FUSION PROCEDURE FOR THE BRAUER ALGEBRA

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Dedicated to L. D. Faddeev on the occasion of his 75th birthday

ABSTRACT. It is shown that all primitive idempotents for the Brauer algebra $\mathcal{B}_n(\omega)$ can be found by evaluating a rational function in several variables that has the form of a product of R -matrix type factors. This provides an analog of the fusion procedure for $\mathcal{B}_n(\omega)$.

§1. INTRODUCTION

It is well known that all primitive idempotents of the symmetric group $\mathfrak{S}_n$ can be obtained by taking certain limit values of the rational function

$$(1.1) \quad \Phi(u_1, \dots, u_n) = \prod_{1 \leq i < j \leq n} \left(1 - \frac{s_{ij}}{u_i - u_j}\right),$$

where $s_{ij} \in \mathfrak{S}_n$ is the transposition of i and j , $u_1, \dots, u_n$ are complex variables, and the product is calculated in the group algebra $\mathbb{C}[\mathfrak{S}_n]$ in the lexicographical order on the pairs (i, j) . This construction, which is commonly referred to as the *fusion procedure*, goes back to Jucys [8] and Cherednik [5]. Detailed proofs were given by Nazarov [16]. A simple version of the fusion procedure was found in [13]; see also [14, Chapter 6] for applications to the Yangian representation theory and more references. In more detail, let T be a standard tableau associated with a partition λ of n , and let $c_k = j - i$ if the element k occupies the cell of the tableau in row i and column j . Then the consecutive evaluations

$$(1.2) \quad \Phi(u_1, \dots, u_n) \Big|_{u_1=c_1} \Big|_{u_2=c_2} \cdots \Big|_{u_n=c_n}$$

are well defined, and this value yields the corresponding primitive idempotent E_T^λ multiplied by the product of the hooks of the diagram of λ .

In this paper we give a similar fusion procedure for the Brauer algebra $\mathcal{B}_n(\omega)$. This algebra was introduced by Brauer in [4] and its structure and representation theory was studied by many authors; see, for instance, Wenzl [22], Nazarov [17], Leduc and Ram [11] and Rui [20]. We refer the reader to the survey paper by Barcelo and Ram [1] for the discussion of the Brauer algebra in the context of combinatorial representation theory and more references. The irreducible representations of $\mathcal{B}_n(\omega)$ are indexed by all partitions of the nonnegative integers $n, n-2, n-4, \dots$. If λ is such a partition, then the *up-down λ -tableaux* T parametrize basis vectors of the corresponding representation; see §2.

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Consider the rational function

$$(1.3) \quad \Psi(u_1, \dots, u_n) = \prod_{1 \leq i < j \leq n} \left(1 - \frac{e_{ij}}{u_i + u_j}\right) \prod_{1 \leq i < j \leq n} \left(1 - \frac{s_{ij}}{u_i - u_j}\right)$$

with the ordered products as in (1.1); the elements $e_{ij}, s_{ij} \in \mathcal{B}_n(\omega)$ are defined in §2 below. This function was first introduced by Nazarov [18, (3.14)] in the context of representations of the classical Lie algebras and twisted Yangians.

Our main result is the following analog of the fusion procedure for the Brauer algebra: given an up-down λ -tableau T , the consecutive evaluations

$$(1.4) \quad (u_1 - c_1)^{p_1} \dots (u_n - c_n)^{p_n} \Psi(u_1, \dots, u_n) \Big|_{u_1=c_1} \Big|_{u_2=c_2} \dots \Big|_{u_n=c_n}$$

are well defined, and this value yields the corresponding primitive idempotent E_T^λ multiplied by a nonzero constant $f(T)$, which is calculated in an explicit form. Here $p_1, \dots, p_n$ are certain integers depending on T , which we call the *exponents* of T , and the c_i are the *contents* of T ; see §2 for precise definitions.

In the particular case where λ is a partition of n , we thus reproduce some closely related results by Nazarov [18]; see, in particular, Propositions 3.2, 3.3 and formulas (3.20)–(3.23) there. In fact, he worked with wider classes of representations of the orthogonal and symplectic groups G_N parametrized by certain skew Young diagrams with n boxes. The natural action of G_N on the tensor power $(\mathbb{C}^N)^{\otimes n}$ commutes with the action of the Brauer algebra $\mathcal{B}_n(\omega)$ for a suitably specialized value of ω . Nazarov's formulas for the idempotents provide remarkable analogs of the Young symmetrizers in an explicit form. Their images in $(\mathbb{C}^N)^{\otimes n}$ yield realizations of the representations of G_N associated with the skew Young diagrams. Note that the corresponding images of the factors in (1.3) are the values of the Yang R -matrix and its transpose; cf. Remark 3.8 below.

If λ is a partition of n , then all exponents p_i are equal to zero, while the constant $f(T)$ takes the same value as for (1.2), thus making this case quite similar to that of the symmetric group. The existence of a special monomorphism $\mathbb{C}[\mathfrak{S}_n] \rightarrow \mathcal{B}_n(\omega)$ [2] can be regarded as an ‘explanation’ of this analogy. If λ is a partition of $n - 2f$ for some $f \geq 1$, then the function (1.3) can have zeros or poles of certain multiplicities at $u_i = c_i$, so that in place of (1.2) we need to take ‘regularized evaluations’ as in (1.4).

The proof of our main theorem (Theorem 3.4) follows the approach of [13] and is based on the construction of the primitive idempotents E_T^λ in terms of the Jucys–Murphy elements for the Brauer algebra. These elements were introduced independently by Nazarov [17] and by Leduc and Ram [11], where analogs of Young's seminormal representations for the Brauer algebra were given. In a more general context of cellular algebras equipped with a family of Jucys–Murphy elements, the construction of the primitive idempotents and seminormal forms was given by Mathas [12].

We expect that a result similar to Theorem 3.4 holds true for the Birman–Murakami–Wenzl algebras, which will be considered in our publications elsewhere; cf. [6, 7].

We are pleased to dedicate this work to L. D. Faddeev on his 75th birthday. The *fusion procedure* originates in the work of his *Leningrad School* on the quantum inverse scattering method. A key role in this method is played by solutions of the Yang–Baxter equation (R -matrices). The fusion procedure is understood as a way to obtain new solutions of this equation out of old ones; see, e.g., [10]. As a result, in the quantum inverse scattering method [21] one obtains a family of commuting transfer matrices that satisfy a number of functional equations (the so-called *fusion relations*). They are similar to the relations for the characters of the quantum linear groups (in the case of Hecke type R -matrices) and for the quantum SO and Sp groups (in the case of Birman–Murakami–Wenzl type R -matrices). The fundamental R -matrices of the Hecke and Birman–Murakami–Wenzl types were discussed in [19].

§2. THE BRAUER ALGEBRA AND ITS REPRESENTATIONS

Let n be a positive integer and ω an indeterminate. An n -diagram d is a collection of $2n$ dots arranged into two rows with n dots in each row connected by n edges such that any dot belongs to only one edge. The product of two diagrams d_1 and d_2 is determined by placing d_1 above d_2 and identifying the vertices of the bottom row of d_1 with the corresponding vertices in the top row of d_2 . Let s be the number of closed loops obtained in this placement. The product $d_1 d_2$ is given by ω^s times the resulting diagram without loops. The *Brauer algebra* $\mathcal{B}_n(\omega)$ is defined as the $\mathbb{C}(\omega)$ -linear span of the n -diagrams with the multiplication defined above. The dimension of the algebra is $1 \cdot 3 \cdots (2n-1)$. The following presentation of $\mathcal{B}_n(\omega)$ is well known; see, e.g., [3].

Proposition 2.1. *The Brauer algebra $\mathcal{B}_n(\omega)$ is isomorphic to the algebra with $2n-2$ generators $s_1, \dots, s_{n-1}, e_1, \dots, e_{n-1}$ and the defining relations*

$$\begin{aligned} s_i^2 &= 1, & e_i^2 &= \omega e_i, & s_i e_i &= e_i s_i = e_i, & i &= 1, \dots, n-1, \\ s_i s_j &= s_j s_i, & e_i e_j &= e_j e_i, & s_i e_j &= e_j s_i, & |i-j| &> 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, & e_i e_{i+1} e_i &= e_i, & e_{i+1} e_i e_{i+1} &= e_{i+1}, \\ s_i e_{i+1} e_i &= s_{i+1} e_i, & e_{i+1} e_i s_{i+1} &= e_{i+1} s_i, & i &= 1, \dots, n-2. \end{aligned}$$

The generators s_i and e_i correspond to the following diagrams, respectively:

The subalgebra of $\mathcal{B}_n(\omega)$ generated over $\mathbb{C}$ by $s_1, \dots, s_{n-1}$ is isomorphic to the group algebra $\mathbb{C}[\mathfrak{S}_n]$ so that s_i can be identified with the transposition $(i, i+1)$. Then for any $1 \leq i < j \leq n$ the transposition $s_{ij} = (i, j)$ can be regarded as an element of $\mathcal{B}_n(\omega)$. Moreover, e_{ij} will denote the element of $\mathcal{B}_n(\omega)$ represented by the diagram in which the i th and j th dots in the top row, as well as the i th and j th dots in the bottom row are connected by an edge, while the remaining edges connect the k th dot in the top row with the k th dot in the bottom row for each $k \neq i, j$. Equivalently, in terms of the presentation of $\mathcal{B}_n(\omega)$ provided by Proposition 2.1,

$$s_{ij} = s_i s_{i+1} \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_{i+1} s_i \quad \text{and} \quad e_{ij} = s_{i,j-1} e_{j-1} s_{i,j-1}.$$

The Brauer algebra $\mathcal{B}_{n-1}(\omega)$ can be regarded as the subalgebra of $\mathcal{B}_n(\omega)$ spanned by all diagrams in which the n th dots in the top and bottom rows are connected by an edge.

The *Jucys–Murphy elements* $x_1, \dots, x_n$ for the Brauer algebra $\mathcal{B}_n(\omega)$ were introduced independently in [11] and [17]; they are given by the formulas

$$x_r = \frac{\omega - 1}{2} + \sum_{k=1}^{r-1} (s_{kr} - e_{kr}), \quad r = 1, \dots, n.$$

The element x_n commutes with the subalgebra of $\mathcal{B}_{n-1}(\omega)$. This implies that the elements $x_1, \dots, x_n$ of $\mathcal{B}_n(\omega)$ commute pairwise. They can be used to construct a complete set of pairwise orthogonal primitive idempotents for the Brauer algebra following the approach of Jucys [9] and Murphy [15]; see also [12] for a generalization to a wider class of cellular algebras. Namely, let λ be a partition of $n-2f$ for some $f \in \{0, 1, \dots, \lfloor n/2 \rfloor\}$. We shall identify partitions with their diagrams, so that if the parts of λ are $\lambda_1, \lambda_2, \dots$, then the corresponding diagram is a left-justified array of rows of unit boxes containing λ_1 boxes in the top row, λ_2 boxes in the second row, etc. The box in row i and column

j of a diagram will be denoted as the pair (i, j) . An *up-down λ -tableau* is a sequence $T = (\Lambda_1, \dots, \Lambda_n)$ of diagrams such that for each $r = 1, \dots, n$ the diagram Λ_r is obtained from Λ_{r-1} by adding or removing one box, where $\Lambda_0 = \emptyset$ is the empty diagram and $\Lambda_n = \lambda$. To each up-down tableau T we attach the corresponding sequence of *contents* $(c_1, \dots, c_n)$, $c_r = c_r(T)$, where

$$c_r = \frac{\omega - 1}{2} + j - i \quad \text{or} \quad c_r = -\left(\frac{\omega - 1}{2} + j - i\right)$$

if Λ_r is obtained by adding the box (i, j) to Λ_{r-1} or by removing this box from Λ_{r-1} , respectively. The primitive idempotents $E_T = E_T^\lambda$ can now be defined by the following recurrence formula (we omit the superscripts indicating the diagrams, because they are determined by the up-down tableaux). Set $\mu = \Lambda_{n-1}$ and consider the up-down μ -tableau $U = (\Lambda_1, \dots, \Lambda_{n-1})$. Let α be the box that is added to or removed from μ to get λ . Then

$$(2.1) \quad E_T = E_U \frac{(x_n - a_1) \dots (x_n - a_k)}{(c_n - a_1) \dots (c_n - a_k)},$$

where $a_1, \dots, a_k$ are the contents of all boxes excluding α , which can be removed from or added to μ to get a diagram. When λ runs over all partitions of n , $n - 2, \dots$ and T runs over all up-down λ -tableaux, the elements $\{E_T\}$ yield a complete set of pairwise orthogonal primitive idempotents for $\mathcal{B}_n(\omega)$. They have the properties

$$(2.2) \quad x_r E_T = E_T x_r = c_r(T) E_T, \quad r = 1, \dots, n.$$

Moreover, given an up-down tableau $U = (\Lambda_1, \dots, \Lambda_{n-1})$, we have the relation

$$(2.3) \quad E_U = \sum_T E_T,$$

summed over all up-down tableaux of the form $T = (\Lambda_1, \dots, \Lambda_{n-1}, \Lambda_n)$; we refer the reader to [11, 12] and [17] for more details. Relation (2.1) admits the following equivalent form:

$$(2.4) \quad E_T = E_U \frac{u - c_n}{u - x_n} \Big|_{u=c_n},$$

where u is a complex variable. This relation is derived from (2.2) and (2.3) exactly as in the case of the symmetric group; see [13].

§3. THE FUSION PROCEDURE

Some combinatorial data extracted from the up-down tableaux will be convenient for the formulations below. Let $U = (\Lambda_1, \dots, \Lambda_{n-1})$ be a μ -tableau; we define two infinite matrices $m(U)$ and $m'(U)$ whose rows and columns are labeled by positive integers and only a finite number of entries in each of the matrices is nonzero. The entry m_{ij} of the matrix $m(U)$ (respectively, the entry m'_{ij} of the matrix $m'(U)$) equals the number of times the box (i, j) was added (respectively, removed) in the sequence of diagrams $(\emptyset = \Lambda_0, \Lambda_1, \dots, \Lambda_{n-1})$. So, the difference $m(U) - m'(U)$ is the matrix all of whose entries are zero except for the ij th matrix elements equal to 1 for which the corresponding boxes (i, j) are contained in the diagram μ .

Example 3.1. For the up-down tableau

$$U = \left(\square, \quad \square\square, \quad \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \quad \begin{smallmatrix} \square & \square \\ & \square \end{smallmatrix}, \quad \square, \quad \begin{smallmatrix} \square & \square \\ & \square \end{smallmatrix}, \quad \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}, \quad \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} \right),$$

the matrices are

$$m(U) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad m'(U) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

where the common zeros in both matrices have been omitted.

Furthermore, for each integer k we define the nonnegative integers $d_k = d_k(U)$ and $d'_k = d'_k(U)$ as the respective sums of the entries of the matrices $m(U)$ and $m'(U)$ on the k th diagonal:

$$d_k = \sum_{j-i=k} m_{ij}, \quad d'_k = \sum_{j-i=k} m'_{ij}.$$

So, in Example 3.1, we have $d_{-1} = d_0 = d_1 = 2$, while $d'_{-1} = d'_0 = d'_1 = 1$ and the remaining values d_k and d'_k are zero.

Finally, for each integer k we introduce the parameters $g_k = g_k(U)$ and $g'_k = g'_k(U)$ by

$$(3.1) \quad g_k = \delta_{k0} + d_{k-1} + d_{k+1} - 2d_k, \quad g'_k = d'_{k-1} + d'_{k+1} - 2d'_k.$$

Now the *exponents* $p_1, \dots, p_n$ of an up-down λ -tableau $T = (\Lambda_1, \dots, \Lambda_n)$ are defined inductively, so that p_r depends only on the first r diagrams $(\Lambda_1, \dots, \Lambda_r)$ of T . Hence, it suffices to define p_n . Taking $U = (\Lambda_1, \dots, \Lambda_{n-1})$, we set

$$(3.2) \quad p_n = 1 - g_{k_n}(U) \quad \text{or} \quad p_n = 1 - g'_{k_n}(U),$$

respectively, if Λ_n is obtained from Λ_{n-1} by adding a box on the diagonal k_n or by removing a box on the diagonal k_n .

Example 3.2. The exponents for the up-down tableau

$$T = \left(\square, \quad \square\square, \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \quad \begin{array}{|c|} \hline \square \\ \hline \end{array}, \quad \square, \quad \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \right)$$

are $p_1 = p_2 = p_3 = 0$, $p_4 = p_5 = 1$, and $p_6 = 2$.

The constants $f(T)$ mentioned in the Introduction are defined inductively by the formula

$$(3.3) \quad f(T) = f(U) \varphi(U, T),$$

where $U = (\Lambda_1, \dots, \Lambda_{n-1})$ and $T = (\Lambda_1, \dots, \Lambda_n)$. Here

$$\varphi(U, T) = \prod_{k \neq k_n} (k_n - k)^{g_k} \prod_k (k_n + k + \omega - 1)^{g'_k}$$

or

$$\varphi(U, T) = \prod_{k \neq k_n} (-k_n + k)^{g'_k} \prod_k (-k_n - k - \omega + 1)^{g_k}$$

if Λ_n is obtained from Λ_{n-1} by adding or removing a box on the diagonal k_n , respectively, where the products are taken over all integers k , while $g_k = g_k(U)$ and $g'_k = g'_k(U)$. Note that only a finite number of the parameters g_k and g'_k are nonzero, so that each product in the above formulas contains only a finite number of factors not equal to 1.

Proposition 3.3. *If $T = (\Lambda_1, \dots, \Lambda_n)$ is an up-down λ -tableau and λ is a partition of n , then all exponents $p_1, \dots, p_n$ of T are equal to zero, and $f(T)$ equals the product of the hooks of λ .*

Proof. Set $U = (\Lambda_1, \dots, \Lambda_{n-1})$ and $\mu = \Lambda_{n-1}$. The nonzero entries of the matrix $m(U)$ are equal to 1; these are the ij th matrix elements such that the corresponding boxes (i, j) are contained in the diagram μ . Furthermore, all entries of the matrix $m'(U)$ are zero. Hence, the parameters $g'_k(U)$ are all zero, while the nonzero values of $g_k(U)$ are equal to ± 1 . The value 1 (respectively, -1) corresponds to those diagonals k where a box can

be added to (respectively, removed from) the diagram μ . This proves that $p_r = 0$ for all r and the claim about $f(T)$ is also verified easily. $\square$

Consider the rational function $\Psi(u_1, \dots, u_n)$ with values in the Brauer algebra $\mathcal{B}_n(\omega)$, defined by (1.3). Now we can prove our main theorem.

Theorem 3.4. *For any up-down tableau $T = (\Lambda_1, \dots, \Lambda_n)$, the consecutive evaluations*

$$(u_1 - c_1)^{p_1} \dots (u_n - c_n)^{p_n} \Psi(u_1, \dots, u_n) \Big|_{u_1=c_1} \Big|_{u_2=c_2} \dots \Big|_{u_n=c_n}$$

are well defined. The corresponding value coincides with $f(T) E_T$.

Proof. The proof of the theorem will follow from a sequence of lemmas.

Lemma 3.5. *The function $\Psi(u_1, \dots, u_n)$ can be written in the equivalent form*

$$(3.4) \quad \Psi(u_1, \dots, u_n) = \prod_{r=2, \dots, n}^{\rightarrow} \left(1 - \frac{e_{r-1, r}}{u_{r-1} + u_r}\right) \dots \times \left(1 - \frac{e_{1, r}}{u_1 + u_r}\right) \left(1 - \frac{s_{1, r}}{u_1 - u_r}\right) \dots \left(1 - \frac{s_{r-1, r}}{u_{r-1} - u_r}\right),$$

where the factors are ordered in accordance with the increasing values of r .

Proof. This follows by using the following easily verified identities for the rational functions in u and v with values in $\mathcal{B}_n(\omega)$: if $i < j < r$, then

$$(3.5) \quad \left(1 - \frac{e_{ir}}{u}\right) \left(1 - \frac{e_{jr}}{v}\right) \left(1 - \frac{s_{ij}}{u-v}\right) = \left(1 - \frac{s_{ij}}{u-v}\right) \left(1 - \frac{e_{jr}}{v}\right) \left(1 - \frac{e_{ir}}{u}\right).$$

If the indices i, j, k, l are distinct, then the elements e_{ij} and e_{kl} of $\mathcal{B}_n(\omega)$ commute. Therefore, we can represent the first product occurring in (1.3) as

$$\prod_{1 \leq i < j \leq n} \left(1 - \frac{e_{ij}}{u_i + u_j}\right) = \prod_{1 \leq i < j \leq n-1} \left(1 - \frac{e_{ij}}{u_i + u_j}\right) \left(1 - \frac{e_{1, n}}{u_1 + u_n}\right) \dots \left(1 - \frac{e_{n-1, n}}{u_{n-1} + u_n}\right).$$

Now, using identities (3.5) repeatedly, we get

$$\begin{aligned} & \left(1 - \frac{e_{1, n}}{u_1 + u_n}\right) \dots \left(1 - \frac{e_{n-1, n}}{u_{n-1} + u_n}\right) \prod_{1 \leq i < j \leq n-1} \left(1 - \frac{s_{ij}}{u_i - u_j}\right) \\ &= \prod_{1 \leq i < j \leq n-1} \left(1 - \frac{s_{ij}}{u_i - u_j}\right) \left(1 - \frac{e_{n-1, n}}{u_{n-1} + u_n}\right) \dots \left(1 - \frac{e_{1, n}}{u_1 + u_n}\right). \end{aligned}$$

Hence, the function (1.3) can be written as

$$(3.6) \quad \begin{aligned} \Psi(u_1, \dots, u_n) &= \Psi(u_1, \dots, u_{n-1}) \left(1 - \frac{e_{n-1, n}}{u_{n-1} + u_n}\right) \dots \\ &\quad \times \left(1 - \frac{e_{1, n}}{u_1 + u_n}\right) \left(1 - \frac{s_{1, n}}{u_1 - u_n}\right) \dots \left(1 - \frac{s_{n-1, n}}{u_{n-1} - u_n}\right), \end{aligned}$$

and the decomposition (3.4) follows by induction on n . $\square$

Lemma 3.5 allows us to use the induction on n to prove the theorem. By the inductive hypothesis, setting $u = u_n$ we get

$$(3.7) \quad \begin{aligned} & (u_1 - c_1)^{p_1} \dots (u_n - c_n)^{p_n} \Psi(u_1, \dots, u_n) \Big|_{u_1=c_1} \Big|_{u_2=c_2} \dots \Big|_{u_{n-1}=c_{n-1}} \\ &= f(U) E_U (u - c_n)^{p_n} \left(1 - \frac{e_{n-1, n}}{c_{n-1} + u}\right) \dots \left(1 - \frac{e_{1, n}}{c_1 + u}\right) \\ &\quad \times \left(1 - \frac{s_{1, n}}{c_1 - u}\right) \dots \left(1 - \frac{s_{n-1, n}}{c_{n-1} - u}\right), \end{aligned}$$

where U is the up-down tableau $(\Lambda_1, \dots, \Lambda_{n-1})$. The next lemma will allow us to simplify this expression.

Lemma 3.6. *We have*

$$(3.8) \quad \begin{aligned} E_U \left(1 - \frac{e_{n-1,n}}{c_{n-1} + u}\right) \cdots \left(1 - \frac{e_{1,n}}{c_1 + u}\right) \left(1 - \frac{s_{1,n}}{c_1 - u}\right) \cdots \left(1 - \frac{s_{n-1,n}}{c_{n-1} - u}\right) \\ = \frac{u - c_1}{u - c_n} \prod_{r=1}^{n-1} \left(1 - \frac{1}{(u - c_r)^2}\right) E_U \frac{u - c_n}{u - x_n}. \end{aligned}$$

Proof. Note that the Jucys–Murphy element x_n commutes with E_U , and the inverses of the expressions occurring in the product are found by

$$\left(1 - \frac{s_{r,n}}{c_r - u}\right)^{-1} \left(1 - \frac{1}{(u - c_r)^2}\right) = \left(1 + \frac{s_{r,n}}{c_r - u}\right)$$

and

$$\left(1 - \frac{e_{r,n}}{c_r + u}\right)^{-1} = \left(1 + \frac{e_{r,n}}{c_r + u - \omega}\right),$$

where we have used the relations $s_{r,n}^2 = 1$ and $e_{r,n}^2 = \omega e_{r,n}$. Hence, relation (3.8) is equivalent to

$$(3.9) \quad \begin{aligned} E_U \left(1 + \frac{s_{n-1,n}}{c_{n-1} - u}\right) \cdots \left(1 + \frac{s_{1,n}}{c_1 - u}\right) \left(1 + \frac{e_{1,n}}{c_1 + u - \omega}\right) \\ \cdots \left(1 + \frac{e_{n-1,n}}{c_{n-1} + u - \omega}\right) = E_U \frac{u - x_n}{u - c_1}. \end{aligned}$$

We embed the Brauer algebra $\mathcal{B}_n(\omega)$ in $\mathcal{B}_m(\omega)$ for some $m \geq n$ and use induction on n to verify the more general identity

$$(3.10) \quad \begin{aligned} E_U \left(1 + \frac{s_{n-1,m}}{c_{n-1} - u}\right) \cdots \left(1 + \frac{s_{1,m}}{c_1 - u}\right) \left(1 + \frac{e_{1,m}}{c_1 + u - \omega}\right) \\ \cdots \left(1 + \frac{e_{n-1,m}}{c_{n-1} + u - \omega}\right) = E_U \frac{u - x_n^{(m)}}{u - c_1}, \end{aligned}$$

where

$$x_n^{(m)} = \frac{\omega - 1}{2} + \sum_{k=1}^{n-1} (s_{km} - e_{km}).$$

By (2.3), $E_U = E_U E_W$, where W is the up-down tableau $(\Lambda_1, \dots, \Lambda_{n-2})$. Hence, using the inductive hypothesis, we can write the left-hand side of (3.10) as

$$\begin{aligned} E_U \left(1 + \frac{s_{n-1,m}}{c_{n-1} - u}\right) E_W \frac{u - x_{n-1}^{(m)}}{u - c_1} \left(1 + \frac{e_{n-1,m}}{c_{n-1} + u - \omega}\right) \\ = \frac{1}{u - c_1} E_U \left(u - x_{n-1}^{(m)} + \frac{s_{n-1,m} (u - x_{n-1}^{(m)})}{c_{n-1} - u} + \frac{(u - x_{n-1}^{(m)}) e_{n-1,m}}{c_{n-1} + u - \omega} \right. \\ \left. + \frac{s_{n-1,m} (u - x_{n-1}^{(m)}) e_{n-1,m}}{(c_{n-1} - u)(c_{n-1} + u - \omega)} \right). \end{aligned}$$

Now we use the following relations in $\mathcal{B}_m(\omega)$, which are valid for $1 \leq r < n-1$:

$$s_{n-1,m} s_{r,m} = s_{r,n-1} s_{n-1,m}, \quad s_{n-1,m} e_{r,m} = e_{r,n-1} s_{n-1,m}$$

and

$$s_{r,m} e_{n-1,m} = e_{r,n-1} e_{n-1,m}, \quad e_{r,m} e_{n-1,m} = s_{r,n-1} e_{n-1,m}.$$

They imply that

$$s_{n-1,m} x_{n-1}^{(m)} = x_{n-1} s_{n-1,m}$$

and

$$x_{n-1}^{(m)} e_{n-1,m} = (\omega - 1 - x_{n-1}) e_{n-1,m}.$$

Together with the relation $E_U x_{n-1} = c_{n-1} E_U$ implied by (2.2), this allows us to bring the left-hand side of (3.10) to the form

$$\frac{1}{u - c_1} E_U \left(u - x_{n-1}^{(m)} - s_{n-1,m} + e_{n-1,m} \right) = E_U \frac{u - x_n^{(m)}}{u - c_1},$$

as required. $\square$

Lemma 3.6 shows that, in order to complete the proof of the theorem, we need to show that the rational function

$$f(U) (u - c_1) \prod_{r=1}^{n-1} \left(1 - \frac{1}{(u - c_r)^2} \right) (u - c_n)^{p_n-1} \cdot E_U \frac{u - c_n}{u - x_n}$$

is regular at $u = c_n$ and that its value equals $f(T) E_T$. Using the parameters (3.1), we can write this expression as

$$f(U) \prod_k \left(u - \frac{\omega - 1}{2} - k \right)^{g_k} \prod_k \left(u + \frac{\omega - 1}{2} + k \right)^{g'_k} (u - c_n)^{p_n-1} \cdot E_U \frac{u - c_n}{u - x_n},$$

where k runs over the set of integers. If the diagram Λ_n is obtained from Λ_{n-1} by adding or removing a box on the diagonal k_n , then the value of the content c_n is given by the respective formulas

$$c_n = \frac{\omega - 1}{2} + k_n \quad \text{or} \quad c_n = -\left(\frac{\omega - 1}{2} + k_n \right).$$

The definition of the exponents (3.2) and the constants $f(T)$ in (3.3) together with (2.4) imply the desired statement. $\square$

The following corollary is immediate from Proposition 3.3 and Theorem 3.4; cf. [13, 18].

Corollary 3.7. *If $T = (\Lambda_1, \dots, \Lambda_n)$ is an up-down λ -tableau and λ is a partition of n , then the consecutive evaluations*

$$\Psi(u_1, \dots, u_n) \Big|_{u_1=c_1} \Big|_{u_2=c_2} \cdots \Big|_{u_n=c_n}$$

are well defined. The corresponding value coincides with $H(\lambda) E_T$, where $H(\lambda)$ is the product of the hooks of λ .

Remark 3.8. In two particular cases where λ is a row- or column-diagram with n boxes, one can write alternative multiplicative expressions associated with the respective tableaux. Namely, the primitive idempotent corresponding to the only up-down (n) -tableau is proportional to

$$\prod_{1 \leq i < j \leq n} \left(1 + \frac{s_{ij}}{j - i} - \frac{e_{ij}}{j - i + \omega/2 - 1} \right),$$

while the primitive idempotent corresponding to the up-down (1^n) -tableau is proportional to

$$\prod_{1 \leq i < j \leq n} \left(1 - \frac{s_{ij}}{j - i} \right),$$

with both products taken in the lexicographical order on the pairs (i, j) . These formulas are easily verified by using the well-known fact that the rational function

$$R_{ij}(u) = 1 - \frac{s_{ij}}{u} + \frac{e_{ij}}{u - \omega/2 + 1}$$

is a solution of the Yang–Baxter equation

$$R_{12}(u) R_{13}(u + v) R_{23}(v) = R_{23}(v) R_{13}(u + v) R_{12}(u);$$

see [23]. These multiplicative formulas for the idempotents do not seem to have natural analogs for general up-down tableaux. Note, however, that the following alternative rational function in the case of $\mathcal{B}_3(\omega)$ can be used instead of $\Psi(u_1, u_2, u_3)$ in the formulation of the fusion procedure:

$$\begin{aligned} \tilde{\Psi}(u_1, u_2, u_3) = & \left(1 - (u_1 - u_2) s_1 + \frac{u_1 - u_2 - 1}{u_1 + u_2} e_1\right) \left(1 - (u_1 - u_3) s_2 + \frac{u_1 - u_3 - 2}{u_2 + u_3} e_2\right) \\ & \times \left(1 - (u_1 - u_2) s_1 + \frac{u_1 - u_2 - 1}{u_1 + u_2} e_1\right). \end{aligned}$$

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