FRACTIONAL MOMENTS OF AUTOMORPHIC L -FUNCTIONS

O. M. FOMENKO

ABSTRACT. Upper and lower bounds for fractional moments of automorphic L -functions are found.

§1. INTRODUCTION

Let

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (\operatorname{Re}(s) > 1)$$

be the Riemann zeta function. Let k be a nonnegative real number, and let $T \geq 2$. Heath-Brown [1] investigated the behavior as $T \rightarrow \infty$ of the mean values

$$I_k(T) = \int_1^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt.$$

Consider the inequalities

$$(1.1) \quad I_k(T) \gg T(\log T)^{k^2}$$

and

$$(1.2) \quad I_k(T) \ll T(\log T)^{k^2}.$$

Heath-Brown proved the following result.

Theorem A (Heath-Brown [1]).

- (i) If $k \geq 0$ is rational, then (1.1) holds.
- (ii) If $k = 1/n > 0$, where n is an integer, then (1.2) holds.
- (iii) Under the Riemann hypothesis, (1.1) holds for all real $k \geq 0$.
- (iv) Under the Riemann hypothesis, (1.2) holds for $0 \leq k \leq 2$.

Note that for $I_1(T)$ and $I_2(T)$, asymptotics are known; these classical facts were proved by Hardy and Littlewood and by Ingham, respectively.

Our purpose in the present paper is to transfer the result of Heath-Brown to some automorphic L -functions.

Consider the space $S_{\mathfrak{z}}(\Gamma)$ of holomorphic cusp forms

$$f(z) = \sum_{n=1}^{\infty} a_f(n) \exp(2\pi i n z)$$

2010 *Mathematics Subject Classification*. Primary 11F03.

Key words and phrases. Symmetric square L -function, fractional moment, convexity.

The author was partly supported by RFBR (project no. 08-01-00233).

of even weight $\varkappa \geq 12$ with respect to the full modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, $z = x + iy$, $y > 0$; let $S_\varkappa(\Gamma)^+$ be the set of all Hecke eigencuspforms with $a_f(1) = 1$ in this space. In the sequel it is always assumed that $f \in S_\varkappa(\Gamma)^+$. Set

$$\lambda_f(n) = a_f(n)/n^{\frac{\varkappa-1}{2}};$$

then $\lambda_f(n)$ is an eigenvalue of the Hecke operator T_n , $n = 1, 2, 3, \dots$. By the classical result by Deligne, for any prime p we have

$$\lambda_f(p) = \alpha_p + \bar{\alpha}_p, \quad \alpha_p \bar{\alpha}_p = 1.$$

We consider the following automorphic L -functions, defined initially on the half-plane $\mathrm{Re}(s) > 1$:

$$1) \quad L(s, f) = \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \bar{\alpha}_p p^{-s})^{-1} = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}$$

(the Hecke L -function of f);

$$\begin{aligned} 2) \quad L(s, \mathrm{sym}^2 f) &= \prod_p (1 - \alpha_p^2 p^{-s})^{-1} (1 - p^{-s})^{-1} (1 - \bar{\alpha}_p^2 p^{-s})^{-1} \\ &= \zeta(2s) \sum_{n=1}^{\infty} \lambda_f(n^2) n^{-s} = \sum_{n=1}^{\infty} c_n n^{-s} \end{aligned}$$

(the Shimura symmetric square L -function of f); and

$$\begin{aligned} 3) \quad L(s, f \times f) &= \prod_p (1 - \alpha_p^2 p^{-s})^{-1} (1 - p^{-s})^{-2} (1 - \bar{\alpha}_p^2 p^{-s})^{-1} \\ &= \zeta(2s) \sum_{n=1}^{\infty} \lambda_f(n)^2 n^{-s} = \sum_{n=1}^{\infty} d_n n^{-s} \end{aligned}$$

(the L -function of the Rankin–Selberg convolution of f with itself).

The L -functions introduced above admit analytic continuation to the entire s -plane; the functions $L(s, f)$ and $L(s, \mathrm{sym}^2 f)$ are entire, and $L(s, f \times f)$ has a simple pole at $s = 1$ as a unique singularity. Each of these functions obeys a functional equation relating its values at s and $1 - s$. Below (up to §5) we need the functional equation for $L(s, \mathrm{sym}^2 f)$:

$$\begin{aligned} &\pi^{-3s/2} \Gamma\left(\frac{s}{2} + \frac{1}{2}\right) \Gamma\left(\frac{s}{2} + \frac{\varkappa-1}{2}\right) \Gamma\left(\frac{s}{2} + \frac{\varkappa}{2}\right) L(s, \mathrm{sym}^2 f) \\ &= \pi^{-3(1-s)/2} \Gamma\left(\frac{1-s}{2} + \frac{1}{2}\right) \Gamma\left(\frac{1-s}{2} + \frac{\varkappa-1}{2}\right) \Gamma\left(\frac{1-s}{2} + \frac{\varkappa}{2}\right) L(1-s, \mathrm{sym}^2 f). \end{aligned}$$

We introduce the mean value

$$I_k(T, \mathrm{sym}^2 f) = \int_1^T \left| L\left(\frac{1}{2} + it, \mathrm{sym}^2 f\right) \right|^{2k} dt$$

and, similarly, the mean values of $L(s, f)$, $L(s, f \times f)$, and the Dedekind zeta function $\zeta_{K_3}(s)$ of a cubic field K_3 . Below (up to §5) we deal only with the L -function $L(s, \mathrm{sym}^2 f)$.

Theorem 1.

(i) *If $k \geq 0$ is rational, then*

$$(1.3) \quad I_k(T, \mathrm{sym}^2 f) \gg T(\log T)^{k^2}.$$

(ii) *Under the Riemann hypothesis for $L(s, \mathrm{sym}^2 f)$, estimate (1.3) is true for all real $k \geq 0$.*

(iii) Under the Riemann hypothesis for $L(s, \text{sym}^2 f)$, we have

$$(1.4) \quad I_k(T, \text{sym}^2 f) \ll T(\log T)^{k^2} \\ \text{for } 0 \leq k < \frac{2}{3}.$$

The proof is presented in §§2–4 and involves the approach of [1] (see also [2]). Similar results for the L -functions $L(s, f)$, $L(s, f \times f)$, and $\zeta_{K_3}(s)$ will be considered in §§5 and 6.

All constants implied by the $\ll$ (or $\gg$) notation may depend on k (and on f , which is regarded as fixed) unless otherwise indicated. However, the constants are independent of the variable σ introduced in §3. The symbols $a_1, a_2, a_3, \dots$ will denote positive constants; $\varepsilon > 0$ is an arbitrarily small fixed number.

§2. DIRICHLET POLYNOMIALS

A multiplicative function $d_k(n)$ arises for real $k \geq 0$ from the relation

$$\zeta^k(s) = \prod_p (1 - p^{-s})^{-k} = \sum_{n=1}^{\infty} d_k(n) n^{-s} \quad (\text{Re}(s) > 1)$$

(for the details, see [1]). We have the formula

$$d_k(n) = \prod_{p^\alpha \parallel n} \frac{k(k+1) \cdots (k+\alpha-1)}{\alpha!}.$$

We fix a branch of $\log L(s, \text{sym}^2 f)$:

$$\log L(s, \text{sym}^2 f) = \sum_p \sum_{j=1}^{\infty} \frac{(\alpha_p^{2j} + 1 + \bar{\alpha}_p^{2j})}{j p^{js}} \quad (\text{Re}(s) > 1)$$

and then use the formula

$$L^k(s, \text{sym}^2 f) = \exp\{k \log L(s, \text{sym}^2 f)\} \quad (\text{Re}(s) > 1)$$

to choose a branch of $L^k(s, \text{sym}^2 f)$.

Let the coefficients $\nu(n, k)$ be defined by the expansion

$$L^k(s, \text{sym}^2 f) = \sum_{n=1}^{\infty} \nu(n, k) n^{-s} \quad (\text{Re}(s) > 1).$$

For $\text{Re}(s) > 1$ we also have

$$L^k(s, \text{sym}^2 f) = \prod_p \sum_{\substack{m \geq 0, \\ n \geq 0, l \geq 0}} \frac{d_k(p^m) d_k(p^n) d_k(p^l) \alpha_p^{2m} \bar{\alpha}_p^{2n}}{p^{(m+n+l)s}},$$

which gives the inequality

$$|\nu(n, k)| \leq d_{3k}(n).$$

Consequently,

$$\nu(n, k) \ll n^\varepsilon.$$

The principal result of this section is as follows.

Lemma 1. *For any fixed real $k \geq 0$, there exists $c'_k > 0$ such that*

$$\left(\sigma - \frac{1}{2}\right)^{-k^2} \ll \sum_{n=1}^N \nu(n, k)^2 n^{-2\sigma} \ll \left(\sigma - \frac{1}{2}\right)^{-k^2}$$

uniformly for

$$\frac{1}{2} + \frac{c'_k}{\log N} \leq \sigma \leq 1.$$

Moreover,

$$(\log N)^{k^2} \ll \sum_{n=1}^N \nu(n, k)^2 n^{-1} \ll (\log N)^{k^2}.$$

We shall use the following result by Odoni.

Theorem B (Odoni [3]). *Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be multiplicative, with $f(n) \geq 0$ for all $n \in \mathbb{N}$. Suppose that there exist constants $a_1, a_2 > 1$ such that $0 \leq f(p^n) \leq a_1 n^{a_2}$ for all n , where p is a prime integer. Suppose also that there are constants $c_0 > 0$ and $0 < \gamma < 1$ such that*

$$\sum_{p \leq x} f(p) = c_0 x (\log x)^{-1} + O(x (\log x)^{-1-\gamma})$$

as $x \rightarrow \infty$. Then there is a constant $A_0 > 0$ such that

$$\sum_{n \leq x} n^{-1} f(n) = A_0 (\log x)^{c_0} + O((\log x)^{c_0-\gamma/2}).$$

We pass to the proof of Lemma 1. With every form $f \in S_{\kappa}(\Gamma)^+$, we associate a form F on $\mathrm{GL}(3)$ with the Fourier coefficients $a(m, n)$ and the L -function

$$L(s, F) = \sum_{n=1}^{\infty} a(1, n) n^{-s}$$

such that

$$L(s, F) = L(s, \mathrm{sym}^2 f).$$

The form F is called the Gelbart–Jacquet lift (the adjoint square lift) of f ; the existence of it was established in [4]. Let $L(s, F \times F)$ be the L -function of the Rankin–Selberg convolution of the form F with itself. Recall that

$$L(s, f) = \prod_p (1 - \alpha p^{-s})^{-1} (1 - \bar{\alpha} p^{-s})^{-1}$$

(for simplicity, we set $\alpha_p = \alpha$); then $L(s, F \times F)$ has the Euler product

$$\begin{aligned} L(s, F \times F) &= \prod_p (1 - \alpha^4 p^{-s})^{-1} (1 - \alpha^2 p^{-s})^{-1} (1 - p^{-s})^{-1} \\ &\quad \times (1 - \alpha^2 p^{-s})^{-1} (1 - p^{-s})^{-1} (1 - \bar{\alpha}^2 p^{-s})^{-1} (1 - p^{-s})^{-1} \\ &\quad \times (1 - \bar{\alpha}^2 p^{-s})^{-1} (1 - \bar{\alpha}^4 p^{-s})^{-1}. \end{aligned}$$

The L -function $L(s, F \times F)$ has a simple pole at $s = 1$, and is analytic elsewhere, and obeys a functional equation relating its values at s and $1 - s$ (see [5]).

Consider the function

$$T(s) = \sum_{n=1}^{\infty} c_n^2 n^{-s}.$$

It is easily seen that

$$T(s) = L(s, F \times F) \cdot B(s),$$

where

$$\begin{aligned} B(s) &= \prod_p X_p(p^{-s}), \\ X_p(t) &= 1 + A_2 t^2 + A_3 t^3 + \cdots. \end{aligned}$$

Using methods of the theory of distribution of primes, we show that there is a region

$$\sigma > 1 - C / \log(|t| + 2), \quad C > 0,$$

that is free from zeros of $L(s, F \times F)$. This yields the asymptotic formula

$$\sum_{p \leq x} c_p^2 \log p = x + O(x \exp(-D\sqrt{\log x})), \quad D > 0, \quad x \rightarrow \infty,$$

which shows that

$$(2.1) \quad \sum_{p \leq x} c_p^2 = x \log^{-1} x + O(x \log^{-2} x).$$

Since $\nu(p, k) = kc_p$, we have

$$\sum_{p \leq x} \nu(p, k)^2 = k^2 x \log^{-1} x + O(k^2 x \log^{-2} x).$$

By Theorem B,

$$(2.2) \quad \sum_{n \leq x} n^{-1} \nu(n, k)^2 = \frac{\exp(B_0 k^2)}{\Gamma(k^2 + 1)} \cdot P \cdot (\log x)^{k^2} + O((\log x)^{k^2 - \gamma/2}),$$

where $0 < \gamma < 1$ is an arbitrary but fixed number, and

$$P = \prod_p \left\{ e^{-\nu(p, k)p^{-1}} \left(1 + \nu(p, k)p^{-1} + \nu(p^2, k)p^{-2} + \cdots \right) \right\}.$$

The constant of the leading term was calculated with the help of Wirsing's result [6]. Now, the first part of Lemma 1 follows easily from (2.2) by summation by parts.

§3. CONVEXITY ESTIMATES

The basis of the further considerations is the following result by Gabriel.

Theorem C (Gabriel [7]). *Let $F(s)$ be regular in the infinite strip $\alpha < \operatorname{Re}(s) < \beta$, and continuous for $\alpha \leq \operatorname{Re}(s) \leq \beta$. Suppose $F(s) \rightarrow 0$ as $|\operatorname{Im}(s)| \rightarrow \infty$ uniformly for $\alpha \leq \operatorname{Re}(s) \leq \beta$. Then for $\alpha \leq \gamma \leq \beta$ and any $q > 0$ we have*

$$\int_{-\infty}^{\infty} |F(\gamma + it)|^q dt \leq \left\{ \int_{-\infty}^{\infty} |F(\alpha + it)|^q dt \right\}^{\frac{\beta - \gamma}{\beta - \alpha}} \left\{ \int_{-\infty}^{\infty} |F(\beta + it)|^q dt \right\}^{\frac{\gamma - \alpha}{\beta - \alpha}}.$$

Consider some applications of Theorem C. Put

$$w(t, T) = \int_T^{2T} \exp\{-2k(t - \tau)^2\} d\tau,$$

$$J(\sigma, T) = \int_{-\infty}^{\infty} |L(\sigma + it, \operatorname{sym}^2 f)|^{2k} w(t, T) dt.$$

In Theorem C, we take

$$F(s) = L(s, \operatorname{sym}^2 f) \exp\{(s - i\tau)^2\},$$

$$\gamma = \frac{1}{2}, \quad \alpha = 1 - \sigma, \quad \beta = \sigma, \quad \text{where} \quad \frac{1}{2} \leq \sigma \leq \frac{3}{4},$$

$$q = 2k > 0 \quad \text{and} \quad \tau \geq 2.$$

By the functional equation and convexity,

$$(3.1) \quad L(\alpha + it, \operatorname{sym}^2 f) \ll |L(\beta + it, \operatorname{sym}^2 f)| (1 + |t|)^{3(\sigma - 1/2)},$$

$$L(s, \operatorname{sym}^2 f) \ll (|t| + 1)^{3/4 + \varepsilon} \quad \left(\operatorname{Re}(s) \geq \frac{1}{2} \right).$$

Using (3.1), we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} |F(\alpha + it)|^{2k} dt \\ & \ll \left(\int_{-\infty}^{\tau/2} + \int_{\tau/2}^{3\tau/2} + \int_{3\tau/2}^{\infty} \right) |L(\sigma + it, \text{sym}^2 f)|^{2k} (1 + |t|)^{6k(\sigma-1/2)} \exp\{-2k(t-\tau)^2\} dt \\ & \ll \tau^{6k(\sigma-1/2)} \int_{-\infty}^{\infty} |L(\sigma + it, \text{sym}^2 f)|^{2k} \exp\{-2k(t-\tau)^2\} dt + \exp(-a_3 k \tau^2). \end{aligned}$$

Lemma C and the above estimate yield

$$\begin{aligned} & \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, \text{sym}^2 f\right) \right|^{2k} \exp\{-2k(t-\tau)^2\} dt \\ & \ll \tau^{3k(\sigma-1/2)} \int_{-\infty}^{\infty} |L(\sigma + it, \text{sym}^2 f)|^{2k} \exp\{-2k(t-\tau)^2\} dt + \exp(-a_4 k \tau^2). \end{aligned}$$

Finally, integrating over $T \leq \tau \leq 2T$, we get the following.

Lemma 2. Suppose $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$, $k > 0$, and $T \geq 2$. Then

$$J\left(\frac{1}{2}, T\right) \ll T^{3k(\sigma-1/2)} J(\sigma, T) + \exp(-a_5 k T^2).$$

Now we take $F(s) = L(s, \text{sym}^2 f) \exp\{(s - i\tau)^2\}$, $\gamma = \sigma$, $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, $q = 2k > 0$, where $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$ and $\tau \geq 2$. Then

$$\begin{aligned} & \int_{-\infty}^{\infty} \left| F\left(\frac{1}{2} + it\right) \right|^{2k} dt \\ & \ll \int_{\tau/2}^{3\tau/2} \left| L\left(\frac{1}{2} + it, \text{sym}^2 f\right) \right|^{2k} \exp\{-2k(t-\tau)^2\} dt + \exp(-a_6 k \tau^2) \end{aligned}$$

and

$$\int_{-\infty}^{\infty} \left| F\left(\frac{3}{2} + it\right) \right|^{2k} \ll 1.$$

Theorem C implies that

$$\begin{aligned} (3.2) \quad & \int_{-\infty}^{\infty} |F(\sigma + it)|^{2k} dt \\ & \ll \left\{ \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, \text{sym}^2 f\right) \right|^{2k} \exp\{-2k(t-\tau)^2\} dt \right\}^{3/2-\sigma} + \exp(-a_7 k \tau^2). \end{aligned}$$

However,

$$\int_{-\infty}^{\infty} |L(\sigma + it, \text{sym}^2 f)|^{2k} \exp\{-2k(t-\tau)^2\} dt \ll \int_{-\infty}^{\infty} |F(\sigma + it)|^{2k} dt + \exp(-a_8 k \tau^2).$$

Combining this with (3.2), integrating over $T \leq \tau \leq 2T$, and using Hölder's inequality, we obtain the following statement.

Lemma 3. Suppose $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$, $k > 0$, and $T \geq 2$. Then

$$J(\sigma, T) \ll T^{\sigma-1/2} J\left(\frac{1}{2}, T\right)^{3/2-\sigma} + \exp(-a_9 k T^2).$$

We need some new notation. We write $k = u/v$, where, in the case of Theorem 1 (i), u and v are positive coprime integers, in the case of (ii) $v = 1$ and $u = k$ is any positive real number, and in the case of (iii) $v = 1$ and $0 < u = k < 2/3$. We may assume that $k > 0$, because for $k = 0$, the theorem is trivial.

Now we define $N = T^{1/2}$ in the cases of (i) and (ii), and $N = T^{(10+3k)/12}$ in the case of (iii). We introduce the following quantities:

$$\begin{aligned} S_N(s) &= \sum_{n \leq N} \nu(n, k) n^{-s}, \\ g(s) &= L(s, \text{sym}^2 f)^u - S_N^v(s), \\ K(\sigma, T) &= \int_{-\infty}^{\infty} |g(\sigma + it)|^{2/v} w(t, T) dt. \end{aligned}$$

In the cases of (ii) and (iii) the Riemann hypothesis for $L(s, \text{sym}^2 f)$ (the extended Riemann hypothesis, ERH for short) is assumed to ensure analytic continuation of $L(s, \text{sym}^2 f)^u$ into the half-plane $\text{Re}(s) > \frac{1}{2}$.

For the proof of the following lemma we need the Montgomery–Vaughan mean value theorem.

Theorem D (Montgomery and Vaughan [8]). *If $\{b_n\}$ is a sequence of complex numbers such that $\sum_{n \geq 1} n|b_n|^2$ is convergent, then*

$$\int_0^T \left| \sum_{n=1}^{\infty} b_n n^{-it} \right|^2 dt = T \sum_{n=1}^{\infty} |b_n|^2 + O\left(\sum_{n=1}^{\infty} n|b_n|^2 \right),$$

where the implied constant is absolute.

Lemma 4. *Suppose $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$, $k = u/v > 0$, and $T \geq 2$. Then*

$$K(\sigma, T) \ll K\left(\frac{1}{2}, T\right)^{(5-4\sigma)/3} \left\{ (TN^{(\varepsilon-3/2)/v})^{(4\sigma-2)/3} + \exp\{-a_{10}kT^2(2\sigma-1)\} \right\},$$

where, in the cases of (ii) and (iii), ERH is assumed.

For the proof, in Theorem C we take

$$\begin{aligned} F(s) &= g(s) \exp\{u(s - i\tau)^2\}, \\ \gamma &= \sigma, \quad \alpha = \frac{1}{2}, \quad \beta = \frac{5}{4}, \quad \text{and} \quad q = \frac{2}{v}. \end{aligned}$$

Observe that

$$\begin{aligned} (3.3) \quad \int_{-\infty}^{\infty} |F(\sigma + it)|^{2/v} dt &\leq \left\{ \int_{-\infty}^{\infty} \left| F\left(\frac{1}{2} + it\right) \right|^{2/v} dt \right\}^{(5-4\sigma)/3} \\ &\quad \times \left\{ \int_{-\infty}^{\infty} \left| F\left(\frac{5}{4} + it\right) \right|^{2/v} dt \right\}^{(4\sigma-2)/3}. \end{aligned}$$

By (3.1), for $\text{Re}(s) \geq \frac{1}{2}$ we have

$$g(s) \ll (T + |t|)^{u+v}.$$

Consequently,

$$\int_{-\infty}^{\infty} \left| F\left(\frac{5}{4} + it\right) \right|^{2/v} dt \ll \int_{\tau/2}^{3\tau/2} \left| F\left(\frac{5}{4} + it\right) \right|^{2/v} dt + T^{2+2k} \exp(-a_{11}k\tau^2).$$

Therefore, (3.3) and the above inequality imply that

$$\begin{aligned} & \int_{-\infty}^{\infty} |F(\sigma + it)|^{2/v} dt \\ & \ll \left\{ \int_{-\infty}^{\infty} \left| F\left(\frac{1}{2} + it\right) \right|^{2/v} dt \right\}^{(5-4\sigma)/3} \left\{ \int_{\tau/2}^{3\tau/2} \left| F\left(\frac{5}{4} + it\right) \right|^{2/v} dt \right\}^{(4\sigma-2)/3} \\ & \quad + \left\{ \int_{-\infty}^{\infty} \left| F\left(\frac{1}{2} + it\right) \right|^{2/v} dt \right\}^{(5-4\sigma)/3} \left\{ T^{2+2k} \exp(-a_{12}k\tau^2) \right\}^{(4\sigma-2)/3}. \end{aligned}$$

We integrate over $T \leq \tau \leq 2T$ and apply Hölder's inequality, obtaining

$$\begin{aligned} (3.4) \quad K(\sigma, T) & \ll K\left(\frac{1}{2}, T\right)^{(5-4\sigma)/3} \\ & \quad \times \left\{ \int_T^{2T} \int_{\tau/2}^{3\tau/2} \left| g\left(\frac{5}{4} + it\right) \right|^{2/v} \exp\{-2k(t-\tau)^2\} dt d\tau \right\}^{(4\sigma-2)/3} \\ & \quad + K\left(\frac{1}{2}, T\right)^{(5-4\sigma)/3} \exp\{-a_{13}kT^2(2\sigma-1)\} \\ & \ll K\left(\frac{1}{2}, T\right)^{(5-4\sigma)/3} \left\{ \int_{T/2}^{3T} \left| g\left(\frac{5}{4} + it\right) \right|^{2/v} dt \right\}^{(4\sigma-2)/3} \\ & \quad + K\left(\frac{1}{2}, T\right)^{(5-4\sigma)/3} \exp\{-a_{13}kT^2(2\sigma-1)\}. \end{aligned}$$

First, we consider cases (ii) and (iii). We have

$$g\left(\frac{5}{4} + it\right) = L^k\left(\frac{5}{4} + it, \text{sym}^2 f\right) - S_N\left(\frac{5}{4} + it\right) = \sum_{n>N} \frac{\nu(n, k)}{n^{5/4+it}}.$$

By Theorem D,

$$\int_{T/2}^{3T} \left| g\left(\frac{5}{4} + it\right) \right|^2 dt \ll T \sum_{n=N}^{\infty} \nu(n, k)^2 n^{-5/2} + \sum_{n=N}^{\infty} \nu(n, k)^2 n^{-3/2} \ll TN^{\varepsilon-3/2}.$$

In the case of (i),

$$\begin{aligned} g(s) & = \left\{ \sum_{n=1}^{\infty} \nu(n, u/v) n^{-s} \right\}^v - \left\{ \sum_{n=1}^N \nu(n, u/v) n^{-s} \right\}^v \\ & = \sum_{n>N} a_n n^{-s} \quad (\text{Re}(s) > 1); \quad 0 \leq |a_n| \leq d_{3u}(n). \end{aligned}$$

We have

$$\int_{T/2}^{3T} \left| g\left(\frac{5}{4} + it\right) \right|^{2/v} dt \ll TN^{(\varepsilon-3/2)/v}.$$

Now, to prove Lemma 4 it remains to use (3.4).

§4. PROOF OF THEOREM 1

We define

$$L(\sigma, T) = \int_{-\infty}^{\infty} |S_N(\sigma + it)|^2 w(t, T) dt$$

for $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$. Note (see [1]) that

$$w(t, T) \ll \exp\{- (T^2 + t^2)k/18\}$$

for $t \leq 0$ and $t \geq 3T$, whence

$$L(\sigma, T) = \int_0^{3T} |S_N(\sigma + it)|^2 w(t, T) dt + O(1).$$

Moreover, $w(t, T) \ll 1$ for all t , and $w(t, T) \gg 1$ for $4T/3 \leq t \leq 5T/3$. Theorem D yields

$$\int_0^{3T} |S_N(\sigma + it)|^2 dt \ll T \sum_{n=1}^N \nu(n, k)^2 n^{-2\sigma}$$

and

$$\int_{4T/3}^{5T/3} |S_N(\sigma + it)|^2 dt \gg T \sum_{n=1}^N \nu(n, k)^2 n^{-2\sigma}.$$

Hence, Lemma 1 shows that

$$(4.1) \quad T \left(\sigma - \frac{1}{2} \right)^{-k^2} \ll L(\sigma, T) \ll T \left(\sigma - \frac{1}{2} \right)^{-k^2}$$

for $\frac{1}{2} + c'_k (\log N)^{-1} \leq \sigma \leq \frac{3}{4}$ and also that

$$(4.2) \quad T(\log T)^{k^2} \ll L\left(\frac{1}{2}, T\right) \ll T(\log T)^{k^2}.$$

Lemma 5. *In the cases of (i) and (ii) we have the estimate*

$$J\left(\frac{1}{2}, T\right) \gg T(\log T)^{k^2},$$

where in case (ii), ERH is assumed. In the case of (iii), under ERH we have the estimate

$$J\left(\frac{1}{2}, T\right) \ll T(\log T)^{k^2}.$$

We start the proof of the first part of the lemma with the trivial inequality

$$|S_N^v(s)|^{2/v} \ll |L(s, \text{sym}^2 f)|^{2k} + |g(s)|^{2/v},$$

whence it follows that

$$(4.3) \quad L(\sigma, T) \ll J(\sigma, T) + K(\sigma, T).$$

Similarly,

$$(4.4) \quad J(\sigma, T) \ll K(\sigma, T) + L(\sigma, T)$$

and

$$(4.5) \quad K\left(\frac{1}{2}, T\right) \ll L\left(\frac{1}{2}, T\right) + J\left(\frac{1}{2}, T\right).$$

First, let $K(\frac{1}{2}, T) \leq T$. Then, by (4.2) and (4.3) with $\sigma = \frac{1}{2}$, we have

$$T(\log T)^{k^2} \ll L\left(\frac{1}{2}, T\right) \ll J\left(\frac{1}{2}, T\right) + K\left(\frac{1}{2}, T\right) \ll J\left(\frac{1}{2}, T\right) + T,$$

and the assertion is proved.

Now, let $K(\frac{1}{2}, T) > T$. By Lemma 4,

$$(4.6) \quad \begin{aligned} K(\sigma, T) &\ll K\left(\frac{1}{2}, T\right) N^{-2\lambda(\sigma-1/2)/v} + T^{(5-4\sigma)/3} \exp\{-a_{10}kT^2(2\sigma-1)\} \\ &\ll K\left(\frac{1}{2}, T\right) N^{-2\lambda(\sigma-1/2)/v}, \end{aligned}$$

where we have set

$$\lambda = 1 - \frac{2\varepsilon}{3}.$$

Recall that $N = T^{1/2}$. Relations (4.3), (4.6), and (4.5) imply that

$$\begin{aligned} L(\sigma, T) &\ll J(\sigma, T) + K(\sigma, T) \ll J(\sigma, T) + K\left(\frac{1}{2}, T\right) T^{-\lambda(\sigma-1/2)/v} \\ &\ll J(\sigma, T) + \left\{ L\left(\frac{1}{2}, T\right) + J\left(\frac{1}{2}, T\right) \right\} T^{-\lambda(\sigma-1/2)/v}. \end{aligned}$$

Thus, either

$$(4.7) \quad L(\sigma, T) \ll L\left(\frac{1}{2}, T\right) T^{-\lambda(\sigma-1/2)/v}$$

or

$$(4.8) \quad L(\sigma, T) \ll J(\sigma, T) + J\left(\frac{1}{2}, T\right) T^{-\lambda(\sigma-1/2)/v}.$$

We write $\sigma - \frac{1}{2} = c(\log T)^{-1}$. Then (4.7) and (4.1), (4.2) yield

$$\begin{aligned} T\left(\sigma - \frac{1}{2}\right)^{-k^2} &= c^{-k^2} T(\log T)^{k^2} \ll L(\sigma, T) \\ &\ll L\left(\frac{1}{2}, T\right) T^{-\lambda(\sigma-1/2)/v} \ll T(\log T)^{k^2} \exp(-\lambda c/v), \end{aligned}$$

whence

$$\exp(\lambda c/v) \leq C_1 c^{k^2}$$

with $C_1 = C_1(k)$. Since this fails if $c = c(k)$ is sufficiently large, the alternative (4.8) holds true. Inequality (4.8) and Lemma 3 imply that

$$L(\sigma, T) \ll T^{\sigma-1/2} J\left(\frac{1}{2}, T\right)^{3/2-\sigma} + J\left(\frac{1}{2}, T\right) T^{-\lambda(\sigma-1/2)/v}.$$

With this value of c in $\sigma - \frac{1}{2} = c(\log T)^{-1}$, the above inequality and (4.1) give

$$\begin{aligned} T(\log T)^{k^2} c^{-k^2} &\ll L(\sigma, T) \\ &\ll J\left(\frac{1}{2}, T\right) \left\{ e^c J\left(\frac{1}{2}, T\right)^{-c(\log T)^{-1}} + \exp(-\lambda c/v) \right\} \ll J\left(\frac{1}{2}, T\right), \end{aligned}$$

whence

$$T(\log T)^{k^2} \ll J\left(\frac{1}{2}, T\right),$$

which proves the first part of the lemma.

We pass to the proof of the second part of the lemma. If $K(\frac{1}{2}, T) \leq T$, then, by (4.4) with $\sigma = \frac{1}{2}$ and (4.2),

$$J\left(\frac{1}{2}, T\right) \ll L\left(\frac{1}{2}, T\right) + K\left(\frac{1}{2}, T\right) \ll T(\log T)^{k^2} + T \ll T(\log T)^{k^2},$$

and the assertion is proved.

Now, let $K(\frac{1}{2}, T) > T$. Recall that $N = T^{(10+3k)/12}$.

Using (4.4), (4.6), and (4.5) we get

$$\begin{aligned} J(\sigma, T) &\ll L(\sigma, T) + K(\sigma, T) \\ &\ll L(\sigma, T) + \left\{ L\left(\frac{1}{2}, T\right) + J\left(\frac{1}{2}, T\right) \right\} T^{-\lambda(\sigma-1/2)(10+3k)/6}. \end{aligned}$$

Thus, either

$$(4.9) \quad J(\sigma, T) \ll J\left(\frac{1}{2}, T\right) T^{-\lambda(\sigma-1/2)(10+3k)/6}$$

or

$$(4.10) \quad J(\sigma, T) \ll L(\sigma, T) + L\left(\frac{1}{2}, T\right) T^{-\lambda(\sigma-1/2)(10+3k)/6}.$$

Take $\sigma - 1/2 = c(\log T)^{-1}$. With the help of (4.9) and Lemma 2, we obtain

$$\begin{aligned} J\left(\frac{1}{2}, T\right) &\ll J\left(\frac{1}{2}, T\right) T^{3k(\sigma-\frac{1}{2})-\lambda(\sigma-1/2)(10+3k)/6} \\ &= J\left(\frac{1}{2}, T\right) \exp\{c(3k - \lambda k/2 - 5\lambda/3)\}. \end{aligned}$$

Since $\lambda = 1 - \frac{2\varepsilon}{3}$ may be chosen arbitrarily close to 1, for $0 < k < \frac{2}{3}$ and sufficiently large c we get a contradiction. Now, combining (4.10) with (4.1), (4.2), and Lemma 2, we prove the second part of Lemma 5.

At this point, it is easy to prove Theorem 1. The properties of $w(t, T)$ imply the inequality

$$J\left(\frac{1}{2}, T\right) \ll I_k(3T, \text{sym}^2 f) + \exp(-a_{14}KT^2),$$

which gives (1.3) with the help of Lemma 5. Next, the properties of $w(t, T)$ and Lemma 5 give

$$I_k(5T/3, \text{sym}^2 f) - I_k(4T/3, \text{sym}^2 f) \ll J\left(\frac{1}{2}, T\right) \ll T(\log T)^{k^2}.$$

Replacing T by $(4/5)^n T$ and summing over n , we deduce (1.4), completing the proof of Theorem 1.

We finish with a remark. Jutila [2] observed that if $k = 1/n$, $n = 1, 2, 3, \dots$, then the implied constants in Theorem A can be taken to be independent of k . Jutila indicated the necessary changes in Heath-Brown's proof; as a corollary, he obtained information on large deviations of $\log |\zeta(\frac{1}{2} + it)|$. To get a result uniform for $k = 1/n$, certain similar changes can also be made in the proof of Theorem 1.

§5 THE AUTOMORPHIC L -FUNCTIONS $L(s, f)$ AND $L(s, f \times f)$

In this section we state analogs of Theorem 1 for $L(s, f)$ and $L(s, f \times f)$.

Theorem 2. (i) *If $k \geq 0$ is rational, then*

$$(5.1) \quad I_k(T, f) \gg T(\log T)^{k^2},$$

where

$$I_k(T, f) = \int_1^T \left| L\left(\frac{1}{2} + it, f\right) \right|^{2k} dt.$$

(ii) *Under the Riemann hypothesis for $L(s, f)$, estimate (5.1) is valid for all real $k \geq 0$.*

(iii) *Under the Riemann hypothesis for $L(s, f)$, we have*

$$I_k(T, f) \ll T(\log T)^{k^2}$$

for $0 \leq k < 1$.

Theorem 3.

(i) If $k \geq 0$ is rational, then

$$(5.2) \quad I_k(T, f \times f) \gg T(\log T)^{2k^2},$$

where

$$I_k(T, f \times f) = \int_1^T \left| L\left(\frac{1}{2} + it, f \times f\right) \right|^{2k} dt.$$

(ii) Under the Riemann hypothesis for $L(s, f \times f)$, estimate (5.2) is valid for all real $k \geq 0$.

(iii) Under the Riemann hypothesis for $L(s, f \times f)$, we have

$$I_k(T, f \times f) \ll T(\log T)^{2k^2}$$

for $0 \leq k < \frac{1}{2}$.

We do not present the proofs because they are much similar to that of Theorem 1.

In connection with Theorems 2 and 3, we make several remarks. We begin with Theorem 2.

1. Good [9] proved that

$$I_1(T, f) = B_1 T \log T + B_2 T + O(T^{2/3+\varepsilon}).$$

2. Part of the results of Theorem 2 was announced earlier by Laurinćikas [10].

3. As an analog of the asymptotics (2.1) we mention the formula

$$\sum_{p \leq x} \lambda_f^2(p) = x \log^{-1} x + O(x \log^{-2} x).$$

This was proved by Moreno [11].

Now we proceed to Theorem 3.

1. An analog of the asymptotics (2.1) is

$$(5.3) \quad \sum_{p \leq x} d_p^2 = 2x \log^{-1} x + O(x \log^{-2} x).$$

For the proof, we can use the identity

$$\lambda_f^4(p) = (\alpha_p^2 + 2 + \bar{\alpha}_p^2)^2 = (\alpha_p^2 + 1 + \bar{\alpha}_p^2)^2 + 1 + 2(\alpha_p^2 + 1 + \bar{\alpha}_p^2),$$

which is equivalent to

$$d_p^2 = c_p^2 + 1 + 2c_p.$$

It can be shown that

$$\sum_{p \leq x} c_p \ll x \log^{-2} x.$$

Therefore, using (2.1), we arrive at (5.3).

2. We define the coefficients $\nu_1(n, k)$ by the expansion (the choice of a branch is similar to that of $L^k(s, \text{sym}^2 f)$)

$$L^k(s, f \times f) = \sum_{n=1}^{\infty} \nu_1(n, k) n^{-s} \quad (\text{Re}(s) > 1).$$

An analog of Lemma 1 looks like this:

for fixed real $k \geq 0$, there exists $c_k'' > 0$ such that

$$\left(\sigma - \frac{1}{2}\right)^{-2k^2} \ll \sum_{n=1}^N \nu_1(n, k)^2 n^{-2\sigma} \ll \left(\sigma - \frac{1}{2}\right)^{-2k^2}$$

uniformly for

$$\frac{1}{2} + \frac{c_k''}{\log N} \leq \sigma \leq 1.$$

Moreover, we have

$$(\log N)^{2k^2} \ll \sum_{n=1}^N \nu_1(n, k)^2 n^{-1} \ll (\log N)^{2k^2}.$$

3. In order to avoid the singularity of $\zeta(s)$ at $s = 1$, Heath-Brown applied yet another convexity theorem of Gabriel [7]. We can employ that theorem in the case of the L -function $L(s, f \times f)$, which is holomorphic in the s -plane except for a simple pole at $s = 1$.

4. Since

$$L(s, f \times f) = \zeta(s) L(s, \text{sym}^2 f),$$

the Riemann hypothesis for $L(s, f \times f)$ is the conjunction of the Riemann hypotheses for $\zeta(s)$ and $L(s, \text{sym}^2 f)$.

5. In [12], it was proved that

$$I_1(T, f \times f) \ll T^{11/6+\varepsilon},$$

which can be deduced from the estimate

$$I_1(T, \text{sym}^2 f) \ll T^{3/2+\varepsilon}.$$

§6. THE DEDEKIND ZETA FUNCTION $\zeta_{K_3}(s)$

Let K_3 be the cubic field over $\mathbb{Q}$ given by adjoining a root of the polynomial $x^3 + ax^2 + bx + c$ of discriminant $D < 0$ with the Galois group S_3 . Consider $\zeta_{K_3}(s)$, the Dedekind zeta function of the field K_3 :

$$\zeta_{K_3}(s) = \sum_{m=1}^{\infty} M(m) m^{-s} \quad (\text{Re}(s) > 1),$$

where $M(m)$ is the number of integral ideals in the field K_3 of norm m . It is known (see [12, 13]) that

$$(6.1) \quad \zeta_{K_3}(s) = \zeta(s) L(s, \varphi),$$

where $\varphi(z)$ is a holomorphic Hecke eigencuspform of weight 1 with respect to the congruence subgroup $\Gamma_0(|D|)$,

$$\varphi(z) = \sum_{m=1}^{\infty} a(m) \exp(2\pi i m z).$$

We have

$$M(m) = \sum_{d|m} a(d);$$

in particular, $M(p) = 1 + a(p)$.

Theorem 4.

(i) *If $k \geq 0$ is rational, then*

$$(6.2) \quad I_k(T, K_3) \gg T(\log T)^{2k^2},$$

where

$$I_k(T, K_3) = \int_1^T \left| \zeta_{K_3} \left(\frac{1}{2} + it \right) \right|^{2k} dt.$$

(ii) *Under the Riemann hypothesis for $\zeta_{K_3}(s)$, estimate (6.2) is valid for all real $k \geq 0$.*

(iii) Under the Riemann hypothesis for $\zeta_{K_3}(s)$, we have

$$I_k(T, K_3) \ll T(\log T)^{2k^2}$$

for $0 \leq k < \frac{2}{3}$.

The proof is similar to that of Theorems 1–3. Several remarks are in order.

1. An analog of the asymptotics (2.1) is

$$\sum_{p \leq x} M(p)^2 = 2x \log^{-1} x + O(x \log^{-2} x).$$

This can be proved with the help of the identity

$$M(p)^2 = 1 + 2a(p) + a^2(p)$$

and some well-known facts.

2. The results of Hardy and Littlewood

$$I_1(T) \sim T \log T$$

and Good [9]

$$I_1(T, \varphi) \sim B_1 T \log T,$$

combined with the Cauchy inequality and (6.1), show that

$$I_{1/2}(T, K_3) \ll T \log T.$$

3. In [14] it was proved that

$$I_1(T, K_3) \ll T^{5/4+\varepsilon}.$$

4. The results of Heath-Brown [15]

$$I_6(T) \ll T^{2+\varepsilon}$$

and Jutila [16]

$$I_3(T, \varphi) \ll T^{2+\varepsilon},$$

combined with the Hölder inequality and (6.1) yield

$$I_2(T, K_3) \ll T^{2+\varepsilon}.$$

REFERENCES

- [1] D. R. Heath-Brown, *Fractional moments of the Riemann zeta function*, J. London Math. Soc. (2) **24** (1981), 65–78. MR0623671 (82h:10052)
- [2] M. Jutila, *On the value distribution of the zeta function on the critical line*, Bull. London Math. Soc. **15** (1983), 513–518. MR0705532 (84j:10049)
- [3] R. W. K. Odoni, *A problem of Rankin on sums of powers of cusp-form coefficients*, J. London Math. Soc. (2) **44** (1991), 203–217. MR1136435 (93d:11048)
- [4] S. Gelbart and H. Jacquet, *A relation between automorphic representations of $GL(2)$ and $GL(3)$* , Ann. Sci. École Norm. Sup. (4) **11** (1978), 471–542. MR0533066 (81e:10025)
- [5] D. Bump and D. Ginzburg, *Symmetric square L -functions on $GL(r)$* , Ann. of Math. (2) **136** (1992), 137–205. MR1173928 (93i:11058)
- [6] E. Wirsing, *Das asymptotische Verhalten von Summen über multiplikative Funktionen*, Math. Ann. **143** (1961), 75–102. MR0131389 (24:A1241)
- [7] R. M. Gabriel, *Some results concerning the integrals of moduli of regular functions along certain curves*, J. London Math. Soc. **2** (1927), 112–117.
- [8] H. L. Montgomery and R. C. Vaughan, *Hilbert's inequality*, J. London Math. Soc. (2) **8** (1974), 73–82. MR0337775 (49:2544)
- [9] A. Good, *The square mean of Dirichlet series associated with cusp forms*, Mathematika **29** (1982), 278–295. MR0696884 (84f:10036)
- [10] A. Laurinćikas, *On moments of zeta-functions associated to certain cusp forms*, Algebra and Number Theory: Modern Problems and Applications (Saratov, 2004), Thesis of Dokl., Saratov, 2004, pp. 136–137. (English) MR2280027 (2007i:11072)

- [11] C. J. Moreno, *The Hoheisel phenomenon for generalized Dirichlet series*, Proc. Amer. Math. Soc. **40** (1973), 47–51. MR0327682 (48:6024)
- [12] A. Sankaranarayanan, *Fundamental properties of symmetric square L -functions. I*, Illinois J. Math. **46** (2002), 23–43. MR1936073 (2004a:11038)
- [13] M. Koike, *Higher reciprocity law, modular forms of weight 1 and elliptic curves*, Nagoya Math. J. **98** (1985), 109–115. MR0792775 (86j:11044)
- [14] O. M. Fomenko, *Mean values connected with the Dedekind zeta function*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **350** (2007), 187–198; English transl., J. Math. Sci. (N.Y.) **150** (2008), no. 3, 2115–2122. MR2722976
- [15] D. R. Heath-Brown, *The twelfth power moment of the Riemann zeta-function*, Quart. J. Math. Oxford (2) **29** (1978), 443–462. MR0517737 (80d:10059)
- [16] M. Jutila, *Lectures on a method in the theory of exponential sums*, Tata Inst. Fund. Res. Lectures on Math. and Phys., vol. 80, Tata Inst. Fund. Res., Bombay, 1987. MR0910497 (89g:11069)
- [17] S. Zmambaga, *On fractional moments of Dirichlet L -functions. II*, Liet. Mat. Rink. **46** (2006), no. 4, 606–621. MR2320367 (2008d:11093)

ST. PETERSBURG BRANCH, STEKLOV MATHEMATICAL INSTITUTE, 27 FONTANKA, ST. PETERSBURG 191023, RUSSIA

E-mail address: fomenko@pdmi.ras.ru

Received 20/MAR/2009

Translated by G. V. KUZ'MINA AND O. M. FOMENKO