EXTREMAL PROPERTIES OF SPHERICAL SEMIDESIGNS

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ABSTRACT. For every even $t \geq 2$ and every set of vectors $\Phi = \{\varphi_1, \dots, \varphi_m\}$ on the sphere S^{n-1} , the notion of the t -potential $P_t(\Phi) = \sum_{i,j=1}^m [\langle \varphi_i, \varphi_j \rangle]^t$ is introduced. It is proved that the minimum value of the t -potential is attained at the spherical semidesigns of order t and only at them. The first result of this type was obtained by B. B. Venkov. The result is extended to the case of sets Φ that do not lie on the sphere S^{n-1} . For the V. A. Yudin potentials $U_k(\Phi)$, $k = 2, 4, \dots, t$, it is shown that they attain the minimal value at the spherical semidesigns of order t and only at them.

§1. NOTATION AND PRELIMINARY INFORMATION

We use the dot product $\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$ of vectors $x, y \in \mathbb{R}^n$ and the norm $\|x\| = \sqrt{\langle x, x \rangle}$.

Let

$$S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$$

be the unit sphere in $\mathbb{R}^n$. In the sequel, we denote by t an even number, $t \geq 2$.

Definition 1. A set of vectors $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_m\} \subset S^{n-1}$ is called a *spherical semidesign of order t* (or, simply, a *spherical t -semidesign*) if there is a constant $A_t > 0$ such that the Waring identity

$$(1) \quad \sum_{i=1}^m [\langle \varphi_i, x \rangle]^t = A_t \|x\|^t, \quad x \in \mathbb{R}^n,$$

is valid.

The constant A_t can be found by successive applications of the Laplace operator Δ to both sides of equation (1). As a result, we obtain $A_t = cm$, where

$$(2) \quad c = \frac{(t-1)!!}{n(n+2) \cdots (n+t-2)}.$$

We also prove that a spherical semidesign of order t is a spherical semidesign of all orders $k = 2, 4, \dots, t$. The spherical p -designs were defined by Delsarte, Goethals, and Seidel in [4] in 1977. The same definition was used by many other authors [2, 4, 11, 12].

Definition 2. A set of points $\Phi = \{\varphi_1, \dots, \varphi_m\} \subset S^{n-1}$ is called a *spherical p -design* if the identity

$$(3) \quad \frac{1}{\sigma_n} \int_{S^{n-1}} Q(x) dS = \frac{1}{m} \sum_{i=1}^m Q(\varphi_i)$$

is valid for all polynomials $Q(x)$ of degree at most p .

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Here $\sigma_n = 2\pi^{n/2}/\Gamma(\frac{n}{2})$ is the area of the sphere S^{n-1} , the order p can be odd, and the polynomial $Q(x)$ is not necessarily homogeneous.

It is easy to establish a relationship between Definitions 1 and 2. If $\Phi = \{\varphi_1, \dots, \varphi_m\}$ is a spherical semidesign of even order t , then

$$\Phi_{2m} = \{\varphi_1, \dots, \varphi_m, -\varphi_1, \dots, -\varphi_m\}$$

is a spherical design of order $t+1$.

Conversely, let the set $\Phi_{2m} \subset S^{n-1}$ be symmetric, i.e., for each vector $\varphi \in \Phi_{2m}$ we have $-\varphi \in \Phi_{2m}$. Then Φ_{2m} can be represented in the form

$$\Phi_{2m} = \Phi \cup -\Phi, \quad \Phi \cap -\Phi = \emptyset.$$

If Φ_{2m} is a spherical $(t+1)$ -design, then Φ is a spherical semidesign of order t .

It should be noted that there exist nonsymmetric spherical designs. Interesting examples were presented in the papers [11, 12] by Yudin.

§2. THE INEQUALITY OF B. B. VENKOV

We consider an arbitrary set $\Phi = \{\varphi_1, \dots, \varphi_m\}$ of vectors on the sphere S^{n-1} . By the t -potential of this set we mean the quantity

$$(4) \quad P_t(\Phi) = \sum_{i=1}^m \sum_{j=1}^m [\langle \varphi_i, \varphi_j \rangle]^t.$$

For $t=2$, the potential $P_2(\Phi)$ is called a *frame*. This potential was introduced in the papers [7, 8].

Theorem 1 (B. B. Venkov [1]). *For every set $\Phi = \{\varphi_1, \dots, \varphi_m\} \subset S^{n-1}$, we have*

$$(5) \quad P_t(\Phi) \geq cm^2,$$

where the constant c is defined by (2). Equality occurs in (5) on the spherical t -semidesigns and only on them.

In [1], only spherical sets Φ were considered, and in this case, the minimum value of $P_t(\Phi)$ is attained at spherical t -designs and only at them. In the case of arbitrary sets Φ on the sphere, the minimal value of $P_t(\Phi)$ is attained at spherical semidesigns of order t . In the present paper, we generalize Theorem 1 to the case of nonunit vectors (see Theorem 2 below).

Example 1. Consider the set $\Phi = \{\varphi_1, \dots, \varphi_6\}$ of six vertices of an icosahedron inscribed in S^2 such that Φ contains no opposite vectors. We calculate the 4-potential $P_4(\Phi)$.

Since $\varphi_i \in S^2$, we obtain $\|\varphi_i\|^2 = 1$, $i = 1, \dots, 6$. By Haantjes's theorem [6], the following relations are valid for the vertices of the icosahedron:

$$\langle \varphi_i, \varphi_j \rangle = \pm \frac{1}{\sqrt{5}} \quad \text{for } i \neq j.$$

Therefore, $P_4(\Phi) = 6 \cdot 1 + 30 \cdot \frac{1}{25} = \frac{36}{5}$. At the same time, for $n=3$, $t=4$, and $m=6$, we have

$$cm^2 = \frac{1 \cdot 3}{3 \cdot 5} \cdot 6^2 = \frac{36}{5}.$$

Since $P_4(\Phi) = cm^2$, Φ is a 4-semidesign. The set $\Phi_{12} = \{\varphi_1, \dots, \varphi_6, -\varphi_1, \dots, -\varphi_6\}$ of all 12 vertices of the icosahedron is a spherical 5-design.

§3. NONSPHERICAL SEMIDESIGNS

Now we consider t -semidesigns in $\mathbb{R}^n$ containing vectors with different lengths. Let t be even, $t \geq 2$.

Definition 3. A set of nonzero vectors $\Phi = \{\varphi_1, \dots, \varphi_m\}$ in $\mathbb{R}^n$ is called a t -semidesign if there is $A_t > 0$ such that

$$(6) \quad \sum_{i=1}^m [\langle \varphi_i, x \rangle]^t = A_t \|x\|^t, \quad x \in \mathbb{R}^n.$$

Lemma 1. Let $t \geq 4$, and let Φ be a t -semidesign. Then the set of vectors

$$\theta_i = \|\varphi_i\|^{2/(t-2)} \varphi_i, \quad i = 1, \dots, m,$$

is a $(t-2)$ -semidesign with the constant

$$A_{t-2} = A_t \frac{t+n-2}{t-1}.$$

Proof. Applying the Laplace operator Δ to both sides of equation (6), we obtain

$$t(t-1) \sum_{i=1}^m [\langle \varphi_i, x \rangle]^{t-2} \|\varphi_i\|^2 = A_t t(t+n-2) \|x\|^{t-2}, \quad x \in \mathbb{R}^n.$$

This implies the claim. $\square$

We put $s = t/2$. Applying the operator Δ $s-1$ times, we arrive at the identity

$$\sum_{i=1}^m [\langle \varphi_i, x \rangle]^2 \|\varphi_i\|^{t-2} = A_2 \|x\|^2, \quad x \in \mathbb{R}^n.$$

Applying the operator Δ yet another time, we obtain

$$2 \sum_{i=1}^m \|\varphi_i\|^t = A_2 \cdot 2n.$$

Hence $A_2 = \mu/n$, where

$$(7) \quad \mu = \sum_{i=1}^m \|\varphi_i\|^t.$$

The recurrence relation in the lemma and the equation $A_2 = \mu/n$ imply that $A_t = c\mu$, where c is defined by (2).

§4. AN INEQUALITY FOR THE t -POTENTIAL

The t -potential $P_t(\Phi)$ is defined as above by formula (4).

Theorem 2. For every set Φ of vectors in the space $\mathbb{R}^n$, we have

$$(8) \quad P_t(\Phi) \geq c \left(\sum_{i=1}^m \|\varphi_i\|^t \right)^2,$$

where the constant c is defined by (2). Equality in (8) is attained at the t -semidesigns and only at them.

For the proof, we need a certain technique involving a dot product in the space of homogeneous polynomials $\mathcal{P}_{n,t}$ of degree t .

For polynomials $f, g \in \mathcal{P}_{n,t}$,

$$f(x) = \sum_{|i|=t} a(i)x^i, \quad g(x) = \sum_{|i|=t} b(i)x^i,$$

where $i = (i_1, \dots, i_n)$ is a vector with nonnegative integral components (multi-index), $|i| = i_1 + \dots + i_n$, and $x^i = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$, we define the dot product

$$[f, g] = \sum_{|i|=t} \frac{a(i)b(i)}{c(i)}, \quad \text{where } c(i) = \frac{t!}{i_1! i_2! \dots i_n!}.$$

Supplied with this dot product, $\mathcal{P}_{n,t}$ becomes a Hilbert space. For every $\varphi \in \mathbb{R}^n$, we consider the polynomial

$$\rho_\varphi(x) = [\langle \varphi, x \rangle]^t = [\varphi_1 x_1 + \dots + \varphi_n x_n]^t.$$

It is easy to verify that for every $\varphi \in \mathbb{R}^n$ and $f \in \mathcal{P}_{n,t}$ we have

$$(9) \quad [\rho_\varphi, f] = f(\varphi);$$

i.e., $\rho_\varphi(x)$ is the reproducing kernel of the space $\mathcal{P}_{n,t}$. We also consider the polynomial $\omega_t(x) = \|x\|^t$ (as above, t is an even number, $t \geq 2$). In [1] and [2], the following relation was proved:

$$(10) \quad [\omega_t, \omega_t] = \frac{1}{c},$$

where c is a constant of the form (2).

Proof. The proof of Theorem 2 follows the idea indicated by B. B. Venkov in [1]. Using the notation μ introduced in (7), we can represent the required inequality in the form $P_t(\Phi) \geq c\mu^2$. By the inequality

$$W := \left[\sum_{i=1}^m \rho_{\varphi_i} - c\mu\omega_t, \sum_{j=1}^m \rho_{\varphi_j} - c\mu\omega_t \right] \geq 0,$$

we obtain

$$W = W_1 - 2c\mu W_2 + c^2\mu^2 \cdot [\omega_t, \omega_t].$$

By (9), we have

$$\begin{aligned} W_1 &= \sum_{i,j=1}^m [\rho_{\varphi_i}, \rho_{\varphi_j}] = \sum_{i,j=1}^m \rho_{\varphi_j}(\varphi_i) = \sum_{i,j=1}^m [\langle \varphi_i, \varphi_j \rangle]^t = P_t(\Phi), \\ W_2 &= \sum_{i=1}^m [\rho_{\varphi_i}, \omega_t] = \sum_{i=1}^m \omega_t(\varphi_i) = \sum_{i=1}^m \|\varphi_i\|^t = \mu. \end{aligned}$$

Consequently, using (10), we see that

$$W = P_t(\Phi) - 2c\mu^2 + c^2\mu^2 \frac{1}{c} = P_t(\Phi) - c\mu^2.$$

Since $W \geq 0$, we have $P_t(\Phi) \geq c\mu^2$, which is equivalent to (8).

Now, we study when equality is attained.

Let the set Φ be a t -semidesign. In (6), we put $x = \varphi_j$ and use the fact that $A_t = c\mu$. We obtain

$$(11) \quad \sum_{i=1}^m [\langle \varphi_i, \varphi_j \rangle]^t = c\mu \|\varphi_j\|^t, \quad j = 1, \dots, m.$$

Performing summation in (11), we obtain $P_t(\Phi) = c\mu^2$.

Conversely, suppose equality in (8) occurs for some set Φ . Then the quantity W is equal to zero. The definition of W shows that

$$\sum_{i=1}^m \rho_{\varphi_i}(x) - c\mu\omega_t(x) = 0, \quad x \in \mathbb{R}^n;$$

i.e., relation (6) is satisfied with the constant $A_t = c\mu$. Therefore, Φ is a t -semidesign. The theorem is proved. $\square$

§5. HEHENBAUER POLYNOMIALS AND A THEOREM OF V. A. YUDIN

We fix a positive integer $n \geq 3$ and consider the weight

$$w_n(u) = (1 - u^2)^{(n-3)/2}, \quad u \in [-1, 1].$$

Let $G_k(u)$ be the Hehenbauer polynomials (ultraspherical polynomials) of degree k . They are orthogonal with weight $w_n(u)$,

$$(12) \quad \int_{-1}^1 G_k(u)G_s(u)w_n(u) du = 0, \quad k \neq s$$

(see [9]). For $k = 0$, we have $G_0(u) \equiv 1$. For $k \geq 1$, the polynomials $G_k(u)$ contain only even or only odd powers of u depending on whether k is even or odd. The addition formula

$$(13) \quad G_k(\langle x, y \rangle) = \sum_{j=1}^{N(k)} Y_j(x)Y_j(y), \quad x, y \in S^{n-1}$$

is valid, where the Y_j , $j = 1, \dots, N(k)$, are spherical functions of order k (see [10]).

Let $\Phi^* = \{\varphi_1^*, \dots, \varphi_m^*\} \subset S^{n-1}$ be a spherical t -semidesign, where t is even, $t \geq 2$.

In [12], Yudin proved a criterion for the sphericity of designs in terms of Hehenbauer polynomials.

Since our definition of a spherical t -semidesign differs from that in [12], it is necessary to change the statement of the theorem.

Theorem 3 (V. A. Yudin). *A set $\Phi^* \in S^{n-1}$ is a spherical t -semidesign if and only if*

$$(14) \quad \sum_{i=1}^m G_k(\langle \varphi_i^*, x \rangle) = 0, \quad x \in S^{n-1}, \quad k = 2, 4, \dots, t.$$

§6. YUDIN POTENTIALS

For every set $\Phi = \{\varphi_1, \dots, \varphi_m\}$ on the sphere S^{n-1} , we define the Yudin potential

$$U_k(\Phi) = \sum_{i=1}^m \sum_{j=1}^m G_k(\langle \varphi_i, \varphi_j \rangle), \quad k = 1, 2, \dots, t.$$

The motivation for the term “potential” is the fact that the expressions $U_k(\Phi)$ are nonnegative. Indeed, by the addition formula (13), we have

$$U_k(\Phi) = \sum_{s=1}^{N(k)} \sum_{i=1}^m Y_s(\varphi_i) \sum_{j=1}^m Y_s(\varphi_j) = \sum_{s=1}^{N(k)} \left[\sum_{i=1}^m Y_s(\varphi_i) \right]^2 \geq 0.$$

Thus, for every set $\Phi \subset S^{n-1}$, the Yudin potentials $U_k(\Phi)$ are nonnegative for all $k = 1, 2, \dots, t$.

In the case where $n = 3$, so that the Hehenbauer polynomials become Legendre polynomials, the following theorem was stated in [13]. In the proof of the “if” part, it is convenient to use Venkov’s theorem.

Theorem 4. *Let t be an even number with $t \geq 2$. A set $\Phi^* = \{\varphi_1^*, \dots, \varphi_m^*\}$ on the sphere S^{n-1} is a spherical t -semidesign if and only if*

$$(15) \quad U_k(\Phi^*) = 0, \quad k = 2, 4, \dots, t.$$

Proof. The “only if” part. For a spherical t -semidesign Φ^* , the Yudin theorem implies equations (14). We substitute $x = \varphi_j^*$ in these equations and sum over $j = 1, \dots, m$. We obtain

$$(16) \quad \sum_{i=1}^m \sum_{j=1}^m G_k(\langle \varphi_i^*, \varphi_j^* \rangle) = 0, \quad k = 2, 4, \dots, t,$$

which is equivalent to (15).

The “if” part. We use Theorem 1. Assume that relations (15) are valid. We calculate the usual t -potential

$$P_t(\Phi^*) = \sum_{i,j=1}^m [\langle \varphi_i^*, \varphi_j^* \rangle]^t$$

and verify that it is equal to cm^2 , where c is defined by formula (2).

The polynomial u^t can be expanded in even Hehenbauer polynomials as follows: $u^t = \sum_{k=0,2,\dots,t} d(k)G_k(u)$. We substitute $u = \langle \varphi_i^*, \varphi_j^* \rangle$ and sum over $i, j \in 1 : m$. Then

$$P_t(\Phi^*) = \sum_{k=0,2,\dots,t} d(k) \sum_{i,j=1}^m G_k(\langle \varphi_i^*, \varphi_j^* \rangle).$$

By (16) and the identity $G_0(u) \equiv 1$, we arrive at the formula

$$(17) \quad P_t(\Phi^*) = d(0)m^2.$$

To calculate $d(0)$, we multiply u^t by G_0 , using the dot product $(,)$ in the space $L_2(-1, 1)$ with weight $w_n(u)$,

$$d(0) = \frac{(u^t, G_0)}{(G_0, G_0)} = \frac{\int_{-1}^1 u^t w_n(u) du}{\int_{-1}^1 w_n(u) du} =: \frac{I(t)}{I(0)}.$$

For even t , the integral $I(t)$ is calculated as follows:

$$I(t) = 2 \int_0^1 u^t w_n(u) du = \int_0^1 x^{(t-1)/2} (1-x)^{(n-3)/2} dx = \frac{\Gamma(\frac{t+1}{2}) \Gamma(\frac{n-1}{2})}{\Gamma(\frac{t+n}{2})}.$$

For $t = 0$, we obtain the integral $I(0)$. As a result, we have

$$d(0) = \frac{\Gamma(\frac{t+1}{2}) \Gamma(\frac{n}{2})}{\sqrt{\pi} \Gamma(\frac{t+n}{2})} = \frac{(t-1)!!}{n(n+2) \cdots (n+t-2)} = c,$$

where c is as in (2). From (17), we conclude that $P_t(\Phi^*) = cm^2$, and, by Venkov’s theorem, Φ^* is a spherical t -semidesign. The theorem is proved. $\square$

Remark 1. For a symmetric spherical t -semidesign of the form

$$\Phi_{2m}^* = \{\varphi_1^*, \dots, \varphi_m^*, -\varphi_1^*, \dots, -\varphi_m^*\},$$

the “only if” part of the theorem can be refined, namely, we have

$$U_k(\Phi_{2m}^*) = 0 \text{ for all } k = 1, 2, 3, \dots, t.$$

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