CYCLICITY OF ELEMENTARY POLYCYCLES WITH FIXED NUMBER OF SINGULAR POINTS IN GENERIC k -PARAMETER FAMILIES

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ABSTRACT. An estimate is found for the number of limit cycles arising from polycycles in generic finite-parameter families of differential equations on the two-sphere. It is proved that if the polycycles have a fixed number of singular points and all the singular points are elementary, then an estimate of cyclicity holds true, which is polynomial in the number of parameters of the family.

§1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

1.1. Motivation and the preceding results. In the present paper, we discuss the *Hilbert–Arnol'd problem*, which is closely related to Hilbert's 16th problem. In the latter problem, it is required to estimate the number of limit cycles of a polynomial vector field on the plane. Arnol'd [1] proposed to consider not polynomial families, but arbitrary *generic*¹ finite-parameter families of smooth differential equations on the two-sphere with compact base of parameters, and he formulated a number of conjectures.

One of them turned out to be incorrect, but nevertheless it led Il'yashenko [2] to the statement of the following problem: *Show that for every such family, the number of limit cycles admits a uniform estimate for all values of the parameter.*

Using compactness arguments going back to Roussarie [7], one can reduce the above problem to estimating the number of cycles arising from bifurcations of the so-called *polycycles*, i.e., separatrix polygons. The latter problem is called the (local) Hilbert–Arnol'd problem. (See Problem 1 below.)

Rigorous definitions and the statement of the problem are presented below.

Definition 1.1. A *polycycle* γ of a vector field on the sphere $\mathbb{S}^2$ is a cyclically ordered collection of vertices, i.e., singular points $p_1, \dots, p_n$ (possibly, with repetition), and arcs of trajectories $\gamma_1, \dots, \gamma_n$ (without repetition) connecting the vertices in the following order: the j th arc connects the vertices p_j and p_{j+1} , where $j = 1, \dots, n$ and $p_{n+1} := p_1$.

A polycycle is said to be *nontrivial* if it contains at least one singular point. The maximum number of cycles that can arise from a perturbation of a polycycle is called the *cyclicity* of the polycycle. The rigorous definition is as follows.

Definition 1.2. Consider a k -parameter family of vector fields on the sphere:

$$\{\dot{x} = v(x, \varepsilon)\}_{\varepsilon \in B^k}, \quad x \in \mathbb{S}^2,$$

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¹Genericity is understood here in the topological sense.

FIGURE 1. A polycycle and the Poincaré map.

where B^k is the unit ball in $\mathbb{R}^k$. Assume that the system has a polycycle γ for $\varepsilon = \varepsilon_* \in B^k$. Let μ be the minimum integer with the following property: there exist neighborhoods U and V such that

$$\mathbb{S}^2 \supseteq U \supset \gamma, \quad B^k \supseteq V \ni \varepsilon_*,$$

and for each $\varepsilon \in V$ the neighborhood U contains at most μ limit cycles of the field $v(\cdot, \varepsilon)$. Then μ is called the *cyclicity* of the polycycle γ in the family $\{v(x, \varepsilon)\}_{\varepsilon \in B^k}$.

Definition 1.3. The *bifurcation number* $B(k)$ is the maximum cyclicity of a nontrivial polycycle in a generic k -parameter family of C^∞ -smooth vector fields.

We observe that the number $B(k)$ depends only on the number of parameters in the family.

Problem 1 (Hilbert–Arnol’d problem). Prove that the bifurcation number $B(k)$ is finite for each finite k , and find an upper bound for $B(k)$.

This problem remains open.

Definition 1.4. A singular point of a vector field v on the two-sphere $\mathbb{S}^2$ is said to be *elementary* if the linearization of v at that point has at least one nonzero eigenvalue. A polycycle is *elementary* if all vertices of the polycycle are isolated elementary singular points.

The Hilbert–Arnol’d problem was solved for the case of an elementary polycycle. Yu. Il’yashenko and S. Yakovenko made the first step, and the next step was made by V. Kaloshin.

Definition 1.5. The maximum cyclicity of a nontrivial elementary polycycle in a generic k -parameter family of C^∞ -smooth vector fields is called the *elementary bifurcation number* and is denoted by $E(k)$.

Theorem 1.6 (Il’yashenko and Yakovenko [4]). *For each positive integer k , the number $E(k)$ is finite.*

Theorem 1.7 (V. Kaloshin [5]). *For each positive integer k , we have*

$$(1.1) \quad E(k) \leq 2^{25k^2}.$$

Theorem 1.7 yields the first explicit estimate for the cyclicity of a polycycle in a family with an arbitrary number of parameters. However, estimate (1.1) appears to be greatly overstated.

1.2. Statement of the main result of the paper. It turns out that if the number of singular points of the system is fixed, then the cyclicity grows not faster than a certain polynomial in the number of parameters of the family. Our main result is Theorem 1.8, which suggests an explicit estimate of such a form.

Theorem 1.8. *The maximum cyclicity of a nontrivial elementary polycycle with n vertices in a generic k -parameter family is bounded by the number*

$$E(n, k) \leq C(n)k^{3n},$$

where $C(n) = 2^{5n^2+20n}$.

In the present paper, we use techniques developed by Il'yashenko jointly with Yakovenko [4] and substantially elaborated by Kaloshin [5].

1.3. Structure of the paper. In §2, we describe a strategy for solving problems of this type and give necessary formulations. In Subsection 2.7, we clarify the difference between the techniques developed by Kaloshin and those developed by Il'yashenko and Yakovenko. There, we also describe the main idea that allowed us to obtain sharper estimates and to prove Theorem 1.8. Finally, the necessary computations are presented in §3.

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§2. EXISTENCE OF THE NUMBER $E(n, k)$ AND ITS ESTIMATE

A complete description of the techniques for solving similar type problems is beyond the scope of the present paper; it can be found in [5] and [4]. In this section, we *describe* the strategy of the proof; largely we follow [5] and [4].

2.1. Plan of the proof. We consider a generic k -parameter family of vector fields on the plane depending on the parameter ε and assume that for $\varepsilon = \varepsilon_*$ the corresponding system possesses a nontrivial polycycle. Here, we use a standard trick and consider the *Poincaré map*. After that, the problem of estimating the number of limit cycles of a differential equation reduces to estimating the number of isolated fixed points of the Poincaré map (see Subsection 2.2).

We look for the fixed points as for solutions of a special system of equations, which is called the *basic system*. (The system also depends on the parameter ε .) The further plan is to use sequentially a number of transformations and bring the basic system to a form admitting an effective estimate for the number of solutions. When doing so, we control the estimate for the number of solutions during each transformation (see Figure 2).

Using finitely smooth changes of coordinates, we bring the initial family of vector fields to a special simple form in neighborhoods of singular points. This allows us to write the equations of the basic system in the normalized form (see Subsection 2.3).

The equations involve singular functions, which are difficult for study. However, solutions of such equations, in turn, satisfy the so-called *Pfaffian equations*, i.e., differential equations with *polynomial coefficients* (see Subsection 2.4). Therefore, the number of solutions of the basic system is estimated by the number of solutions of a new *functional-Pfaffian* system, in which equations with singular functions are replaced by Pfaffian differential equations. This allows us to eliminate the singular functions.

Below, we use a trick known as *Khovanskii's reduction*. As a result, the functional-Pfaffian system is replaced by a (finite) collection of purely functional systems having a special form and containing no singular equations. Furthermore, the number of solutions of the initial system does not exceed the sum of the numbers of solutions of the systems in the collection (see Subsection 2.5).

An analog of Bezout's theorem is applicable to such systems: this is the Bezout–Kaloshin theorem. The latter allows us to give an explicit estimate for the number of

FIGURE 2. Strategy of the proof.

solutions and, simultaneously, for the number of limit cycles in the initial family (see Subsection 2.6).

In order to complete the proof, it only remains to calculate the degrees of the equations in the systems to which the Bezout–Kaloshin theorem is applied. This is done in §3.

2.2. The basic system. We denote by n the number of vertices of a nontrivial elementary polycycle, and we denote by k the number of parameters of the family. Our problem is to estimate the number of limit cycles arising from a given polycycle in terms of n and k .

A polycycle γ is said to be *monodromic* if the phase space contains a half-interval Σ such that an end of Σ lies on γ , Σ is transversal to the vector field, and the *Poincaré map* (the first return map; see Figure 3) is defined on Σ . Such a half-interval Σ is briefly called a *transversal* to the polycycle γ .

We assume that the polycycle is monodromic² and consider the corresponding Poincaré map Π .

Let Σ_j^{in} and Σ_j^{out} be transversals to the polycycle in the vicinity of the j th vertex ($j = 1, \dots, n$) such that the correspondence mappings

$$\Delta_j : \Sigma_j^{\text{in}} \rightarrow \Sigma_j^{\text{out}} \quad \text{and} \quad f_j : \Sigma_j^{\text{out}} \rightarrow \Sigma_{j+1}^{\text{in}}$$

are defined (where we put $\Sigma_{n+1}^{\text{in}} := \Sigma_1^{\text{in}}$) and the mapping Π is the composition

$$\Pi = f_n \circ \Delta_n \circ \dots \circ f_1 \circ \Delta_1$$

(see Figure 3).

²Remark 2.9 translates the construction to the case of nonmonodromic polycycles.

FIGURE 3. The Poincaré map of the polycycle.

If closed trajectories arise from a perturbation of the polycycle, then they correspond to fixed points of the mapping Π . They can be found as solutions of the *basic system*:

$$(2.1) \quad \begin{cases} y_j = \Delta_j(x_j, \varepsilon), \\ x_{j+1} = f_j(y_j, \varepsilon), \quad j = 1, \dots, n, \\ x_{n+1} = x_1, \end{cases}$$

where x_j is the coordinate on the “input” transversal Σ_j^{in} of the j th vertex, y_j is the coordinate on the “output” transversal Σ_j^{out} of the j th vertex, and

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$$

is the vector of parameters.

2.3. Normal forms of local families. If an elementary polycycle can generate limit cycles, then topological arguments imply that the only singular points on the polycycle are saddles and saddle-nodes. Using finitely smooth changes of variables, we bring local families presenting perturbations of saddles and saddle-nodes to *normal forms* described in [3]. In this section, we formulate the necessary facts.

The saddles split into two classes having distinct normal forms: *resonant* and *non-resonant* saddles; the first case corresponds to the rational quotient $\frac{n}{m}$ of eigenvalues (in this case, we speak about a resonance of type $n : m$). The second case corresponds to an irrational quotient of eigenvalues.

Let μ be a positive integer, and let $\lambda = (\lambda_0, \dots, \lambda_\mu)$ be a multidimensional parameter. We introduce the following notation:

$$(2.2) \quad \begin{aligned} u &= x^m y^n, \\ P_\mu(u, \lambda) &= \pm u^\mu (1 + \lambda_\mu u^\mu) + W_{\mu-1}(u, \lambda), \\ Q_\mu(x, \lambda) &= x^{\mu+1} + \lambda_\mu x^{2\mu+1} + W_{\mu-1}(x, \lambda), \end{aligned}$$

where

$$W_{\mu-1}(x, \lambda) = \lambda_0 + \dots + \lambda_{\mu-1} x^{\mu-1}.$$

The normal form of a resonant saddle in a finite-parameter family of vector fields looks like this:

$$(2.3) \quad \begin{cases} \dot{x} = x \left(\frac{n}{m} + P_\mu(u, \lambda) \right), \\ \dot{y} = -y \end{cases}$$

(see [4, Table 1]). The number μ depends on the initial saddle and is called the *multiplicity* of the saddle; we denote by S_μ saddles of multiplicity μ .

Remark 2.1. Actually, the normal form (2.3) occurs when the germ of the vector field for $\varepsilon = \varepsilon_*$ has no nontrivial formal first integral (see [3]). This condition determines a set of infinite codimension in the space of smooth hyperbolic resonant germs of vector fields.

For a singular point of saddle-node type, we denote by $\mu + 1$ the usual multiplicity of the singular point (i.e., the maximum number of singular points into which the singular point “scatters” under a small perturbation).

In a finite-parameter family of vector fields, the normal form of a saddle-node of multiplicity $\mu + 1$ looks like this:

$$(2.4) \quad \begin{cases} \dot{x} = Q_\mu(x, \lambda), \\ \dot{y} = -y \end{cases}$$

(see [4, Table 1]).

Remark 2.2. Actually, the normal form (2.4) occurs when the Lojasiewicz condition is fulfilled (when approaching the singular point, the length of the vector field decreases not faster than a certain power of the distance to the point); see [4]. This condition determines a set of infinite codimension in the space of smooth saddle-node germs of vector fields.

Using the normal forms indicated above, we may assume that the basic system (2.1) is normalized, i.e., each function Δ_j corresponds to a singular point written in the normal form.

With the help of the normalized basic system, a k -parameter family of vector fields determines a k -parameter family of collections of finitely smooth functions f_j . As a result, we obtain the following proposition.

Proposition 2.3. *Suppose that the functions f_j in the basic system (2.1) are generic. Then the maximum cyclicity in Theorem 1.8 does not exceed the number of solutions of system (2.1) that are close to zero.*

Remark 2.4. A precise description of the set of generic collections of finitely smooth functions f_j will be given below; see Definition 2.17.

2.4. Pfaffian equations and the functional-Pfaffian system. The normal forms (2.3) and (2.4) can be integrated in quadratures, and the correspondence mappings for them can be computed explicitly. These mappings are determined by transcendental functions and in general are not everywhere defined (see, e.g., [4, §1.1]). However, the mappings satisfy *Pfaffian equations*, i.e., differential equations with *polynomial* coefficients (loc. cit.). We can replace the singular equations

$$y_j - \Delta_j(x_j, \varepsilon) = 0$$

by the Pfaffian equations satisfied by their left-hand sides. We observe that the number of solutions of the resulting system (the latter is called the *functional-Pfaffian system*) does not exceed the number of solutions of the corresponding basic system (2.1).

We now present the necessary formulations.

Lemma 2.5 (Il'yashenko and Yakovenko [4]). *The correspondence mappings for a non-degenerate saddle satisfy the Pfaffian equations presented in Table 1, where S_0 denotes a nonresonant saddle and S_μ denotes a saddle of multiplicity μ ; see formula (2.2) for the definition of P_μ .*

TABLE 1. Pfaffian equations for saddles

Type	Pfaffian equations
S_0	$x dy - \lambda y dx = 0$
S_μ	$P_\mu(x^m, \lambda) [y P_\mu(y^n, \lambda) dx - (\frac{n}{m} + P_\mu(y^n, \lambda)) x P_\mu(x^m, \lambda) dy] = 0$

There are two types of correspondence mappings for saddle-nodes: along the central manifold (see Figure 4) and from the hyperbolic manifold to the central one (or backwards). The correspondence mappings related to a saddle-node of multiplicity μ are denoted by D_μ^c and D_μ^h .

Lemma 2.6 (Il'yashenko and Yakovenko [4]). *The correspondence mappings for the saddle-node satisfy the equations presented in Table 2; see formula (2.2) for the definition of Q_μ .*

TABLE 2. Pfaffian equations for saddle-nodes

Type	Pfaffian equations
D_μ^c	$x(x dy - y dx) = 0$
D_μ^h	$Q_\mu(x, \lambda) dy - y dx = 0$

Remark 2.7. The factors $P_\mu(x^m, \lambda)$ and x in the Pfaffian equations for S_μ (see Table 1) and D_μ^c (see Table 2), respectively, are added in order to fulfill the topological requirements of Khovanskii's procedure (see [4, Lemma 3.1] for the details).

Remark 2.8. Obviously, the correspondence mappings for saddles of both types and saddle-nodes of type D_μ^h are defined for the unperturbed system and also for any sufficiently small perturbation of the system. This is not true for saddle-nodes of type D_μ^c : the correspondence mapping is defined only for certain perturbations of the initial system. (See Figure 4: the correspondence mapping is defined only for case c).)

FIGURE 4. The correspondence mapping for a saddle-node of type D_μ^c : a) for a precritical value of the parameter; b) for a critical value of the parameter; c) for a postcritical value of the parameter.

Remark 2.9. The basic system for a monodromic polycycle makes sense for all values of the parameter for which all correspondence mappings are defined. However, the corresponding Pfaffian equations still make sense for any perturbations. Therefore, passage to the Pfaffian system can yield “redundant” solutions (not corresponding to limit cycles), but we do not lose the “useful” solutions (corresponding to limit cycles).

As has already been noted, we formally replace the singular equations in the basic system by the corresponding Pfaffian equations of the form

$$A_j dx_j + B_j dy_j = 0$$

taken from Tables 1 and 2, obtaining the *functional-Pfaffian system*

$$(2.5) \quad \begin{cases} \omega_j = 0, \\ F_j(x, y, \varepsilon) = 0, \end{cases} \quad j = 1, \dots, n,$$

$$\begin{aligned} \omega_j &:= A_j dx_j + B_j dy_j, \quad F_j(x, y, \varepsilon) := x_{j+1} - f_j(y_j, \varepsilon), \\ (x, y) &:= (x_1, y_1, \dots, x_n, y_n) \in (\mathbb{R}^{2n}, 0), \quad \varepsilon \in (\mathbb{R}^k, 0). \end{aligned}$$

This system should be interpreted as follows: we take an arbitrary integral manifold Γ of the system consisting of the Pfaffian equations in the system (2.5) and find the intersection of Γ with the level set $F^{-1}(0)$, where

$$F : (\mathbb{R}^{2n}, 0) \rightarrow \mathbb{R}^n$$

is the mapping with the coordinate functions F_j . The solutions of the basic system (2.1) yield a solution of the functional-Pfaffian system for a certain specific integral manifold Γ . Estimating the number of isolated intersection points for an *arbitrary* manifold Γ , we obtain an upper bound for the number of solutions of the basic system.

Limit cycles correspond to isolated fixed points of the Poincaré map, so that we must estimate only the number of isolated solutions of (2.1). The latter number does not exceed the number of *nondegenerate* solutions of a similar system with a small right-hand side (see [4, Lemma 3.2]). In turn, these solutions correspond to *transversal* intersections of Γ with a typical fiber $F^{-1}(a)$ for a small $a \in (\mathbb{R}^n, 0)$ (see [4, Lemma 3.3]).

Since the integral manifold and the level set have complementary dimensions, it follows that the transversal intersection always consists of isolated points, which are called *regular solutions* of system (2.5). We must estimate their number from above, and the estimate must be uniform with respect to all integral manifolds Γ and all sufficiently small values of parameters.

In the further arguments, it is convenient to regard the parameters ε and

$$\lambda = (\lambda^1, \dots, \lambda^n), \quad \lambda^j \in \mathbb{R}^{\mu_j+1},$$

as additional independent variables. For this, we add to the system new equations of the form

$$\varepsilon = \varepsilon^* \in \mathbb{R}^k, \quad \lambda = \lambda^* \in \mathbb{R}^m,$$

where $m := n + \sum \mu_j$.

Furthermore, the degrees of the variables x and y in the Pfaffian equation for the resonant saddle S_μ (see Table 1) are unbounded because of the presence of the resonance $m : n$. The unbounded degrees are an obstruction to our research, since the degrees of the Pfaffian equations enter the final estimate. However, this is easy to correct by adding new variables,

$$z = x^m \quad \text{and} \quad w = y^n,$$

and Pfaffian equations for z and w . As a result, we obtain three Pfaffian equations for each resonant saddle:

$$(2.6) \quad \begin{cases} P_\mu(z, \lambda) [yP_\mu(w, \lambda) dx - (\frac{n}{m} + P_\mu(w, \lambda)) xP_\mu(z, \lambda) dy] = 0, \\ x dz - mz dx = 0, \\ y dw - nw dy = 0. \end{cases}$$

The first equation is said to be *basic*, and the other two are said to be *additional*. The equations in Table 2 are also said to be *basic*.

So, now we replace the functional-Pfaffian system (2.5) by a new system of the same type with the following distinctions:

- (1) each Pfaffian equation for S_μ in Table 1 is replaced by the three equations (2.6);
- (2) the parameters ε and λ in the expression (2.2) for Q_μ are included into the number of the variables, and the system contains equations for the parameters:

$$\varepsilon - \varepsilon^* = b_\varepsilon \quad \text{and} \quad \lambda - \lambda^* = b_\lambda,$$

where the right-hand sides are small.

We look for an estimate not only for systems with *zero* right-hand side, but also for those with a *small* right-hand side. The estimate must be uniform with respect to the right-hand side.

The modified functional-Pfaffian system can be written as follows:

$$(2.7) \quad \begin{cases} \Omega = 0 \rightsquigarrow n + 2s, \\ F_f = a \rightsquigarrow n, \\ F = b \rightsquigarrow k + m. \end{cases}$$

Here, the line $\Omega = 0$ denotes Pfaffian equations taken from Tables 1 and 2, and from system (2.6) in the case of a resonant saddle. The line $F_f = a$ denotes functional equations of the form

$$x_{j+1} - f_j(y_j, \varepsilon) = a_j,$$

while the line $F = b$ denotes equations for the parameters ε and the additional parameters λ . The numbers to the right of the sign “ $\rightsquigarrow$ ” mean a bound for the number of the equations of the type indicated. The phase space of the system has dimension

$$\mathbf{m} := 2n + 2s + k + m.$$

2.5. Khovanskii’s procedure. A detailed description of Khovanskii’s procedure [6] is beyond the scope of the present paper. In this section, we mention only the most necessary facts (see [5, §3.3]).

Khovanskii’s procedure consists in sequential replacement of the Pfaffian equations (one at each step) by functional equations in accordance with a certain algorithm. Each step of the procedure consists in passing from a system of the form (2.7) to one or two similar systems with a smaller number of Pfaffian equations. Furthermore, the number of solutions of the initial system does not exceed the sum of the numbers of solutions of the new systems.

As a result of the procedure, we pass from a single functional-Pfaffian system to a collection of purely functional systems of a special form, for which the number of solutions is estimated with the help of an analog of Bezout’s theorem. Furthermore, the sum of the bounds for the numbers of solutions of the systems estimates the number of solutions of the initial functional-Pfaffian system.

In more detail, the algorithm is as follows. At each step of the procedure, one of the Pfaffian equations in the initial system is replaced either by a functional *contact equation* or by an equation with a *covering function*.

The contact equation has the form

$$(2.8) \quad \mathcal{F} := * \omega_1 \wedge \cdots \wedge \omega_{n+2s} \wedge dF_{f_1} \wedge \cdots \wedge dF_{f_l} \wedge dF_1 \wedge \cdots \wedge dF_k = \epsilon$$

(see [5, §3.3 and formula (47)]), where the ω_j are the left-hand sides of the Pfaffian equations,

$$F_{f_j} := y_j - f_j(x_j, \epsilon), \quad F_j := \epsilon_j - b_j,$$

and the number ϵ is small.

The equation with a covering function has the form

$$(2.9) \quad \rho := \rho_1 \cdots \rho_n \rho_\epsilon = \epsilon$$

(see [5, formula (64)]), where

$$\rho_\epsilon = \sum (r^2 - \epsilon_i^2)$$

for a certain small r , the functions ρ_j are taken from Table 3 in accordance with the type of the j th singular point, and ϵ is a small number.

TABLE 3. Covering functions

Type of the singular point	Covering function
S_0	$xy(r-x)(r-y)\tilde{\rho}$
S_μ	$xyzw(r-x)(r-y)(r-z)(r-w)P_\mu^2(z, \lambda)\tilde{\rho}$
D_μ^c	$x^2(r^2 - x^2)(r^2 - y^2)\tilde{\rho}$
D_μ^h	$y(r-y)(r^2 - x^2)Q_\mu(x, \lambda)\tilde{\rho}$

Explanations to Table 3. The table presents covering functions for various types of singular points. Here, r is a small parameter, $z := x^m$, and $w := y^n$. Finally, $\tilde{\rho}$ is a polynomial of the form

$$(2.10) \quad \tilde{\rho}(\lambda) := (r^2 - \lambda_1^2) \cdots (r^2 - \lambda_{\mu-1}^2)(r^2 - (c - \lambda_\mu)^2),$$

where c is a real number called the *localization parameter* of the family. (See [3] for more details.) The rest of the notation was introduced before (see Subsection 2.3).

Thus, two new systems arise, each containing one differential equation less than the original system. After the replacement of one equation, the next equation is replaced in a similar way. However, if at a certain step the system already contains an equation with a covering function, then afterwards the Pfaffian systems are replaced by contact equations only (see the plan).

Plan of Khovanskiĭ's procedure: Little circles denote the systems that we consider at each step, the arrows marked by $*$ denote the replacement of one of the Pfaffian equations by a contact equation, and ρ denotes the replacement of one of the Pfaffian equations by an equation with a covering function.

Khovanskiĭ's procedure stops when all Pfaffian equations in the system are replaced. Since the number of Pfaffian equations is equal to $n + 2s$, we obtain $n + 2s + 1 \leq 4n$ systems as a result.

Assertion 2.10 (See Theorem 7 and Corollary 4 in [5]). *After the passage from a single system to one or two new systems (as indicated above), the total number of solutions of all resulting systems majorizes the number of solutions of the original system.*

Remark 2.11. Let $\mathcal{G}_{\mathbf{f}} = a$ be a system obtained at a certain step of Khovanskii's procedure. Taking (2.8) and (2.9) into account, we easily show by induction that each component $\mathcal{G}_i$ is either a polynomial in the variables, the parameters, the functions f_j , and their derivatives of various orders, or a differential form with polynomial coefficients.

Remark 2.11 motivates the following definition.

Definition 2.12. Consider an equation in the system involved in Khovanskii's procedure. If the equation is Pfaffian, it has the form $\omega = 0$. The *K-degree* of the equation is the maximum degree of the coefficients of the form ω . If the equation is functional, then its *K-degree* is the degree of the equation as of a polynomial in the variables, the functions f_j , and their derivatives.

Remark 2.13. The K-degree of any Pfaffian equation is equal to the usual degree of the equation; the same is also true for any covering function. The K-degree of each component of the functional equation $F_f = a$, as well as the degree of the differential of the component, is equal to unity. The K-degree of each component of the equation for the parameters is also equal to unity, while the K-degree of the differential of the component is zero.

Applying Khovanskii's procedure to system (2.7), we obtain at most $4n$ functional systems, and we can estimate the K-degree of each system (see §3). The number of solutions of the initial system does not exceed the sum of the numbers of solutions of the final systems. Should the latter systems be only polynomial (with respect to the variables), to estimate the number of solutions it would have sufficed to apply the classical Bezout theorem.

It turns out that, under certain conditions on the right-hand side of the functional system, an analog of Bezout's theorem holds (see Theorem 2.15). This analog yields an estimate of the number of solutions for *typical Cartesian* mappings by the product of the K-degrees of all equations in the system.

Using Theorem 2.15, we estimate the number of solutions of the systems obtained by applying Khovanskii's procedure. After that, we multiply the result by the number of the systems, thus estimating the number of solutions of the initial system.

2.6. Bezout–Kaloshin theorem on the estimate for the number of solutions and its application. We formulate the Bezout–Kaloshin theorem not in full generality, but in the form in which we shall use it. The complete formulation can be found in [5, Theorem 12].

Let

$$\mathcal{G}_{\mathbf{f}} = a$$

be a certain system obtained by Khovanskii's procedure. Here, we have

$$\mathcal{G}_{\mathbf{f}} := (P^1, \dots, P^{\mathbf{m}}),$$

where the P^i are polynomials in the variables, the functions f_j , and their derivatives. We observe that if $i \neq j$, then the functions f_i and f_j depend on distinct collections of variables.

Let $f_i \in C_{\mathcal{I}_i}^p$, where $p > \mathbf{m} + 1$ and $\mathcal{I}_i$ denotes the collection of the variables corresponding to the function f_i .

Definition 2.14. For any positive integer d and small positive δ , the set

$$K^{d,\delta} := \{(a_1, \dots, a_{\mathbf{m}}) : 0 < |a_1| < \delta, 0 < |a_{j+1}| < |a_1 \cdots a_j|^d, j = 1, \dots, \mathbf{m} - 1\}$$

is called the *Kaloshin cone*.

Theorem 2.15 (Bezout–Kaloshin; see [5]). *There exist an open, everywhere dense set F_P in the function space*

$$C_{\mathcal{I}_1}^p \times \cdots \times C_{\mathcal{I}_n}^p$$

and an integer d such that for each $\mathbf{f} \in F_P$ there are a characteristic size $r_0 > 0$ and a number $\delta > 0$ for which the number of regular preimages of an arbitrary point $a = (a_1, \dots, a_{\mathbf{m}})$ in the Kaloshin cone $K^{d,\delta}$ admits the estimate

$$\#\{x \in U_r : \mathcal{G}_{\mathbf{f}} = a\} \leq \prod_{i=1}^{\mathbf{m}} \deg P^i,$$

where $0 < r \leq r_0$, and $U_r \subset \mathbb{R}^{\mathbf{m}}$ is the r -cube centered at the origin.

The following lemma, the proof of which is contained in §4 of the paper [5], allows us to apply the Bezout–Kaloshin theorem.

Lemma 2.16. *We may assume that the right-hand sides $a := (a_1, \dots, a_{\mathbf{m}})$ of the functional systems obtained by Khovanskii’s procedure belong to the Kaloshin cone in the statement of Theorem 2.15.*

As was already noted at the end of Subsection 2.5, after applying the Bezout–Kaloshin theorem it only remains to estimate the degrees of the systems obtained by Khovanskii’s procedure.

Now we can rigorously formulate what the term *genericity* means.

We order the systems obtained by Khovanskii’s procedure in an arbitrary way and denote by F_{P_j} the open, everywhere dense set in Theorem 2.15 for the j th system.

Definition 2.17. A collection $\mathbf{f} = (f_1, \dots, f_n)$ consisting of C^p -smooth functions, $p > \mathbf{m} + 1$, is said to be *generic* if $\mathbf{f} \in F_{P_j}$ for all $j = 1, \dots, \mathbf{m}$.

2.7. Differences in approaches. Il’yashenko and Yakovenko [4] investigated the number of the regular preimages of points in the image (see system (2.5) and its description on p. 564) without any restrictions on the points. Using Khovanskii’s method and Gabrielov’s theorem, they proved that $E(k)$ is finite for any positive integer k .

In [5], Kaloshin considered only preimages of special points of the image (see the condition on a_i in the statement of Theorem 2.15). This allowed him to prove an analog of Bezout’s theorem (see Theorem 2.15), which simplified the analysis the result of which was finding an explicit estimate for the number $E(k)$.

In the present paper, we use the same method. However, a more accurate combinatorial analysis yields an explicit estimate for $E(n, k)$. The main idea of the calculation is to control the growth of the degrees at each step of Khovanskii’s procedure and to give a common estimate for the degrees of the equations in the systems at each step.

The difference between our paper and Kaloshin’s paper [5] consists in the method of calculating the degrees at the penultimate step in the plan of the proof (the dotted arrow in Figure 2).

§3. UPPER ESTIMATE FOR THE NUMBER $E(n, k)$

In this section, we perform the computations needed for the proof of Theorem 1.8.

We recall our notation:

k is the number of parameters of the family (the codimension);

n is the number (with repetitions) of vertices in the polycycle;

s is the number of resonant saddles;

μ_j is the multiplicity of the j th singular point; and

$m := n + \sum \mu_j$ is the dimension of the space of parameters λ .

We consider the functional-Pfaffian system

$$(3.1) \quad \begin{cases} \Omega = 0 & \sim n + 2s \leq 3n, \\ F_f = a & \sim n, \\ F = b & \sim k + m. \end{cases}$$

Applying Khovanskii's procedure to this system, we obtain $4n$ functional systems as a result. By the Bezout–Kaloshin theorem, in order to estimate the number of solutions to such a system, it suffices to estimate the product of the K-degrees of the equations in the system.

For this, we first estimate the K-degrees of system (3.1), of the contact function for this system, and of the covering function. After that, using the results obtained, we estimate the growth of the K-degrees in Khovanskii's procedure (see Lemma 3.4). Finally, we use the latter estimate to obtain the required result.

Remark 3.1. The degree of the polynomial coefficients of the equations in the group $\Omega = 0$ of the system (2.7) (also see (3.1)) does not exceed $6\mu + 4$.

Proof. For instance, we consider the Pfaffian equation for the resonant saddle S_μ (see Table 1). Since $\deg P_\mu = 2\mu + 1$ (see (2.2)), the degree of the Pfaffian equation in the variables $x, y, z = x^m, w = y^n$, and λ is equal to $6\mu + 4$. The remaining equations are treated in a similar way. $\square$

Lemma 3.2. *The following assertions hold true for the functional-Pfaffian system (3.1):*

- 1) *the K-degree of the contact function $\mathcal{F}$ does not exceed $6k + 7n$, and*
- 2) *the K-degree of the differential $d\mathcal{F}$ does not exceed that of $\mathcal{F}$.*

Proof. 1) We recall the expression for the contact function (see Subsection 2.5):

$$\mathcal{F} = *\omega_1 \wedge \cdots \wedge \omega_{n+2s} \wedge dF_{f_1} \wedge \cdots \wedge dF_{f_t} \wedge dF_1 \wedge \cdots \wedge dF_k.$$

Here, the ω_j are the left-hand sides of the Pfaffian equations,

$$F_{f_j} \equiv y_j - f_j(x_j, \varepsilon) \quad \text{and} \quad F_j = \varepsilon_j - b_j.$$

The forms ω_j corresponding to the additional equations of resonant saddles have unit K-degree, and their number is $2s \leq 2n$. In accordance with Remark 3.1, the K-degree of the remaining forms ω_j is estimated by $6\mu_j + 4$, and their number is n . Observe that

$$\sum \mu_j \leq k.$$

The forms dF_{f_j} have K-degree 1, while the forms dF_j have K-degree equal to zero. Combining all estimates, we obtain the first assertion of the lemma.

2) The assertion concerning the K-degrees of the differential holds true not only for the contact function but also for *any* polynomial in the variables, the functions f_j , and their derivatives.

Indeed, differentiation of a polynomial with respect to any variable reduces the degree of the polynomial by one. Differentiation with respect to the variables x_j , ε , and λ can also yield a factor of the form $\partial f_j / \partial x_j$, for example:

$$F = f_1(x_1, y_1, \varepsilon), \quad \frac{\partial F}{\partial x_1} = \frac{\partial f_1(x_1, y_1, \varepsilon)}{\partial x_1}.$$

As a result, the K-degree of the polynomial is preserved. This completes the proof of the lemma. $\square$

Remark 3.3. We easily see that:

- (1) the degree of any covering function in Table 3 is estimated by the expression

$$(3.2) \quad \deg \rho_j \leq 6\mu + 10;$$

- (2) the degree of the covering function in Khovanskii's procedure (see equation (2.9)) is estimated by the expression

$$(3.3) \quad \deg \rho \leq 8k + 10n.$$

Proof. Inequality (3.2) follows from the relations

$$\deg P_\mu = 2\mu + 1, \quad \deg Q_\mu = 2\mu + 2, \quad \text{and} \quad \deg \tilde{\rho}_\mu = 2\mu$$

(see (2.2) and (2.10)). Inequality (3.3) follows from (3.2), (2.9), and $\sum \mu_j \leq k$. $\square$

Lemma 3.4 (On the exponential growth of degrees). *The K-degree of the contact equation and the covering function (if the latter occurs) at the j th step of Khovanskii's procedure does not exceed $2^j D$, where $D := 18k$.*

Proof. We observe that, by the inequality³ $n \leq k$, for any positive integer k we have the inequality

$$(3.4) \quad D \geq \max(6k + 7n, 8k + 10n).$$

For $j = 1$, the lemma follows from inequality (3.4), Lemma 3.2, and Remark 3.1.

Now, assuming that the claim holds true for all $j < j_0$, we prove it for j_0 . Observe that the sum of the K-degrees of the equations in the initial system does not exceed D . At the j_0 th step, we have two possibilities: the next form is replaced by either the covering function or the contact equation.

In the first case, once again, the claim follows from Remark 3.1. We consider the contact equation at the j_0 th step of Khovanskii's procedure. By construction, the K-degree of the equation does not exceed the sum of the K-degrees of the equations in the system at the preceding step, for which the inductive hypothesis holds true. This sum has the form

$$\sum_{j=1}^{j_0-1} 2^j D + D = 2^{j_0} D. \quad \square$$

Proof of Theorem 1.8. Let Π be the product of the K-degrees of the equations in each system obtained as a result of Khovanskii's procedure. Using the lemma on the exponential growth of degrees, we write the following chain of inequalities:

$$\Pi \leq \prod_{j=1}^{n+2s} 2^j D \leq \prod_{j=1}^{3n} 2^j D \leq 2^{5n^2+2n} D^{3n} \leq 2^{5n^2+17n} \cdot k^{3n}.$$

³Indeed, a saddle-node is a degeneration of codimension one, while coincidence of the separatrices of two saddles is a degeneration of the same codimension. Therefore, a polycycle with n vertices is a degeneration of codimension at least n .

In total, Khovanskii's procedure yields at most $4n$ systems. Applying the Bezout–Kaloshin theorem to them, we obtain the announced result: we have

$$E(n, k) \leq 2^{5n^2+20n} k^{3n}.$$

Theorem 1.8 is proved. □

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