EXTENDED QUADRATIC ALGEBRA AND A MODEL OF THE EQUIVARIANT COHOMOLOGY RING OF FLAG VARIETIES

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To Ludwig Dmitrievich Faddeev on the occasion of his 75th birthday

ABSTRACT. For a root system of type A , a certain extension of the quadratic algebra invented by S. Fomin and the first author is introduced and studied, which makes it possible to construct a model for the equivariant cohomology ring of the corresponding flag variety. As an application, a generalization of the equivariant Pieri rule for double Schubert polynomials is described. For a general finite Coxeter system, an extension of the corresponding Nichols–Woronowicz algebra is constructed. In the case of finite crystallographic Coxeter systems, a construction is presented of an extended Nichols–Woronowicz algebra model for the equivariant cohomology of the corresponding flag variety.

§1. INTRODUCTION

In the paper [6], S. Fomin and the first author introduced and studied a model for the cohomology ring of flag varieties of type A as a commutative subalgebra generated by the so-called Dunkl elements in a certain noncommutative quadratic algebra $\mathcal{E}_n$. An advantage of the approach developed in [6] is that it admits a simple generalization suitable for a description of the (small) quantum cohomology ring of flag varieties, as well as the (quantum) Schubert polynomials. The constructions of [6] were generalized to other finite root systems by the authors in [9]. A principal constituent of the above constructions is the Dunkl elements. The basic properties of the Dunkl elements are as follows:

- 1) they are pairwise commuting;
- 2) in the so-called Calogero–Moser representation [6, 9], they correspond to the *truncated* (i.e., without differential part) rational Dunkl operators [4];
- 3) in the crystallographic case they correspond — after applying the so-called Bruhat representation [6, 9] — to the Monk formula in the cohomology ring of the flag variety in question;
- 4) in the crystallographic case, subtraction-free expressions of Schubert polynomials calculated at the Dunkl elements in the algebra $\widetilde{\mathcal{BE}}(\Delta)$ should provide a combinatorial rule for describing the Schubert basis structural constants, i.e., the intersection numbers of Schubert classes.

In the case of classical root systems Δ , the first author [7] defined a certain extension $\widetilde{\mathcal{BE}}(\Delta)$ of the algebra $\mathcal{BE}(\Delta)$, together with a pairwise commuting family of elements, called the Dunkl elements, which after applying the Calogero–Moser representation

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coincide exactly with the rational Dunkl operators. We recall that the Calogero–Moser representation of the algebra $\widetilde{\mathcal{BE}}(\Delta)$ is a representation given by the divided difference operators $\partial_\alpha := (1 - s_\alpha)/\alpha$ acting on the symmetric algebra $S(V)$, where V is a reflection representation of the Weyl group corresponding to the root system Δ , and s_α is a reflection with respect to the root $\alpha \in \Delta$.

One of our main objectives in this paper is to study the commutative subalgebra generated by the Dunkl elements in the extended algebra $\widetilde{\mathcal{BE}}(\Delta)$ in the case of type A root systems. Postnikov [13] proved a Pieri-type formula for the elementary symmetric polynomials evaluated at the Dunkl elements, which had been originally conjectured in [6]. In Theorem 2.1 we show an analog of the Pieri-type formula in the extended algebra $\widetilde{\mathcal{BE}}(\Delta)$ of type A . As a consequence, it is shown that the commutative subalgebra generated by the Dunkl elements is isomorphic to the equivariant cohomology ring $H_T^*(Fl_n)$, where Fl_n is the flag variety parametrizing the full flags in the vector space $\mathbb{C}^n$, and $T = (\mathbb{C}^\times)^n$ is the torus acting on $\mathbb{C}^n$ diagonally.

In §3 we construct the Bruhat representation of the algebra $\mathcal{E}_n(R)[t]$ and study some properties of the former. The existence of Bruhat’s representation of the algebra $\mathcal{E}_n(R)[t]$ plays a crucial role in applications to the equivariant Schubert calculus and constitutes an important step in the construction of the model of $H_T^*(Fl_n)$. Our formula in Theorem 2.1 describes the Pieri formula for the equivariant cohomology ring $H_T^*(Fl_n)$ via the Bruhat representation. The Pieri formula for double Schubert polynomials was studied in [15, Chapter 5]. A similar rule in the cohomology ring $H_T^*(Fl_n)$ was stated and proved in [14]. We shall show how to compute the multiplication by the special Schubert classes via the Bruhat representation in Example 3.1, where an example is also given to see that the result of the computation based on our formula coincides with that in [14].

In §4 we introduce and study the “quantum” deformation $\mathcal{E}_n^q$ and $\mathcal{E}_n^q(R)[t]$ of the algebras $\mathcal{E}_n$ and $\mathcal{E}_n(R)[t]$. It is remarkable that the description of the Pieri-type formula based on the quadratic algebra $\mathcal{E}_n^q$ gives the multiplication rule in the quantum cohomology ring $QH^*(Fl_n)$ under a natural quantum deformation of the algebra $\mathcal{E}_n$ presented in Definition 4.1. As shown in §4, our formula also covers a similar formula in the equivariant quantum cohomology ring $QH_T^*(Fl_n)$.

Another objective in our paper is to construct a certain extension of the Nichols–Woronowicz model for the coinvariant algebra of a finite Coxeter group W . It is conjectured that the Fomin–Kirillov quadratic algebra $\mathcal{E}_n$ is isomorphic to the Nichols–Woronowicz algebra associated with a certain kind of Yetter–Drinfeld module defined by the data of the root system of type A_{n-1} . More generally, the algebra $\mathcal{BE}(\Delta)$ is a lift of the Nichols–Woronowicz algebra for the root system Δ . Recall that the Nichols–Woronowicz algebra model for the cohomology ring of flag varieties was invented by Y. Bazlov [2]. In §4 we introduce a certain extension $\widetilde{\mathcal{B}}_W$ of the Nichols–Woronowicz algebra $\mathcal{B}_W$ and construct a commutative subalgebra in the extended Nichols–Woronowicz algebra. Our second main result states that, for crystallographic root systems and $t = 0$, the commutative subalgebra of $\widetilde{\mathcal{B}}_W$ in question is isomorphic to the T -equivariant cohomology ring of the corresponding flag variety.

§2. EXTENSION OF THE QUADRATIC ALGEBRA

Definition 2.1. The algebra $\mathcal{E}_n$ is an associative algebra generated by the symbols $[i, j]$, $1 \leq i, j \leq n$, $i \neq j$, subject to the relations

- (0) $[i, j] = -[j, i]$,
- (1) $[i, j]^2 = 0$,
- (2) $[i, j][k, l] = [k, l][i, j]$ if $\{i, j\} \cap \{k, l\} = \emptyset$,
- (3) $[i, j][j, k] + [j, k][k, i] + [k, i][i, j] = 0$.

We consider the extension $\mathcal{E}_n\langle R\rangle[t]$ of the quadratic algebra $\mathcal{E}_n$ by the polynomial ring $R[t] = \mathbb{Z}[x_1, \dots, x_n][t]$ defined by the following commutation relations:

- (A) $[i, j]x_k = x_k[i, j]$, for $k \neq i, j$,
- (B) $[i, j]x_i = x_j[i, j] + t$, $[i, j]x_j = x_i[i, j] - t$, for $i < j$,
- (C) $[i, j]t = t[i, j]$.

Note that the $\mathbf{S}_n$ -invariant subalgebra $R^{\mathbf{S}_n}[t]$ of $R[t]$ is contained in the center of the algebra $\mathcal{E}_n\langle R\rangle[t]$. Let $e_k(x_1, \dots, x_n)$, $1 \leq k \leq n$, stand for the elementary symmetric polynomial of degree k in the variables $x_1, \dots, x_n$. By definition, we put $e_0(x_1, \dots, x_n) = 1$, and $e_k(x_1, \dots, x_n) = 0$ if $k < 0$.

Definition 2.2. (1) We define the $R[t]$ -algebra $\tilde{\mathcal{E}}_n[t]$ by

$$\tilde{\mathcal{E}}_n[t] = \mathcal{E}_n\langle R\rangle[t] \otimes_{R^{\mathbf{S}_n}} R.$$

More explicitly, $\tilde{\mathcal{E}}_n[t]$ is an algebra over the polynomial ring $\mathbb{Z}[y_1, \dots, y_n]$ generated by the symbols $[i, j]$, $1 \leq i, j \leq n$, $i \neq j$, and $x_1, \dots, x_n, t$ satisfying the relations in the definition of the algebra $\mathcal{E}_n\langle R\rangle[t]$, together with the identification $e_i(x_1, \dots, x_n) = e_i(y_1, \dots, y_n)$ for $i = 1, \dots, n$. Let $\tilde{\mathcal{E}}_{n,t_0}$ denote the specialization of $\tilde{\mathcal{E}}_n[t]$ at $t = t_0$.

- (2) The Dunkl elements $\theta_i \in \tilde{\mathcal{E}}_n[t]$, $i = 1, \dots, n$, are defined by the formula

$$\theta_i = \theta_i^{(n)} = x_i + \sum_{j \neq i} [i, j].$$

Remark 2.1. Note that the x_i 's do not commute with the Dunkl elements, but only symmetric polynomials in the x_i 's do. Moreover, we need the R -algebra structure of $\tilde{\mathcal{E}}_n[t]$ to construct the model of the T -equivariant cohomology ring $H_T^*(Fl_n)$, which is an algebra over $H_T^*(pt) \cong R$. For this reason, we need the second copy of $R = \mathbb{Z}[y_1, \dots, y_n]$, where the y_i 's are assumed to belong to the center of the algebra $\tilde{\mathcal{E}}_n[t]$, and $f(x_1, \dots, x_n) = f(y_1, \dots, y_n)$ for any symmetric polynomial f .

Lemma 2.1. *The Dunkl elements commute with each other.*

Proof. This follows from the fact that

$$(x_i + x_j)[i, j] = [i, j](x_i + x_j). \quad \square$$

Theorem 2.1 (Pieri formula in the algebra $\mathcal{E}_n\langle R\rangle[t]$). *For $k \leq m \leq n$, we have*

$$e_k(\theta_1^{(n)}, \dots, \theta_m^{(n)}) = \sum_{r \geq 0} (-t)^r N(m - k, 2r) \left\{ \sum_{S, I = \{i_a\}, (j_a)} X_S \cdot [i_1, j_1] \cdots [i_{|I|}, j_{|I|}] \right\},$$

where

$$N(a, 2b) = (2b - 1)!! \binom{a + 2b}{2b},$$

$X_S := \prod_{s \in S} x_s$, and the second summation runs over the triples

$$(S, I = \{i_1, \dots, i_{|I|}\}, (j_a)_{a=1}^{|I|})$$

such that $S \subset \{1, \dots, m\}$; I is a subset of $\{1, \dots, m\} \setminus S$; $|I| + |S| + 2r = k$; $1 \leq i_a \leq m < j_a \leq n$; and $j_1 \leq \dots \leq j_{|I|}$.

Proof. Let $\mathcal{A}$ be a subset of $\{1, \dots, n\}$; we put $m := |\mathcal{A}|$, $d := n - m$, and $\{1, \dots, n\} \setminus \mathcal{A} = \{j_1 < \dots < j_d\}$. Denote by $E_k(\mathcal{A})$ the right-hand side of the formula, i.e.,

$$E_k(\mathcal{A}) := \sum_{r \geq 0} (-t)^r N(m - k, 2r) \sum_{S \subset \mathcal{A}} X_S \sum_{(*)} [s_1, t_1] \cdots [s_{k-2r-|S|}, t_{k-2r-|S|}],$$

where $(*)$ stands for the following conditions: the elements $s_1, \dots, s_{k-2r-|S|} \in \mathcal{A} \setminus S$ are

distinct, $t_1, \dots, t_{k-2r-|S|} \in \{1, \dots, n\} \setminus \mathcal{A}$, and $t_1 \leq \dots \leq t_{k-2r-|S|}$. It will suffice to prove the recursive formula

$$E_k(\mathcal{A} \cup \{j = j_1\}) = E_k(\mathcal{A}) + E_{k-1}(\mathcal{A}) \left(x_j + \sum_{s \neq j} [j, s] \right).$$

For a subset $I = \{i_1, \dots, i_l\} \subset \{1, \dots, n\}$ and $p \notin I$, we use the symbol

$$\langle\langle I|p \rangle\rangle = \sum_{w \in \mathbf{S}_l} [i_{w(1)}, p] \cdots [i_{w(l)}, p]$$

as defined in [13]. We also use the symbol $I_1 \cdots I_d \subset_m I$, which means that $I_1, \dots, I_d \subset I$ are disjoint and $\#I_1 + \dots + \#I_d = m$. We have the following decompositions:

$$\begin{aligned} E_k(\mathcal{A}) &= \sum_{r \geq 0} (-t)^r N(m-k, 2r) \sum_{S \subset \mathcal{A}} X_S \sum_{I_1 \cdots I_d \subset_{k-2r-|S|} \mathcal{A} \setminus S} \langle\langle I_1|j_1 \rangle\rangle \cdots \langle\langle I_d|j_d \rangle\rangle \\ &= \sum_{r \geq 0} (-t)^r N(m-k, 2r) (A_1^r + A_2^r), \\ E_k(\mathcal{A} \cup \{j\}) &= \sum_{r \geq 0} (-t)^r N(m-k+1, 2r) \sum_{S \subset \mathcal{A}} X_S \sum_{I_2 \cdots I_d \subset_{k-2r-|S|} \mathcal{A} \cup \{j\} \setminus S} \langle\langle I_2|j_2 \rangle\rangle \cdots \langle\langle I_d|j_d \rangle\rangle \\ &= \sum_{r \geq 0} (-t)^r N(m-k+1, 2r) (B_1^r + B_2^r + B_3^r), \\ E_{k-1}(\mathcal{A}) \sum_{s \neq j} [j, s] &= \sum_{r \geq 0} (-t)^r N(m-k+1, 2r) \sum_{S \subset \mathcal{A}} X_S \sum_{I_1 \cdots I_d \subset_{k-1-2r-|S|} \mathcal{A} \setminus S} \langle\langle I_1|j_1 \rangle\rangle \cdots \langle\langle I_d|j_d \rangle\rangle \sum_{s \neq j} [j, s] \\ &= \sum_{r \geq 0} (-t)^r N(m-k+1, 2r) (C_1^r + C_2^r + C_3^r + C_4^r), \\ E_{k-1}(\mathcal{A}) x_j &= \sum_{r \geq 0} (-t)^r N(m-k+1, 2r) \sum_{S \subset \mathcal{A}} X_S \sum_{I_1 \cdots I_d \subset_{k-1-2r-|S|} \mathcal{A} \setminus S} \langle\langle I_1|j_1 \rangle\rangle \cdots \langle\langle I_d|j_d \rangle\rangle x_j \\ &= \sum_{r \geq 0} (-t)^r N(m-k+1, 2r) (D_1^r + D_2^r), \end{aligned}$$

where A_i^r , B_i^r , C_i^r , D_i^r are defined as follows.

- A_1^r is the sum of terms with $I_1 = \emptyset$; A_2^r is the sum of terms with $I_1 \neq \emptyset$.
- B_1^r is the sum of terms with $j \notin S \cup I_2 \cup \dots \cup I_d$; B_2^r is the sum of terms with $j \in I_2 \cup \dots \cup I_d$; B_3^r is the sum of terms with $j \in S$.
- C_1^r is the sum of terms with $s \in \mathcal{A} \setminus (S \cup I_1 \cup \dots \cup I_d)$; C_2^r is the sum of terms with $s \in I_2 \cup \dots \cup I_d \cup \mathcal{A}^c$; C_3^r is the sum of terms with $s \in S$; C_4^r is the sum of terms with $s \in I_1$.
- D_1^r is the sum of terms with $I_1 = \emptyset$; D_2^r is the sum of terms with $I_1 \neq \emptyset$.

The same arguments as in [13] show that $A_1^r = B_1^r$, $A_2^r + C_1^r = 0$, $B_2^r = C_2^r$, and

$C_4^r = 0$. It is also easy to check that $B_3^r = D_1^r$. Now we have

$$\begin{aligned}
& E_k(\mathcal{A}) + E_{k-1}(\mathcal{A}) \left(x_j + \sum_{s \neq j} [j, s] \right) - E_k(\mathcal{A} \cup \{j\}) \\
&= \sum_{r \geq 0} (-t)^r \left(N(m-k, 2r)(A_1^r + A_2^r) - N(m-k+1, 2r)(B_1^r - C_1^r - C_3^r - D_2^r) \right) \\
&= \sum_{r \geq 1} (-t)^r \left(N(m-k, 2r) - N(m-k+1, 2r) \right) (A_1^r + A_2^r) \\
&\quad + \sum_{r \geq 0} (-t)^r N(m-k+1, 2r) (C_3^r + D_2^r).
\end{aligned}$$

From the commutation relation $[i, j]x_j = x_i[i, j] - t$, we have

$$\begin{aligned}
D_2^r &= \sum_{S \subset \mathcal{A}} X_S \sum_{\substack{I_1 \cdots I_d \subset_{k-1-2r-|S|} \mathcal{A} \setminus S \\ I_1 = \{a_1, \dots, a_{|I_1|}\}}} \\
&\quad \times \sum_{w \in \mathbf{S}_{|I_1|}} x_{a_{w(|I_1|)}} [a_{w(1)}, j] \cdots [a_{w(|I_1|)}, j] \langle\langle I_2 | j_2 \rangle\rangle \cdots \langle\langle I_d | j_d \rangle\rangle \\
&\quad - t \sum_{S \subset \mathcal{A}} X_S \sum_{\substack{I_1 \cdots I_d \subset_{k-1-2r-|S|} \mathcal{A} \setminus S \\ I_1 = \{a_1, \dots, a_{|I_1|}\}}} \sum_{w \in \mathbf{S}_{|I_1|}} [a_{w(1)}, j] \cdots [a_{w(|I_1|-1)}, j] \langle\langle I_2 | j_2 \rangle\rangle \cdots \langle\langle I_d | j_d \rangle\rangle \\
&= \sum_{S \subset \mathcal{A}} \sum_{s \notin S} X_{S \cup \{s\}} \sum_{I_1 \cdots I_d \subset_{k-1-2r-(|S|+1)} \mathcal{A} \setminus S \cup \{s\}} \langle\langle I_1 | j_1 \rangle\rangle [s, j] \langle\langle I_2 | j_2 \rangle\rangle \cdots \langle\langle I_d | j_d \rangle\rangle \\
&\quad - (m-k+2r+2)t \sum_{S \subset \mathcal{A}} X_S \sum_{I_1 \cdots I_d \subset_{k-2-2r-|S|} \mathcal{A} \setminus S} \langle\langle I_1 | j_1 \rangle\rangle \cdots \langle\langle I_d | j_d \rangle\rangle \\
&= -C_3^r + (-t)(m-k+2r+2)(A_1^{r+1} + A_2^{r+1}).
\end{aligned}$$

Hence,

$$\begin{aligned}
& (-t)^{r+1} (N(m-k, 2(r+1)) - N(m-k+1, 2(r+1))) (A_1^{r+1} + A_2^{r+1}) \\
&= -(-t)^{r+1} (2r+1)!! \frac{(m-k+2r+2)!}{(2r+1)!(m-k+1)!} (A_1^{r+1} + A_2^{r+1}) \\
&= -(-t)^r (2r-1)!! \frac{(m-k+2r+1)!}{(2r)!(m-k+1)!} \cdot (-t)(m-k+2r+2) (A_1^{r+1} + A_2^{r+1}) \\
&= -(-t)^r N(m-k+1, 2r) (C_3^r + D_2^r).
\end{aligned}$$

This implies the desired result. $\square$

Example 2.1. Let us check the coefficients of t and t^2 in the expression of $\theta_1 \theta_2 \theta_3 \theta_4 \in \mathcal{E}_5(R)[t]$ by direct computation. It is easily seen that

$$\begin{aligned}
\theta_1 \theta_2 \theta_3 &= -t(x_1 + x_2 + x_3 + [14] + [15] + [24] + [25] + [34] + [35]) \\
&\quad + x_1 x_2 x_3 + x_1 x_2 ([34] + [35]) + x_1 x_3 ([24] + [25]) + x_2 x_3 ([14] + [15]) \\
&\quad + x_1 ([24][34] + [24][35] + [25][35] + [34][24] + [34][25] + [35][25]) \\
&\quad + x_2 ([14][34] + [14][35] + [15][35] + [34][14] + [34][15] + [35][15]) \\
&\quad + x_3 ([14][24] + [14][25] + [15][25] + [24][14] + [24][15] + [25][15]) \\
&\quad + \sum_{\substack{\{i_1, i_2, i_3\} = \{1, 2, 3\}, \\ 4 \leq j_1 \leq j_2 \leq j_3 \leq 5}} [i_1 j_1][i_2 j_2][i_3 j_3].
\end{aligned}$$

We multiply this expression by θ_4 from the right. The contributions to the coefficients of t and t^2 may come only from the following terms:

- $-t(x_1 + x_2 + x_3 + [14] + [15] + [24] + [25] + [34] + [35])\theta_4$
 $= 3t^2 - t(x_1x_4 + x_2x_4 + x_3x_4 + x_1([42] + [43] + [45])$
 $+ x_2([41] + [43] + [45]) + x_3([41] + [42] + [45]) + x_4([15] + [25] + [35])$
 $+ [15][41] + [24][41] + [25][41] + [34][41] + [35][41]$
 $+ [14][42] + [15][42] + [25][42] + [34][42] + [35][42]$
 $+ [14][43] + [15][43] + [24][43] + [25][43] + [35][43]$
 $+ [14][45] + [15][45] + [24][45] + [25][45] + [34][45] + [35][45]),$
- $x_1([24][34] + [24][35] + [25][35] + [34][24] + [34][25] + [35][25])x_4$
 $= -tx_1([24] + [34] + [25] + [35]) + \dots,$
- $x_2([14][34] + [14][35] + [15][35] + [34][14] + [34][15] + [35][15])x_4$
 $= -tx_2([14] + [34] + [15] + [35]) + \dots,$
- $x_3([24][14] + [24][15] + [25][15] + [14][24] + [14][25] + [15][25])x_4$
 $= -tx_3([24] + [14] + [25] + [15]) + \dots,$
- $x_1x_2([34] + [35])x_4 = -tx_1x_2 + x_1x_2(x_3[34] + x_4[35]),$
- $x_2x_3([14] + [15])x_4 = -tx_2x_3 + x_2x_3(x_1[14] + x_4[15]),$
- $x_1x_3([24] + [25])x_4 = -tx_1x_3 + x_1x_3(x_2[24] + x_4[25]),$
- $\sum_{\substack{\{i_1, i_2, i_3\}=\{1,2,3\}, \\ 4 \leq j_1 \leq j_2 \leq j_3 \leq 5}} [i_1j_1][i_2j_2][i_3j_3]x_4$
 $= -t([14][24] + [14][35] + [25][35] + [24][14] + [24][35] + [15][35]$
 $+ [34][14] + [34][25] + [15][25] + [34][24] + [34][15] + [25][15]$
 $+ [14][34] + [14][25] + [35][25] + [24][34] + [24][15] + [35][15]) + \dots.$

By using the relations $[15][41] + [14][45] = [45][15]$, $[25][42] + [24][45] = [45][25]$, $[35][43] + [34][45] = [45][35]$, we obtain finally

$$\begin{aligned} \theta_1\theta_2\theta_3\theta_4 = 3t^2 - t & \left(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \right. \\ & + x_1([25] + [35] + [45]) + x_2([15] + [35] + [45]) \\ & + x_3([15] + [25] + [45]) + x_4([15] + [25] + [35]) \\ & + [15][25] + [15][35] + [15][45] + [25][35] + [25][45] + [35][45] \\ & \left. + [25][15] + [35][15] + [45][15] + [35][25] + [45][25] + [45][35] \right) + \dots. \end{aligned}$$

It is easy to show that the formula for $\theta_1\theta_2\theta_3\theta_4$ stated in Theorem 2.1 produces the same expression.

The following is a special case of the formula in Theorem 2.1 for $m = n$.

Corollary 2.1. *In the algebra $\tilde{\mathcal{E}}_n[t]$, we have the relations*

$$e_k(\theta_1^{(n)}, \dots, \theta_n^{(n)}) = e_k(y_1, \dots, y_n) + \sum_{r \geq 1} (-t)^r (2r-1)!! \binom{n-k+2r}{2r} e_{k-2r}(y_1, \dots, y_n),$$

$$1 \leq k \leq n.$$

In §3 it will be shown that the above relations describe a complete set of relations among the Dunkl elements in $\tilde{\mathcal{E}}_n[t]$.

§3. BRUHAT REPRESENTATION

We recall the definition of the Bruhat representation of the algebra $\mathcal{E}_n$ on the group ring of the symmetric group $\mathbb{Z}\langle \mathbf{S}_n \rangle = \bigoplus_{w \in \mathbf{S}_n} \mathbb{Z} \cdot \underline{w}$. The operator σ_{ij} , $i < j$, is defined as follows:

$$\sigma_{ij}(\underline{w}) = \begin{cases} \underline{wt_{ij}} & \text{if } l(wt_{ij}) = l(w) + 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $t_{ij} \in \mathbf{S}_n$ is the transposition of i and j . Then the Bruhat representation of $\mathcal{E}_n$ is defined by $[i, j] \cdot \underline{w} := \sigma_{ij}(\underline{w})$.

Now we extend the Bruhat representation to that of the algebra $\mathcal{E}_n\langle R \rangle[t]$ defined on

$$R[t]\langle \mathbf{S}_n \rangle = \bigoplus_{w \in \mathbf{S}_n} \mathbb{Z}[y_1, \dots, y_n][t] \cdot \underline{w}.$$

We define the divided difference operator ∂_{ij} on $\mathbb{Z}[y_1, \dots, y_n][t]$ as the $\mathbb{Z}[t]$ -linear operator given by $\partial_{ij} := (1 - t_{ij})/(y_i - y_j)$. For $f(y) \in \mathbb{Z}[y_1, \dots, y_n][t]$ and $w \in \mathbf{S}_n$, we define the $\mathbb{Z}[t]$ -linear operators $\tilde{\sigma}_{ij}$, $i < j$, and ξ_k as follows:

$$\begin{aligned} \tilde{\sigma}_{ij}(f(y)\underline{w}) &= \begin{cases} t(\partial_{w(i)w(j)}f(y))\underline{w} + f(y)\underline{wt_{ij}} & \text{if } l(wt_{ij}) = l(w) + 1, \\ t(\partial_{w(i)w(j)}f(y))\underline{w} & \text{otherwise,} \end{cases} \\ \xi_k(f(y)\underline{w}) &= (y_{w(k)}f(y))\underline{w}. \end{aligned}$$

Proposition 3.1. *The algebra $\mathcal{E}_n\langle R \rangle[t]$ acts $\mathbb{Z}[t]$ -linearly on $\mathbb{Z}[y][t]\langle \mathbf{S}_n \rangle$ via $[ij] \mapsto \tilde{\sigma}_{ij}$ and $x_k \mapsto \xi_k$.*

Proof. We check compatibility with the defining relations of the algebra $\tilde{\mathcal{E}}_n[t]$. We show this only for relations (1), (3), and (B). The rest is easy to check.

We start with relation (1). We have

$$\begin{aligned} \tilde{\sigma}_{ij}^2(f(y)\underline{w}) &= \tilde{\sigma}_{ij}(t(\partial_{w(i)w(j)}f(y))\underline{w} + f(y)\sigma_{ij}(\underline{w})) \\ &= t^2(\partial_{w(i)w(j)}^2f(y))\underline{w} + t(\partial_{w(i)w(j)}f(y))\sigma_{ij}(\underline{w}) \\ &\quad + t(\partial_{w(j)w(i)}f(y))\sigma_{ij}(\underline{w}) + f(y)\sigma_{ij}^2(\underline{w}). \end{aligned}$$

Since $\partial_{w(i)w(j)}^2 = 0$, $\sigma_{ij}^2 = 0$ and $\partial_{w(i)w(j)} = -\partial_{w(j)w(i)}$, we get $\tilde{\sigma}_{ij}^2 = 0$.

For relation (3), we have

$$\begin{aligned} \tilde{\sigma}_{ij}\tilde{\sigma}_{jk}(f(y)\underline{w}) &= \tilde{\sigma}_{ij}(t(\partial_{w(j)w(k)}f(y))\underline{w} + f(y)\sigma_{jk}(\underline{w})) \\ &= t^2(\partial_{w(i)w(j)}\partial_{w(j)w(k)}f(y))\underline{w} + t(\partial_{w(j)w(k)}f(y))\sigma_{ij}(\underline{w}) \\ &\quad + t(\partial_{w(i)w(k)}f(y))\sigma_{jk}(\underline{w}) + f(y)\sigma_{ij}\sigma_{jk}(\underline{w}). \end{aligned}$$

We also obtain $\tilde{\sigma}_{jk}\tilde{\sigma}_{ki}(f(y)\underline{w})$ and $\tilde{\sigma}_{ki}\tilde{\sigma}_{ij}(f(y)\underline{w})$ by the cyclic permutation of i, j, k . The 3-term relations

$$\partial_{w(i)w(j)}\partial_{w(j)w(k)} + \partial_{w(j)w(k)}\partial_{w(k)w(i)} + \partial_{w(k)w(i)}\partial_{w(i)w(j)} = 0$$

and

$$\sigma_{ij}\sigma_{jk} + \sigma_{jk}\sigma_{ki} + \sigma_{ki}\sigma_{ij} = 0$$

show the desired identity

$$\tilde{\sigma}_{ij}\tilde{\sigma}_{jk} + \tilde{\sigma}_{jk}\tilde{\sigma}_{ki} + \tilde{\sigma}_{ki}\tilde{\sigma}_{ij} = 0.$$

Finally, we check relation (B). We have

$$\begin{aligned}
 \tilde{\sigma}_{ij}\xi_i(f(y)\underline{w}) &= \tilde{\sigma}_{ij}(y_{w(i)}f(y)\underline{w}) \\
 &= t\partial_{w(i)w(j)}(y_{w(i)}f(y))\underline{w} + (y_{w(i)}f(y))\sigma_{ij}(\underline{w}) \\
 &= t(f(y)\underline{w}) + t(y_{w(j)}\partial_{w(i)w(j)}f(y))\underline{w} + y_{wt_{ij}(j)}\sigma_{ij}(\underline{w}) \\
 &= \xi_j\tilde{\sigma}_{ij}(f(y)\underline{w}) + t(f(y)\underline{w}).
 \end{aligned}$$

This completes the proof of the fact that the Bruhat representation is well defined.

Let AH_n^0 denote the subalgebra in $\mathcal{E}_n(R)[t]$ generated by the elements t , $h_1 := [1, 2]$, $h_2 := [2, 3], \dots, h_{n-1} := [n-1, n]$, and $x_1, \dots, x_n$. $\square$

Lemma 3.1. *The module $R[t]\langle \mathbf{S}_n \rangle$ is generated by $\underline{\text{id.}}$ over AH_n^0 .*

Proof. Consider a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$, $s_i = (i, i+1)$, of a permutation $w \in \mathbf{S}_n$; then

$$h_{i_1} \cdots h_{i_l}(\underline{\text{id.}}) = \underline{w}.$$

Therefore, $R[t]\langle \mathbf{S}_n \rangle = AH_n^0 \cdot \underline{\text{id.}}$ $\square$

Theorem 3.1. *The subalgebra AH_n^0 of $\mathcal{E}_n(R)[t]$ is isomorphic to the nil degenerate affine Hecke algebra $\mathcal{AH}_n^0$ of type $A_{n-1}^{(1)}$, i.e., the $\mathbb{Z}[t]$ -algebra given by two sets of generators $g_1, \dots, g_{n-1}$ and $x_1, \dots, x_n$ subject to the following set of defining relations:*

$$\begin{aligned}
 g_i^2 &= 0; \quad g_i g_j = g_j g_i \quad \text{if } |i-j| > 1; \quad g_i g_j g_i = g_j g_i g_j \quad \text{if } |i-j| = 1; \\
 x_i x_j &= x_j x_i, \quad x_k g_i = g_i x_k \quad \text{if } k \neq i, i+1; \quad g_i x_i - x_{i+1} g_i = t.
 \end{aligned}$$

Proof. First, it is easy to check that the elements $t, h_1, \dots, h_{n-1}, x_1, \dots, x_n$ do satisfy the relations listed above. Hence, we have a surjective homomorphism ρ from the nil degenerate affine Hecke algebra $\mathcal{AH}_n^0$ to AH_n^0 , given by $\rho(g_i) = h_i$. Now we are going to construct a basis in the algebra $\mathcal{AH}_n^0$. Let $w \in \mathbf{S}_n$ be a permutation and $w = s_{i_1} \cdots s_{i_k}$ a reduced decomposition of it. Since the elements $g_1, \dots, g_{n-1}$ satisfy the Coxeter relations, the element $g_w := g_{i_1} \cdots g_{i_k}$ is well defined. On the other hand, using relations among the elements $x_1, \dots, x_n$ and $g_1, \dots, g_{n-1}$ in the algebra $\mathcal{AH}_n^0$, we can write any element of $\mathcal{AH}_n^0$ as a linear combination of elements $t^a x^m g_w$, where $w \in \mathbf{S}_n$, $m \in \mathbb{Z}_{\geq 0}^n$, and $a \in \mathbb{Z}_{\geq 0}$. Lemma 3.1 implies that $R[t]\langle \mathbf{S}_n \rangle = \rho(\mathcal{AH}_n^0) \cdot \underline{\text{id.}}$, so that the elements $\{t^a x^m g_w \mid a \in \mathbb{Z}_{\geq 0}, w \in \mathbf{S}_n, m \in \mathbb{Z}_{\geq 0}^n\}$ must be linearly independent over $\mathbf{Z}$. This yields the injectivity of ρ , and we conclude that the homomorphism ρ is an isomorphism. $\square$

Let us consider the double Schubert polynomials $\mathfrak{S}_w(x, y)$, $w \in \mathbf{S}_n$, introduced by Lascoux and Schützenberger [10]. By definition, the polynomial $\mathfrak{S}_{w_0}(x, y)$ for the maximal element $w_0 \in \mathbf{S}_n$ is given by $\mathfrak{S}_{w_0}(x, y) := \prod_{i+j \leq n} (x_i - y_j)$. For $w \in \mathbf{S}_n$, take a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$. The divided difference operator $\partial_w := \partial_{i_1} \cdots \partial_{i_l}$ is well defined, thanks to the Coxeter relations. We define the double Schubert polynomial for w by $\mathfrak{S}_w(x, y) := \partial_{w^{-1}w_0}^{(x)} \mathfrak{S}_{w_0}(x, y)$, where $\partial_w^{(x)}$ means a divided difference operator on x variables. The Schubert polynomials are defined to be the specializations $\mathfrak{S}_w(x) := \mathfrak{S}_w(x, 0)$ of the double Schubert polynomials.

Theorem 3.2. *Let $\mathfrak{S}_w(x, y)$ be the double Schubert polynomial corresponding to $w \in \mathbf{S}_n$. When $t = 0$, we have*

$$\mathfrak{S}_w(\theta, y)(\underline{\text{id.}}) = \underline{w}.$$

Proof. This follows from the Monk formula for the double Schubert polynomials (see, e.g., [12, Exercise 2.7.2]) and the relation

$$\begin{aligned} (\theta_i - y_{w(i)})(\underline{w}) &= \xi_i(\underline{w}) + \sum_{j \neq i} \sigma_{ij}(\underline{w}) - y_{w(i)}\underline{w} \\ &= \sum_{\substack{j > i \\ l(wt_{ij})=l(w)+1}} \frac{wt_{ij}}{l(wt_{ij})=l(w)+1} - \sum_{\substack{j < i \\ l(wt_{ij})=l(w)+1}} \frac{wt_{ij}}{l(wt_{ij})=l(w)+1}. \end{aligned}$$

Let $w \in \mathbf{S}_n$, let r be the maximal descent of w , and s the greatest integer such that $w(s) < w(r)$. By applying the Monk formula to the product $x_r \mathfrak{S}_u(x, y)$, $u = wt_{rs}$, we get the transition formula

$$\mathfrak{S}_w(x, y) = (x_r - y_{u(r)})\mathfrak{S}_u(x, y) + \sum_{v \in S(w, r)} \mathfrak{S}_v(x, y),$$

where $S(w, r)$ is the set of permutations of the form $ut_{jr} = wt_{rs}t_{jr}$ with $j < r$ and of the same length as w . Similarly, we have

$$\underline{w} = (\theta_r - y_{u(r)})\underline{u} + \sum_{v \in S(w, r)} \underline{v}.$$

Let r' be the maximal descent of $v = wt_{rs}t_{jr} \in S(w, r)$. Since $l(v) = l(wt_{rs}) + 1$, we have $v(r) = wt_{rs}(j) < wt_{rs}(r) = w(s) < w(r)$. Hence, we have either $r' < r$ or $\{r' = r$ and $v(r) < w(r)\}$. Then our assertion is proved by induction on the Bruhat ordering, the maximal descent r and $w(r)$. $\square$

Remark 3.1. Only in the case where $t = 0$ does the Bruhat representation of the algebra $\mathcal{E}_n\langle R \rangle[t]$ admit a $\mathbb{Z}[y][t]$ -linear extension up to that of the algebra $\tilde{\mathcal{E}}_{n,0}$. In this case, the Dunkl elements commute with multiplication by the y_i 's.

Proposition 3.2. *The list of relations given in Corollary 2.1 describes a complete set of relations among the Dunkl elements $\theta_1^{(n)}, \dots, \theta_n^{(n)}$ in the algebra $\tilde{\mathcal{E}}_n[t]$. In other words, the following surjective homomorphism φ between $\mathbb{Z}[t][y_1, \dots, y_n]$ -algebras is an isomorphism:*

$$\begin{aligned} \varphi: \mathbb{Z}[t][y_1, \dots, y_n][z_1, \dots, z_n]/J_t^n &\rightarrow \mathbb{Z}[t][y_1, \dots, y_n][\theta_1, \dots, \theta_n] \subset \tilde{\mathcal{E}}_n[t], \\ z_i &\mapsto \theta_i, \end{aligned}$$

where the ideal J_t is generated by the polynomials

$$e_k(z_1, \dots, z_n) - e_k(y_1, \dots, y_n) - \sum_{r \geq 1} (-t)^r (2r-1)!! \binom{n-k+2r}{2r} e_{k-2r}(y_1, \dots, y_n)$$

for $k = 1, \dots, n$.

Proof. For the (single) Schubert polynomial $\mathfrak{S}_w(\theta_1, \dots, \theta_n) \in \mathcal{E}_n\langle R \rangle[t]$ in the Dunkl elements, we have

$$\mathfrak{S}_w(\theta_1, \dots, \theta_n)(\underline{\text{id}}.)|_{t=0} = \underline{w} + (\text{linear combination of } \underline{v} \text{ with } l(v) < l(w)).$$

Consequently,

$$\begin{aligned} \mathfrak{S}_w(\theta_1, \dots, \theta_n)(\underline{\text{id}}.) &= \mathfrak{S}_w(\theta_1, \dots, \theta_n)(\underline{\text{id}}.)|_{t=0} + (\text{linear combination of } \underline{v} \text{ with } l(v) < l(w)) \\ &= \underline{w} + (\text{linear combination of } \underline{v} \text{ with } l(v) < l(w)). \end{aligned}$$

Hence, the polynomials $\mathfrak{S}_w(\theta_1, \dots, \theta_n)$ are linearly independent in $\mathcal{E}_n\langle R \rangle[t]$ over $R^{\mathbf{S}_n}$.

Let $R_{\mathbf{S}_n}$ be the coinvariant algebra of $\mathbf{S}_n$. Since $R = R^{\mathbf{S}_n} \otimes R_{\mathbf{S}_n}$, the polynomials $\mathfrak{S}_w(z)$, $w \in \mathbf{S}_n$, form a $\mathbb{Z}[z]^{\mathbf{S}_n}$ -basis of $\mathbb{Z}[z]$. In particular, any polynomial $f(z_1, \dots, z_n) \in \mathbb{Z}[z]$ can be expressed as

$$f(z_1, \dots, z_n) = \sum_{w \in \mathbf{S}_n} \phi_w(z) \mathfrak{S}_w(z),$$

where $\phi_w(z) \in \mathbb{Z}[z]^{\mathbf{S}_n} = \mathbb{Z}[e_1(z), \dots, e_n(z)]$. It follows that the image of $f(z_1, \dots, z_n)$ in $\mathbb{Z}[t][y][z_1, \dots, z_n]/J_n^t$ is a linear combination of $\mathfrak{S}_w(z)$'s over $\mathbb{Z}[t][y]$. Since the elements $\mathfrak{S}_w(\theta_1, \dots, \theta_n)$ in $\tilde{\mathcal{E}}_n[t]$ are linearly independent over $\mathbb{Z}[t][y]$, the homomorphism φ is an isomorphism. $\square$

Corollary 3.1. *The subalgebra of $\tilde{\mathcal{E}}_{n,0}$ generated by the Dunkl elements $\theta_1, \dots, \theta_n$ over $H_T^*(\text{pt}) = \mathbb{Z}[y_1, \dots, y_n]$ is isomorphic to the T -equivariant cohomology ring $H_T^*(Fl_n)$.*

Proof. Let $(0 = U_0 \subset U_1 \subset \dots \subset U_n)$ be the universal flag over Fl_n . Corollary 2.1 shows that the natural map $z_i := -c_1^T(U_i/U_{i-1}) \mapsto \theta_i$, $y_i \mapsto y_i$ gives rise to a surjective homomorphism

$$\pi : H_T^*(Fl_n) \rightarrow \mathbb{Z}[y_1, \dots, y_n][\theta_1, \dots, \theta_n] \subset \tilde{\mathcal{E}}_{n,0}.$$

On the other hand, Proposition 3.2 implies that the homomorphism π is injective. $\square$

Example 3.1. When $t = 0$, our formula in Theorem 2.1 specializes to the Pieri formula in $H_T^*(Fl_n)$ under the identification in Corollary 3.1. Let $\{\Omega_w\}_{w \in \mathbf{S}_n}$ be the Schubert basis of $H_T^*(Fl_n)$, and let $z_i := -c_1^T(U_i/U_{i-1})$. Then our formula in $H_T^*(Fl_n)$ can be written as follows:

$$e_k(z_1, \dots, z_m) \cdot \Omega_w = \sum_{S \subset \{1, \dots, m\}} \prod_{i \in S} y_{w(i)} \sum_{(*)} \Omega_{wt_{i_1 j_1} \dots t_{i_l j_l}},$$

where $(*)$ stands for the following conditions: $|S| + l = k$; $1 \leq i_a \leq m < j_a \leq n$ for $1 \leq a \leq l$; $i_1, \dots, i_l$ are distinct; $j_1 \leq \dots \leq j_l$; there exists a path $w \rightarrow wt_{i_l j_l} \rightarrow wt_{i_l j_l} t_{i_{l-1} j_{l-1}} \rightarrow \dots \rightarrow wt_{i_1 j_1} \dots t_{i_1 j_1}$ in the Bruhat ordering of $\mathbf{S}_n$. For the cyclic permutation $[m, k] := s_{m-k+1} s_{m-k+2} \dots s_m$, the corresponding double Schubert polynomial is given as follows (see [12, Proposition 2.6.7]):

$$\mathfrak{S}_{[m,k]}(x, y) = \sum_{j=0}^k e_{k-j}(x_1, \dots, x_m) h_j(-y_1, \dots, -y_{m-k+1}),$$

where h_j is the complete symmetric polynomial of degree j , so that the above formula gives the multiplication rule for $\Omega_{[m,k]}$ in $H_T^*(Fl_n)$.

Now we consider an example from [14]. Let $[5, 5] = s_1 s_2 s_3 s_4 s_5$ be a permutation in $\mathbf{S}_9$. For $w = s_3 s_4 s_5 s_7 s_6 s_5$ and $u = wt_{26} t_{16} t_{59}$, we compute the coefficient $p_{[5,5],w}^u$ of Ω_u in the expansion of $\Omega_{[5,5]} \cdot \Omega_w$ by using our formula. It is easily seen that there exists a unique path of length 3 from w to u , namely,

$$w \rightarrow wt_{59} \rightarrow wt_{59} t_{26} \rightarrow u = wt_{59} t_{26} t_{16}.$$

Hence, the action of $\mathfrak{S}_{[m,k]}(\theta, y)$ on $\underline{w}$ under the Bruhat representation at $t = 0$ is given by

$$\begin{aligned} \mathfrak{S}_{[m,k]}(\theta, y)(\underline{w}) &= \sum_{j=0}^5 (-y_1)^j e_{5-j}(\theta_1, \dots, \theta_5) \underline{w} \\ &= (\dots + (x_3 x_4 - y_1(x_3 + x_4) + y_1^2)[16][26][59] + \dots) \underline{w} \\ &= \dots + (y_1 - y_4)(y_1 - y_6) \underline{u} + \dots, \end{aligned}$$

and we get $p_{[5,5],w}^\nu = (\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1 + \cdots + \alpha_5)$, $\alpha_i = y_i - y_{i+1}$. This coincides with the result computed in [14, Example 4.8].

§4. QUANTIZATION

In this section we discuss a quantized version of our formula obtained in §2, which gives a multiplication rule in the quantum cohomology ring $QH_T^*(Fl_n)$ and a characterization of the quantum Schubert polynomials.

Definition 4.1. The algebra $\mathcal{E}_n^{\mathbf{q}}$ is a $\mathbb{Z}[q_{ij} = q_{ji} \mid 1 \leq i < j \leq n]$ -algebra determined by the same generators and relations as in the definition of the algebra $\mathcal{E}_n$ except that relation (1) is replaced by

$$(1)' \quad [i, j]^2 = q_{ij}$$

for $1 \leq i < j \leq n$. The algebra $\mathcal{E}_n^q$ is defined as a $\mathbb{Z}[q_1, \dots, q_{n-1}]$ -algebra obtained from $\mathcal{E}_n^{\mathbf{q}}$ by the specialization

$$q_{ij} = \begin{cases} q_i & \text{if } i = j - 1, \\ 0 & \text{if } i < j - 1. \end{cases}$$

The extension $\mathcal{E}_n^{\mathbf{q}}\langle R \rangle[t]$ (respectively, $\mathcal{E}_n^q\langle R \rangle[t]$) of the algebra $\mathcal{E}_n^{\mathbf{q}}$ (respectively, $\mathcal{E}_n^q$) is also defined by relations (A), (B) and (C).

In the algebra $\mathcal{E}_n^{\mathbf{q}}\langle R \rangle[t]$, we have a formula similar to Theorem 2.1. In order to state the formula, we need the quantum elementary symmetric polynomials $e_k^{\mathbf{q}}$ defined by the recursive formula

$$\begin{aligned} e_k^{\mathbf{q}}(X_i \mid i \in I \cup \{j\}) &= e_k^{\mathbf{q}}(X_i \mid i \in I) + X_j e_{k-1}^{\mathbf{q}}(X_i \mid i \in I) + \sum_{a \in I} q_{aj} e_{k-2}^{\mathbf{q}}(X_i \mid i \in I \setminus \{a\}), \\ e_0^{\mathbf{q}}(X_i \mid i \in I) &= 1, \quad e_k^{\mathbf{q}}(\emptyset) = 0, \quad k > 0, \end{aligned}$$

where I is a subset of $\{1, \dots, n\}$ and $j \notin I$. The Dunkl elements $\theta_i = \theta_i^{(n)}$ in $\mathcal{E}_n^{\mathbf{q}}\langle R \rangle[t]$ are defined by the same formula as in the classical case.

Theorem 4.1. For $k \leq m \leq n$, we have the following formula in the algebra $\mathcal{E}_n^{\mathbf{q}}\langle R \rangle[t]$:

$$e_k^{\mathbf{q}}(\theta_1^{(n)}, \dots, \theta_m^{(n)}) = \sum_{r \geq 0} (-t)^r N(m - k, 2r) \left\{ \sum_{S, I = \{i_1, \dots, i_{|I|}\}, (j_a)} X_S \cdot [i_1, j_1] \cdots [i_{|I|}, j_{|I|}] \right\},$$

where the second summation is over the triples $(S, I = \{i_1, \dots, i_{|I|}\}, (j_a)_{a=1}^{|I|})$ such that $S \subset \{1, \dots, m\}$, I is a subset of $\{1, \dots, m\} \setminus S$, $|I| + |S| + 2r = k$, $1 \leq i_a \leq m < j_a \leq n$, and $j_1 \leq \cdots \leq j_{|I|}$.

Proof. We use the same symbols as in the proof of Theorem 2.1. We shall show that $E_k(\mathcal{A})$ satisfies the recursive relation

$$E_k(\mathcal{A} \cup \{j\}) = E_k(\mathcal{A}) + E_{k-1}(\mathcal{A}) \left(x_j + \sum_{s \neq j} [j, s] \right) + \sum_{\nu \in \mathcal{A}} q_{\nu j} E_{k-2}(\mathcal{A} \setminus \{\nu\})$$

in the algebra $\mathcal{E}_n^{\mathbf{q}}\langle R \rangle[t]$. All the arguments in the proof of Theorem 2.1 work, except that $C_4^r = 0$; therefore,

$$E_k(\mathcal{A} \cup \{j\}) - E_k(\mathcal{A}) - E_{k-1}(\mathcal{A}) \left(x_j + \sum_{s \neq j} [j, s] \right) = - \sum_{r \geq 0} (-t)^r N(m - k + 1, 2r) C_4^r.$$

The following cyclic relation [13, Lemma 5.3] is fulfilled in the algebra $\mathcal{E}_n^q(R)[t]$:

$$\begin{aligned} & \sum_{k=1}^m [a, i_k][a, i_{k+1}] \cdots [a, i_m] \cdot [a, i_1] \cdots [a, i_{k-1}][a, i_k] \\ &= \sum_{k=1}^m q_{ak}[i_k, i_{k+1}][i_k, i_{k+2}] \cdots [i_k, i_m][i_k, i_1] \cdots [i_k, i_{k-1}], \end{aligned}$$

where $1 \leq a, i_1, \dots, i_m \leq n$ are distinct. By using this cyclic relation, we obtain

$$\begin{aligned} & - \sum_{r \geq 0} (-t)^r N(m-k+1, 2r) C_4^r \\ &= - \sum_{r \geq 0} (-t)^r N(m-k+1, 2r) \sum_{S \subset \mathcal{A}} X_S \sum_{I_1 \cdots I_d \subset_{k-1-2r-|S|} \mathcal{A} \setminus S} \langle I_1 | j_1 \rangle \cdots \langle I_d | j_d \rangle \sum_{s \in I_1} [j, s] \\ &= \sum_{r \geq 0} (-t)^r N(m-k+1, 2r) \sum_{S \subset \mathcal{A}} X_S \sum_{\nu \in \mathcal{A} \setminus S} \sum_{(*)} q_{j\nu} \langle I_1 | j_1 \rangle \cdots \langle I_d | j_d \rangle \langle I_{d+1} | \nu \rangle \\ &= \sum_{\nu \in \mathcal{A}} q_{j\nu} \sum_{r \geq 0} (-t)^r N((m-1)-(k-2), 2r) \sum_{S \subset \mathcal{A} \setminus \{\nu\}} X_S \sum_{(*)} \langle I_1 | j_1 \rangle \cdots \langle I_{d+1} | j_{d+1} \rangle \\ &= \sum_{\nu \in \mathcal{A}} q_{j\nu} E_{k-2}(\mathcal{A} \setminus \{\nu\}), \end{aligned}$$

where $(*)$ means the condition $I_1 \cdots I_{d+1} \subset_{k-2-2r-|S|} \mathcal{A} \setminus (S \cup \{\nu\})$. This completes the proof. $\square$

The Bruhat representation for $\mathcal{E}_n$ is deformed to the quantum Bruhat representation for $\mathcal{E}_n^q$. Define the quantum Bruhat operator σ_{ij}^q , $i < j$, acting on $\mathbb{Z}[q_1, \dots, q_{n-1}](\mathbf{S}_n) = \bigoplus_{w \in \mathbf{S}_n} \mathbb{Z}[q_1, \dots, q_{n-1}] \cdot \underline{w}$ as follows:

$$\sigma_{ij}^q(\underline{w}) = \begin{cases} q_{ij} \underline{wt_{ij}} & \text{if } l(wt_{ij}) = l(w) - 2(j-i) + 1, \\ \underline{wt_{ij}} & \text{if } l(wt_{ij}) = l(w) + 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $f(y) \in \mathbb{Z}[y_1, \dots, y_n][t]$ and $w \in \mathbf{S}_n$, we define the $\mathbb{Z}[q_1, \dots, q_{n-1}][t]$ -linear operators $\tilde{\sigma}_{ij}^q$ by

$$\tilde{\sigma}_{ij}^q(f(y)\underline{w}) = t(\partial_{w(i)w(j)} f(y))\underline{w} + f(y)\sigma_{ij}^q(\underline{w}).$$

In the same way as in the proof of Proposition 3.1, we can check that the quantum extended Bruhat representation $[ij] \mapsto \tilde{\sigma}_{ij}^q$, $x_k \mapsto \xi_k$ of the algebra $\mathcal{E}_n^q(R)[t]$ is well defined.

Theorem 4.2. *Let $\mathfrak{S}_w^q(x, y)$ be the quantum double Schubert polynomial corresponding to $w \in \mathbf{S}_n$ (see [3] and [8]). When $t = 0$, we have*

$$\mathfrak{S}_w^q(\theta, y)(\underline{\text{id.}}) = \underline{w}$$

under the quantum extended Bruhat representation.

Proof. This follows from the quantum Monk formula [5] for the quantum Schubert polynomials. $\square$

Corollary 4.1. *The quantum double Schubert polynomials $\mathfrak{S}_w^q(x, y)$ are characterized by the following conditions:*

- (1) $\mathfrak{S}_w^q(x, y)|_{q=0} = \mathfrak{S}_w(x, y)$;
- (2) $\mathfrak{S}_w^q(x, y)$ is a linear combination of $\mathfrak{S}_v(x, y)$ with $v \leq w$ over $\mathbb{Z}[q_1, \dots, q_{n-1}]$;
- (3) $\mathfrak{S}_w^q(\theta, y)(\underline{\text{id.}}) = \underline{w}$ under the quantum extended Bruhat representation at $t = 0$.

Proof. Let $\{P_w(x, y)\}_{w \in \mathbf{S}_n}$ be a family of polynomials that satisfy the above properties (1), (2) and (3) of the quantum Schubert polynomials. We shall show that $P_w(x, y) = \mathfrak{S}_w^q(x, y)$ for all $w \in \mathbf{S}_n$. Let $S = 1 + (q_1, \dots, q_{n-1})$ be a multiplicative system of $\mathbb{Z}[q_1, \dots, q_{n-1}]$ consisting of polynomials of the form $1 + \sum_{i=1}^{n-1} q_i a_i(q)$, $a_i(q) \in \mathbb{Z}[q_1, \dots, q_{n-1}]$. Properties (1) and (2) imply that

$$\mathfrak{S}_w^q(x, y) = \sum_{v \leq w} b_w^v(q) \mathfrak{S}_v(x, y), \quad b_w^v(q) \in \mathbb{Z}[q_1, \dots, q_{n-1}], \quad b_w^w(q) \in S.$$

Then it is easy to check that

$$\mathfrak{S}_w(x, y) = \sum_{v \leq w} c_w^v(q) \mathfrak{S}_v^q(x, y), \quad c_w^v(q) \in S^{-1} \mathbb{Z}[q_1, \dots, q_{n-1}],$$

by induction on the Bruhat ordering. Similarly, we also have

$$\mathfrak{S}_w(x, y) = \sum_{v \leq w} d_w^v(q) P_v(x, y), \quad d_w^v(q) \in S^{-1} \mathbb{Z}[q_1, \dots, q_{n-1}].$$

Hence,

$$\sum_{v \leq w} (c_w^v(q) \mathfrak{S}_v^q(x, y) - d_w^v(q) P_v(x, y)) = 0$$

for all $w \in \mathbf{S}_n$. From property (3), we obtain

$$\sum_{v \leq w} (c_w^v(q) - d_w^v(q)) \underline{v} = 0$$

for all $w \in \mathbf{S}_n$, whence $c_w^v(q) = d_w^v(q)$ for all $w \in \mathbf{S}_n$ and $v \leq w$. Using these identities, we can conclude that $P_w(x, y) = \mathfrak{S}_w^q(x, y)$, again by induction on the Bruhat ordering. $\square$

§5. NICHOLS–WORONOWICZ MODEL

The model of the equivariant cohomology ring $H_T^*(Fl_n)$ in the algebra $\tilde{\mathcal{E}}_n$ has a natural interpretation in terms of the Nichols–Woronowicz algebra. The Nichols–Woronowicz approach leads us to the uniform construction for arbitrary root systems. For the definition of the Nichols–Woronowicz algebra, see, e.g., [1, 2, 11] and [16].

We denote by $\mathcal{B}_W$ the Nichols–Woronowicz algebra associated with the Yetter–Drinfeld module

$$V = \bigoplus_{\alpha \in \Delta} \mathbf{R}[\alpha] / ([\alpha] + [-\alpha])$$

over the finite Coxeter group W of the root system Δ . Let $\mathfrak{h}$ be the reflection representation of W and $R = \text{Sym} \mathfrak{h}^*$ the ring of polynomial functions on $\mathfrak{h}$. Consider the extension $\mathcal{B}_W \langle R \rangle [t]$ of the algebra $\mathcal{B}_W$ by the polynomial ring $R[t]$ defined by the commutation relation

$$[\alpha]x = s_\alpha(x)[\alpha] + t(x, \alpha) \quad \text{for } x \in \mathfrak{h}^*.$$

Definition 5.1. We define the R -algebra $\tilde{\mathcal{B}}_W$ by

$$\tilde{\mathcal{B}}_W = \mathcal{B}_W \langle R \rangle [t] \otimes_{R^W} R.$$

Choosing W -invariant constants $(c_\alpha)_\alpha$, we consider a linear map $\mu : \mathfrak{h}^* \rightarrow \tilde{\mathcal{B}}_W$ defined as

$$\mu(x) = x + \sum_{\alpha \in \Delta_+} c_\alpha(x, \alpha) [\alpha]$$

for $x \in \mathfrak{h}^*$.

Proposition 5.1. $[\mu(x), \mu(y)] = 0$, $x, y \in \mathfrak{h}^*$.

Proof. Let $\mu_0(x) := \sum_{\alpha \in \Delta_+} c_\alpha(x, \alpha)[\alpha]$. Here we may normalize the length of the roots to have $(\alpha, \alpha) = 1$, $\alpha \in \Delta$. The commutativity $[\mu_0(x), \mu_0(y)] = 0$ was shown in [2]. The commutativity between $\mu(x)$ and $\mu(y)$ follows from

$$\begin{aligned} & \mu_0(x)y + x\mu_0(y) \\ &= \sum_{\alpha \in \Delta_+} c_\alpha(x, \alpha)s_\alpha(y)[\alpha] + t \sum_{\alpha \in \Delta_+} c_\alpha(x, \alpha)(y, \alpha) + \sum_{\alpha \in \Delta_+} c_\alpha(y, \alpha)x[\alpha] \\ &= \sum_{\alpha \in \Delta_+} c_\alpha(x, \alpha)y[\alpha] - \sum_{\alpha \in \Delta_+} 2c_\alpha(x, \alpha)(y, \alpha)\alpha[\alpha] \\ & \quad + t \sum_{\alpha \in \Delta_+} c_\alpha(x, \alpha)(y, \alpha) + \sum_{\alpha \in \Delta_+} c_\alpha(y, \alpha)x[\alpha] \\ &= \sum_{\alpha \in \Delta_+} c_\alpha(x, \alpha)y[\alpha] + t \sum_{\alpha \in \Delta_+} c_\alpha(x, \alpha)(y, \alpha) + \sum_{\alpha \in \Delta_+} c_\alpha(y, \alpha)s_\alpha(x)[\alpha] \\ &= y\mu_0(x) + \mu_0(y)x. \end{aligned}$$

Proposition 5.1 shows that the linear map μ extends to a homomorphism of algebras

$$\mu : R \rightarrow \mathcal{B}_W \langle R \rangle [t].$$

Denote by $\tilde{\mu}$ the composition of the homomorphisms

$$R \otimes_{\mathbf{R}} R \xrightarrow{\mu \otimes 1} \mathcal{B}_W \langle R \rangle [t] \otimes_{\mathbf{R}} R \rightarrow \tilde{\mathcal{B}}_W.$$

In Theorem 5.1 below we shall show that the image of the algebra homomorphism $\tilde{\mu}$ at $t = 0$ is isomorphic to the algebra $R \otimes_{R^W} R$. The proof is based on the correspondence between the twisted derivation D_α and the divided difference operator $\partial_\alpha := (1 - s_\alpha)/\alpha$, which acts on the first tensor component of $R \otimes_{R^W} R$ and extends linearly with respect to the second tensor component. We define the operator D_α as the twisted derivation on $\tilde{\mathcal{B}}_W$ determined by the conditions

- (1) $D_\alpha(x) = 0$ for $x \in R$,
- (2) $D_\alpha([\beta]) = \delta_{\alpha, \beta}$ for $\alpha, \beta \in \Delta_+$,
- (3) $D_\alpha(fg) = D_\alpha(f)g + s_\alpha(f)D_\alpha(g)$.

The operator D_α is linear with respect to R on the second component. □

Proposition 5.2.

$$\bigcap_{\alpha \in \Delta_+} \text{Ker}(D_\alpha) = R[t] \otimes_{R^W} R.$$

Proof. Since $\mathcal{B}_W \langle R \rangle [t]$ is isomorphic to $R[t] \otimes_{\mathbf{R}} \mathcal{B}_W$ as a right $\mathcal{B}_W$ -module, any element $\omega \in \mathcal{B}_W \langle R \rangle [t]$ can be written as

$$\omega = f_1\varphi_1 + \cdots + f_k\varphi_k,$$

where $f_1, \dots, f_k \in R[t]$ are linearly independent, and $\varphi_1, \dots, \varphi_k \in \mathcal{B}_W$. We have

$$D_\alpha(\omega) = s_\alpha(f_1)D_\alpha(\varphi_1) + \cdots + s_\alpha(f_k)D_\alpha(\varphi_k)$$

by the twisted Leibniz rule. If $D_\alpha(\omega) = 0$, we have $D_\alpha(\varphi_1) = \cdots = D_\alpha(\varphi_k) = 0$. Hence, if $\omega \in \bigcap_{\alpha \in \Delta_+} \text{Ker}(D_\alpha)$, then φ_i belongs to the homogeneous part $\mathcal{B}_W^0 \cong \mathbf{R}$ of degree zero for $i = 1, \dots, k$. This means that $\omega \in R[t]$. □

Proposition 5.3. For $x \in R \otimes_{\mathbf{R}} R$,

$$D_\alpha(\tilde{\mu}(x)) = c_\alpha \tilde{\mu}(\partial_\alpha(x)).$$

Proof. If $x = \beta \otimes 1$, $\beta \in \Delta$, we can check the relation

$$D_\alpha(\tilde{\mu}(\beta \otimes 1)) = c_\alpha(\beta, \alpha) = c_\alpha \tilde{\mu}(\partial_\alpha(\beta)).$$

Hence, we have $D_\alpha(\tilde{\mu}(x)) = c_\alpha \tilde{\mu}(\partial_\alpha(x))$ for $x \in \mathfrak{h}^* \otimes R$. On the other hand, the two sides satisfy the same twisted Leibniz rule, whence it follows that $D_\alpha(\tilde{\mu}(x)) = c_\alpha \tilde{\mu}(\partial_\alpha(x))$ for $x \in R \otimes R$. $\square$

Theorem 5.1. *If $t = 0$ and the constants $(c_\alpha)_\alpha$ are generic, the image of the homomorphism $\tilde{\mu}$ is isomorphic to the algebra $R \otimes_{R^W} R$. In particular, when W is the Weyl group, it is isomorphic to the T -equivariant cohomology ring $H_T^*(G/B)$ of the corresponding flag variety G/B .*

Proof. If $x \in R^W \otimes_{\mathbf{R}} R$, we have $D_\alpha(\tilde{\mu}(x)) = 0$ for every $\alpha \in \Delta_+$ by Proposition 5.3. This implies, by Proposition 5.2, that $\tilde{\mu}(x) \in R^W \otimes_{R^W} R$. When $t = 0$, $\tilde{\mu}(x)$ coincides with the element of R obtained by replacing all the symbols $[\alpha]$ with zero in $\tilde{\mu}(x)$. Hence, the homomorphism $\tilde{\mu}$ factors through $R \otimes_{R^W} R \rightarrow \tilde{\mathcal{B}}_W$.

We take a reduced decomposition $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_l}}$ of an element $w \in W$. Then the operator $\partial_w := \partial_{\alpha_{i_1}} \cdots \partial_{\alpha_{i_l}}$ is independent of the choice of reduced decompositions, thanks to the Coxeter relation. We also define $D_w := D_{\alpha_{i_1}} \cdots D_{\alpha_{i_l}}$. We introduce a family $\{X_w\}_{w \in W}$ of polynomials by $X_{w_0} := |W|^{-1} \prod_{\alpha \in \Delta_+} \alpha$ and $X_w := \partial_{w^{-1}w_0} X_{w_0}$. The family $\{X_w\}_{w \in W}$ gives a linear basis of the coinvariant algebra R_W . We can see that

$$CT(\partial_w X_v) = \begin{cases} 1 & \text{if } w = v, \\ 0 & \text{otherwise,} \end{cases}$$

where CT stands for the part of degree zero. Since a linear basis $\{X_w\}_{w \in W}$ of the coinvariant algebra of W gives an R^W -basis of R , and $D_w \tilde{\mu}(X_v \otimes 1) = c_{\alpha_{i_1}} \cdots c_{\alpha_{i_l}} \tilde{\mu}(\partial_w X_v \otimes 1)$, it is easy to check that $R \otimes_{R^W} R \rightarrow \tilde{\mathcal{B}}_W$ is injective. $\square$

Remark 5.1. Our construction is not a straightforward application of the functor $(-)\otimes_{R^W} R$ to the one given in [2] even when $t = 0$. In fact, the defining relations of the algebra $\mathcal{B}_W(R)[t]$ involve a nontrivial commutation relation

$$[\alpha]x = s_\alpha(x)[\alpha] + t(x, \alpha).$$

Moreover, in the formula

$$\mu(x) = x + \sum_{\alpha \in \Delta_+} c_\alpha(x, \alpha)[\alpha],$$

the first term on the right-hand side does not occur in the nonequivariant case.

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