widehat{w_0})^{-1}]_+^{-1}. \end{aligned}$$

Proof. We use the definition of $(w_1, w_2)_v$ -maps, equation (8.9), and the fact that conjugating every element of the Borel subgroup $B_\pm$ by $\widehat{w_0}$ gives an element of the opposite Borel subgroup $B_\mp$ to deduce the relations

$$(9.4) \quad \begin{aligned} \rho_{t,(e,e)_{w_0}}(b_1, bH) &= \underbrace{b_1 b \, t \widehat{w_0} [b \widehat{w_0}]_0^{-1} \widehat{w_0}^{-1} [b_1 \widehat{w_0}^{-1}]_0^{-1}}_{\in B} \underbrace{[b_1 \widehat{w_0}^{-1}]_0^{-1}}_{\in N_-} \quad \text{and} \\ \rho_{t,(w_0,e)_{w_0}}(c_1, bH) &= \underbrace{[[c_1 \widehat{w_0}]]_+}_{\in N_+} \underbrace{[[c_1 \widehat{w_0}]]_{\leq 0} \widehat{w_0}^{-1} b t \widehat{w_0} (c_1 [b \widehat{w_0}]_{\leq 0})^{-1}}_{\in B_-}. \end{aligned}$$

Moreover, it is clear that the unipotent part in the above identities does not depend on the element $bH \in N$ relative to the choice of w_2 . Therefore, the lemma is true for every $w_2 \in W$. $\square$

We recall the map Λ_t given by (8.11).

Proposition 9.6. *Let $b_1, b \in B$ and $t \in H$. The triplet $(h, n, n_-) \in H \times N \times N_-$ such that*

$$nhn_-^{-1} = \rho_{t,(e,e)_{w_0}}(b_1, bH)$$

is given by the following formulas:

$$(9.5) \quad \begin{cases} h &= [b_1 b]_0 \, t \widehat{w_0} [b \widehat{w_0}]_0^{-1} \widehat{w_0}^{-1} [b_1 \widehat{w_0}^{-1}]_0^{-1}, \\ n &= [(\Lambda_t(b_1, \zeta^{1,w_0}(b)H) \, \widehat{w_0})^{-1}]_+^{-1}, \\ n_- &= [b_1 \widehat{w_0}^{-1}]_-. \end{cases}$$

Proof. The negative unipotent part of (9.5) is given by the first identity in Lemma 9.5, its diagonal part is given by the first equation in (9.4), and its positive unipotent part is given by Theorem 5.34, Lemma 8.11, and the second identity in Lemma 9.5. $\square$

9.3. Evaluations related to (G^*, π_{G^*}) . We are going to give the cluster combinatorics on (G^*, π_{G^*}) . For that, we start with giving evaluation maps for the elements $h \in H$ in (9.5).

Proposition 9.7. *The identity $[[\widehat{\text{ev}}_{\mathbf{i}}(\mathbf{x})]]_0 = \text{ev}_{\mathbf{1}}(\mathbf{X})$ is valid for every double word $\mathbf{i} = \mathbf{i}_1 \mathbf{i}_2 \in D_e(w_0)$ and every cluster $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}]\mathfrak{R}}$ if and only if $\mathbf{X} = (X_1, \dots, X_l)$ is the set of monomials given by*

$$X_i = x_{(N^{\mathbf{i}}(\mathbf{i}))} \prod_{k=1}^l \prod_{\ell=1}^{\ell(w_0)} \prod_{\substack{j_1 < N^{\mathbf{i}_\ell}(\mathbf{i}_{1\ell}) \\ j_2 < N^{\mathbf{i}_\ell}(\mathbf{i}_{2\ell})}} \left(-x_{\binom{i_\ell}{j_1}}^{-1} x_{\binom{i_\ell}{N^{\mathbf{i}_\ell}(\mathbf{i}_{1\ell})+j_2}} \right)^{(A^{-1})_{ki} \langle \alpha_{i_\ell}^\vee, w_0^{-1} \omega_k \rangle}.$$

Because of its length, the proof of Proposition 9.7 is postponed to Subsection 9.4. However, we stress that the same kind of a monomial formula would have been obtained by choosing a trivial double word $\mathbf{i} = \mathbf{i}_1 \mathbf{i}_2 \in D_{w_0}(w_0)$.

Example 9.8. As usual, set $\mathfrak{g} = A_2$ and take $t \in H$. We consider a cluster $\mathbf{x} \in \mathcal{X}_{[121 \ 121]_{\mathfrak{R}}}(t)$ and the related elements

$$b_1 = \text{ev}_{121}^{\text{red}}(x_{(1)}^{(1)}, x_{(1)}^{(1)}, x_{(2)}^{(2)}), \quad b = \text{ev}_{121}^{\text{red}}(x_{(2)}^{(2)}, x_{(3)}^{(1)}, x_{(2)}^{(2)}), \quad \text{and} \quad t = \text{ev}_1(t_1, t_2).$$

Using Example 5.30, Lemma 5.14, Lemma 5.15, Proposition 5.16, and equation (9.5), or simply the formula above, we get

$$\begin{aligned} [[\widehat{\text{ev}}_{121 \ 121}(\mathbf{x})]]_0 &= \text{ev}_1 \circ \text{om} \left(\left(x_{(1)}^{-1} x_{(1)}^{(1)}, x_{(2)}^{-1} x_{(1)}^{-1} \right), \left(x_{(2)}^{(2)} x_{(3)}^{(1)}, x_{(2)}^{(2)} x_{(3)}^{-1} \right), (t_1, t_2) \right) \\ &= \text{ev}_1 \left(x_{(1)}^{-1} x_{(1)}^{(1)} x_{(3)}^{(1)} x_{(2)}^{(2)} t_1, x_{(2)}^{-1} x_{(1)}^{-1} x_{(2)}^{(2)} x_{(3)}^{-1} t_2 \right). \end{aligned}$$

We then focus on the evaluation maps relative to the elements $n \in N$ and $n_- \in N_-$ in equation (9.5). We recall the involutions $\star$ and $\circ$ on double words and seed $\mathcal{X}$ -tori given in Subsection 5.2.

Lemma 9.9. *For every $\mathbf{i} \in R(1, w_0)$, $\mathbf{j} \in R(w_0, 1)$, and $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$, $\mathbf{y} \in \mathcal{X}_{\mathbf{j}}$, we have*

$$[\text{ev}_{\mathbf{i}}(\mathbf{x})\widehat{w}_0^{-1}]_- = \tau_{\mathbf{i}^\star}(\mathbf{x}^\star) \quad \text{and} \quad [[\text{ev}_{\mathbf{j}}(\mathbf{y})\widehat{w}_0]]_+^{-1} = \tau_{\mathbf{j}^\circ}(\mathbf{y}^\circ).$$

Proof. The first relation comes from Lemma 5.14 and Lemma 9.3. Equation (8.9) and Corollary 5.16 imply that the left-hand side of the second relation is equal to $[\text{ev}_{\mathbf{j}^\circ}(\mathbf{y}^\circ)\widehat{w}_0]_+$. Then, Lemma 9.3 and Lemma 5.15 lead to the right-hand side of the second relation. $\square$

Lemma 9.10. *Let $\mathbf{i}, \mathbf{j} \in D(w_0)$, let $\mathbf{i}$ be a $(e, e)_{w_0}$ -word and $\mathbf{j}$ a $(w_0, e)_{w_0}$ -word, and let $\mathfrak{s}, \mathfrak{s}'$ be $\mathcal{X}$ -splits relative to the $(w_1, w_2)_{w_0}$ -decompositions $\mathbf{i} \rightarrow (\mathbf{i}_1, \mathbf{i}_2)$ and $\mathbf{j} \rightarrow (\mathbf{j}_1, \mathbf{j}_2)$, respectively. Then*

$$[[\widehat{\text{ev}}_{\mathbf{i}}(\mathbf{x})]]_-^{-1} = \tau_{\mathbf{i}_1^\star}(\mathbf{x}_{(1)}^\star) \quad \text{and} \quad [[\widehat{\text{ev}}_{\mathbf{j}}(\mathbf{y})]]_+^{-1} = \tau_{\mathbf{j}_1^\circ}(\mathbf{y}_{(1)}^\circ).$$

Proof. These relations are derived from Lemmas 7.8, 9.5, and 9.9. $\square$

Now we can get the cluster combinatorics on (G^*, π_{G^*}) . With any double word $\mathbf{i} \in D(w_0)$ we associate double words $\mathbf{i}_e \in D_e(w_0)$ and $\mathbf{i}_{w_0} \in D_{w_0}(w_0)$ that are, respectively, an (e, w_0) -trivial double word and a (w_0, e) -trivial double word. Moreover, let $\mathbf{i}_{e+} \in R(1, w_0)$ and $\mathbf{i}_{w_0-} \in R(w_0, 1)$ denote, respectively, the positive part of $\mathbf{i}_e$ and the negative part of $\mathbf{i}_{w_0}$, according to the definition of Subsection 5.4, and let $\wp_e : \mathcal{X}_{[\mathbf{i}_e]_{\mathfrak{R}}} \rightarrow \mathcal{X}_{\mathbf{i}_{e+}}^{\text{red}}$ and $\wp_{w_0} : \mathcal{X}_{[\mathbf{i}_{w_0}]_{\mathfrak{R}}} \rightarrow \mathcal{X}_{\mathbf{i}_{w_0-}}^{\text{red}}$ be the corresponding canonical projections on seed $\mathcal{X}$ -tori. Finally, recall the birational Poisson isomorphisms $\widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}_e}$ and $\widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}_{w_0}}$ defined in Subsection 8.3. With any $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$, we associate the clusters

$$\mathbf{x}_e = \wp_e \circ \widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}_e}(\mathbf{x}) \quad \text{and} \quad \mathbf{x}_{w_0} = \wp_{w_0} \circ \widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}_{w_0}}(\mathbf{x}).$$

The following lemma is deduced from Proposition 9.7, Lemma 9.10, and Theorem 8.12.

Lemma 9.11. *For every $w \in W$, every double word $\mathbf{i} \in D_w(w_0)$, and every $\mathbf{x} \in \mathcal{X}_w$, we have*

$$\begin{cases} [[\widehat{\text{ev}}_{\mathbf{i}}(\mathbf{x})]]_0 &= \text{ev}_1(\mathbf{X}_e), \\ [[\widehat{\text{ev}}_{\mathbf{i}}(\mathbf{x})]]_+ &= \tau_{\mathbf{i}_{w_0}}(\mathbf{x}_{w_0}^\circ)^{-1}, \\ [[\widehat{\text{ev}}_{\mathbf{i}}(\mathbf{x})]]_-^{-1} &= \tau_{\mathbf{i}_e^\star}(\mathbf{x}_e^\star); \end{cases}$$

where $\mathbf{X}_e = (X_1, \dots, X_l)$ is the set of monomials given in Proposition 9.7 applied to the cluster $\widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}_e}(\mathbf{x})$.

Now, we recall that the map $\phi : (G^*, \pi_{G^*}) \rightarrow (BB_-, \pi_*)$ given by the formula $(nh, n_-h^{-1}) \mapsto nh^2n_-^{-1}$ is not an isomorphism but a covering of degree 2^l . Example 9.8 in particular shows that in the general case we cannot expect to directly obtain rational

evaluations for the dual Poisson–Lie group (G^*, π_{G^*}) , because of this covering $h \mapsto h^2$ on the Cartan subgroup H of G . The remedying idea is to take covers on cluster variables, which mimic ϕ . Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ be a seed; the seed $\mathcal{X}$ -torus denoted $\mathcal{X}_{\mathbf{I}/2}$ is the torus $(\mathbb{C}_{\neq 0})^{|I|}$ given with the Poisson bracket

$$\{x_i, x_j\} = \frac{\hat{\varepsilon}_{ij}}{4} x_i x_j,$$

where $\{x_i \mid i \in I\}$ still denote the standard coordinates on the factors. In particular, the following map is a Poisson covering of degree $2^{|I|}$:

$$(9.6) \quad \mathbf{c}_{\mathcal{X}} : \mathcal{X}_{\mathbf{I}/2} \longrightarrow \mathcal{X}_{\mathbf{I}} : (x_1, \dots, x_{|I|}) \longmapsto (x_1^2, \dots, x_{|I|}^2).$$

Thus, Lemma 9.11 and the fact that the maps $\mathbf{c}_{\mathcal{X}}$ and ϕ are Poisson coverings whose degrees are some powers of 2 lead us to the following result.

Theorem 9.12. *Let $\mathbf{i} \in D(w_0)$. The following evaluation map $\text{Ev}_{\mathbf{i}}$ is a Poisson covering of degree 2^n , for some $n \leq \dim G$, onto a Zariski open set of G^* :*

$$(9.7) \quad \begin{aligned} \text{Ev}_{\mathbf{i}} : \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}^{1/2} &\rightarrow (G^*, \pi_{G^*}) : \mathbf{x} \mapsto (\text{ev}_{\mathbf{i}}^+(\mathbf{x}), \text{ev}_{\mathbf{i}}^-(\mathbf{x})), \quad \text{where} \\ \begin{cases} \text{ev}_{\mathbf{i}}^+(\mathbf{x}) = \tau_{\mathbf{i}_{w_0}}(\mathbf{c}_{\mathcal{X}}(\mathbf{x})_{w_0}^{\odot})^{-1} \text{ev}_{\mathbf{1}}(\mathbf{X}_e), \\ \text{ev}_{\mathbf{i}}^-(\mathbf{x}) = \tau_{\mathbf{i}_e}(\mathbf{c}_{\mathcal{X}}(\mathbf{x})_e^*)^{-1} \text{ev}_{\mathbf{1}}(\mathbf{X}_e)^{-1}, \end{cases} \end{aligned}$$

and the set $\mathbf{X}_e = (X_1, \dots, X_l)$ is the same as in Lemma 9.11.

Remark 9.13. A careful study of the cluster variables appearing in the monomial formulas describing $\mathbf{X}_e = (X_1, \dots, X_l)$ and given in Proposition 9.7 leads to the choice of a subcovering of the covering $\mathbf{c}_{\mathcal{X}}$ that minimizes the value n in the previous theorem.

Example 9.14. In the case where $\mathfrak{g} = A_2$, the heuristics of Theorem 9.12 is illustrated by Figure 27, where we have used the notation $G^* = (G_+^*, G_-^*)$ to abbreviate the description (2.10). In particular, if we choose the double word $\mathbf{i} = 121121$, then we can take $\mathbf{i}_e = \mathbf{i}$ and $\mathbf{i}_{w_0} = \bar{2}\bar{1}\bar{2}121$. Therefore, the following elements are associated with any $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$:

$$\mathbf{x}_e = \wp_e(\mathbf{x}) \quad \text{and} \quad \mathbf{x}_{w_0} = \wp_{w_0} \circ \Xi_{s_1} \circ \mu_{\bar{2}\bar{1}\bar{1}121 \rightarrow \bar{2}\bar{1}\bar{1}121} \circ \Xi_{s_2} \circ \mu_{12\bar{2}121 \rightarrow \bar{2}12121} \circ \Xi_{s_1}(\mathbf{x}),$$

whereas $\mathbf{X}_e$ was already given in Example 9.8.

FIGURE 30. Evaluations related to (G^*, π_{G^*}) in the case where $\mathfrak{g} = A_2$.

Of course, these evaluation maps on (G^*, π_{G^*}) are compatible with the cluster combinatorics already developed. The following theorem is derived from Theorem 8.12 and the definitions (9.6) and (9.7) of the covering $\mathfrak{c}_{\mathcal{X}}$ and the evaluation map Ev_i .

Theorem 9.15. *The following diagram is commutative for any double words $\mathbf{i}, \mathbf{j} \in D(w_0)$:*

$$\begin{array}{ccc}
 & \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}^{1/2}} & \xrightarrow{\mathfrak{c}_{\mathcal{X}}} \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}} \\
 \text{Ev}_{\mathbf{i}} \swarrow & & \downarrow \hat{\mu}_{\mathbf{i} \rightarrow \mathbf{j}} \\
 (G^*, \pi_{G^*}) & & \mathcal{X}_{[\mathbf{j}]_{\mathfrak{R}}^{1/2}} \xrightarrow{\mathfrak{c}_{\mathcal{X}}} \mathcal{X}_{[\mathbf{j}]_{\mathfrak{R}}} \\
 \text{Ev}_{\mathbf{j}} \nwarrow & &
 \end{array}$$

9.4. Proof of Proposition 9.7. The main ingredient in the proof of Proposition 9.7 is the factorization theorem [FZ99, Theorem 1.10 and formula (1.21)] of Fomin and Zelevinsky. Here, we mainly follow the exposition in [KZ02]. Let $\tilde{G}$ be the simply connected cover of G , let $\tilde{B}, \tilde{B}_-$ denote the Borel subgroups of $\tilde{G}$ such that their images in G are B and B_- , respectively, and let $\tilde{H} = \tilde{B} \cap \tilde{B}_-$. In the same way, for every $u, v \in W$, we denote by $\tilde{G}^{u,v}$ the double Bruhat cell in $\tilde{G}$ whose image in G is $G^{u,v}$. For $x \in \tilde{B}\tilde{B}_-$ and a fundamental weight ω_i , define $\bar{\Delta}_i(x) = [x]_0^{\omega_i}$. In [FZ99] it was shown that $\bar{\Delta}_i$ extends to a regular function on $\tilde{G}$. For type A_n (when $\tilde{G} = \text{SL}(n+1, \mathbb{C})$), this is simply the principal $(i \times i)$ minor of a matrix x . For any pair $u, v \in W$, the corresponding *generalized minor* is the regular function on $\tilde{G}$ given by

$$\Delta_{u\omega_i, v\omega_i}(x) = \bar{\Delta}_i(\hat{u}^{-1}x\hat{v}).$$

In [FZ99] it was shown that these functions are well defined; i.e., they depend only on the weights $u\omega_i$ and $v\omega_i$ and do not depend on the particular choice of u and v . For $i = 1, \dots, l$, we denote $\varepsilon(i) = +1$ and $\varepsilon(\bar{i}) = -1$, and recall that $|i| = |\bar{i}| = i$. In what follows, we fix $u, v \in W$ and a double reduced word $\mathbf{i}$ of (u, v) . We append l entries $i_{m+1}, \dots, i_{m+l}$ to $\mathbf{i}$ by setting $i_{m+j} = \bar{j}$. For $k = 1, \dots, m$, we set

$$u_{\geq k} = \prod_{\substack{\ell=m, \dots, k \\ \varepsilon(i_{\ell})=-1}} s_{|i_{\ell}|}, \quad v_{< k} = \prod_{\substack{\ell=1, \dots, k-1 \\ \varepsilon(i_{\ell})=+1}} s_{|i_{\ell}|},$$

where the notation implies that the index ℓ in the first (respectively, second) product is decreasing (respectively, increasing). We also set $u_{\geq k} = e$, $v_{< k} = v$ for $k = m+1, \dots, m+l$. For example, if $\mathbf{i} = 1\bar{2}\bar{2}3\bar{3}2\bar{1}$, then $u_{\geq 4} = s_1s_3$, $v_{< 4} = s_1s_2$. For every $k = 1, \dots, m+l$, we set $\gamma^k = u_{\geq k}\omega_{|i_k|}$, $\delta^k = v_{< k}\omega_{|i_k|}$, and introduce a regular function M_k on $\tilde{G}^{u,v}$ by setting

$$(9.8) \quad M_k(x) = \Delta_{\gamma^k, \delta^k}(x'),$$

where x' is the twist of x given by (5.15). We refer to the family $M_1, \dots, M_{m+l}$ as *twisted minors* associated with a reduced word $\mathbf{i}$. Their significance stems from the following result (see [FZ99, Theorems 1.2, 1.9, 1.10 and formula (1.21)]). Recalling the notation given in (5.11), we define, for every double word $\mathbf{i} = i_1 \dots i_m$, the map $x_{\mathbf{i}} : \mathbb{C}_{\neq 0}^m \rightarrow G$ by

$$x_{\mathbf{i}}(\mathbf{t}) = x_{i_1}(t_1) \cdots x_{i_m}(t_m), \quad \text{where } \mathbf{t} = (t_1, \dots, t_m).$$

Theorem 9.16 ([KZ02, Theorem 2.3]). *The map $x_{\mathbf{i}} : \tilde{H} \times \mathbb{C}^m \rightarrow \tilde{G}$ given by*

$$x_{\mathbf{i}}(a; t_1, \dots, t_m) = ax_{i_1}(t_1) \cdots x_{i_m}(t_m)$$

restricts to a biregular isomorphism between a complex torus $\tilde{H} \times (\mathbb{C} - \{0\})^m$ and a Zariski open subset $U_{\mathbf{i}} = \{x \in \tilde{G}^{u,v} : M_k(x) \neq 0 \text{ for } 1 \leq k \leq m+l\}$ of the double Bruhat cell $\tilde{G}^{u,v}$. Furthermore, for $k = 1, \dots, m+l$ and $x = x_{\mathbf{i}}(a; t_1, \dots, t_m) \in U_{\mathbf{i}}$, we have

$$(9.9) \quad M_k(x) = a^{-u\gamma^k} \prod_{\substack{1 \leq \ell < k \\ \varepsilon(i_\ell) = -1}} t_\ell^{\langle \alpha_{|i_\ell|}^\vee, u_{\geq \ell}^{-1} \gamma^k \rangle} \prod_{\substack{k \leq \ell \leq m \\ \varepsilon(i_\ell) = +1}} t_\ell^{\langle \alpha_{|i_\ell|}^\vee, v_{< \ell+1}^{-1} \delta^k \rangle}.$$

We are going to use this theorem in the proof of Proposition 9.7. First, we note that for $u = w_0$ and $v = e$, we have $\gamma^k = \omega_k$ and $\delta^k = \omega_k$ for every $k > \ell(w_0)$. Now, observe that if b_1 and b belong to the double Bruhat cell $\tilde{G}^{e, w_0}$, then the elements

$$b_1^\star = \widehat{w_0}^{-1} b_1 \widehat{w_0} \quad \text{and} \quad b^\circ = \widehat{w_0}^{-1} b^{-1} \widehat{w_0}$$

belong to the double Bruhat cell $\tilde{G}^{w_0, e}$. Therefore, using the definition of the twist map $x \mapsto x'$, formula (9.8), and the fact that $a^\theta = a^{-1}$ for every $a \in \tilde{H}$, we obtain the following identities for every $k > \ell(w_0)$:

$$\begin{aligned} M_k(b_1^\star) &= ([\widehat{w_0}^{-1} b_1^\star]_{\geq 0}^\theta)^{\omega_{i_k}} = [b_1 \widehat{w_0}^{-1}]_0^{-\omega_k} \quad \text{and} \\ M_k(b^\circ)^{-1} &= (\widehat{w_0} [b \widehat{w_0}]_0^{-1} \widehat{w_0}^{-1})^{\omega_k}. \end{aligned}$$

Taking $\mathbf{i} \in R(w_0, 1)$ to parameterize $b_1^\star = ax_{\mathbf{i}}(t_1^\star, \dots, t_{\ell(w_0)}^\star)$ and $b^\circ = a'x_{\mathbf{i}}(t_1^\circ, \dots, t_{\ell(w_0)}^\circ)$, and applying formula (9.9), we get

$$\begin{aligned} [b_1 \widehat{w_0}^{-1}]_0^{-\omega_k} &= a^{\omega_{k^\star}} \prod_{\ell} t_\ell^{\langle \alpha_{|i_\ell|}^\vee, w_0_{\geq \ell}^{-1} \omega_k \rangle} \quad \text{and} \\ (\widehat{w_0} [b \widehat{w_0}]_0^{-1} \widehat{w_0}^{-1})^{\omega_k} &= a'^{-\omega_{k^\star}} \prod_{\ell} t_\ell^{\circ - \langle \alpha_{|i_\ell|}^\vee, w_0_{\geq \ell}^{-1} \omega_k \rangle}. \end{aligned}$$

So, we have the following identity:

$$(9.10) \quad ([b_1]_0 [b_1 \widehat{w_0}^{-1}]_0^{-1} [b]_0 \widehat{w_0} [b \widehat{w_0}]_0^{-1} \widehat{w_0}^{-1})^{\omega_k} = \prod_{\ell} (t_\ell^\star t_\ell^{\circ -1})^{\langle \alpha_{|i_\ell|}^\vee, w_0_{\geq \ell}^{-1} \omega_k \rangle}.$$

Now, we fix some $\mathbf{j} = j_1 \dots j_{\ell(w_0)} \in R(1, w_0)$ and evaluate some $b_1, b \in N \cap G^{e, w_0}$ by $\mathbf{z}, \mathbf{z}' \in \mathcal{X}_{\mathbf{i}}$, i.e., $b_1 = \text{ev}_{\mathbf{j}}(\mathbf{z})$ and $b = \text{ev}_{\mathbf{j}}(\mathbf{z}')$. We also denote $\mathbf{j}_\ell := j_1 \dots j_\ell$ for every $\ell \leq \ell(w_0)$. The relations $\mathbf{x}_{\mathbf{j}}(t_1, \dots, t_{\ell(w_0)}) = \text{ev}_{\mathbf{j}}(\mathbf{z})$ and $\mathbf{x}_{\mathbf{j}}(t'_1, \dots, t'_{\ell(w_0)}) = \text{ev}_{\mathbf{j}}(\mathbf{z}')$ imply

$$t_\ell = \prod_{j < N^{j_\ell}(\mathbf{j}_\ell)} z_{\binom{j_\ell}{j}} \quad \text{and} \quad t'_\ell = \prod_{j < N^{j_\ell}(\mathbf{j}_\ell)} z'_{\binom{j_\ell}{j}}.$$

Moreover, since $b \in N$, we have $\pi_{\mathbf{i}}(\mathbf{z}') = 1$. Thus, adding Lemma 5.14, formula (5.6), and Proposition 5.16 to the previous identities gives

$$t_\ell^\star t_\ell^{\circ -1} = - \prod_{j < N^{j_\ell}(\mathbf{j}_\ell)} \left(z_{\binom{j_\ell}{j}} z'_{\binom{j_\ell}{N^{j_\ell}(\mathbf{j}) - j}} \right)^{-1} = - \prod_{j < N^{j_\ell}(\mathbf{j}_\ell)} z_{\binom{j_\ell}{j}}^{-1} z'_{\binom{j_\ell}{j}}.$$

In fact, it is easy to show that the same kind of formula can be obtained when the evaluations of b_1 and b are done (respectively) for any $\mathbf{i}_1$ and $\mathbf{i}_2$ in $R(1, w_0)$. Then the setting $\mathbf{i} = \mathbf{i}_1 \mathbf{i}_2 \in D_e(w_0)$ and the relation $b_1 b H = \text{ev}_{\mathbf{i}}^{\text{red}}(\mathbf{x})$ imply that $\mathbf{x}^{\text{red}} = \mathbf{m}(\mathbf{z}, \mathbf{z}'^{\text{red}})$. Finally, we apply Proposition 9.6, identity (2.3), and formula (9.10) to end the proof of

Proposition 9.7:

$$\begin{aligned} X_i &= x_{(N^i(i))} \prod_{k=1}^l \prod_{\ell=1}^{\ell(w_0)} \prod_{\substack{j < N^i \ell(\mathbf{i}_{1\ell}) \\ j' < N^i \ell(\mathbf{i}_{2\ell})}} \left(-z_{\binom{i_\ell}{j}}^{-1} z'_{\binom{i_\ell}{j'}} \right)^{(A^{-1})_{ki} \langle \alpha_{i_\ell}^\vee, w_0^{-1} \omega_k \rangle} \\ &= x_{(N^i(i))} \prod_{k=1}^l \prod_{\ell=1}^{\ell(w_0)} \prod_{\substack{j_1 < N^i \ell(\mathbf{i}_{1\ell}) \\ j_2 < N^i \ell(\mathbf{i}_{2\ell})}} \left(-x_{\binom{i_\ell}{j_1}}^{-1} x_{\binom{i_\ell}{j_2}} \right)^{(A^{-1})_{ki} \langle \alpha_{i_\ell}^\vee, w_0^{-1} \omega_k \rangle}. \end{aligned}$$

§10. AN ELEMENTARY APPROACH FOR THE CASE WHERE $G = \mathrm{SL}(2, \mathbb{C})$

To fix the ideas, we consider with full details all the evaluation maps met before, and the related cluster combinatorics, in the simplest case: $G = \mathrm{SL}(2, \mathbb{C})$. We thus start by recalling the construction of Fock and Goncharov for $(\mathrm{SL}(2, \mathbb{C}), \pi_G)$ and successively consider the models $(\mathrm{SL}(2, \mathbb{C}), \pi_*)$ and $(\mathrm{SL}(2, \mathbb{C})^*, \pi_{G^*})$ for dual Poisson-Lie groups, and, as a conclusion, we give the quantization of this elementary construction by considering the cluster combinatorics associated with the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. This section is written to be as self-contained as possible.

10.1. Elementary Lie data. We recall that the complex simple Lie group

$$(10.1) \quad \mathrm{SL}(2, \mathbb{C}) = \left\{ \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} : t_{11}t_{22} - t_{12}t_{21} = 1, \quad t_{ij} \in \mathbb{C} \right\}$$

has its Lie algebra $\mathfrak{g}$ equal to the set $\mathfrak{sl}(2, \mathbb{C})$ of complex (2×2) -matrices with zero trace. The Chevalley generators $\{e_1, f_1, h_1\}$ and their related basis $\{e_1, f_1, h^1\}$ are then given by the following matrices:

$$\begin{aligned} e_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & f_1 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ h_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & h^1 &= \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}. \end{aligned}$$

Using the exponential map $\exp : \mathfrak{g} \rightarrow G$, which, in this case, takes a matrix $M \in \mathfrak{g}$ to the usual matrix $\sum_{n=0}^{\infty} \frac{M^n}{n!} \in G$, we get the following generators of G , the last two being associated with every nonzero complex number x :

$$\begin{aligned} E^1 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & F^1 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\ H_1(x) &= \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, & H^1(x) &= \begin{pmatrix} x^{1/2} & 0 \\ 0 & x^{-1/2} \end{pmatrix}. \end{aligned}$$

In particular, these generators of the diagonal subgroup H of G satisfy the relation $H^1(x^2) = H_1(x)$ for every complex number $x \in \mathbb{C}_{\neq 0}$, which agrees with formula (2.3), because here the Cartan matrix A is simply the number 2. We stress, however, that the generator $H^1(x)$ is generally ill-defined on $\mathrm{SL}(2, \mathbb{C})$, because $\mathrm{SL}(2, \mathbb{C})$ is not of adjoint type, but simply connected. The related adjoint group is $\mathrm{PGL}(2, \mathbb{C})$, and $H^1(x)$ is well defined on $\mathrm{PGL}(2, \mathbb{C})$, because of the identity

$$H^1(x) = \begin{pmatrix} x^{1/2} & 0 \\ 0 & x^{-1/2} \end{pmatrix} \stackrel{\mathrm{PGL}(2, \mathbb{C})}{=} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, since there is only one simple root α_1 , the Weyl group W contains only two elements $\{1, s_1\}$ and the different double reduced words are the double words $1, \bar{1}, 1\bar{1}, \bar{1}1$ without

forgetting the trivial double word $\mathbf{1}$ associated with the unity element of the direct product $W \times W$. Finally, the r -matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$ associated with $\mathfrak{sl}(2, \mathbb{C})$ and its related elements $r_{\pm} \mathfrak{g} \otimes \mathfrak{g}$ are given by the following formulas:

$$(10.2) \quad \begin{aligned} r &= e_1 \wedge f_1, \quad r_+ = \frac{1}{4} h_1 \otimes h_1 + e_1 \otimes f_1 \quad \text{and} \\ r_- &= -\frac{1}{4} h_1 \otimes h_1 - f_1 \otimes e_1. \end{aligned}$$

10.2. Cluster $\mathcal{X}$ -varieties related to $(\mathrm{SL}(2, \mathbb{C}), \pi_G)$. The evaluation maps of Fock and Goncharov associated with the previous double reduced words mentioned above are as follows:

$$\begin{aligned} \mathrm{ev}_{\mathbf{1}}(x_0) &= H^1(x_0) \in G^{1,1} = \begin{pmatrix} x_0^{1/2} & 0 \\ 0 & x_0^{-1/2} \end{pmatrix}, \\ \mathrm{ev}_{\mathbf{1}}(y_0, y_1) &= H^1(y_0) E^1 H^1(y_1) \in G^{1, w_0} \\ &= \begin{pmatrix} y_0^{1/2} y_1^{1/2} & y_0^{1/2} y_1^{-1/2} \\ 0 & y_0^{-1/2} y_1^{-1/2} \end{pmatrix} = \begin{pmatrix} 1 & y_0 \\ 0 & 1 \end{pmatrix} H^1(y_0 y_1), \\ \mathrm{ev}_{\bar{\mathbf{1}}}(z_0, z_1) &= H^1(z_0) F^1 H^1(z_1) \in G^{w_0, 1} \\ &= \begin{pmatrix} z_0^{1/2} z_1^{1/2} & 0 \\ z_0^{-1/2} z_1^{1/2} & z_0^{-1/2} z_1^{-1/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z_0^{-1} & 1 \end{pmatrix} H^1(z_0 z_1), \\ \mathrm{ev}_{\mathbf{1}\bar{\mathbf{1}}}(u_0, u_1, u_2) &= H^1(u_0) E^1 H^1(u_1) F^1 H^1(u_2) \in G^{w_0, w_0} \\ &= \begin{pmatrix} u_0^{1/2} u_1^{1/2} u_2^{1/2} + u_0^{1/2} u_1^{-1/2} u_2^{1/2} & u_0^{1/2} u_1^{-1/2} u_2^{-1/2} \\ u_0^{-1/2} u_1^{-1/2} u_2^{1/2} & u_0^{-1/2} u_1^{-1/2} u_2^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} 1 + u_1^{-1} & u_0 \\ u_0^{-1} u_1^{-1} & 1 \end{pmatrix} H^1(u_0 u_1 u_2), \\ \mathrm{ev}_{\bar{\mathbf{1}}\mathbf{1}}(v_0, v_1, v_2) &= H^1(v_0) F^1 H^1(v_1) E^1 H^1(v_2) \in G^{w_0, w_0} \\ &= \begin{pmatrix} v_0^{1/2} v_1^{1/2} v_2^{1/2} & v_0^{1/2} v_1^{1/2} v_2^{-1/2} \\ v_0^{-1/2} v_1^{1/2} v_2^{1/2} & v_0^{-1/2} v_1^{-1/2} v_2^{-1/2} + v_0^{-1/2} v_1^{1/2} v_2^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & v_0 v_1 \\ v_0^{-1} & v_1 + 1 \end{pmatrix} H^1(v_0 v_1 v_2). \end{aligned}$$

Again, the reader annoyed with rational powers is free to replace $\mathrm{SL}(2, \mathbb{C})$ by $\mathrm{PGL}(2, \mathbb{C})$. We remark, however, that for every $u, v \in W$ and every double reduced word $\mathbf{i} \in R(u, v)$, the associated reduced evaluation maps $\mathrm{ev}_{\mathbf{i}}^{\mathrm{red}} : \mathcal{X}_{\mathbf{i}}^{\mathrm{red}} \rightarrow G^{u,v}/H$ described in Subsection 4.1 are well-defined birational isomorphisms both on $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{PGL}(2, \mathbb{C})$. Moreover, we notice that it is also possible to construct the last two evaluation maps from the others, using the amalgamated product. Indeed, according to formula (3.4), we get the relations

$$\begin{aligned} u_0 &:= y_0, & u_1 &:= y_1 z_0, & u_2 &:= z_1 & \text{and} \\ v_0 &:= z_0, & v_1 &:= z_1 y_0, & v_2 &:= y_1. \end{aligned}$$

On the other hand, if $\mathrm{ev}_{\mathbf{1}\bar{\mathbf{1}}}(u_0, u_1, u_2) = \mathrm{ev}_{\bar{\mathbf{1}}\mathbf{1}}(v_0, v_1, v_2)$, then we have the following relations between the u_i and v_j :

$$(10.3) \quad \begin{cases} v_0 = u_0(1 + u_1), \\ v_1 = u_1^{-1}, \\ v_2 = u_2(1 + u_1), \end{cases} \quad \text{and} \quad \begin{cases} u_0 = v_0(1 + v_1^{-1})^{-1}, \\ u_1 = v_1^{-1}, \\ u_2 = v_2(1 + v_1^{-1})^{-1}. \end{cases}$$

Now, for every $i, j \in [1, 2]$, let t_{ij} be the coordinate function associated with (10.1). Applying formula (10.2) to the Sklyanin bracket (2.8), we can see that the standard Poisson bracket on the Poisson-Lie group G is given by the formulas

$$\begin{cases} \{t_{11}, t_{12}\}_G = \frac{1}{2}t_{11}t_{12}, & \{t_{11}, t_{21}\}_G = \frac{1}{2}t_{11}t_{21}, \\ \{t_{11}, t_{22}\}_G = t_{12}t_{21}, & \{t_{12}, t_{21}\}_G = 0, \\ \{t_{12}, t_{22}\}_G = \frac{1}{2}t_{12}t_{22}, & \{t_{21}, t_{22}\}_G = \frac{1}{2}t_{21}t_{22}. \end{cases}$$

We quickly check that the maps ev_1 , ev_1 , and $\text{ev}_{\bar{1}}$ are Poisson when the matrices (respectively, quivers) establishing the Poisson structure on the seed $\mathcal{X}$ -tori are given by

$$\varepsilon(\mathbf{1}) = (0), \quad \varepsilon(1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon(\bar{1}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the amalgamation procedure leads to the following matrices (respectively, quivers) establishing the Poisson structures on the associated seed $\mathcal{X}$ -tori, for which the maps $\text{ev}_{1\bar{1}}$ and $\text{ev}_{\bar{1}1}$ are Poisson:

$$(10.4) \quad \varepsilon(1\bar{1}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \varepsilon(\bar{1}1) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Looking at them, we see that the expressions in (10.3) describe respectively the cluster transformation $\mu_{1\bar{1} \rightarrow \bar{1}1} : \mathcal{X}_{1\bar{1}} \rightarrow \mathcal{X}_{\bar{1}1}$ associated with the variable u_1 and the cluster transformation $\mu_{\bar{1}1 \rightarrow 1\bar{1}} : \mathcal{X}_{\bar{1}1} \rightarrow \mathcal{X}_{1\bar{1}}$ associated with v_1 :

$$(v_0, v_1, v_2) = \mu_{\binom{1}{1}}(u_0, u_1, u_2) \quad \text{and} \quad (u_0, u_1, u_2) = \mu_{\binom{1}{1}}(v_0, v_1, v_2).$$

We also note that both are mutations and that there is no other direction of mutation. Therefore, we get the following summary:

$$\mathcal{X}_1 \xrightarrow{\text{ev}_1} (G^{1,1}, \pi_G), \quad \mathcal{X}_1 \xrightarrow{\text{ev}_1} (G^{1,w_0}, \pi_G),$$

$$\begin{array}{ccc} & \mathcal{X}_{1\bar{1}} & \\ & \uparrow & \searrow \text{ev}_{1\bar{1}} \\ \mathcal{X}_1 \xrightarrow{\text{ev}_{\bar{1}}} (G^{w_0,1}, \pi_G), & \mu_{\binom{1}{1}} & (G^{w_0,w_0}, \pi_G). \\ & \downarrow & \nearrow \text{ev}_{\bar{1}1} \\ & \mathcal{X}_{\bar{1}1} & \end{array}$$

10.3. Cluster $\mathcal{X}$ -varieties related to $(\text{SL}(2, \mathbb{C}), \pi_*)$. Still applying formula (10.2), but this time to the Semenov-Tian-Shansky Poisson bracket given by Proposition 2.2, and still using the previous coordinate functions t_{ij} , it is easy to prove that, in the matrix case, the Poisson bracket on (G, π_*) is given by the following relations:

$$\begin{cases} \{t_{11}, t_{12}\}_* = t_{12}t_{22}, & \{t_{11}, t_{21}\}_* = -t_{21}t_{22}, \\ \{t_{11}, t_{22}\}_* = 0, & \{t_{12}, t_{21}\}_* = t_{11}t_{22} - t_{22}^2, \\ \{t_{12}, t_{22}\}_* = t_{12}t_{22}, & \{t_{21}, t_{22}\}_* = -t_{21}t_{22}. \end{cases}$$

10.3.1. Evaluation maps for $(\mathrm{SL}(2, \mathbb{C}), \pi_*)$. It is easy to check that the evaluation $\mathrm{ev}_1^{\mathrm{dual}} : \mathcal{X}_{[1]_{\mathfrak{R}}} \rightarrow (G, \pi_*)$, parameterizing the union F_{w_0} over $t \in H$ of the varieties F_{t, s_1} , given by (2.11), is Poisson. Indeed, it is given by the expression

$$\begin{aligned} \mathrm{ev}_1^{\mathrm{dual}}(x_0, t) &= H^1(x_0)E^1 \widehat{w_0} H_1(t)(F^1)^{-1}H^1(x_0^{-1}) \in F_{w_0} \\ &= \begin{pmatrix} t + t^{-1} & -x_0 t^{-1} \\ x_0^{-1}t & 0 \end{pmatrix}. \end{aligned}$$

Then the evaluations $\mathrm{ev}_{\bar{1}\bar{1}}^{\mathrm{dual}} : \mathcal{X}_{[\bar{1}\bar{1}]_{\mathfrak{R}}} \rightarrow BB_-$ and $\mathrm{ev}_{1\bar{1}}^{\mathrm{dual}} : \mathcal{X}_{[1\bar{1}]_{\mathfrak{R}}} \rightarrow BB_-$, parameterizing the variety BB_- , are obtained by the following computation:

$$\begin{aligned} \mathrm{ev}_{\bar{1}\bar{1}}^{\mathrm{dual}}(y_0, y_1, t) &= H^1(y_0)F^1 \mathrm{ev}_1^{\mathrm{dual}}(y_1, t) (F^1)^{-1}H^1(y_0^{-1}) \\ &= \begin{pmatrix} t^{-1}(1 + y_1) + t & -y_0 y_1 t^{-1} \\ y_0^{-1}(t(1 + y_1^{-1}) + t^{-1}(1 + y_1)) & -y_1 t^{-1} \end{pmatrix}, \\ \mathrm{ev}_{1\bar{1}}^{\mathrm{dual}}(\tilde{y}_0, \tilde{y}_1, t) &= \begin{pmatrix} t^{-1/2}(1 + \tilde{y}_1^{-1}) + t & -t^{-1}\tilde{y}_0(1 + \tilde{y}_1^{-1}) \\ \tilde{y}_0^{-1}(t + t^{-1}\tilde{y}_1^{-1}) & -\tilde{y}_1^{-1}t \end{pmatrix}. \end{aligned}$$

It is straightforward to check that $\mu_{[\bar{1}\bar{1}]_{\mathfrak{R}} \rightarrow [1\bar{1}]_{\mathfrak{R}}} : (y_0, y_1, t) \mapsto (\tilde{y}_0, \tilde{y}_1, t)$.

These evaluations are particular cases of the twisted evaluations described in Subsection 5.4. The remaining twisted evaluations $\hat{\mathrm{ev}}_{11}, \hat{\mathrm{ev}}_{1\bar{1}} : \mathcal{X}_{[11]_{\mathfrak{R}}} \rightarrow BB_-$ and $\hat{\mathrm{ev}}_{\bar{1}\bar{1}} : \mathcal{X}_{[\bar{1}\bar{1}]_{\mathfrak{R}}} \rightarrow BB_-$, which are described in Subsection 7.2, also parameterize the variety BB_- . They are given by the formulas

$$\begin{aligned} \hat{\mathrm{ev}}_{11}(z_0, z_1, t) &= \hat{\mathrm{ev}}_{\bar{1}\bar{1}}(z_0, z_1, t) = H^1(z_0)E^1 \hat{\mathrm{ev}}_1(z_1, t) (E^1)^{-1}H^1(z_0^{-1}) \\ &= \begin{pmatrix} (1 + z_1^{-1})t + t^{-1} & -z_0((1 + z_1^{-1})t + (1 + z_1)t^{-1}) \\ z_0^{-1}z_1^{-1}t & -z_1^{-1}t \end{pmatrix}, \\ \hat{\mathrm{ev}}_{\bar{1}\bar{1}}(y_0, y_1, t) &= \mathrm{ev}_{\bar{1}\bar{1}}^{\mathrm{dual}}(y_0, y_1, t) \\ &= \begin{pmatrix} t^{-1}(1 + y_1) + t & -t^{-1}y_0 y_1 \\ y_0^{-1}(t(1 + y_1^{-1}) + t^{-1}(1 + y_1)) & -y_1 t^{-1} \end{pmatrix}. \end{aligned}$$

It is easy to check that all these maps are Poisson when the matrices (respectively, quivers) establishing the Poisson structure on the related seed $\mathcal{X}$ -tori are given by the matrices (respectively, quivers):

$$(10.5) \quad \eta(11) = \eta(1\bar{1}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \eta(\bar{1}\bar{1}) = \eta(1\bar{1}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\begin{array}{ccc} \circ \longleftrightarrow \bullet & \circ & \circ \longleftrightarrow \bullet \quad \circ \end{array}$$

Therefore, the truncation map (4.5) provides a way to pass from the Poisson structures defined by (10.4) to the Poisson structures defined by (10.4). We get a cluster $\mathcal{X}$ -variety, denoted $\mathcal{X}_{e \leq e}$, for the variety F_{w_0} , and two isomorphic cluster $\mathcal{X}$ -varieties for the variety BB_- , denoted $\mathcal{X}_e$ and $\mathcal{X}_{w_0}$, and associated with the cluster variables (y_0, y_1, t) and (z_0, z_1, t) , respectively.

10.3.2. Remarks about evaluation maps for $(\mathrm{PGL}(2, \mathbb{C}), \pi_*)$. The careful reader will observe that the evaluation maps related to $(\mathrm{SL}(2, \mathbb{C}), \pi_*)$ we have just obtained slightly differ from the twisted evaluation maps of §7. Again, this is because the Lie group $\mathrm{SL}(2, \mathbb{C})$ is not of adjoint type. According to Remark 7.14, the corresponding

evaluation maps for $G = \mathrm{PGL}(2, \mathbb{C})$ look like this:

$$\begin{aligned}
\underline{\mathrm{ev}}_1^{\mathrm{dual}}(x_0, t) &= H^1(x_0)E^1\widehat{w}_0H^1(t)(F^1)^{-1}H^1(x_0^{-1}) \in F_{w_0} \\
&= \begin{pmatrix} t^{1/2} + t^{-1/2} & -x_0t^{-1/2} \\ x_0^{-1}t^{1/2} & 0 \end{pmatrix}, \\
\underline{\mathrm{ev}}_{\bar{1}\bar{1}}^{\mathrm{dual}}(y_0, y_1, t) &= H^1(y_0)F^1 \underline{\mathrm{ev}}_1^{\mathrm{dual}}(y_1, t) (F^1)^{-1}H^1(y_0^{-1}) \\
&= \begin{pmatrix} t^{-1/2}(1+y_1) + t^{1/2} & -y_0y_1t^{-1/2} \\ y_0^{-1}(t^{1/2}(1+y_1^{-1}) + t^{-1/2}(1+y_1)) & -y_1t^{-1/2} \end{pmatrix}, \\
\underline{\mathrm{ev}}_{\bar{1}\bar{1}}^{\mathrm{dual}}(\tilde{y}_0, \tilde{y}_1, t) &= \begin{pmatrix} t^{-1/2}(1+\tilde{y}_1^{-1/2}) + t^{1/2} & -t^{-1/2}\tilde{y}_0(1+\tilde{y}_1^{-1}) \\ \tilde{y}_0^{-1}(t^{1/2} + t^{-1/2}\tilde{y}_1^{-1}) & -\tilde{y}_1^{-1}t^{1/2} \end{pmatrix}, \\
\hat{\underline{\mathrm{ev}}}_{11}(z_0, z_1, t) &= \hat{\underline{\mathrm{ev}}}_{11}(z_0, z_1, t) \\
&= H^1(z_0)E^1 \hat{\underline{\mathrm{ev}}}_1(z_1, t) (E^1)^{-1}H^1(z_0^{-1}) \\
&= \begin{pmatrix} (1+z_1^{-1})t^{1/2} + t^{-1/2} & -z_0((1+z_1^{-1})t^{1/2} + (1+z_1)t^{-1/2}) \\ z_0^{-1}z_1^{-1}t^{1/2} & -z_1^{-1}t^{1/2} \end{pmatrix}, \\
\hat{\underline{\mathrm{ev}}}_{\bar{1}\bar{1}}^{\mathrm{dual}}(y_0, y_1, t) &= \underline{\mathrm{ev}}_{\bar{1}\bar{1}}^{\mathrm{dual}}(y_0, y_1, t) \\
&= \begin{pmatrix} t^{-1/2}(1+y_1) + t^{1/2} & -t^{-1/2}y_0y_1 \\ y_0^{-1}(t^{1/2}(1+y_1^{-1}) + t^{-1/2}(1+y_1)) & -y_1t^{-1/2} \end{pmatrix}.
\end{aligned}$$

10.3.3. How to use the saltation map. If the evaluations $\underline{\mathrm{ev}}_{\bar{1}\bar{1}}^{\mathrm{dual}}(y_0, y_1, t)$ and $\hat{\underline{\mathrm{ev}}}_{11}(z_0, z_1, t)$ parameterize the same element, we quickly check with the expressions above that the map $\varphi : (y_0, y_1, t) \mapsto (z_0, z_1, t)$ is given by

$$(10.6) \quad \begin{cases} z_0 = y_0(1+y_1^{-1})^{-1}(1+y_1^{-1}t^2)^{-1}, \\ z_1 = t^2y_1^{-1}. \end{cases}$$

Before linking the map φ with the cluster combinatorics we have developed, we stress (again) that saltations are really needed in the story because, since the variable t is a Casimir function, we cannot expect to obtain a formula such as $z_1 = t^2y_1^{-1}$ by only cluster transformations. Now, we note that the cover $\mathfrak{p}_{\mathcal{X}} : \mathcal{X}_{[\bar{1}\bar{1}]\mathfrak{R}} \rightarrow \mathcal{X}_{[\bar{1}\bar{1}]\mathfrak{R}}$ is of degree 2 and is given by the formula

$$\mathfrak{p}_{\mathcal{X}} : (y_1, y_2, t) \mapsto (y_1, y_2, t^2).$$

We are going to prove that $\mathfrak{p}_{\mathcal{X}} \circ \varphi = \Xi_{s_1} \circ \mathfrak{p}_{\mathcal{X}}$, where Ξ_{s_1} denotes the birational Poisson isomorphism given by Corollary 8.9. For this, first we describe the saltation Ξ_1 given by (8.8). It is associated with the following generalized cluster transformation, acting on every element $(x_0, x_1, x_2) \in \mathcal{X}_{\bar{1}\bar{1}}$:

$$\begin{aligned}
\bar{\mu}_{\bar{1}\bar{1}} \circ \zeta_1 \circ \mu_{\bar{1}\bar{1} \rightarrow \bar{1}\bar{1}}(x_0, x_1, x_2) &= \mu_{(1)} \circ \mu_{(2)} \circ \mu_{(1)}(x_0, x_1, x_2) \\
&= \mu_{(1)} \circ \mu_{(2)}(x_0(1+x_1), x_1^{-1}, x_2(1+x_1)) \\
&= \mu_{(1)}(x_0(1+x_1), x_1^{-1}, x_2^{-1}(1+x_1)^{-1}) \\
&= (x_0, x_1, x_2^{-1}x_1^{-1}).
\end{aligned}$$

Therefore,

$$\Xi_1(x_0, x_1, t) = (x_0, x_1^{-1}t^{-1}, t),$$

because

$$\Xi_1 \circ \mathbf{t}_{\binom{1}{2}(t)} = \mathbf{t}_{\binom{1}{1}(t)} \circ \mu_{\binom{1}{1}} \circ \mu_{\binom{1}{2}} \circ \mu_{\binom{1}{1}}.$$

We get the following formula for the birational isomorphism Ξ_{s_1} :

$$\begin{aligned} \Xi_{s_1}(y_0, y_1, t) &= \mu_{[\bar{1}\bar{1}]_{\mathfrak{R}} \rightarrow [\bar{1}\bar{1}]_{\mathfrak{R}}} \circ \Xi_1 \circ \mu_{[\bar{1}\bar{1}]_{\mathfrak{R}} \rightarrow [\bar{1}\bar{1}]_{\mathfrak{R}}}(y_0, y_1, t) \\ &= \mu_{[\bar{1}\bar{1}]_{\mathfrak{R}} \rightarrow [\bar{1}\bar{1}]_{\mathfrak{R}}} \circ \Xi_1(y_0(1 + y_1^{-1})^{-1}, y_1^{-1}, t) \\ (10.7) \quad &= \mu_{[\bar{1}\bar{1}]_{\mathfrak{R}} \rightarrow [\bar{1}\bar{1}]_{\mathfrak{R}}}(y_0(1 + y_1^{-1})^{-1}, y_1 t^{-1}, t) \\ &= (y_0(1 + y_1^{-1})^{-1}(1 + y_1^{-1}t)^{-1}, y_1^{-1}t, t). \end{aligned}$$

Now it is clear that $\mathbf{p}_{\mathcal{X}} \circ \varphi = \Xi_{s_1} \circ \mathbf{p}_{\mathcal{X}}$. Moreover, the way the cluster varieties $\mathcal{X}_e$ and $\mathcal{X}_{s_1}$ are related by the saltation Ξ_1 is described by the elementary W -permutohedron pictured in Figure 31. We note, for the reader who prefers to deal with $\mathrm{PGL}(2, \mathbb{C})$, that things are

FIGURE 31. Saltation and cluster $\mathcal{X}$ -varieties for $\mathfrak{g} = A_1$.

much simpler for him, because the covering $\mathbf{p}_{\mathcal{X}}$ is then the identity map on seed $\mathcal{X}$ -tori, while the formulas for the saltation and cluster transformations remain unchanged.

10.4. Evaluation maps for $(\mathrm{SL}(2, \mathbb{C})^*, \pi_{G^*})$. First, we recall that the set $\mathrm{SL}(2, \mathbb{C})^*$ has the following description:

$$\begin{aligned} (10.8) \quad \mathrm{SL}(2, \mathbb{C})^* &= \left\{ \left(\begin{pmatrix} t_{11}^+ & t_{12}^+ \\ 0 & t_{22}^+ \end{pmatrix}, \begin{pmatrix} t_{11}^- & 0 \\ t_{21}^- & t_{22}^- \end{pmatrix} \right) : \right. \\ &\quad \left. t_{11}^- = (t_{11}^+)^{-1}, \quad t_{22}^- = (t_{22}^+)^{-1}, \quad t_{11}^+ t_{22}^+ = 1, \quad t_{ij}^\pm \in \mathbb{C} \right\}. \end{aligned}$$

Now, recall its Poisson structure π_{G^*} . Again, we denote by $t_{ij}^\pm$ the corresponding coordinate functions. The Poisson bracket we are looking for on $\mathrm{SL}(2, \mathbb{C})^*$ is given by

$$\begin{cases} \{t_{11}^\pm, t_{12}^\pm\}_{G^*} = \pm t_{11}^\pm t_{12}^\pm, & \{t_{11}^\pm, t_{21}^\pm\}_{G^*} = \mp t_{11}^\pm t_{21}^\pm, \\ \{t_{11}^\pm, t_{22}^\pm\}_{G^*} = 0, & \{t_{12}^\pm, t_{21}^\pm\}_{G^*} = t_{22}^\pm t_{11}^\pm, \\ \{t_{12}^\pm, t_{22}^\pm\}_{G^*} = \pm t_{12}^\pm t_{22}^\pm, & \{t_{21}^\pm, t_{22}^\pm\}_{G^*} = \mp t_{21}^\pm t_{22}^\pm. \end{cases}$$

Because of the Poisson covering $\phi : (\mathrm{SL}(2, \mathbb{C})^*, \pi_{G^*}) \rightarrow (\mathrm{SL}(2, \mathbb{C}), \pi_*)$ of degree 2, we use the previous evaluation maps on $(\mathrm{SL}(2, \mathbb{C}), \pi_*)$ to get the evaluation maps on $(\mathrm{SL}(2, \mathbb{C})^*, \pi_{G^*})$. For that, we introduce the following notation for every nonzero complex number x :

$$E^1(x) = H^1(x)E^1H^1(x^{-1}) \quad \text{and} \quad F^1(x) = H^1(x^{-1})F^1H^1(x).$$

Now, if the evaluations $\mathrm{ev}_{11}^{\mathrm{dual}}(y_0, y_1, t)$ and $\widehat{\mathrm{ev}}_{11}(z_0, z_1, t)$ give one and the same element $nh^2n_-^{-1}$ such that n (respectively, n_-) is an upper (respectively, lower) triangular matrix with the number 1 on the diagonal entries and h is a diagonal matrix, then we can

evaluate h , n , and n_- in the following way, using for example a computation in the spirit of Lemma 9.5:

$$\begin{cases} h = H^1(-t^{-1}z_1) = H^1(-ty_1^{-1}), \\ n = \left[\left[F^1(y_0) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \right]_+ = E^1(y_0), \\ n_- = \left[E^1(z_0) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right]_- = F^1(z_0^{-1}). \end{cases}$$

Since the map Ξ_{s_1} is such that $\mathfrak{p}_{\mathcal{X}} \circ \varphi = \Xi_{s_1} \circ \mathfrak{p}_{\mathcal{X}}$, where $\varphi : (y_0, y_1, t) \mapsto (z_0, z_1, t)$, these formulas are in agreement with Theorem 9.12, which states that

$$\begin{cases} \text{ev}_{11}^+(\mathbf{y}) = E^1(y_0)H^1(-ty_1^{-1}), \\ \text{ev}_{11}^-(\mathbf{y}) = F^1(\Xi_{s_1}(\mathbf{y})_0^{-1})H^1(-ty_1^{-1})^{-1}, \end{cases}$$

and

$$\begin{cases} \text{ev}_{11}^+(\mathbf{z}) = E^1(\Xi_{s_1}^{-1}(\mathbf{z})_0)H^1(-t^{-1}z_1), \\ \text{ev}_{11}^-(\mathbf{z}) = F^1(z_0^{-1})H^1(-t^{-1}z_1)^{-1}. \end{cases}$$

Finally, we get the following description of G^* , using matrices similar to that given by (10.8), which involves our cluster variables and their map $\varphi : (y_0, y_1, t) \mapsto (z_0, z_1, t)$ given by (10.6):

$$\left(\begin{pmatrix} (-ty_1^{-1})^{1/2} & (-ty_1^{-1})^{-1/2}y_0 \\ 0 & (-ty_1^{-1})^{-1/2} \end{pmatrix}, \begin{pmatrix} (-ty_1^{-1})^{-1/2} & 0 \\ (-ty_1^{-1})^{-1/2}z_0^{-1} & (-ty_1^{-1})^{1/2} \end{pmatrix} \right).$$

Thus, we are in an optimal position for the quantization process. Indeed, the R -matrix associated with the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ associated with the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and the related quantum group $\mathcal{F}_q(\text{SL}(2, \mathbb{C})^*)$ are given, respectively, by the formulas

$$(10.9) \quad \begin{aligned} \mathcal{R} &= q^{\frac{1}{2}H \otimes H} E \otimes F, \\ L^+ &= \begin{pmatrix} (qK)^{1/2} & (qK)^{1/2}(q - q^{-1})F \\ 0 & (qK)^{-1/2} \end{pmatrix}, \\ L^- &= \begin{pmatrix} (qK)^{-1/2} & 0 \\ (qK)^{-1/2}(q - q^{-1})E & (qK)^{1/2} \end{pmatrix}. \end{aligned}$$

10.5. Quantum evaluation maps for algebra $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$. The dual Poisson-Lie group (G^*, π_{G^*}) is the semiclassical limit of the quantum group $\mathcal{F}_q(G^*)$, which is isomorphic (as a Hopf algebra) to the very famous quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$. As a conclusion to this work, we give the quantum picture for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. For technical reasons, we consider the quantized universal enveloping algebra $\mathcal{U}_{q^{-1}}(\mathfrak{g})$ instead of $\mathcal{U}_q(\mathfrak{g})$. It is the $\mathbb{C}(q)$ -algebra generated by E , F , and K with relations

$$KE = q^{-2}EK, \quad KF = q^2FK, \quad \text{and} \quad EF - FE = \frac{K - K^{-1}}{q^{-1} - q}.$$

We define the quantum tori $\mathcal{X}_{[\bar{1}\bar{1}]}^q$ and $\mathcal{X}_{[11]}^q$ as the $\mathbb{C}(q)$ -algebra generated (respectively) by the elements Y_0, Y_1, T and Z_0, Z_1, T with the q -commutation relations

$$\begin{aligned} Y_0Y_1 &= q^2Y_1Y_0 & \text{and} & & Z_0Z_1 &= q^{-2}Z_1Z_0; \\ Y_0T &= TY_0 & \text{and} & & Z_0T &= TZ_0; \\ TY_1 &= Y_1T & \text{and} & & TZ_1 &= Z_1T. \end{aligned}$$

In particular, it is clear that the seed $\mathcal{X}$ -tori $\mathcal{X}_{[\bar{1}\bar{1}]}^q$ and $\mathcal{X}_{[11]}^q$, whose Poisson structures are given by (10.4), are the semiclassical limits of these quantum tori. Luckily, the

quantum evaluation maps for $\mathcal{U}_{q^{-1}}(\mathfrak{g})$ come without effort from the semiclassical evaluation maps we have obtained for $(\mathrm{SL}(2, \mathbb{C})^*, \pi_{G^*})$. (We do not forget to switch q into q^{-1} in formula (10.9), according to the switch $\mathcal{U}_q(\mathfrak{g}) \leftrightarrow \mathcal{U}_{q^{-1}}(\mathfrak{g})$.) In particular, it is straightforward to check that the following map is an algebra morphism:

$$\begin{cases} \mathrm{ev}^{q^{-1}}(K) = -qTY_1^{-1} = -qT^{-1}Z_1, \\ \mathrm{ev}^{q^{-1}}(F) = -(q^{-1} - q)^{-1}T^{-1}Y_1Y_0, \\ \mathrm{ev}^{q^{-1}}(E) = (q^{-1} - q)^{-1}Z_0^{-1} \end{cases}$$

if

$$\begin{cases} Z_0 = Y_0(1 + qY_1^{-1})^{-1}(1 + qY_1^{-1}T^2)^{-1}, \\ Z_1 = T^2Y_1^{-1}. \end{cases}$$

Moreover, the link between the Y_i and the Z_j is given by quantizing the birational Poisson isomorphism φ . To see that, we use the quantization formulas of [FG07a]: in the case where $|\varepsilon_{ik}| = 1$, we get the following *quantum mutations*:

$$(10.10) \quad X_{\mu_k^q(i)} = \begin{cases} X_k^{-1} & \text{if } i = k, \\ X_i X_k^{[\varepsilon_{ik}]_+} (1 + qX_k)^{-\varepsilon_{ik}} & \text{if } i \neq k. \end{cases}$$

Therefore, quantization of the computation (10.7) gives us the following identities, using a still mysterious map denoted Ξ_1^q :

$$\begin{aligned} \Xi_{s_1}^q(Y_0, Y_1, T) &= \mu_{[\bar{1}\bar{1}]_{\mathfrak{R}} \rightarrow [\bar{1}\bar{1}]_{\mathfrak{R}}}^q \circ \Xi_1^q \circ \mu_{[\bar{1}\bar{1}]_{\mathfrak{R}} \rightarrow [\bar{1}\bar{1}]_{\mathfrak{R}}}^q(Y_0, Y_1, T) \\ &= \mu_{[\bar{1}\bar{1}]_{\mathfrak{R}} \rightarrow [\bar{1}\bar{1}]_{\mathfrak{R}}}^q \circ \Xi_1^q(Y_0(1 + qY_1^{-1})^{-1}, Y_1^{-1}, T) \\ &= \mu_{[\bar{1}\bar{1}]_{\mathfrak{R}} \rightarrow [\bar{1}\bar{1}]_{\mathfrak{R}}}^q(Y_0(1 + qY_1^{-1})^{-1}, Y_1T^{-1}, T) \\ &= (Y_0(1 + qY_1^{-1})^{-1}(1 + qY_1^{-1}t)^{-1}, Y_1^{-1}T, T). \end{aligned}$$

Moreover, we have $\mathbf{p}_{\mathcal{X}} \circ \varphi^q = \Xi_{s_1}^q \circ \mathbf{p}_{\mathcal{X}}$, where φ^q is the quantization of the map φ , i.e., $\varphi^q : (Y_0, Y_1, T) \rightarrow (Z_0, Z_1, T)$. We stress, however, that there is still something strange in this story. Indeed, by keeping the tropicalization formula, we therefore also get tropical quantum mutations from (10.10); we use them to introduce the quantum saltation $\tilde{\Xi}_1^q$, defined by intertwining generalized quantum cluster transformations with truncation maps, by mimicking the previous computation. Thus, we get

$$\begin{aligned} \mu_{(1)}^q \circ \mu_{(2)}^q \circ \mu_{(1)}^q(X_0, X_1, X_2) &= \mu_{(1)}^q \circ \mu_{(2)}^q(X_0(1 + qX_1), X_1^{-1}, X_2(1 + qX_1)) \\ &= \mu_{(1)}^q(X_0(1 + qX_1), X_1^{-1}, X_2^{-1}(1 + qX_1)^{-1}) \\ &= (X_0, X_1, qX_2^{-1}X_1^{-1}). \end{aligned}$$

Therefore,

$$\tilde{\Xi}_1^q(X_0, X_1, T) = (X_0, qX_1^{-1}T^{-1}, T)$$

because

$$\tilde{\Xi}_1^q \circ \mathbf{t}_{(2)}^{(1)}(t) = \mathbf{t}_{(1)}^{(1)}(t) \circ \mu_{(1)}^q \circ \mu_{(2)}^q \circ \mu_{(1)}^q.$$

But, unfortunately, it is clear that $\tilde{\Xi}_1^q \neq \Xi_1^q$.

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REFERENCES

- [AM94] A. Yu. Alekseev and A. Z. Malkin, *Symplectic structures associated to Lie–Poisson groups*, Comm. Math. Phys. **162** (1994), no. 1, 147–173. MR1272770 (95g:58074)
- [BFZ05] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky, *Cluster algebras. III. Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), no. 1, 1–52. MR2110627 (2005i:16065)
- [BZ01] Arkady Berenstein and Andrei Zelevinsky, *Tensor product multiplicities, canonical bases and totally positive varieties*, Invent. Math. **143** (2001), no. 1, 77–128. MR1802793 (2002c:17005)
- [BB05] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter groups*, Grad. Texts in Math., vol. 231, Springer, New York, 2005. MR2133266 (2006d:05001)
- [B] Renaud Brahami, *Cluster $\mathcal{X}$ -varieties and dual Poisson–Lie groups. II* (in preparation).
- [EL07] Sam Evens and Jiang-Hua Lu, *Poisson geometry of the Grothendieck resolution of a complex semisimple group*, Mosc. Math. J. **7** (2007), no. 4, 613–642. (English) MR2372206 (2008k:53181)
- [FG06a] Vladimir Fock and Alexander Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. No. 103 (2006), 1–211. MR2233852 (2009k:32011)
- [FG06b] ———, *Cluster $\mathcal{X}$ -varieties, amalgamation, and Poisson–Lie groups*, Algebraic Geometry and Number Theory, Progr. Math., vol. 253, Birkhäuser Boston, Boston, MA, 2006, pp. 27–68. MR2263192 (2008b:22009)
- [FG07a] ———, *Cluster ensembles, quantization and the dilogarithm*, arXiv:math/0311245.
- [FG07b] ———, *The quantum dilogarithm and unitary representations quantized cluster varieties*, arXiv:math/0702397.
- [FR07] Sergey Fomin and Nathan Reading, *Root systems and generalized associahedra*, Geometric Combinatorics, IAS/Park City Math. Ser., vol. 13, Amer. Math. Soc., Providence, RI, 2007, pp. 63–131. MR2383126
- [FST] Sergey Fomin, Michael Shapiro, and Dylan Thurston, *Cluster algebras and triangulated surfaces. Pt. I: Cluster complexes*, arXiv:math/0608367.
- [FZ02] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529. MR1887642 (2003f:16050)
- [FZ03a] ———, *Cluster algebras. II. Finite type classification*, Invent. Math. **154** (2003), no. 1, 63–121. MR2004457 (2004m:17011)
- [FZ07] ———, *Cluster algebras. IV. Coefficients*, Compos. Math. **143** (2007), no. 1, 112–164. MR2295199 (2008d:16049)
- [FZ99] ———, *Double Bruhat cells and total positivity*, J. Amer. Math. Soc. **12** (1999), no. 2, 335–380. MR1652878 (2001f:20097)
- [FH91] W. Fulton and J. Harris, *Representation theory*, Grad. Texts in Math., vol. 129, Springer-Verlag, New York, 1991. MR1153249 (93a:20069)
- [GSV03] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein, *Cluster algebras and Poisson geometry*, Mosc. Math. J. **3** (2003), no. 3, 899–934. (English) MR2078567 (2005i:53104)
- [HKKR00] T. Hoffmann, J. Kellendonk, N. Kutz, and N. Reshetikhin, *Factorization dynamics and Coxeter–Toda lattices*, Comm. Math. Phys. **212** (2000), no. 2, 297–321. MR1772248 (2001g:37098)
- [KZ02] Mikhail Kogan and Andrei Zelevinsky, *On symplectic leaves and integrable systems in standard complex semisimple Poisson–Lie groups*, Int. Math. Res. Not. **2002**, no. 32, 1685–1702. MR1916837 (2003k:53107)
- [LW90] Jiang-Hua Lu and Alan Weinstein, *Poisson Lie groups, dressing transformations, and Bruhat decompositions*, J. Differential Geom. **31** (1990), no. 2, 501–526. MR1037412 (91c:22012)

- [P05] A. Postnikov, *Permutohedra, associahedra, and beyond*, [arXiv:math/0507163v1](#). Int. Math. Res. Not. **2009**, no. 6, 1026–1106. MR2487491 (2010g:05399)
- [R03] N. Reshetikhin, *Integrability of characteristic Hamiltonian systems on simple Lie groups with standard Poisson Lie structure*, Comm. Math. Phys. **242** (2003), 1–29. MR2018267 (2004m:37108)
- [STS85] M. Semenov-Tian-Shansky, *Dressing transformations and Poisson group actions*, Publ. Res. Inst. Math. Sci. **21** (1985), no. 6, 1237–1260. MR0842417 (88b:58057)
- [STS93] ———, *Poisson–Lie groups. The quantum duality principle and the twisted quantum double*, Teoret. Mat. Fiz. **93** (1992), no. 2, 302–329; English transl., Theoret. and Math. Phys. **93** (1992), no. 2, 1292–1307 (1993). MR1233548 (94e:58007)

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