LINEARLY CONTROLLED ASYMPTOTIC DIMENSION OF THE FUNDAMENTAL GROUP OF A GRAPH-MANIFOLD

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ABSTRACT. We prove the estimate $\ell\text{-asdim } \pi_1(M) \leq 7$ for the linearly controlled asymptotic dimension of the fundamental group of any 3-dimensional graph-manifold M . As applications, we show that the universal cover $\widetilde{M}$ of M is an absolute Lipschitz retract and admits a quasisymmetric embedding into the product of 8 metric trees.

§1. INTRODUCTION

A motivation of the present work is a result of Bell and Dranishnikov [3, Theorem 1'] that the asymptotic dimension of a graph-group whose vertex groups have finite asymptotic dimensions is also finite. The fundamental groups of graph-manifolds are graph-groups. However, it is unclear whether there exists an analog of the Bell–Dranishnikov result for the linearly controlled asymptotic dimension. A construction of suitable coverings for the universal cover of a graph-manifold, which we present below, allows us to prove the following statement.

Theorem 1. *For the fundamental group of a graph-manifold M , taken with any word metric, and for its linearly controlled asymptotic dimension, we have $\ell\text{-asdim } \pi_1(M) \leq 7$.*

Corollary 1. *Suppose $\widetilde{M}$ is the universal cover of a graph-manifold M . Then we have $3 \leq \dim_{\text{AN}} \widetilde{M} \leq 7$, where $\dim_{\text{AN}}$ is the Assouad–Nagata dimension, M being taken with any Riemannian metric and $\widetilde{M}$ with the metric lifted from M .*

The next two corollaries are obtained as applications of Corollary 1 and [9, Theorem 1.3, Theorem 1.5].

Corollary 2. *Let $\widetilde{M}$ be the universal cover of a graph-manifold M taken with the Riemannian distance d lifted from M . Then for all sufficiently small $p \in (0, 1)$, there exists a bi-Lipschitz embedding of $(\widetilde{M}, d^p)$ into the product of 8 metric trees; in particular, $(\widetilde{M}, d)$ admits a quasisymmetric embedding into the product of 8 metric trees.*

Corollary 3. *The universal cover $\widetilde{M}$ of a graph-manifold is an absolute Lipschitz retract; i.e., given a metric space X , there is $C > 0$ such that for every subset $A \subset X$ and every λ -Lipschitz map $f : A \rightarrow \widetilde{M}$, $\lambda > 0$, there exists a $C\lambda$ -Lipschitz extension $\bar{f} : X \rightarrow \widetilde{M}$ of f .*

§2. PRELIMINARIES

In this section we recall some basic definitions and notation. Let X be a metric space. We denote by $|xy|$ the distance between $x, y \in X$, and $d(U, V) := \inf\{|uv| : u \in U, v \in V\}$ is the distance between $U, V \subset X$. We write $B_r(x) = \{x' \in X : |xx'| \leq r\}$ for the ball

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with center x and radius r . A map $f : X \rightarrow Y$ is said to be *quasi-isometric* if there exist $\lambda \geq 1, C \geq 0$ such that

$$\frac{1}{\lambda}|xy| - C \leq |f(x)f(y)| \leq \lambda|xy| + C$$

for each $x, y \in X$. Metric spaces X and Y are *quasi-isometric* if there is a quasi-isometric map $f : X \rightarrow Y$ such that $f(X)$ is a net in Y . In this case f is called a *quasi-isometry*.

We say that a family $\mathcal{U}$ of subsets of X is a *covering* if for each point $x \in X$ there is a subset $U \in \mathcal{U}$ such that $x \in U$. A family $\mathcal{U}$ of sets is *disjoint* if each two sets $U, V \in \mathcal{U}$ are disjoint. The union $\mathcal{U} = \bigcup \{U^\alpha : \alpha \in \mathcal{A}\}$ of disjoint families U^α is said to be *n-colored*, where $n = |\mathcal{A}|$ is the cardinality of $\mathcal{A}$.

Also recall that a family $\mathcal{U}$ is *D-bounded* if the diameter of every $U \in \mathcal{U}$ does not exceed D , $\text{diam } U \leq D$. An *n-colored* family of sets $\mathcal{U}$ is *r-disjoint* if for every color $\alpha \in \mathcal{A}$ and each two sets $U, V \in \mathcal{U}^\alpha$ we have $d(U, V) \geq r$.

The linearly controlled asymptotic dimension is a version of Gromov's asymptotic dimension asdim .

Definition 1 (Roe [10]). The *linearly controlled asymptotic dimension* of a metric space X , ℓ - $\text{asdim } X$, is the smallest integer n such that for any sufficiently large real R there exists an $(n + 1)$ -colored, R -disjoint, and CR -bounded covering of the space X , where the number $C > 0$ is independent of R .

A *tripod* in a geodesic metric space X is a union of three geodesic segments $xt \cup yt \cup zt$ that have only one common point t . We say that t is the *center* of the tripod. A geodesic metric space X is called a *metric tree* if each triangle in it is a tripod (possibly, degenerate).

Let T be a metric tree. We recall the well-known construction of a 2-colored, r -disjoint, $3r$ -bounded covering of T (see [10]). Fix some vertex $v \in T$. Suppose $k \geq 1$ is an integer and consider the annuli

$$A_k := \{x \in T : kr \leq |vx| \leq (k + 1)r\}$$

of width r centered at v . For $x, y \in A_k$, we say that $x \sim_k y$ if $|zv| \geq (k - \frac{1}{2})r$, where z is the center of the tripod with vertices x, y, v . It is easily checked (see [10]) that $\sim_k$ is an equivalence relation on A_k . We shall use the following 2-colored covering: $\mathcal{U} := \mathcal{U}^1 \cup \mathcal{U}^2$, where $\mathcal{U}^1$ consists of the ball $B_r(v)$ and the equivalence classes for $\sim_k$ corresponding to all even k . The family $\mathcal{U}^2$ consists of the equivalence classes for $\sim_k$ corresponding to all odd k . In [10] it was shown that this covering is r -disjoint and $3r$ -bounded. We call it the *standard (r, v) -covering* of T .

We also use a modified covering, where the ball $B_r(v)$ is replaced by the ball $B_d(v)$ with $0 < d \leq r$. The annuli A_k are replaced by the annuli

$$B_k := \{x \in T : (k - 1)r + d \leq |vx| \leq kr + d\}$$

of width r and centered at v . Now, for each integer $k > 1$, and also for $k = 1$ and $d \geq r/2$, on B_k we introduce the following equivalence relation: $x \sim_k y$ if and only if $|zv| \geq (k - \frac{3}{2})r + d$. If $k = 1$ and $d < r/2$, then we say that all points in B_1 are $\sim_1$ -equivalent. As before, it is easy to check that the modified covering is r -disjoint and $3r$ -bounded. We call this covering the *standard (r, d, v) -covering* of T .

Let G be a finitely generated group and $S \subset G$ a finite symmetric generating set for G ($S^{-1} = S$). Recall that a *word metric* on the group G (with respect to S) is the metric determined by the norm $\|\cdot\|_S$, where for each $g \in G$ its norm $\|g\|_S$ is the smallest number of elements of S whose product is g . In this paper we consider only finitely generated groups with a word metric.

Definition 2. By a *graph-manifold* we mean a closed three-dimensional manifold that admits a finite cover by a connected orientable manifold glued from blocks along boundary tori in a regular way. By a *block* we mean a trivial S^1 -fibration over a surface with negative Euler characteristic and nonempty boundary. A gluing of two blocks M_1 and M_2 (possibly, $M_1 = M_2$) along boundary tori $T_1 \subset \partial M_1$ and $T_2 \subset \partial M_2$ is said to be *regular* if the S^1 -fibers coming up from T_1 and T_2 are not homotopic on the glued torus.

Fix a Riemannian metric on a graph-manifold M . The following fact is well known [5].

Lemma 1. *Let Y be a compact metric space with length metric. Let X be a universal cover of Y with the metric lifted from Y . Then X is quasi-isometric to the group $\pi_1(Y)$ with any word metric.*

§3. GRAPH-MANIFOLD MODEL

3.1. Description of the model. Consider the hyperbolic plane $\mathbb{H}_\kappa^2$ with a curvature $-\kappa$ ($\kappa > 0$) such that the side of a rectangular equilateral hexagon θ in the plane $\mathbb{H}_\kappa^2$ has length 1. Let ρ be the distance between the middle points of sides that have a common adjacent side, and let δ be the diameter of θ . It is clear that $\rho \leq 2$ and $\delta \leq 3$.

We mark each second side of θ (three sides are marked) and consider a set H_0 defined as follows. Take the subgroup G_θ of the isometry group of $\mathbb{H}_\kappa^2$ generated by reflections in (three) marked sides of θ , and let H_0 be the orbit of θ with respect to G_θ . Then H_0 is a convex subset in $\mathbb{H}_\kappa^2$ divided into hexagons that are isometric to θ . Furthermore, the boundary of H_0 has infinitely many connected components each of which is a geodesic line in $\mathbb{H}_\kappa^2$.

The graph T_0 dual to the decomposition of H_0 into hexagons is the standard binary tree whose vertices are all of degree three. Any metric space isometric to H_0 will be called a θ -tree. Given a vertex p of T_0 , we denote by θ_p the corresponding hexagon in H_0 .

Every boundary line ℓ of H_0 is the union of segments of length 1, each of which is a side of a hexagon in H_0 . Thus, ℓ determines a sequence of vertices of T_0 connected consecutively by edges that form a line $\bar{\ell}$ in T_0 .

Consider the line $\mathbb{R}$ and its partition into segments $\mathbb{R} = \bigcup \{[i, i+1] : i \in \mathbb{Z}\}$. Then the graph S_0 dual to this partition is an infinite tree with all vertices having degree 2; in particular, S_0 is homeomorphic to $\mathbb{R}$. Given a vertex q of S_0 , we denote by $I_q \subset H_0$ the corresponding segment.

Remark 1. In what follows, we shall view T_0 and S_0 as metric spaces with metrics such that the length of each edge is equal to 1. Then these metric spaces are metric trees. We denote the set of vertices in T_0 by $V(T_0)$, and the set of vertices in S_0 by $V(S_0)$.

Remark 2. Given distinct vertices $p, q \in V(T_0)$ in T_0 and $x \in \theta_p, y \in \theta_q$, we have

$$(\star) \quad |pq| - 1 \leq |xy| \leq (|pq| - 1)\rho + 2\delta.$$

Similarly, given distinct $p, q \in V(S_0)$ and $x \in I_p, y \in I_q$, we have

$$(\star\star) \quad |pq| - 1 \leq |xy| \leq |pq| + 1.$$

Remark 3. Note that, for each connected subset $U \subset T_0$, the set $U' := \bigcup \{\theta_p : p \in U \cap V(T_0)\}$ is convex. Similarly, for each connected subset $V \subset S_0$, the set $V' := \bigcup \{I_q : q \in V \cap V(S_0)\}$ is convex.

Now, we describe the following metric space X .

Construction. Let T be an infinite tree, all vertices of which have countably infinite degree. Denote by $V(T)$ the set of all vertices of T and by $E(T)$ the set of all edges of T . For each $v \in V(T)$, we consider the metric product $X_v := H_0 \times \mathbb{R}$ and fix a bijection b_v between the set E_v of all edges adjacent to v and all the boundary lines in the factor H_0 . Note that the partition of H_0 into hexagons and the partition of $\mathbb{R}$ into segments give rise to a partition of X_v into sets isometric to $\theta \times [0, 1]$; we call such sets *bricks*. Furthermore, each boundary plane $\sigma \subset X_v$ is obtained as the product of $\mathbb{R}$ and a boundary line ℓ in the θ -tree factor, and the partition of X_v into bricks gives rise to a partition of σ into squares with side length 1, to be called *the grid* on the plane σ . Next, for each edge uv in $E(T)$, we consider the boundary planes $\sigma_u = b_u(uv) \times \mathbb{R}_u \subset X_u$ and $\sigma_v = b_v(uv) \times \mathbb{R}_v \subset X_v$, respectively, and glue them by an isometry that glues the grids on σ_u and σ_v , flipping the factors $\mathbb{R}_u, \mathbb{R}_v$ to the factors $b_v(uv), b_u(uv)$, respectively. We call the resulting space the *model* and denote it by X .

Recall that a complete geodesic metric space Y is called a CAT(0)-space if the following is true: for each triple of points $x, y, z \in Y$ and a point $t \in yz$, if points $\bar{x}, \bar{y}, \bar{z}$ on the plane $\mathbb{R}^2$ are such that $|xy| = |\bar{x}\bar{y}|$, $|yz| = |\bar{y}\bar{z}|$, $|xz| = |\bar{x}\bar{z}|$ and the point $\bar{t} \in [\bar{y}\bar{z}]$ satisfies $|\bar{y}\bar{t}| = |\bar{y}t|$, then $|xt| \leq |\bar{x}\bar{t}|$. A CAT(0)-space is also called a *Hadamard space*. The spaces H_0 and $\mathbb{R}$ are CAT(0)-spaces, so that the X_v , $v \in V(T)$, and X are also CAT(0)-spaces (see, e.g., [5]).

3.2. The model X as the universal cover of a closed graph-manifold. Here we construct a closed graph-manifold Q (with C^1 -smooth Riemannian nonpositively curved metric) whose universal cover $\tilde{Q}$ is isometric to X .

Let ω be the union of adjacent hexagons $\theta_1, \theta_2 \subset H_0$. Denote by $\tau : \theta_1 \rightarrow \theta_2$ the reflection with respect to the common side $\theta_1 \cap \theta_2$ and consider the surface P_0 obtained from ω by gluing the marked sides matched by τ . Then P_0 is a surface of constant curvature $-\kappa$ with geodesic boundary. The boundary ∂P_0 consists of 3 components γ_1, γ_2 , and γ_3 of length 2. The hexagon vertices subdivide every γ_i , $i = 1, 2, 3$, into a pair of segments of length 1.

Let us mark the images of the hexagons' vertices (so, there will be 6 marked points). Note that the universal cover of P_0 is isometric to H_0 . We consider the circle S_2^1 of length 2 and mark two diametrically opposite points s_1 and s_2 on it. For each boundary component γ_i ($i \in \{1, 2, 3\}$), consider an isometry $\Phi_i : \gamma_i \rightarrow S_2^1$ taking the marked points into the marked points.

Let $P := P_0 \times S_2^1$. Consider an isometric copy P' of P . All the elements of P' will be denoted by the same letters but with a prime. Let Q be the manifold obtained by gluing P and P' along the boundary tori $\gamma_i \times S_2^1 \subset \partial P$ and $S_2^1 \times \gamma'_i \subset \partial P'$ in accordance with the isometries

$$\Phi_i \times \Phi_i'^{-1} : \gamma_i \times S_2^1 \longrightarrow S_2^1 \times \gamma'_i, \quad i \in \{1, 2, 3\}.$$

Note that Q is a graph-manifold, the universal cover $\tilde{Q}$ of which consists of blocks isometric to $H_0 \times \mathbb{R}$. Any two of these blocks either are disjoint or intersect over a plane covering a torus $\gamma_i \times S_2^1$, $i \in \{1, 2, 3\}$. This plane is divided by the lines covering the circles $\{x_i\} \times S_2^1$ and $\gamma_i \times \{s_i\}$ (where $x_i \in \gamma_i$ and $s_i \in S_2^1$ are marked points) into squares with side 1. It is clear that $\tilde{Q}$ is isometric to X . We recall the following result [2, Theorem 2.1].

Theorem 2. *Any two graph-manifolds have bi-Lipschitz homeomorphic universal covers; in particular, their fundamental groups are quasi-isometric.*

This theorem shows that for any graph-manifold M its fundamental group $\pi_1(M)$ is quasi-isometric to the fundamental group $\pi_1(Q)$ of the manifold Q , and, by Lemma 1,

$\pi_1(Q)$ is quasi-isometric to the model X . Therefore, since ℓ -asdim is a quasi-isometry invariant (see, e.g., [9]), we have ℓ -asdim $\pi_1(M) = \ell$ -asdim X . So, to prove Theorem 2 it suffices to show that ℓ -asdim $X \leq 7$. For this, for each $R \in \mathbb{N}$, $R > 10$, we construct an R -disjoint, CR -bounded, 8-colored covering of the model X , where $C = 88$.

§4. COVERINGS OF THE MODEL

We briefly describe the construction of the required covering of X . We fix $R \in \mathbb{N}$, $R > 10$. First, for every block X_u , $u \in V(T)$, we construct a 4-colored, $2R$ -disjoint, and $15R$ -bounded covering $\mathcal{W}_u$ that is the product of standard 2-colored coverings of the factors in the decomposition $X_u = H_0 \times \mathbb{R}$. For different neighboring blocks, these coverings are compatible (see Lemma 2) and, taken together, form a 4-colored covering $\tilde{\mathcal{W}}$ of X . The covering $\tilde{\mathcal{W}}$ is R -disjoint (see Proposition 1 below), and this is the main result of the current section. However, all the members of $\tilde{\mathcal{W}}$ have infinite diameter. Fortunately, every member W of $\tilde{\mathcal{W}}$ has a tree-like structure, and in Subsection 5.2 we shall construct a 2-colored covering of W similar to the standard 2-colored covering of a tree. A typical W is disconnected, and every connected component of it is convex. For technical reasons, it is convenient to start with extending every $W \in \tilde{\mathcal{W}}$ to a convex $\bar{W} \supset W$, and then to cover $\bar{W}$ by a 2-colored $\mathcal{X}_W$. The extension procedure is described in Subsection 5.1. Convexity plays an important role in proving the properties of $\mathcal{X}_W$. After that, we forget about the extension $\bar{W}$ and restrict $\mathcal{X}_W$ to W . In that way, we produce the required 8-colored covering $\mathcal{X}$ of X .

4.1. Constructing a covering of a block. Put $N := 2R + 1$. Consider an arbitrary block $X_v = H_0 \times \mathbb{R} \subset X$. We choose a brick $\theta_p \times I_q \subset X_v$, where $p \in V(T_0)$ and $q \in V(S_0)$. Consider the standard (N, p) -covering $\mathcal{U} = \mathcal{U}^1 \cup \mathcal{U}^2$ of the tree T_0 (see §2). We build a 2-colored covering $\tilde{\mathcal{U}} = \tilde{\mathcal{U}}^1 \cup \tilde{\mathcal{U}}^2$ of the θ -tree as follows: for each set $U \in \mathcal{U}^i$, $i \in \{1, 2\}$, we define the set

$$\tilde{U} := \bigcup \{\theta_{p'} : p' \in U \cap V(T_0)\}$$

and consider the families

$$\tilde{\mathcal{U}}^i := \{\tilde{U} : U \in \mathcal{U}^i\}, \quad i \in \{1, 2\}.$$

It is clear that the family $\tilde{\mathcal{U}}$ is a 2-colored covering of the θ -tree. Recall that ρ is the distance between the middle points of two sides of the hexagon θ that have a common adjacent side, and δ is the diameter of θ .

Since the covering $\mathcal{U}$ is N -disjoint and $3N$ -bounded, inequality $(\star)$ implies that the covering $\tilde{\mathcal{U}}$ is $2R$ -disjoint and $((6R + 2)\rho + 2\delta)$ -bounded. Since $\rho \leq 2$, $\delta \leq 3$, and $R > 10$, we have $(6R + 2)\rho + 2\delta \leq 13R$, whence $\tilde{\mathcal{U}}$ is $2R$ -disjoint and $13R$ -bounded.

Fixing a natural $d \leq N$, we consider the standard (N, d, q) -covering $\mathcal{V} = \mathcal{V}^1 \cup \mathcal{V}^2$ of the tree S_0 (see §2). We build a 2-colored covering $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}^1 \cup \tilde{\mathcal{V}}^2$ of the line $\mathbb{R}$ as follows: for each set $V \in \mathcal{V}^i$, $i \in \{1, 2\}$, we consider the set

$$\tilde{V} := \bigcup \{I_{p'} : p' \in V \cap V(S_0)\}$$

and the families

$$\tilde{\mathcal{V}}^i := \{\tilde{V} : V \in \mathcal{V}^i\}, \quad i \in \{1, 2\}.$$

Clearly, the family $\tilde{\mathcal{V}}$ is a 2-colored covering of $\mathbb{R}$. Note that, since the family $\mathcal{V}$ is N -disjoint and $3N$ -bounded, inequality $(\star\star)$ implies that the covering $\tilde{\mathcal{V}}$ is $2R$ -disjoint and $(6R + 2)$ -bounded. Since $R > 10$, we have $6R + 2 \leq 7R$, so that the covering $\tilde{\mathcal{V}}$ is $2R$ -disjoint and $7R$ -bounded.

Consider the 4-colored covering

$$\mathcal{W} := \bigcup \{\mathcal{W}^{(i,j)} : (i,j) \in \{1,2\} \times \{1,2\}\}$$

of the block X_v , where

$$\mathcal{W}^{(i,j)} := \{U \times V : U \in \tilde{\mathcal{U}}^i, V \in \tilde{\mathcal{V}}^j\}.$$

Since the coverings $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{V}}$ are $2R$ -disjoint, so is the covering $\mathcal{W}$, and since the covering $\tilde{\mathcal{U}}$ is $13R$ -bounded and $\tilde{\mathcal{V}}$ is $7R$ -bounded, the covering $\mathcal{W}$ is $R\sqrt{7^2 + 13^2}$ -bounded, whence it is $15R$ -bounded. This covering of the block X_v will be called the (N, d) -covering with the initial brick $\theta_p \times I_q$.

Consider the block X_v with the fixed initial brick $\theta_{p_v} \times I_{q_v}$, $p_v \in V(T_0)$, $q_v \in V(S_0)$, and its boundary plane $\sigma = \ell \times \mathbb{R}_v$, where ℓ is a boundary line of the θ -tree factor of X_v . Let $\bar{\ell}$ be the line in the tree T_0 that corresponds to ℓ (see Subsection 3.1), and let $p' \in \bar{\ell}$ be the vertex nearest to p_v . We denote $n := |p_v p'|$ and put $k \equiv n \pmod{N}$, $0 \leq k < N$.

Let X_u be the block glued to σ , let $1 \leq d \leq N$ be a natural number, and let $I_{q_u} \times \theta_{p_u}$ be the brick in X_u having the common boundary square with the brick $\theta_{p'} \times I_{q_v}$. We represent the boundary plane σ as the product of the line $\mathbb{R}$ and an appropriate boundary line ℓ' in the θ -tree factor of the block X_u , $\sigma = \mathbb{R}_u \times \ell'$. Consider the line $\bar{\ell}'$ in the tree T_0 that corresponds to the line ℓ' . Then the vertex p_u belongs to $\bar{\ell}'$. We choose a vertex p'_u of T_0 such that the vertex p_u is the point of the line $\bar{\ell}'$ nearest to p'_u and the distance $|p_u p'_u|$ is equal to $N - d$.

Lemma 2. *The (N, d) -covering $\mathcal{W}_v$ with the initial brick $\theta_{p_v} \times I_{q_v}$ of the block X_v and the $(N, N - k)$ -covering $\mathcal{W}_u$ with the initial brick $I_{q_u} \times \theta_{p'_u}$ of the block X_u agree on the common boundary plane σ of the blocks X_u and X_v ; i.e., for each pair $(i, j) \in \{1, 2\} \times \{1, 2\}$, any two sets $V \in \mathcal{W}_v^{(i,j)}$ and $U \in \mathcal{W}_u^{(j,i)}$ either are disjoint, or their intersections with σ coincide.*

Proof. Indeed, since the coverings $\mathcal{W}_v$ and $\mathcal{W}_u$ are products of coverings of the corresponding factors, it suffices to check that the coverings of the factors agree. We have $\sigma = \ell \times \ell'$, and the $\mathbb{R}$ -factor of the block X_u is glued to the line ℓ . Let $\bar{\ell}$ be the line corresponding to the line ℓ in the tree T_0 . If we view the line $\bar{\ell}$ as an isometric copy of the tree S_0 , then the standard (N, p_v) -covering of the tree T_0 induces a covering $\mathcal{U}'$ of the line $\bar{\ell}$, and this is a standard $(N, N - k, p')$ -covering. Therefore, since the side of the hexagon $\theta_{p'}$ lying on ℓ is glued to the segment I_{q_u} of the $\mathbb{R}$ -factor of the block X_u , the standard $(N, N - k, q_u)$ -covering of this $\mathbb{R}$ -factor coincides with the covering $\mathcal{U}'$. The case of the other factors is similar. $\square$

4.2. Constructing compatible coverings of blocks. Recall that T is the canonical simplicial tree, all the vertices of which have infinite countable degree. We consider the metric on T such that all the edges have length 1. We fix an arbitrary vertex $v \in V(T)$ and call it the *root* of the tree T . We define the *level* function $l : V(T) \rightarrow \mathbb{Z}$ by $l(v') := |vv'|$. The coverings of blocks of the type discussed earlier, which agree with the coverings of the neighboring blocks on the common boundary planes (see Subsection 4.1), will be constructed by induction on the level m .

The base of induction : $m = 0$.

Let $\theta_p \times I_q$ be an arbitrary brick in X_v . Consider the (N, N) -covering with the initial brick $\theta_p \times I_q$ of the block X_v . We denote this covering by $\mathcal{W}_v$.

The induction step: $m \rightarrow m + 1$.

Suppose that we have already built the coverings $\mathcal{W}_{v'}$ compatible for all blocks $X_{v'}$ with $l(v') \leq m$. Consider an arbitrary vertex $u \in V(T)$ such that $l(u) = m + 1$. The block X_u has precisely one boundary plane σ along which it is glued with a block X_w such that

$l(w) = m$. X_w is already covered by $\mathcal{W}_w$, which is an (N, d_w) -covering with the initial brick $\theta_{p_w} \times I_{q_w}$. Consider the plane $\sigma = \ell \times \mathbb{R}$, the product of a boundary line ℓ in the θ -tree and the factor $\mathbb{R}$. Let $\bar{\ell}$ be the line in T_0 corresponding to ℓ (see Subsection 3.1). We take the vertex p' in $\bar{\ell}$ that is the nearest to p_w and denote $n_w := |p_w p'|$. Let $d_u \equiv n_w \pmod{N}$, $0 \leq d_u < N$. Let $I_{q_u} \times \theta_{p_u}$ be a brick in X_u that has a common boundary square with the brick $\theta_{p'} \times I_{q_w}$. We represent the boundary plane σ as the product of the line $\mathbb{R}$ and a boundary line in the θ -tree factor of the block X_u , $\sigma = \mathbb{R} \times \ell'$. Consider the line $\bar{\ell}'$ in T_0 that corresponds to the line ℓ' . It is clear that the vertex p_u belongs to $\bar{\ell}'$. We choose a vertex p'_u in T_0 such that the vertex p_u is the point nearest to p'_u on the line $\bar{\ell}'$ and the distance $|p_u p'_u|$ is equal to $N - d_u$. Let $\mathcal{W}_u$ denote the $(N, N - d_u)$ -covering of the block X_u with the initial brick $I_{q_u} \times \theta_{p'_u}$. Lemma 2 shows that the coverings $\mathcal{W}_w$ and $\mathcal{W}_u$ of the blocks X_w and X_u (respectively) agree on the boundary plane σ . $\mathcal{W}_u$ is a covering of the type we need; the induction step is completed.

4.3. Constructing a covering of the model. Consider the following 4 families of subsets of the model X . For each pair of numbers $(i, j) \in \{1, 2\} \times \{1, 2\}$, we denote

$$\mathcal{W}^{(i,j)} := \{U : U \in \mathcal{W}_u^{(i,j)}, l(u) \text{ is even}\} \cup \{U : U \in \mathcal{W}_u^{(j,i)}, l(u) \text{ is odd}\}.$$

Moreover, for each pair $(i, j) \in \{1, 2\} \times \{1, 2\}$, we consider the following relation $\sim_{(i,j)}$ on the family $\mathcal{W}^{(i,j)}$: $U \sim_{(i,j)} V$ if and only if there are sets $U_0 = U, U_1, \dots, U_{n+1} = V$ in the family $\mathcal{W}^{(i,j)}$ such that $U_j \cap U_{j+1} \neq \emptyset$ for all $j \in \{0, \dots, n\}$. Clearly, $\sim_{(i,j)}$ is an equivalence relation.

Proposition 1. *Suppose $W, U \in \mathcal{W}^{(i,j)}$ and $d(W, U) < R$. Then $W \sim_{(i,j)} U$.*

We need two auxiliary facts (see Lemmas 3 and 4) for the proof of Proposition 1.

Lemma 3. *Given $U \in \mathcal{W}^{(i,j)}$, $U \subset X_u$, assume that a boundary plane $\sigma \subset X_u$ meets U , $U \cap \sigma \neq \emptyset$, and separates a point $x \in X$ and U . Then $d(x, U) = d(x, U \cap \sigma)$.*

Proof. Without loss of generality, we assume that $x \notin \sigma$. Then, since the hyperplane $\sigma \subset X$ is convex, the segment xy meets σ over a point for every $y \in U$, $xy \cap \sigma = t$.

Recall that we represent the block X_u as the product $X_u = H_0 \times \mathbb{R}$ and also $U = \tilde{U}_\theta \times \tilde{U}_\mathbb{R}$, where $\tilde{U}_\theta, \tilde{U}_\mathbb{R}$ are members of the appropriate coverings of H_0 and $\mathbb{R}$, respectively. Then $t = (t_\theta, t_R) \in \ell \times \mathbb{R} = \sigma$, where $\ell \in H_0$ is a boundary line, and $y = (y_\theta, y_R)$, where $y_\theta \in \tilde{U}_\theta$, $y_R \in \tilde{U}_\mathbb{R}$. For $t' = (t_\theta, y_R) \in \sigma$, we have

$$|xt'| \leq |xt| + |tt'| \leq |xt| + |ty| = |xy|.$$

If $t_\theta \in \tilde{U}_\theta$, then $t' \in U \cap \sigma$ because $y_R \in \tilde{U}_\mathbb{R}$, whence $|xy| \geq d(x, U \cap \sigma)$. Thus, in what follows we assume that $t_\theta \notin \tilde{U}_\theta$.

Let $\theta_p \times I_q$ be the initial brick in $\mathcal{W}_u^{(i,j)}$. Passing to the tree T_0 , we denote by U_0 the set of all vertices q' in $V(T_0)$ such that $\theta_{q'} \subset \tilde{U}_\theta$. Then, by the construction of $\mathcal{W}_u^{(i,j)}$, U_0 is the vertex set of a member of the standard (N, p) -covering of the tree T_0 . We denote this member by U' . There is $k \in \mathbb{N} \cup \{0\}$ such that U' lies in the N -annulus

$$A_k = \{q \in T_0 : kN \leq |qp| \leq (k+1)N\}$$

centered at p . We denote by $\bar{\ell}$ the line in T_0 corresponding to ℓ and pick a vertex $p_0 \in \bar{\ell}$ of T_0 such that $t_\theta \in \theta_{p_0}$. There are at most two such vertices, and we assume that $p_0 \in U'$ if $t_\theta \in \tilde{U}_\theta$. Then the assumption $t_\theta \notin \tilde{U}_\theta$ is equivalent to the fact that $p_0 \notin U'$.

Since $y_\theta \in \tilde{U}_\theta$, there is a vertex p_y of T_0 such that $y_\theta \in \theta_{p_y}$ and $p_y \in U'$. We show that $\text{dist}(p_0, \bar{\ell} \cap U') \leq |p_0 p_y|$ in T_0 . A (unique) point $p' \in \bar{\ell}$ closest to p splits $\bar{\ell}$ into rays $\bar{\ell}_i$, $i = 1, 2$. Each of the rays $\bar{\ell}_1, \bar{\ell}_2$ intersects the annulus A_k over a segment, and at least one of them lies in U' . We assume without loss of generality that $\bar{\ell}_1 \cap A_k \subset U'$.

There is a shortest segment $\bar{\gamma} \subset \bar{\ell}$ between p_0 and $\bar{\ell} \cap U'$. If $p_0 \in \bar{\ell}_1$, then $\bar{\gamma} \subset \bar{\ell}_1$ is a shortest path in T_0 between p_0 and U' . Thus, the length $|\bar{\gamma}|$ is equal to $d(p_0, U')$, whence $d(p_0, \bar{\ell} \cap U') \leq |p_0 p_y|$.

It remains to consider the case where $p_0 \in \bar{\ell}_2$ and $\bar{\ell}_2$ is disjoint with U' (the latter happens precisely when $k > 1$ and $|pp'| \leq (k - \frac{1}{2})N$). As above, we have $|p'p_y| \geq d(p', U') = d(p', \bar{\ell} \cap U')$. On the other hand, any path in T_0 between p_0 and U' passes through p' . Therefore, $|p_0 p_y| = |p_0 p'| + |p' p_y| \geq |\bar{\gamma}| = d(p_0, \bar{\ell} \cap U')$.

The geodesic segment $ty \subset X_u$ projects to the geodesic segment $t_\theta y_\theta$ in the factor H_0 of X_u . While moving from t_θ to y_θ along $t_\theta y_\theta$, we pass through the segment $p_0 p_y \subset T_0$, so that $|t_\theta y_\theta| \geq |p_0 p_y| - 1$ by $(\star)$.

Let $\gamma = t'z \subset \ell \times y_R \subset \sigma$ be the segment between t' and $z \in U \cap \sigma$ that corresponds to $\bar{\gamma} \subset \bar{\ell}$. The length of γ is at most the number of *interior* vertices of $\bar{\gamma}$ plus one, $|\gamma| \leq |\bar{\gamma}| - 1$. Since $|p_0 p_y| \geq |\bar{\gamma}|$, we obtain $|t_\theta y_\theta| \geq |\gamma|$. It follows that $|ty| \geq |tz|$ because both triangles $tt'y$ and $tt'z$ are flat with right angles at t' . Then, moreover, we have $|xy| \geq |xz| \geq d(x, U \cap \sigma)$. $\square$

Lemma 4. *Suppose σ is the common plane of the blocks X_u and $X_{u'}$ such that $|vu'| < |vu|$, where v is the root of T . Assume that $U \in \mathcal{W}^{(i,j)}$, $U \subset X_u$, U is disjoint with σ , $U \cap \sigma = \emptyset$. Then $d(U, \sigma) \geq R$.*

Proof. Let $\theta_p \times I_q$ be the initial brick in $\mathcal{W}_u^{(i,j)}$. Recall that $U = \tilde{U}_\theta \times \tilde{U}_R$, where $\tilde{U}_\theta$ and $\tilde{U}_R$ are members of the appropriate coverings of the θ -tree H_0 and $\mathbb{R}$, respectively. We denote by U_0 the set of all vertices $q' \in V(T_0)$ such that the hexagon $\theta_{q'}$ belongs to $\tilde{U}_\theta$. Then U_0 is the vertex set of a member U' of the standard (N, p) -covering of the tree T_0 . There is $k \in \mathbb{N} \cup \{0\}$ such that U' is a subset of the N -annulus

$$A_k = \{q \in T_0 : kN \leq |qp| \leq (k+1)N\}$$

centered at p .

Representing $\sigma = \ell \times \mathbb{R}$, where $\ell \subset H_0$ is an appropriate boundary line, we let $\bar{\ell} \subset T_0$ be the line corresponding to ℓ . Our assumption on σ and the construction of the family $\mathcal{W}^{(i,j)}$ imply that $d(p, \bar{\ell}) \leq N$, and thus, $\bar{\ell}$ meets every annulus A_k , $k \geq 0$. By assumption, $\bar{\ell}$ is disjoint with $\tilde{U}_\theta$, whence $\tilde{U}_\theta$ is disjoint with any hexagon of H_0 that meets $\bar{\ell}$. Thus, $\bar{\ell}$ is disjoint with U' . Since $\bar{\ell} \cap A_k \neq \emptyset$ is covered by members of the standard (N, p) -covering of T_0 having the same color $i \in \{1, 2\}$ as U' , and the covering is N -disjoint, we have $d(U', \bar{\ell} \cap A_k) \geq N$. Therefore, $d(U', \bar{\ell}) \geq N/2$. Moreover, since $N = 2R + 1$ is odd and the distance $d(U', \bar{\ell})$ is an integer, we actually have $d(U', \bar{\ell}) \geq (N + 1)/2 = R + 1$. Using $(\star)$, we obtain $d(\tilde{U}_\theta, \ell) \geq R$. Thus, $d(U, \sigma) \geq R$. $\square$

Proof of Proposition 1. Suppose U lies in the block X_u and W lies in the block X_w . We prove the proposition by induction on $m = l(u) + l(w)$.

The base of induction : $m = 0$.

The assumption $d(U, W) < R$ implies that $U = W$, because different members of the family $\mathcal{W}^{(i,j)}$ are N -disjoint.

The induction step: $m \rightarrow m + 1$.

Assuming that the assertion is true whenever $l(u) + l(w) \leq m$, we consider the case where $l(u) + l(w) = m + 1$. Without loss of generality, $l(u) \leq l(w)$. Then $l(w) \geq 1$. Thus there exists a boundary plane σ in X_w such that $l(w') = l(w) - 1$ for the block $X_{w'}$ that is glued to X_w over σ .

We have $d(W, \sigma) < R$, because U and W are separated by σ . Therefore, $W \cap \sigma \neq \emptyset$, because X_w and σ satisfy the conditions of Lemma 4. By Lemma 3, $d(x, W) = d(x, W \cap \sigma)$ for every $x \in U$, whence $d(U, W \cap \sigma) < R$. There is $W' \in \mathcal{W}^{(i,j)}$, $W' \subset X_{w'}$, such that

$W' \cap \sigma = \sigma \cap W$ by the construction of the family $\mathcal{W}^{(i,j)}$; in particular, W' is equivalent to W . Then $d(W', U) < R$, and thus, W' is equivalent to U by the inductive hypothesis. Hence, W is equivalent to U . $\square$

Given $(i, j) \in \{1, 2\}^2$, for each equivalence class $\widetilde{W}$ of the equivalence relation $\sim_{(i,j)}$ on the family $\mathcal{W}^{(i,j)}$, we consider the set $W := \bigcup \{U : U \in \widetilde{W}\}$ and denote by $\widetilde{W}^{(i,j)}$ the family of all such sets. Proposition 1 implies that the covering

$$\widetilde{W} = \bigcup \{\widetilde{W}^{(i,j)} : (i, j) \in \{1, 2\}^2\}$$

of X is 4-colored and R -disjoint. However, $\widetilde{W}$ is unbounded, since the diameter of each of its members is infinite.

§5. PROOF OF THEOREM 1

5.1. Extension of the covering members. Let W be a member of the covering $\widetilde{W}$. The sets $W_u = X_u \cap W$, $u \in V(T)$, while nonempty, form the equivalence class $\widetilde{W}$ that corresponds to W . We have $W_u = U \times V$, where $U \subset H_0$ and $V \subset \mathbb{R}$ are members of the coverings $\widetilde{U}^i$ and $\widetilde{V}^j$ (respectively) for some pair $(i, j) \in \{1, 2\} \times \{1, 2\}$.

Recall that U is the union of the hexagons labeled by all vertices of the tree T_0 (see Subsection 3.1 for the definition of T_0) that lie in a member U' of the standard (N, p) -covering of T_0 . There is $k \in \mathbb{N} \cup \{0\}$ such that U' is contained in the N -annulus

$$A_k = \{q : kN \leq |qp| \leq (k+1)N\} \subset T_0$$

centered at p .

Now, we construct an extension $\bar{U}'$ of U' as follows. For each $x \in U'$, consider the geodesic segment $\gamma_x = x'x \subset px$ of length $N/2$ (if $|px| < N/2$, then $k = 0$, $U' = B_N(p)$, and we define $\bar{U}' := U'$). We put

$$\bar{U}' := \bigcup \{\gamma_x : x \in U'\} \subset T_0$$

and define

$$\bar{U} := \bigcup \{\theta_v : v \in \bar{U}' \cap V(T_0)\} \subset H_0.$$

Note that the set $\bar{U}'$ is connected. Indeed, for each $x, y \in \bar{U}'$, there exist $x_1, y_1 \in U'$ such that $x \in \gamma_{x_1}$, $y \in \gamma_{y_1}$. Let s be the center of the tripod x_1, y_1, p . The definition of the set U' implies that $|ps| \geq (k - \frac{1}{2})N$. There are $x' \in px_1$ and $y' \in py_1$ such that $|px'| = |py'| = kN$. Then x_1x' , $y_1y' \subset U'$ and either $s \in U'$, or $|sx'|, |sy'| \leq N/2$. In each case s is connected with x and y by paths in $\bar{U}'$, so that the set $\bar{U}'$ is connected. Then $\bar{U}$ is convex by Remark 3.

Recall that V is the union of the intervals labeled by all the vertices of the tree S_0 (homeomorphic to $\mathbb{R}$; see Subsection 3.1 for the definition of S_0) lying in a member $V' \subset S_0$ of the standard (N, d, q) -covering of S_0 for some natural $d \leq N$.

We construct an extension $\bar{V}'$ of V' in a similar way. For each $x \in V'$, consider the segment $\gamma_x := x'x \subset qx$ of length $N/2$ (if $|qx| < N/2$, then either $V' = B_d(q)$ and we define $\bar{V}' := V'$, or $V' = B_{d+N}(q) \setminus \text{Int } B_d(q)$ and $d < N/2$, and then we define $\bar{V}' := B_{d+N}(q)$). Consider the set

$$\bar{V}' := \bigcup \{\gamma_x : x \in V'\} \subset S_0$$

and put

$$\bar{V} := \bigcup \{I_v : v \in \bar{V}' \cap V(S_0)\} \subset \mathbb{R}.$$

Since $\bar{V}'$ is a segment in S_0 , it is connected. Then $\bar{V}$ is convex by Remark 3.

Therefore, $\bar{W}_u := \bar{U} \times \bar{V}$ is a convex subset of X_u . Moreover, $\bar{W}_u$ is compact by construction. We shall call this set the *extension* of the set W_u . Note that, as in Lemma 2, the extensions of the sets W_u and W_w agree for the blocks X_u and X_w having the common boundary plane σ ; i.e., their intersections with σ coincide. The union of the extensions of all block components W_u will be called the *extension* of the set W .

We recall the well-known fact that for each point x in a Hadamard space X and for each closed convex subset $A \subset X$ there exists a unique point $p_x \in A$ such that $|xp_x| = d(x, A)$ (the point p_x is called the metric projection of x to the set A). The map sending $x \in X$ to its metric projection $p_x \in A$ is called the *projection* to A and is a 1-Lipschitz map.

Proposition 2. *For each member W of the covering $\tilde{W}$, its extension $\bar{W}$ is convex.*

Proof. Given $x, y \in \bar{W}$, we prove that the geodesic segment xy lies in $\bar{W}$. Let $X_u, X_v \subset X$ ($u, v \in V(T)$) be the blocks containing x and y , respectively. If $u = v$, then $x, y \in \bar{W}_u$, which is convex. Thus, we assume that $u \neq v$. The segment xy intersects consecutively the boundary planes $\sigma_1, \dots, \sigma_n$ of the blocks $X_0 = X_u, \dots, X_n = X_w$ ($\sigma_i = X_{i-1} \cap X_i$ for each $i \in \{1, \dots, n\}$) at $z_i \in \sigma_i$, $i \in \{1, \dots, n\}$. The sets $\bar{W}_i := X_i \cap \bar{W}$ are convex for all $i \in \{0, \dots, n\}$. Note that for each $i \in \{1, \dots, n\}$ the projections of z_i to $\bar{W}_{i-1}$ and to $\bar{W}_i$ coincide. Indeed, let $t \in \bar{W}_i$ be the nearest point to z_i . By the construction of $\bar{W}_i$, the metric projection of $\bar{W}_i$ to σ_i coincides with $\bar{W}_i \cap \sigma_i$; in particular, the projection t_i of t to σ_i belongs to $\bar{W}_i$. Since σ_i is convex in the Hadamard space X , we have $|z_i t_i| \leq |z_i t|$. Therefore, the point in $\bar{W}_i$ nearest to z_i lies in $\bar{W}_i \cap \sigma_i$. By a similar argument with $\bar{W}_{i-1}$, we show that the projections of z_i to $\bar{W}_{i-1}$ and to $\bar{W}_i$ coincide. We denote this point by z'_i . By the convexity of $\bar{W}_i$, we have $|z'_i z'_{i+1}| \leq |z_i z_{i+1}|$ for every $i = 0, \dots, n$, where $z_0 = z'_0 = x$, $z_{n+1} = z'_{n+1} = y$; i.e., the length of the curve $\gamma := xz'_1 \cup \dots \cup z'_n y$ does not exceed that of xy , whence $\gamma = xy$. Thus, $xy \subset \bar{W}$. $\square$

5.2. Additional colors. Consider the extension $\bar{W}$ of a member W of the covering $\tilde{W}$. Note that for each block X_u with $X_u \cap W \neq \emptyset$, the set $\bar{W}_u := X_u \cap \bar{W}$ is convex, compact, and $17R$ -bounded. The last claim follows from the fact that $W_u = X_u \cap W$ is $15R$ -bounded, and each point $x \in \bar{W}_u$ lies at a distance of at most R from W_u .

Remark 4. For $u, w \in V(T)$, any segment $xy \subset X$ with $x \in W_u$ and $y \in W_w$ intersects $W_{u'}$ if and only if the segment $uw \subset T$ contains u' .

Put $c = 18$. Like the standard coverings of a metric tree (see §2), for every member W of the covering $\tilde{W}$ we construct a 2-colored, R -disjoint, and $c'R$ -bounded covering of the extension $\bar{W}$, where $c' := 5c - 2$.

Given $x \in X$, $u \in V(T)$, we write $x\bar{W}_u$ for the geodesic segment $xx' \subset X$, where x' is the metric projection of x to $\bar{W}_u$. This notation is consistent, because $\bar{W}_u$ is convex, so that x' is unique. We fix $v \in V(T)$ such that $W \cap X_v \neq \emptyset$. For every $k \in \mathbb{N} \cup \{0\}$ consider the “annulus”

$$A_k := \bigcup \{ \bar{W}_u : ckR \leq d(\bar{W}_v, \bar{W}_u) \leq c(k+1)R \} \subset \bar{W}.$$

We define a relation $\sim_k$ on the set A_k : $x \sim_k y$ for every $x, y \in A_0$, and for each natural k , let $x \sim_k y$ if and only if there exists a vertex $u \in V(T)$ such that the geodesic segments $x\bar{W}_v$ and $y\bar{W}_v$ intersect the set $\bar{W}_u$, and

$$d(\bar{W}_u, \bar{W}_v) \geq (c(k-1) + 1/2)R.$$

Note that $x \sim_k y$ for each $x, y \in \bar{W}_u \subset A_k$.

We prove that $\sim_k$ is an equivalence relation. Indeed, suppose $x \sim_k y$ and $y \sim_k z$. For any $x \in \bar{W}_{v_x}, y \in \bar{W}_{v_y}$, and $z \in \bar{W}_{v_z}$, there are $u \in V(T)$ and $w \in V(T)$ occurring in the

definition of $\sim_k$ for the pairs $\{x, y\}$ and $\{y, z\}$, respectively. Note that u and w lie on the segment $v_y v \subset T$. Without loss of generality, suppose that $|uv| \leq |wv|$. Then the segment $z\bar{W}_v$ intersects the set $\bar{W}_u$; therefore, the points x and z are also equivalent.

Consider the following 2-colored covering of the set $\bar{W}$: the sets of the first color are all the equivalence classes of $\sim_k$ with k even, and the sets of the second color are all the equivalence classes of $\sim_k$ with k odd.

We prove that this covering of $\bar{W}$ is $c'R$ -bounded. Indeed, assume that points $x \in \bar{W}_{v_x} \subset A_k$, $y \in \bar{W}_{v_y} \subset A_k$ are equivalent, and let $u \in V(T)$ be a vertex such that the segments $x\bar{W}_v$, $y\bar{W}_v$ intersect the set $\bar{W}_u$ and

$$d(\bar{W}_u, \bar{W}_v) \geq (c(k-1) + 1/2)R.$$

For $z, t \in \bar{W}_u$ with $z \in x\bar{W}_v$, and $t \in y\bar{W}_v$, we have $|z\bar{W}_v|, |t\bar{W}_v| \geq (c(k-1) + 1/2)R$. Then $|xz| = |x\bar{W}_v| - |z\bar{W}_v| \leq (2c - \frac{1}{2})R$, and, similarly, $|yt| \leq (2c - \frac{1}{2})R$. Since $\bar{W}_u$ is $(c-1)R$ -bounded, we obtain

$$|xy| \leq |xz| + |zt| + |ty| \leq (5c - 2)R = c'R.$$

Now, we prove that the resulting covering of $\bar{W}$ is R -disjoint. Let $U_1 \subset A_k$ and $U_2 \subset A_l$ be sets of the same color.

First, assume that $|k - l| \geq 2$. Without loss of generality, suppose $k \geq l + 2$. If $d(U_1, U_2) < R$, then $|xy| < R$ for some $x \in \bar{W}_{v_x} \subset U_1$ and $y \in \bar{W}_{v_y} \subset U_2$. Note that $d(\bar{W}_v, \bar{W}_{v_y}) \leq c(l+1)R$, whence $|zt| \leq c(l+1)R$ for some $z \in \bar{W}_{v_x}$, $t \in \bar{W}_v$. Then

$$|xt| \leq |xy| + |yz| + |zt| < R + (c-1)R + c(l+1)R = c(l+2)R \leq ckR.$$

This contradicts the inequality $ckR \leq d(\bar{W}_v, \bar{W}_{v_x})$.

Now we assume that $k = l$, $\bar{W}_a \subset U_1$, and $\bar{W}_b \subset U_2$. It suffices to prove that $d(\bar{W}_a, \bar{W}_b) \geq R$. Let $u \in V(T)$ be the center of the tripod $abv \subset T$. Then any geodesic segment going from $\bar{W}_a$ or $\bar{W}_b$ to $\bar{W}_v$ passes through $\bar{W}_u$, and

$$d(\bar{W}_u, \bar{W}_v) < (c(k-1) + 1/2)R.$$

We show that $d(\bar{W}_a, \bar{W}_u) \geq R/2$. Indeed, suppose this is not true. Then there are $x_1 \in \bar{W}_a$, $x_2, z_2 \in \bar{W}_u$, $z_1 \in \bar{W}_v$ such that $|x_1 x_2| < R/2$, $|z_1 z_2| < (c(k-1) + 1/2)R$. Thus,

$$|x_1 z_1| \leq |x_1 x_2| + |x_2 z_2| + |z_2 z_1| < R/2 + (c-1)R + (c(k-1) + 1/2)R = ckR,$$

which contradicts the inclusion $\bar{W}_a \subset A_k$.

Similarly, $d(\bar{W}_b, \bar{W}_u) \geq R/2$. Since $u \in ab \subset T$, we have

$$d(\bar{W}_a, \bar{W}_b) \geq d(\bar{W}_a, \bar{W}_u) + d(\bar{W}_u, \bar{W}_b) \geq R.$$

We restrict the constructed covering of $\bar{W}$ to $W \subset \bar{W}$; i.e., for each member of the covering we consider its intersection with W . The new covering of W is again 2-colored, R -disjoint, and $c'R$ -bounded with $c' = 5c - 2 = 88$. We denote this covering by $\mathcal{X}_W$, $\mathcal{X}_W = \mathcal{X}_W^1 \cup \mathcal{X}_W^2$.

For each triple of numbers $(i, j, k) \in \{1, 2\}^3$, consider the following 8-colored covering of X :

$$\mathcal{X} := \bigcup \{\mathcal{X}_W^k : W \in \tilde{\mathcal{W}}^{(i,j)}, (i, j, k) \in \{1, 2\}^3\}.$$

The covering $\mathcal{X}$ is $c'R$ -bounded because so are the families $\mathcal{X}_W^k$, $k = 1, 2$. We prove that $\mathcal{X}$ is R -disjoint. Indeed, let $U \in \mathcal{X}_{W_1}^k, V \in \mathcal{X}_{W_2}^k$ be two different sets of the same color for some triple of numbers $(i, j, k) \in \{1, 2\}^3$, where $W_1, W_2 \in \tilde{\mathcal{W}}^{(i,j)}$. If $W_1 \neq W_2$, then $d(U, V) \geq d(W_1, W_2) \geq R$, because $U \subset W_1$, $V \subset W_2$, and the families $\tilde{\mathcal{W}}^{(i,j)}$ are R -disjoint. If $W_1 = W_2$, then $d(U, V) \geq R$ because $\mathcal{X}_{W_1}^k$ is R -disjoint. This completes the proof of Theorem 1.

§6. APPLICATIONS

We recall some definitions.

Definition 3 (Assouad [1]). *The Assouad–Nagata dimension $\dim_{\text{AN}} X$ of a metric space X is the smallest integer n such that for each $R > 0$ there exists an $(n + 1)$ -colored, R -disjoint, and CR -bounded covering of X , where the number $C > 0$ is independent of R .*

Definition 4 (Buyalo [6]). *The linearly controlled dimension $\ell\text{-dim } X$ of a metric space X is the smallest integer n such that for each $R \leq 1$ there exists an $(n + 1)$ -colored, R -disjoint, and CR -bounded covering of the space X , where the number $C > 0$ is independent of R .*

It is easily seen that $\dim_{\text{AN}} X = \max\{\ell\text{-dim } X, \ell\text{-asdim } X\}$.

Recall that $\widetilde{M}$ is the universal cover of the graph-manifold M . A remark to [9, Proposition 2.7] implies that $\ell\text{-dim } \widetilde{M} = \dim \widetilde{M}$, whence $3 = \dim \widetilde{M} \leq \dim_{\text{AN}} \widetilde{M}$. Theorem 1 shows that $\dim_{\text{AN}} \widetilde{M} \leq 7$, which gives Corollary 1. However, by the Bell–Dranishnikov result (see [3]), $\text{asdim } \widetilde{M} = 3$, which supports the following conjecture.

Conjecture. *Let $\widetilde{M}$ be the universal cover of a graph-manifold. Then $\dim_{\text{AN}} \widetilde{M} = 3$.*

Corollary 2 follows from Corollary 1 and [9, Theorem 1.3].

Definition 5. A metric space X is said to be *Lipschitz n -connected*, $n \geq 0$, if for each $m \in \{0, \dots, n\}$ there exists $\gamma > 0$ such that every λ -Lipschitz map $f : S^m \rightarrow X$ admits a $\gamma\lambda$ -Lipschitz extension $\tilde{f} : B^{m+1} \rightarrow X$, where S^m and B^{m+1} are the unit sphere and the unit disk in $\mathbb{R}^{m+1}$, respectively.

In [4] it was proved that the Hadamard spaces are Lipschitz n -connected for each $n \geq 0$. Thus, the model space X is Lipschitz n -connected for each $n \geq 0$. By Corollary 1, the space X has finite Assouad–Nagata dimension. Now, Corollary 3 follows from [9, Theorem 1.5].

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