MULTIPLICITY OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR AN EQUATION WITH THE p -LAPLACIAN IN A THREE-DIMENSIONAL SPHERICAL LAYER

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ABSTRACT. The equation $-\Delta_p u = u^{q-1}$ with zero Dirichlet condition on the boundary is considered in a three-dimensional spherical layer. The existence of arbitrarily many distinct positive solutions in a sufficiently thin layer is proved.

INTRODUCTION

For $1 < p < \infty$, we consider the boundary-value problem

$$(1) \quad -\Delta_p u = u^{q-1} \text{ in } \Omega_R, \quad u > 0 \text{ in } \Omega_R, \quad u = 0 \text{ on } \partial\Omega_R,$$

where $\Omega_R = B_R \setminus B_{R-1}$ is a spherical layer in $\mathbb{R}^n$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian, and $p < q < p^*$, where p^* is the limit Sobolev embedding exponent, which is determined from the relation $\frac{1}{p^*} = \left(\frac{1}{p} - \frac{1}{n}\right)_+$.

In the paper [1], it was proved that, for $n \neq 3$, in a sufficiently thin layer, the boundary-value problem (1) admits many solutions. Namely, for any natural K , there exists an $R_0 = R_0(p, q, K)$ such that for all $R > R_0$ we can find K nonequivalent (i.e., those that cannot be obtained from one another by rotations) solutions of the boundary-value problem (1). Previously such results were obtained in the paper [2] for $p = 2$. In the two-dimensional case, these results were derived in [3] for $p = 2$ and in [4] for an arbitrary p such that $1 < p < \infty$. In those papers, the solutions of (1) were obtained as the minima of the functional

$$(2) \quad J[u] = \int_{\Omega_R} |\nabla u|^p dx \Big/ \left(\int_{\Omega_R} |u|^q dx \right)^{\frac{p}{q}}$$

on function spaces with a certain symmetry group.

For $n = 3$ this method does not work; in [5] it was proved that this approach cannot give more than five distinct solutions at all. The existence of many solutions of the boundary-value problem (1) for $n = 3$ and $p = 2$ was established in the paper [6]. The solutions of (1) were obtained as the minimum points of the functional (2) under special additional restrictions.

In the present paper, we establish the existence of many solutions of problem (1) for $n = 3$ and arbitrary p . Henceforth, by a solution we mean a generalized solution in $\dot{W}_p^1(\Omega_R)$.

Several auxiliary statements are given in §1; in §2 we study the boundary-value problem (1).

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We introduce some notation: $f = o_R(1)$ means that $\lim_{R \rightarrow \infty} f = 0$. All functions with subscript R have support in Ω_R and are assumed to be extended to $\mathbb{R}^n \setminus \Omega_R$ by zero. The components of a vector $x \in \mathbb{R}^n$ have superscripts in the notation, for example, $x^{(2)}$. For an arbitrary closed subgroup $\mathcal{G}$ of the group $\mathcal{O}(n)$ of rotations, we denote

$$B_{\mathcal{G}}(x, \rho) = \{gy \mid y \in B(x, \rho), g \in \mathcal{G}\},$$

where $B(x, \rho)$ is the open ball of radius ρ with center x .

§1. AUXILIARY STATEMENTS

Throughout this section, $n \geq 2$ is an arbitrary natural number.

Lemma 1 (Principle of concentrated compactness, a generalization of Lemma 3.3 in [6]). *Let $G(s)$ be a positive function. Consider a sequence u_R of $\mathcal{G}$ -invariant functions, and assume that $\int_{\Omega_R} G(u_R)$ is finite for any R . Then (after passing to a subsequence) one of the following three conditions is fulfilled:*

1) (Concentration). *For any $\varepsilon > 0$, there exist a sequence $\{x_R\} \subset \mathbb{R}^n$ and a radius $\rho > 0$ such that*

$$(3) \quad \int_{B_{\mathcal{G}}(x_R, \rho)} G(u_R) dx \geq (1 - \varepsilon) \int_{\Omega_R} G(u_R) dx \text{ for any } R.$$

2) (Fuzzification). *For all $\rho > 0$, we have*

$$(4) \quad \lim_{R \rightarrow \infty} \sup_{y \in \mathbb{R}^n} \int_{B_{\mathcal{G}}(y, \rho)} G(u_R) dx = 0.$$

3) (Separation). *There exists $\lambda \in (0, 1)$ such that for any $\varepsilon > 0$, there exist $\rho > 0$ and R_0 with the property that for any $R \geq R_0$ there are a sequence of points $\{x_R\} \subset \mathbb{R}^n$ and a sequence of numbers $\rho'(R)$ tending to infinity and satisfying*

$$(5) \quad \begin{aligned} & \left| \lambda \int_{\Omega_R} G(u_R) dx - \int_{B_{\mathcal{G}}(x_R, \rho)} G(u_R) dx \right| \\ & + \left| (1 - \lambda) \int_{\Omega_R} G(u_R) dx - \int_{\mathbb{R}^n \setminus B_{\mathcal{G}}(x_R, \rho'(R))} G(u_R) dx \right| \\ & < \varepsilon \int_{\Omega_R} G(u_R) dx. \end{aligned}$$

Proof. Without loss of generality, we may assume that $\int_{\mathbb{R}^n} G(u_R) = 1$. Consider the functions

$$h(\rho, R) = \sup_{x \in \mathbb{R}^n} \int_{B_{\mathcal{G}}(x, \rho)} G(u_R) dx \quad \text{and} \quad i(\rho) = \limsup_{R \rightarrow \infty} h(\rho, R);$$

$i(\rho)$ is a monotone nondecreasing and bounded function of ρ , and so we may consider $j = \lim_{\rho \rightarrow \infty} i(\rho)$.

If $j = 0$, then $i(\rho) = 0$ for any $\rho > 0$, i.e., condition (4) is fulfilled.

If $j = 1$, then for any $\varepsilon > 0$, there is ρ such that $i(\rho) \geq 1 - \frac{\varepsilon}{3}$, i.e., there exists a subsequence R_m with

$$\sup_{x \in \mathbb{R}^n} \int_{B_{\mathcal{G}}(x, \rho)} G(u_{R_m}) dx \geq 1 - \frac{2\varepsilon}{3},$$

and so we can find x_R such that $\int_{B_{\mathcal{G}}(x_R, \rho)} G(u_{R_m}) dx \geq 1 - \varepsilon$. In this case, condition (3) is satisfied (concentration).

Now, let $j = \lambda \in (0, 1)$. We fix an arbitrary $\varepsilon > 0$. Then there exists ρ_0 such that $i(\rho) \geq \lambda - \frac{\varepsilon}{6}$ if $\rho > \rho_0$. We fix such a $\rho > \rho_0$. Then $\limsup_{R \rightarrow \infty} h(\rho, R) \geq \lambda - \frac{\varepsilon}{6}$ implies

that there is a subsequence R_m tending to infinity and such that $h(\rho, R_m) \geq \lambda - \frac{2\varepsilon}{6}$, and thus there is a sequence of points x_m with

$$(6) \quad \int_{B_G(x_m, \rho)} G(u_{R_m}) dx \geq \lambda - \frac{\varepsilon}{2}.$$

On the other hand, $i(\rho) \leq \lambda$ for each ρ . Take a sequence ρ_m tending to infinity. Then for any m there exists $R_{0,m}$ such that for every $R > R_{0,m}$ we have $h(\rho_m, R) \leq \lambda + \frac{\varepsilon}{2}$, which means that

$$(7) \quad \int_{B_G(x, \rho_m)} G(u_{R_m}) dx \leq \lambda + \frac{\varepsilon}{2}$$

for each $x \in \mathbb{R}^n$. Therefore,

$$\int_{B_G(x_m, \rho)} G(u_{R_m}) dx \leq \int_{B_G(x_m, \rho_m)} G(u_{R_m}) dx \leq \lambda + \frac{\varepsilon}{2},$$

whence

$$\left| \lambda - \int_{B_G(x_m, \rho)} G(u_{R_m}) dx \right| \leq \frac{\varepsilon}{2}.$$

Moreover, from (7) we obtain

$$\int_{\mathbb{R}^n \setminus B_G(x, \rho_m)} G(u_{R_m}) dx \geq 1 - \lambda - \frac{\varepsilon}{2}.$$

On the other hand, (6) implies that

$$\int_{\mathbb{R}^n \setminus B_G(x, \rho_m)} G(u_{R_m}) dx \leq \int_{\mathbb{R}^n \setminus B_G(x, \rho)} G(u_{R_m}) dx \leq 1 - \lambda + \frac{\varepsilon}{2}.$$

Thus,

$$\left| 1 - \lambda - \int_{\mathbb{R}^n \setminus B_G(x, \rho_m)} G(u_{R_m}) dx \right| \leq \frac{\varepsilon}{2}. \quad \square$$

Remark 1 (Concentration on an orbit). Let x_R be a sequence of points in $\mathbb{R}^n$. Then similar arguments show that there exists a subsequence for which one of the following two statements is true:

1) for any $\rho > 0$,

$$(8) \quad \lim_{R \rightarrow \infty} \int_{B_G(x_R, \rho)} G(u_R) dx = 0;$$

2) there exists $\lambda \in (0, 1]$ such that for any $\varepsilon > 0$ there are ρ and $\rho'(R)$ tending to infinity such that inequality (5) is fulfilled.

Remark 2. Two sequences of points x_R and y_R are said to be *equivalent* if there is $C > 0$ such that the inequality $\min_{g \in G} |x_R - gy_R| < C$ holds for every R . Obviously, an application of Remark 1 to the sequences x_R and y_R yields either (8) or (5) with one and the same λ . For this reason, in the sequel we do not distinguish between equivalent sequences.

Lemma 2. Let u_R be a bounded sequence in $\dot{W}_p^1(\mathbb{R}^n)$, $p < q < p^*$, and

$$\lim_{R \rightarrow \infty} \sup_{x \in \omega} \int_{B(x, \rho)} |u_R|^q dx = 0$$

for some $\rho > 0$, where ω is an open set in $\mathbb{R}^3$ such that for all $x_0 \in \omega$ we have

$$|\omega \cap B(x_0, \rho)| \geq A\rho^3$$

with A independent of ρ and x_0 . Then $\int_{\omega} |u_R|^q dx \rightarrow 0$ as $R \rightarrow \infty$.

Proof. By the Hölder inequality,

$$\lim_{R \rightarrow \infty} \sup_{x \in \omega} \int_{B(x, \rho)} |u_R|^p dx = 0.$$

We fix $l > 0$ and denote

$$A_{R,l} = \{x \in \mathbb{R}^n \mid u_R > l\}.$$

By assumption, $|A_{R,l} \cap B(x, \rho) \cap \omega|$ tends to zero as $R \rightarrow \infty$ uniformly with respect to $x \in \omega$. By inequality (3.5) in [7, Chapter 2],

$$\begin{aligned} & \int_{A_{R,l} \cap B(x, \rho) \cap \omega} |u_R - l|^p dx \\ & \leq \beta \left(\frac{\rho^n}{|B(x, \rho) \setminus (A_{R,l} \cap \omega)|} \right)^p |A_{R,l} \cap B(x, \rho) \cap \omega|^{\frac{n}{n-1}} \int_{A_{R,l} \cap B(x, \rho) \cap \omega} |\nabla u_R|^p dx, \end{aligned}$$

where β depends only on p, n , and q . Hence, by the assumption $|A_{R,l} \cap B(x, \rho) \cap \omega| \rightarrow 0$, we obtain

$$(9) \quad l^p |A_{R,2l} \cap B(x, \rho) \cap \omega| \leq 2\beta |A_{R,l} \cap B(x, \rho) \cap \omega|^{\frac{n}{n-1}} \int_{A_{R,l} \cap B(x, \rho) \cap \omega} |\nabla u_R|^p dx$$

for R sufficiently large. We cover ω by balls $\{B(x_i, \rho)\}$ in such a way that any point $x \in \omega$ be contained in at most $2n$ balls. Then, summing over all balls of this covering, we have

$$l^p |A_{R,2l} \cap \omega| \leq 4n\beta \sup_i |A_{R,l} \cap B(x_i, \rho) \cap \omega|^{\frac{n}{n-1}} \int_{A_{R,l} \cap \omega} |\nabla u_R|^p dx.$$

Hence,

$$l^p |A_{R,2l} \cap \omega| \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Therefore,

$$l^p |A_{R,l} \cap \omega| \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

and since $\int_{\Omega_R} |\nabla u_R|^p dx$ is bounded, we have

$$\begin{aligned} \int_{A_{R,l} \cap \omega} |u_R|^p dx & \leq 2^{p-1} \int_{A_{R,l} \cap \omega} (|u_R - l|^p + l^p) dx \\ & \leq C_2 \sup_i |A_{R,l} \cap B(x_i, \rho) \cap \omega|^{\frac{n}{n-1}} + C_1 l^p |A_{R,l} \cap \omega|. \end{aligned}$$

Thus,

$$(10) \quad \int_{A_{R,l} \cap \omega} |u_R|^p dx \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Using inequality (2.10) in [7, Chapter 2], we get

$$\begin{aligned} (11) \quad & \int_{\omega} |u_R|^q dx = \int_{\omega \setminus A_{R,l}} |u_R|^q dx + \int_{A_{R,l} \cap \omega} |u_R|^q dx \\ & \leq l^{q-p} \int_{\omega} |u_R|^p dx + C \left(\int_{A_{R,l} \cap \omega} |u_R|^p dx \right)^{\gamma_1} \left(\int_{\omega} |u_R|^{q_1} dx \right)^{1-\gamma_1} \\ & \leq l^{q-p} \int_{\omega} |u_R|^p dx + C \left(\int_{A_{R,l} \cap \omega} |u_R|^p dx \right)^{\gamma_1} \left(\int_{\omega} |u_R|^p dx \right)^{\gamma_2} \left(\int_{\omega} |\nabla u_R|^p dx \right)^{\gamma_3} \end{aligned}$$

(here $\gamma_i = \gamma_i(p, q) > 0$, $i = 1, 2, 3$, and $q < q_1 < p^*$).

Taking into account the Friedrichs inequality and the boundedness of $\|\nabla u_R\|_p$, we see that $\int_{\omega} |u_R|^p dx \leq \int_{\omega} |\nabla u_R|^p dx \leq c$. The last summand tends to zero because the first of

its factors tends to zero by inequality (10), and the second and third are bounded by the assumption of the lemma. Thus,

$$\lim_{R \rightarrow \infty} \int_{\omega} |u_R|^q dx \leq cl^{q-p},$$

where the constant c does not depend on l . Therefore, choosing a sufficiently small l , for any $\varepsilon > 0$ we have $\lim_{R \rightarrow \infty} \int_{\omega} |u_R|^q dx < \varepsilon$, i.e., $\lim_{R \rightarrow \infty} \int_{\omega} |u_R|^q = 0$. $\square$

Remark 3. Let u_R be a bounded sequence in $\dot{W}_p^1(\mathbb{R}^n)$, and suppose that $\int_{\mathbb{R}^n} |u_R|^q dx$ is bounded away from zero with respect to R for some $p < q < p^*$. Then

$$\liminf_{R \rightarrow \infty} \sup_{x \in \mathbb{R}^n} \int_{B(x, \rho)} |u_R|^q dx > 0 \quad \text{for all } \rho > 0.$$

§2. SOLUTION OF PROBLEM (1) FOR $n = 3$

Let k be a fixed natural number. Consider the group T_k generated by the orthogonal matrices

$$\mathcal{R}_k = \begin{pmatrix} \cos \frac{2\pi}{k} & \sin \frac{2\pi}{k} & 0 \\ -\sin \frac{2\pi}{k} & \cos \frac{2\pi}{k} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix};$$

$\mathcal{R}_k$ is rotation through the angle $\frac{2\pi}{k}$, and $\mathcal{T}$ is reflection with respect to the equator. Also, we use the group T_{∞} generated by $\mathcal{T}$ and by all rotations about the axis $x^{(3)} = 0$. Note that if a point y does not lie on the equator and is not a pole, then $T_{\infty}y$ consists of two parallels symmetric about the equator, and y lies on one of them. For brevity, we denote the set $B_{T_k}(x, \rho)$ by $B_k(x, \rho)$. Let $\mathcal{L}_k$ stand for the space of all T_k -invariant functions in $\dot{W}_p^1(\Omega_R)$.

Consider the set

$$(12) \quad \widetilde{\mathcal{L}}_k = \left\{ v \in \mathcal{L}_k \mid \int_{\Omega_R} |v|^q dx = 1, \int_{\Omega_R} \chi_{\theta} |v|^q dx \leq \delta \right\},$$

where

$$(13) \quad \chi_{\theta}(r, \varphi, \theta) = \begin{cases} 0 & \text{if } |\theta - \frac{\pi}{2}| < \frac{\pi}{4}, \\ 1 & \text{if } |\theta - \frac{\pi}{2}| \geq \frac{\pi}{4}, \end{cases}$$

and $\delta \in (0, 1)$ is a fixed number. The boundary of the support of χ_{θ} will be called the tropic.

We seek the minimum of the functional (2) on the set (12). Following [6], we prove that for sufficiently large R the minimum is attained at a function v_R satisfying the strict inequality $\int_{\Omega_R} \chi_{\theta} |v_R|^q dx < \delta$.

Lemma 3. *The functional (2) attains its minimum on the set (12).*

Proof. The set $\widetilde{\mathcal{L}}_k$ is weakly closed in $\dot{W}_p^1(\Omega_R)$, because the embedding of $\dot{W}_p^1(\Omega_R)$ in $L_q(\Omega_R)$ is compact. On the set $\widetilde{\mathcal{L}}_k$, the functional $J[u]$ coincides with the coercive convex functional $\|\nabla u\|_p^p$, so that it attains its minimum. $\square$

Remark 4. $|v_R|$ furnishes the minimum of the functional (2) in the same class of functions as v_R does. Consequently, $|v_R| \geq 0$ is also a minimizer, and in the sequel we assume that $v_R \geq 0$.

Lemma 4. *Let v_R be a sequence of T_k -invariant functions bounded in $\dot{W}_p^1(\Omega_R)$, let $\|v_R\|_q = 1$, and let*

$$(14) \quad \int_{\Omega_R} \chi_\theta v_R^q dx \leq \delta.$$

Then for any $\varepsilon > 0$ there exists a sequence of T_k -invariant functions W_R that has at most three concentration points, satisfies $1 - \varepsilon \leq \|W_R\|_q \leq 1$, $\|\nabla W_R\|_p \leq \|\nabla v_R\|_p(1 + o_R(1))$, and inequality (14) holds for the functions W_R .

Proof. Since the sequence v_R is bounded in $L_q(\mathbb{R}^3)$, one of the three statements, concentration, separation, and fuzzification with the function $G(s) = s^q$, is fulfilled for v_R .

Lemma 2 implies that fuzzification is impossible. Applying to v_R symmetrization with respect to φ on the interval $(-\frac{\pi}{k}, \frac{\pi}{k})$, we do not change $\|v_R\|_q$ and do not increase $\|\nabla v_R\|_p$. Thus, we may assume that all concentration points lie on the plane $\{x^{(2)} = 0\}$ in the quadrant $P = \{x^{(2)} = 0, x^{(1)} \geq 0, x^{(3)} \geq 0\}$.

Consider the sequence $y_R = (R \sin \frac{\pi}{4}, 0, R \cos \frac{\pi}{4})$. For this sequence we consider two cases described in Remark 1. In the first case, there are $\rho > 0$ and a sequence $\rho'(R)$ tending to infinity such that inequality (5) is fulfilled. We introduce a T_k -invariant cutoff function σ :

$$\sigma(x) = \begin{cases} 1 & \text{for } x \in B_k(y_R, \rho), \\ 1 & \text{for } x \notin B_{T_\infty}(y_R, \rho'(R)), \\ 0 & \text{for } x \in B_{T_\infty}\left(y_R, \rho'(R) - \frac{\rho'(R) - \rho}{4}\right) \setminus B_k\left(y_R, \rho'(R) + \frac{\rho'(R) - \rho}{4}\right), \end{cases}$$

$$|\nabla \sigma| \leq \frac{8}{\rho'(R) - \rho}.$$

This cutoff function leaves intact a neighborhood of the concentration orbit $T_k y_R$ and a domain distant from the tropic. Without loss of generality, we may assume that $\rho'(R) \leq \sqrt{R}$.

Then $\int_{\Omega_R} (\sigma v_R)^q dx \geq 1 - \varepsilon$ and

$$\begin{aligned} & \int_{\Omega_R} |\nabla(\sigma v_R)|^p dx \\ & \leq \int_{\Omega_R} |\nabla v_R|^p dx + C_1 \left(\int_{\Omega_R} |\nabla v_R| |\nabla \sigma|^{p-1} v_R^{p-1} \sigma dx + \int_{\Omega_R} v_R^p |\nabla \sigma|^p dx \right). \end{aligned}$$

We estimate the last term:

$$(15) \quad \begin{aligned} \int_{\Omega_R} v_R^p |\nabla \sigma|^p dx & \leq C_2 (\rho'(R) - \rho)^{-p} \int_{\text{supp } \nabla \sigma} v_R^p dx \\ & \leq C_3 (\rho'(R) - \rho)^{\left(\frac{3}{p} - \frac{3}{q} - 1\right)p} \left(\int_{\Omega_R} v_R^q dx \right)^{\frac{p}{q}} \rightarrow 0 \end{aligned}$$

as $R \rightarrow \infty$ because $\rho'(R) - \rho \rightarrow \infty$ and $p < q < p^*$.

Then, by the Hölder inequality, the second term also tends to zero, and we have $\int_{\Omega_R} \nabla(\sigma v_R)^p dx = \int_{\Omega_R} |\nabla v_R|^p dx(1 + o_R(1))$.

In the second case, for any $\varepsilon > 0$ there is a sequence $\rho'(R)$ tending to infinity such that $\int_{B_k(y_R, \rho'(R))} v_R^q dx < \varepsilon$. In this case, we consider the following T_∞ -invariant cutoff function σ :

$$\sigma(x) = \begin{cases} 1 & \text{for } x \notin B_{T_\infty}(y_R, \rho'(R)), \\ 0 & \text{for } x \in B_{T_\infty}\left(y_R, \frac{\rho'(R)}{2}\right), \end{cases} \quad |\nabla \sigma| \leq \frac{4}{\rho'(R)}.$$

As in the preceding case, we have

$$(16) \quad \int_{\Omega_R} (\sigma v_R)^q dx \geq 1 - \varepsilon \quad \text{and} \quad \int_{\Omega_R} \nabla(\sigma v_R)^p dx = \int_{\Omega_R} |\nabla v_R|^p dx (1 + o_R(1)).$$

In both cases, the support of σ contains the equatorial and two circumpolar connected components, and relations (16) hold true.

Consider two T_∞ -invariant cutoff functions: σ_1 , which is equal to σ on the connected components of $\text{supp } \sigma$ that contain the poles and is equal to zero everywhere except for these components, and σ_2 , which is equal to σ on the connected component of $\text{supp } \sigma$ that contains the equator and is equal to zero everywhere except for this component. Consider also the cutoff function $\sigma_3 = \sigma - \sigma_1 - \sigma_2$. The function σ_3 is T_k -invariant, and its support has $2k$ connected components in the first case. In the second case, $\sigma_3 \equiv 0$.

The cutoff functions σ_1 , σ_2 , and σ_3 single out the circumpolar, equatorial, and tropical concentration domains, respectively. We construct a function W_R that has at most one concentration point in any of these domains.

Assume that the condition $\int_{\Omega_R} (\sigma_1 v_R)^q dx \geq \lambda > 0$ is satisfied for the sequence $\sigma_1 v_R$. In each connected component of $\text{supp } \sigma_1 v_R$, we carry out spherical symmetrization with center at a pole and denote by $\tilde{w}_R$ the function obtained. By symmetrization properties, we have $\|\tilde{w}_R\|_q^q = \|\sigma_1 v_R\|_q^q$ and $\|\nabla \tilde{w}_R\|_p \leq \|\nabla(\sigma_1 v_R)\|_p$. We apply Lemma 1 to the sequence $\tilde{w}_R$. Lemma 2 shows that degeneration is impossible. Moreover, the sequence $\tilde{w}_R$ has no concentration points other than the pole $N_R = (0, 0, R)$. Applying Lemma 2 to the set $\mathbb{R}^3 \setminus B_k(N_R, 2\rho)$, we conclude that $\int_{\mathbb{R}^3 \setminus B_k(N_R, 2\rho)} \tilde{w}_R^q dx \rightarrow 0$ as $R \rightarrow \infty$. Thus, there exists a $\rho_1 > 0$ such that

$$\left| \int_{\Omega_R} \tilde{w}_R^q dx - \int_{B_k(N_R, \rho_1)} \tilde{w}_R^q dx \right| < \varepsilon \int_{\Omega_R} \tilde{w}_R^q dx.$$

Now, consider a T_∞ -invariant cutoff function τ_1 such that

$$\tau_1(x) = \begin{cases} 1 & \text{for } x \in B_k(N_R, \rho_1 + \sqrt{R}), \\ 0 & \text{for } x \notin B_k(N_R, \rho_1 + 2\sqrt{R}), \end{cases} \quad |\nabla \tau_1| \leq \frac{4}{\sqrt{R}}.$$

As above, we obtain

$$\|\tau_1 \tilde{w}_R\|_q^q \geq (1 - \varepsilon) \|\tilde{w}_R\|_q^q \quad \text{and} \quad \|\nabla(\tau_1 \tilde{w}_R)\|_p^p \leq \|\nabla \tilde{w}_R\|_p^p (1 + o_R(1)).$$

Now we set $w_R = \tau_1 \tilde{w}_R$. If $\int_{\Omega_R} (\sigma_1 v_R)^q dx \rightarrow 0$ as $R \rightarrow \infty$, then, to unify the notation, we set $w_R \equiv 0$. We also note that $\int_{\Omega_R} \chi_\theta w_R^q dx \leq \delta$.

Now we consider $\sigma_2 v_R$. As before, there is no loss of generality in assuming that the concentration points lie only on P . Consider the set

$$Q = \left\{ g \left(r \cos \theta \cos \frac{\pi}{k}, r \cos \theta \sin \frac{\pi}{k}, r \sin \theta \right) \mid g \in T_k, r \in [R-1, R], \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4} \right\}.$$

In any finite neighborhood of $B(Q, \rho)$, the integral $\int_{B(Q, \rho)} (\sigma_2 v_R)^q dx$ tends to zero, so that we can choose a sequence D_R tending to infinity and such that $\int_{B(Q, D_R)} (\sigma_2 v_R)^q dx < \frac{\varepsilon}{2}$. Consider a cutoff function τ_2 with the properties

$$\tilde{\tau}_2 = \begin{cases} 1 & \text{for } x \notin B(Q, D_R), \\ 0 & \text{for } x \in B\left(Q, \frac{D_R}{2}\right), \end{cases} \quad |\nabla \tilde{\tau}_2| \leq \frac{4}{D_R},$$

and the cutoff function $\tau_2 = \frac{1}{2} \prod_{g \in T_k} \tilde{\tau}_2(gx)$. Arguing as before, we get

$$\|\tau_2 \sigma_2 v_R\|_q^q \geq (1 - \varepsilon) \|\sigma_2 v_R\|_q^q \quad \text{and} \quad \|\nabla(\tau_2 \sigma_2 v_R)\|_p^p = \|\nabla(\sigma_2 v_R)\|_p^p (1 + o_R(1)).$$

Introducing the cutoff function τ_3 that is equal to τ_2 on the connected component of $\text{supp } \tau_2$ intersecting P and is equal to zero at all remaining points, we apply spherical symmetrization to $\tau_3 \sigma_2 v_R$ and denote by $\tilde{w}'_R$ the function obtained. We have

$$\int_{\Omega_R} (\tilde{w}'_R)^q dx = \frac{1}{k} \int_{\Omega_R} (\tau_2 \sigma_2 v_R)^q dx \quad \text{and} \quad \int_{\Omega_R} |\nabla \tilde{w}'_R|^p dx \leq \frac{1}{k} \int_{\Omega_R} |\nabla (\tau_2 \sigma_2 v_R)|^p dx.$$

We apply to $\tilde{w}'_R$ the lemma on concentration (Lemma 1). As before, fuzzification and dichotomy are impossible; thus, condition (3) is satisfied. Consider a cutoff function τ_4 such that

$$\tau_4 = \begin{cases} 1 & \text{for } x \in B((R, 0, 0), \rho), \\ 0 & \text{for } x \notin B\left((R, 0, 0), \rho + \frac{\rho'(R) - \rho}{2}\right), \end{cases} \quad |\nabla \tau_4| \leq \frac{4}{\rho'(R) - \rho}$$

and the T_k -invariant cutoff function $\tau_5 = \frac{1}{2} \sum_{g \in T_k} \tau_4(gx)$. As before, we may assume that $\rho + \frac{\rho'(R) - \rho}{2} \leq \sqrt{R}$, and we have

$$\|\tau_4 \tilde{w}'_R\|_q^q \geq (1 - \varepsilon) \|\tilde{w}'_R\|_q^q \quad \text{and} \quad \|\nabla(\tau_4 \tilde{w}'_R)\|_p^p = \|\nabla \tilde{w}'_R\|_p^p (1 + o_R(1)).$$

We set $w'_R = \tau_4 \tilde{w}'_R$ and note that the supports of $w'_R(x)$ and $w'_R(\mathcal{R}_k^j x)$ do not intersect if j is not a multiple of k .

In the case where $\sigma_3 \neq 0$, we consider an additional cutoff function that distinguishes one connected component of $\text{supp } \sigma_3$:

$$\tau_6 = \begin{cases} \sigma & \text{for } x \in B\left(y_R, \rho'(R) + \frac{\rho'(R) - \rho}{4}\right), \\ 0 & \text{for } x \notin B\left(y_R, \rho'(R) + \frac{\rho'(R) - \rho}{4}\right), \end{cases} \quad |\nabla \tau_6| \leq \frac{8}{\rho'(R) - \rho}.$$

Then $\sigma_3 = \sum_{g \in T_k} \tau_6(gx)$. The rotation $\mathcal{S}$ about the axis $x^{(2)}$ through the angle $\frac{\pi}{8}$ takes the point N_R to the point $(R \cos \frac{3\pi}{8}, 0, R \sin \frac{3\pi}{8})$. To $\tau_6 v_R(\mathcal{S}x)$ we apply spherical symmetrization with center at the point $\mathcal{S}y_R$, and we denote by $w''_R(x)$ the function obtained. If $\sigma_3 = 0$, we set $w''_R \equiv 0$ to unify the notation.

The graph of w_R represents two caps with centers at the poles, the graph of w'_R is a cap with center on the equator, and the graph of w''_R is a cap with center between the equator and the tropic.

Now we consider the function

$$W_R(x) = w_R(x) + \frac{1}{2} \sum_{g \in T_k} w'_R(gx) + \sum_{g \in T_k} w''_R(gx).$$

The properties of the functions w_R , w'_R , and w''_R show that $\|W_R\|_q^q \geq 1 - 4\varepsilon$, $\|\nabla W_R\|_p \leq \|v_R\|_p(1 + o_R(1))$, and $\int_{\Omega_R} \chi_\theta W_R^q dx \leq \delta$. $\square$

Lemma 5. *Let $\delta = \frac{1}{k}$, and let v_R minimize the functional J on $\widetilde{\mathcal{L}}_k$. Then for sufficiently large R we have*

$$(17) \quad \int_{\Omega_R} \chi_\theta |v_R|^q dx < \delta.$$

Proof. Assume the contrary: let the condition $\int_{\Omega_R} \chi_\theta v_R^q dx = \delta$ be valid for the sequence v_R . Suppose W_R is constructed in the same way as in Lemma 4.

We put

$$\begin{aligned} a_R &= \frac{1}{2} \int_{\Omega_R} |\nabla w_R|^p dx; & \alpha_R &= \frac{1}{2} \int_{\Omega_R} w_R^q dx, \\ b_R &= \int_{\Omega_R} |\nabla w'_R|^p dx; & \beta_R &= \int_{\Omega_R} w'^q_R dx, \\ c_R &= \int_{\Omega_R} |\nabla w''_R|^p dx; & \gamma_R &= \int_{\Omega_R} w''^q_R dx, \end{aligned}$$

where w_R , w'_R , and w''_R are as in Lemma 4. By construction, we have

$$(18) \quad 1 - \varepsilon \leq 2\alpha_R + k\beta_R + 2k\gamma_R \leq 1, \quad 2\alpha_R \leq \delta, \quad 2\alpha_R + 2k\gamma_R \geq \delta - \varepsilon.$$

Now we show that we can reduce the value of the functional. For this, among the functions $\chi_{\{x_{(3)} > 0\}} w_R$, w'_R , and w''_R we choose a function at which the functional J has the smallest value. Namely, consider the sequences α_R , β_R , and γ_R . They cannot tend to zero simultaneously; therefore, after passing to subsequences, we may assume that one of them is bounded away from zero. Among the ratios $a_R/\alpha_R^{\frac{p}{q}}$, $b_R/\beta_R^{\frac{p}{q}}$, and $c_R/\gamma_R^{\frac{p}{q}}$, we choose those with denominators bounded away from zero, and among them we take the minimum one.

If this is α_R , then we set $m_R = a_R$, $\mu_R = \alpha_R$, and $\tilde{u}_R = \chi_{\{x_{(3)} > 0\}} w_R(\mathcal{S}^4 x)$. If this is β_R , then we set $m_R = b_R$, $\mu_R = \beta_R$, and $\tilde{u}_R = w'_R(x)$. If this is γ_R , then we put $m_R = c_R$, $\mu_R = \gamma_R$, and $\tilde{u}_R = w''_R(\mathcal{S}x)$. Next, we consider the function

$$\begin{aligned} u_R(x) &= \left(\frac{\alpha_R}{\mu_R}\right)^{\frac{1}{q}} (\tilde{u}_R(\mathcal{S}_1^{-4}x) + \tilde{u}_R(\mathcal{S}_1^4x)) \\ &\quad + \frac{1}{2} \left(\frac{\beta_R}{\mu_R}\right)^{\frac{1}{q}} \sum_{g \in T_k} \tilde{u}_R(gx) + \left(\frac{\gamma_R}{\mu_R}\right)^{\frac{1}{q}} \sum_{g \in T_k} \tilde{u}_R(g\mathcal{S}_1^{-1}x). \end{aligned}$$

Thus, we have replaced all caps by caps of the best shape that have the same mass.

We have

$$\begin{aligned} \int_{\Omega_R} u_R^q dx &= 2\alpha_R + k\beta_R + 2k\gamma_R \leq 1, \\ \int_{\Omega_R} |\nabla u_R|^p dx &= \left(2\alpha_R^{\frac{p}{q}} + k\beta_R^{\frac{p}{q}} + 2k\gamma_R^{\frac{p}{q}}\right) \frac{m_R}{\mu_R^{\frac{p}{q}}} \leq \int_{\Omega_R} |\nabla W_R|^p dx, \end{aligned}$$

and

$$\left(2\alpha_R^{\frac{p}{q}} + k\beta_R^{\frac{p}{q}} + 2k\gamma_R^{\frac{p}{q}}\right) \frac{m_R}{\mu_R^{\frac{p}{q}}} \leq J[u_R] \leq J[W_R].$$

We retain only the caps on the equator, i.e., we consider the function

$$U_R(x) = \frac{1}{2} \sum_{g \in T_k} \tilde{u}_R(gx).$$

We have

$$\int_{\Omega_R} U_R^q dx = k\mu_R, \quad \int_{\Omega_R} |\nabla U_R|^p dx = km_R$$

and $J[U_R] = k^{1-\frac{p}{q}} m_R / \mu_R^{\frac{p}{q}}$.

Under conditions (18), the minimum value of the function $2\alpha_R^{\frac{p}{q}} + k\beta_R^{\frac{p}{q}} + 2k\gamma_R^{\frac{p}{q}}$ is equal to $2^{1-\frac{p}{q}} \delta^{\frac{p}{q}} + k^{1-\frac{p}{q}} (1 - \varepsilon - \delta)^{\frac{p}{q}}$. Therefore,

$$J[U_R] \leq \frac{1}{\left(\frac{2}{k}\right)^{1-\frac{p}{q}} \delta^{\frac{p}{q}} + (1 - \varepsilon - \delta)^{\frac{p}{q}}} J[u_R].$$

Since $\delta = \frac{1}{k}$, we have $\left(\frac{2}{k}\right)^{1-\frac{p}{q}} \delta^{\frac{p}{q}} + (1-\varepsilon-\delta)^{\frac{p}{q}} = \frac{2^{1-\frac{p}{q}}}{k} + \left(1 - \frac{1}{k} - \varepsilon\right)^{1-\frac{p}{q}} \geq 1 + \frac{2^{1-\frac{p}{q}}-1}{k} - \varepsilon \geq 1 + \frac{2^{1-\frac{p}{q}}-1}{2k}$ for sufficiently small ε . Then

$$\begin{aligned} J[U_R] &\leq \frac{1}{1 + \frac{2^{1-\frac{p}{q}}-1}{2k}} J[u_R] \\ &\leq \frac{1}{1 + \frac{2^{1-\frac{p}{q}}-1}{2k}} J[W_R] \leq \frac{1}{(1-\varepsilon)^{\frac{p}{q}} \left(1 + \frac{2^{1-\frac{p}{q}}-1}{2k}\right)} J[v_R](1 + o_R(1)). \end{aligned}$$

Since $\int_{\Omega_R} \chi_\theta U_R^q dx = 0$, for sufficiently small ε we obtain a contradiction with the minimality of $J[v_R]$. $\square$

Remark 5. As a by-product, we have proved that the sequence v_R may have only one concentration orbit lying on the equator.

Theorem 1. *Let $\delta = \frac{1}{k}$. If R is sufficiently large, then the function v_R constructed in Lemma 3, upon multiplication by an appropriate constant, is a positive generalized solution of the boundary-value problem (1).*

Proof. By Lemma 5, the restriction $\int_{\Omega_R} \chi_\theta v_R^q dx \leq \frac{1}{k}$ is inactive for large R . Therefore, v_R is a local minimum on the set $\{u \in \dot{W}_p^1(\Omega_R) \mid \int_{\Omega_R} |u|^q dx = 1\}$. By the Euler–Lagrange theorem, for some $\lambda \in \mathbb{R}$ we have

$$(19) \quad \int_{\Omega_R} ((|\nabla v_R|^{p-2} \nabla v_R, \nabla h) - \lambda |v_R|^{q-2} v_R h) dx = 0$$

for all $h \in \mathcal{L}_k$.

By the symmetric criticality principle (see [8]), v_R is a solution of the Dirichlet problem

$$(20) \quad -\Delta_p u = \lambda u^{q-1} \text{ in } \Omega_R; \quad u = 0 \text{ on } \partial\Omega_R.$$

It remains to note that $v_R \geq 0$ is super- p -harmonic, and, by the Harnack inequality (see [9]), it is positive in Ω_R . Since the two sides of equation (20) have different degrees of homogeneity, the function $u = cv_R$ with an appropriate constant c is a solution of (1). $\square$

Theorem 2. *For any natural N , there exists $R_0 = R_0(p, q, N)$ such that for any $R > R_0$, the problem (1) has at least N nonequivalent (i.e., such that they cannot be taken to one another by rotations) positive solutions.*

Proof. The boundary-value problem (1) has a T_k -invariant solution for any $1 \leq k \leq N$. As was proved in Lemma 5, for a sequence of such solutions condition (3) is satisfied, and the distance from x_R to the equator is bounded. Without loss of generality, we may assume that $x_R = (R, 0, 0)$. Moreover, for any $\varepsilon > 0$ there is a $\rho > 0$ such that for all $R > R_0(p, q, k)$ we have $|\int_{B(x_R, \rho)} v_R^q dx - \frac{1}{k}| < \varepsilon$. Therefore, for different k the v_R are distinct provided that R is sufficiently large. $\square$

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