INVERTIBILITY THRESHOLD FOR H^∞ TRACE ALGEBRAS, AND EFFECTIVE MATRIX INVERSIONS

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*Dedicated to the memory of M. Sh. Birman,
from whom both of us learned a lot (and not only in mathematics)*

ABSTRACT. Given δ , $0 < \delta < 1$, a Blaschke sequence $\sigma = \{\lambda_j\}$ is constructed such that every function $f \in H^\infty$ satisfying $\delta < \delta_f = \inf_{\lambda \in \sigma} |f(\lambda)| \leq \|f\|_\infty \leq 1$ is invertible in the trace algebra $H^\infty|\sigma$ (with a norm estimate of the inverse depending on δ_f only), but there exists f with $\delta = \delta_f \leq \|f\|_\infty \leq 1$ that is not. As an application, a counterexample to a stronger form of the Bourgain–Tzafriri restricted invertibility conjecture for bounded operators is exhibited, where an “orthogonal (or unconditional) basis” is replaced by a “summation block orthogonal basis”.

§1. INTRODUCTION

The paper deals with numerical control of inverses (condition numbers) for functions $T = f(A)$ of large matrices in terms of the lower spectral parameter

$$\delta = \delta(T) = \min |\lambda_j(T)|.$$

Precisely, our problem is the following. Given a sequence $\sigma = \{\lambda_j\}$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane, we consider all normalized matrices (or Hilbert space operators) A , $\|A\| \leq 1$, such that $\sigma(A) \subset \sigma$ (counting multiplicities), and look for a numerical function $c(\delta) = c(\delta, \sigma)$ bounding the inverses:

$$\|T^{-1}\| \leq c(\delta)$$

for all $T = f(A)$ with $\delta \leq |\lambda_j(T)| \leq \|T\| \leq 1$, where the $\lambda_j(T)$ mean eigenvalues of $T = f(A)$. The best possible upper bound $c(\delta)$ is denoted by $c_1(\delta) = c_1(\delta, \sigma)$,

$$c_1(\delta, \sigma) = \sup \left\{ \|T^{-1}\| : T = f(A), \delta \leq |\lambda_j(T)| \leq \|T\| \leq 1, \sigma(A) \subset \sigma, \|A\| \leq 1 \right\}.$$

Here f can be a polynomial (if A is a finite matrix) or an H^∞ -function (if A is a Hilbert space contraction). Recall that

$$H^\infty = \left\{ f : f \text{ is holomorphic on } \mathbb{D} \text{ and } \|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| < \infty \right\}.$$

Since $\delta \mapsto c_1(\delta, \sigma)$, $0 < \delta < 1$, is a monotone decreasing function, we can define a *critical constant* (or, an *invertibility threshold*) $\delta_1 = \delta_1(\sigma)$, $0 \leq \delta_1 \leq 1$, by the following

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properties:

$$\begin{aligned} 0 < \delta < \delta_1 &\implies c_1(\delta) = \infty, \\ \delta_1 < \delta \leq 1 &\implies c_1(\delta) < \infty. \end{aligned}$$

The number δ_1 can be viewed as a threshold of bounded invertibility or as a threshold for an operator algebra to be *closed under inversion*: operators T in our collection with “*disperse*” spectral data (i.e., $\inf_j |\lambda_j(T)| < \delta_1$, $\|T\| = 1$) are, in general, not invertible, whereas those with “*flat*” spectral data (i.e., $\delta_1 < \delta \leq |\lambda_j(T)| \leq \|T\| \leq 1$) are invertible.

The principal result of this paper is a construction of a Blaschke sequence σ with a value of the critical constant $\delta_1(\sigma) = \delta_1$, $0 \leq \delta_1 \leq 1$ given in advance (see §2 below).

The case where $\delta_1 = 0$ was considered in [GMN]; moreover, the same paper contains necessary and sufficient conditions for $\delta_1(\sigma) = 0$, which reduces to the so-called weak (Carleson) embedding property (WEP). See the statement of the result at the end of this Introduction.

It should be mentioned that, strictly speaking, for a function algebra A on a set σ , the property to be *closed under inversion* (i.e., the property $f \in A$, $\inf_{z \in \sigma} |f(z)| > 0 \implies 1/f \in A$) does not imply that $\delta_1(\sigma, A) = 0$ (this fact was mentioned already in [GMN]; the constants $c_1(\sigma, A)$ and $\delta_1(\sigma, A)$ are defined for A in a similar way). Indeed, for an arbitrary Blaschke sequence $\sigma = \{\lambda_j\}$, the trace algebra $A = C_a(\mathbb{D})|_\sigma$ of the disk algebra $C_a(\mathbb{D}) = H^\infty \cap C(\mathbb{D})$, is always *closed under inversion*, whereas $c_1(\delta, C_a(\mathbb{D})|_\sigma) = c_1(\delta, H^\infty|_\sigma)$ for every δ , $0 < \delta < 1$, and hence $\delta_1(\sigma, C_a(\mathbb{D})|_\sigma) = \delta_1(\sigma, H^\infty|_\sigma)$, but the algebra $H^\infty|_\sigma$ may fail to be closed under inversion (i.e., possibly $\delta_1(\sigma, C_a(\mathbb{D})|_\sigma) > 0$). These properties were shown in [GMN].

The constant $c_1(\delta, \sigma)$ has the meaning of “the best estimate for the worst case”, bounding inverse matrices in terms of the lower spectral parameter δ . Moreover, we can describe it in two other ways, at least in the case of the “simple spectrum” (the points λ_j of the sequence σ are pairwise different). First, it is the optimal upper bound for inverses in the *trace algebra*

$$H^\infty|_\sigma = \{a : \sigma \rightarrow \mathbb{C} : \text{there exists } f \in H^\infty \text{ such that } a = f|_\sigma\}$$

endowed with the trace norm $\|a\| = \inf\{\|f\|_\infty : a = f|_\sigma\}$. Examples of such algebras with a given bounded invertibility threshold δ_1 (§2 below) are, probably, of interest for the H^∞ interpolation theory.

Second, in the definition of $c_1(\delta, \sigma)$, we can restrict ourselves simply to one (“the worst”) contraction A and the algebra generated by the H^∞ -functions of it. This is the so-called *model contraction* M_B , which can be defined as follows. Given a Blaschke product $B = B_\sigma$,

$$B_\sigma = \prod_{j \geq 1} b_{\lambda_j},$$

where $b_\lambda = \frac{\lambda - z}{1 - \bar{\lambda}z} \cdot \frac{|\lambda|}{\lambda}$, $\lambda \in \mathbb{D}$, and $\sigma = \{\lambda_j\}$, $\sum_j (1 - |\lambda_j|) < \infty$ (the Blaschke condition), we set

$$M_B^* f = \frac{f - f(0)}{z}, \quad f \in K_B,$$

where $K_B = H^2 \ominus BH^2$ (the orthogonal complement of BH^2 in H^2) and H^2 stands for the standard Hardy space on the disk,

$$H^2 = \left\{ f = \sum_{k \geq 0} a_k z^k : \sum_{k \geq 0} |a_k|^2 = \|f\|_2^2 < \infty \right\}.$$

It is well known (and easy to verify, see [Nik1, Nik2]) that

$$M_B K_B \subset K_B, \quad \|M_B\| = 1, \quad \text{and} \quad \sigma(M_B) = \text{clos}\{\lambda_j : j = 1, 2, \dots\}.$$

Moreover, $\|M_B^{-1}\| = 1/B(0)$. It is also known that for every matrix A with $\|A\| \leq 1$ and $\sigma(A) \subset \sigma$, we have $\|f(A)\| \leq \|f(M_B)\|$ for every function f . It follows that the question about invertibility and norm control for inverses can be reduced merely to functions $f(M_B)$ of the model operator, and the inverse $f(M_B)^{-1}$, if it exists, is again an H^∞ -function of M_B .

The above discussion readily implies the following.

1) If the set $\{\lambda \in \sigma : \delta \leq |\lambda| \leq \delta'\}$ is infinite for some $0 < \delta \leq \delta' < 1$, then $c_1(\delta, \sigma) = \infty$.

2) If σ is a sequence tending to the unit circle (i.e., the set $\{\lambda \in \sigma : |\lambda| \leq \delta\}$ is finite for every $\delta < 1$) and $\sum_{\lambda \in \sigma} (1 - |\lambda|) = \infty$, then $c_1(\delta, \sigma) = \infty$ for every δ , $0 < \delta < 1$.

These properties show that, in fact, *the Blaschke condition is necessary in order that our problems* (to find or to estimate $c_1(\delta, \sigma)$ and $\delta_1(\sigma)$) *be nontrivial*. In what follows we always assume this condition (if otherwise is not stated explicitly). Now, we can give the following expression for $c_1(\delta, \sigma)$.

Lemma 1. *Let σ be a Blaschke subset of the unit disk $\mathbb{D}$. Then*

$$c_1(\delta, \sigma) = c_1(\delta, H^\infty|\sigma) = c_1(\delta, H^\infty/BH^\infty)$$

for every δ , $0 < \delta < 1$, where $B = B_\sigma$ and

$$\begin{aligned} c_1(\delta, H^\infty/BH^\infty) &= \sup \left\{ \left\| \frac{1}{f} \right\|_{H^\infty/BH^\infty} : \|f\|_\infty \leq 1, \delta \leq \|f(\lambda)\| \text{ for } \lambda \in \sigma \right\} \\ &= \sup \left\{ \inf \{ \|g\|_\infty : gf + hB = 1 \} : \delta \leq \|f(\lambda)\| \leq \|f\|_\infty \leq 1 \text{ for } \lambda \in \sigma \right\}, \end{aligned}$$

and $\|h\|_{H^\infty/BH^\infty}$ denotes $\inf \{ \|g\|_\infty : g(\lambda) = h(\lambda) \text{ for } \lambda \in \sigma \}$.

Proof. For every matrix A , $\|A\| \leq 1$, and $f \in H^\infty$, the von Neumann inequality entails

$$\|f(A)\| \leq \|f\|_\infty.$$

Since $B(A) = 0$, $B = B_\sigma$, we see that if $\sigma(A) \subset \sigma$, then

$$\|f(A)\| \leq \inf_{g \in H^\infty} \|f + Bg\|_\infty = \|f\|_{H^\infty/BH^\infty}.$$

This implies $f(A)^{-1} = h(A)$ and $\|f(A)^{-1}\| \leq \|h\|_{H^\infty}$ for every solution h of the equation $fh + Bk = 1$, whence $\|f(A)^{-1}\| \leq \left\| \frac{1}{f} \right\|_{H^\infty/BH^\infty}$. Thus, $c_1(\delta, \sigma) \leq c_1(\delta, H^\infty/BH^\infty)$.

On the other hand, there exists an “extremal operator” (matrix) for which equality occurs in the above calculus inequality. Indeed, if A is the “model operator” M_B mentioned above, then $\|h(M_B)\| = \|h\|_{H^\infty/BH^\infty}$ for every $h \in H^\infty$ (Sarason’s commutant lifting theorem, see, e.g., [Nik1] or [Nik2]). Hence, $c_1(\delta, \sigma) \geq c_1(\delta, H^\infty/BH^\infty)$. $\square$

Finally, we quote the principal result of [GMN].

Theorem 2 ([GMN]). *Let $\sigma = \{\lambda_j\}$ be a Blaschke sequence in the disk $\mathbb{D}$. The statements below are equivalent.*

- 1) $\delta_1(H^\infty|\sigma) = 0$.
- 2) *The following weak embedding property is fulfilled: for every $\varepsilon > 0$ there exists C such that*

$$\sum_{j \geq 1} \frac{(1 - |\lambda_j|^2)(1 - |z|^2)}{|1 - \bar{\lambda}_j z|^2} \leq C$$

for every $z \in \mathbb{D} \setminus \bigcup_{\lambda \in \sigma} \{\zeta : |b_\lambda(\zeta)| < \varepsilon\}$.

- 3) *For every $\varepsilon > 0$, there exists η such that $|B(z)| \leq \eta$ implies $\inf_{\lambda \in \sigma} |b_\lambda(z)| \leq \varepsilon$; here B is the corresponding Blaschke product $B = \prod_{\lambda \in \sigma} b_\lambda$.*

Moreover, if $\eta(\varepsilon) = \max\{\eta\}$ over all η admitted in 3), then

$$\frac{1}{\eta(\delta)} \leq c_1(\delta, H^\infty|\sigma) \leq \frac{a}{\eta(\delta/3)^2} \log \frac{1}{\eta(\delta/3)}$$

for every δ , $0 < \delta < 1$; $a > 0$ is a numerical constant.

The paper is organized as follows. §2 contains our principal result: for any δ with $0 < \delta < 1$, there exists a Blaschke product $B = B_\sigma$ such that $\delta_1(\sigma, H^\infty|\sigma) = \delta$. We also exhibit an upper estimate for $c_1(\delta, H^\infty|\sigma)$ with $\delta_1 < \delta \leq 1$. Since the problem (and our result) on the invertibility threshold are conformally invariant, we shall change the variable and work (in §2) in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ instead of the unit disk $\mathbb{D}$.

In §3, we use the above result in order to give a counterexample to a stronger form of the so-called (Bourgain–Tzafriri) restricted invertibility conjecture. This conjecture claims (see [CCLV] and comments in §3) that, for every unconditional normalized basic sequence $\{x_j\}_{j \in J}$ in a Hilbert space H and every bounded operator $T : H \rightarrow H$ with $\inf_{j \in J} \|Tx_j\| > 0$, there exists a partition $J = \bigcup_{i=1}^r J_i$ such that all restrictions $T|_{H_{J_i}}$, $i = 1, \dots, r$, are left invertible; here $H_{J'} = \operatorname{span}\{x_j : j \in J'\}$ for every $J' \subset J$. The conjecture is still open (June 2010). A stronger form (which is disproved in §3) claims the same property but for all summation basis sequences $\{x_j\}$.

§2. ALGEBRAS $H^\infty|\sigma$ WITH A GIVEN CONSTANT δ_1

We start with some geometrical considerations. In this section, the symbol b_λ always means the Blaschke factor with the zero λ in the upper half-plane, i.e.,

$$b_\lambda(z) = \frac{z - \lambda}{z - \bar{\lambda}} \cdot \frac{|1 + \lambda^2|}{1 + \lambda^2}.$$

Lemma 3. *The rectangle*

$$\left\{ z : \frac{\sqrt{1 + \varepsilon^2}}{\sqrt{1 + \varepsilon^2} + \sqrt{2}\varepsilon} \leq \frac{\operatorname{Im} z}{\operatorname{Im} \lambda} \leq \frac{\sqrt{1 + \varepsilon^2}}{\sqrt{1 + \varepsilon^2} - \sqrt{2}\varepsilon}, \quad \frac{|\operatorname{Re}(z - \lambda)|}{\operatorname{Im} \lambda} \leq \frac{\sqrt{2}\varepsilon}{\sqrt{1 - \varepsilon^2}} \right\}$$

is inscribed into the circle $\{z : |b_\lambda(z)| \leq \varepsilon\}$.

Proof. Put

$$a = \frac{\operatorname{Re}(z - \lambda)}{\operatorname{Im} \lambda} \quad \text{and} \quad b = \frac{\operatorname{Im} z}{\operatorname{Im} \lambda},$$

then

$$|b_\lambda(z)|^2 = \left| \frac{z - \lambda}{z - \bar{\lambda}} \right|^2 = \frac{a^2 + (b - 1)^2}{a^2 + (b + 1)^2}.$$

We must check that the vertices of the rectangle are on the circle mentioned above, i.e., we need to check that

$$\frac{a^2 + (b - 1)^2}{a^2 + (b + 1)^2} = \varepsilon^2$$

if

$$a = \pm \frac{\sqrt{2}\varepsilon}{\sqrt{1 - \varepsilon^2}}, \quad b = \frac{\sqrt{1 + \varepsilon^2}}{\sqrt{1 + \varepsilon^2} \pm \sqrt{2}\varepsilon}.$$

We verify the required identity in the form $(a^2 + b^2 + 1)(1 - \varepsilon^2) = 2b(1 + \varepsilon^2)$:

$$\begin{aligned} (a^2 + b^2 + 1)(1 - \varepsilon^2) &= \left(\frac{2\varepsilon^2}{1 - \varepsilon^2} + \frac{1 + \varepsilon^2}{(\sqrt{1 + \varepsilon^2} \pm \sqrt{2}\varepsilon)^2} + 1 \right) (1 - \varepsilon^2) \\ &= 2\varepsilon^2 + (1 + \varepsilon^2) \frac{\sqrt{1 + \varepsilon^2} \mp \sqrt{2}\varepsilon}{\sqrt{1 + \varepsilon^2} \pm \sqrt{2}\varepsilon} + 1 - \varepsilon^2 \\ &= (1 + \varepsilon^2) \frac{2\sqrt{1 + \varepsilon^2}}{\sqrt{1 + \varepsilon^2} \pm \sqrt{2}\varepsilon} = 2b(1 + \varepsilon^2). \end{aligned} \quad \square$$

Now, we use the Frostman shifts of an inner function Θ :

$$\Theta_c \stackrel{\text{def}}{=} \frac{\Theta + c}{1 + \bar{c}\Theta},$$

which are Blaschke products for almost all c with $|c| < 1$. In some cases it is easy to check that Θ_c is a Blaschke product *for all* $c \neq 0$. For example, this is the case for $\Theta = e^{iaz}$, $a > 0$. Indeed, since the inner function Θ_c is analytic in a neighborhood of any real point, it can only have a singular factor with a mass at infinity. But there is no such factor because $\lim_{y \rightarrow +\infty} \Theta_c(iy) = c \neq 0$. For more details, see, e.g., [Gar] or [Nik1].

Lemma 4. *Let z_k be the zeros of the Blaschke product*

$$B_{\alpha, \gamma} = \frac{e^{\pi i \gamma z} + e^{-\pi \alpha}}{1 + e^{\pi(i\gamma z - \alpha)}} = \prod_{k=-\infty}^{\infty} b_{z_k},$$

i.e., $z_k = (2k + 1 + i\alpha)/\gamma$, $z_k \in \mathbb{Z}$. Then the strip

$$S_{\alpha, \gamma} = \left\{ z : \frac{\alpha\sqrt{1 + \alpha^2}}{\sqrt{1 + \alpha^2} + 1} \leq \gamma \operatorname{Im} z \leq \frac{\alpha\sqrt{1 + \alpha^2}}{\sqrt{1 + \alpha^2} - 1} \right\}$$

lies in the set

$$\bigcup_{k=-\infty}^{\infty} \{z : |b_{z_k}(z)| < \varepsilon\}$$

provided $\varepsilon > 1/\sqrt{1 + 2\alpha^2}$.

Proof. We apply Lemma 3 with $\lambda = z_k$ and $\varepsilon = 1/\sqrt{1 + 2\alpha^2}$. Then the sides of the rectangle of Lemma 3 are

$$\frac{\sqrt{1 + \varepsilon^2}}{\sqrt{1 + \varepsilon^2} \pm \sqrt{2}\varepsilon} = \frac{\sqrt{1 + \alpha^2}}{\sqrt{1 + \alpha^2} \pm 1} \quad \text{and} \quad \frac{\sqrt{2}\varepsilon}{\sqrt{1 - \varepsilon^2}} = \frac{1}{\alpha},$$

i.e., this rectangle looks like this:

$$\left\{ z : \frac{\sqrt{1 + \alpha^2}}{\sqrt{1 + \alpha^2} + 1} \leq \frac{\gamma}{\alpha} \operatorname{Im} z \leq \frac{\sqrt{1 + \alpha^2}}{\sqrt{1 + \alpha^2} - 1}, |\operatorname{Re}(z - z_k)| \leq \frac{1}{\gamma} \right\}.$$

It is clear that the union of these rectangles gives precisely the required strip. $\square$

Remark 1. Observe that the set

$$S_{\alpha, \gamma} \setminus \bigcup_{k=-\infty}^{\infty} \left\{ z : |b_{z_k}(z)| < \frac{1}{\sqrt{1 + 2\alpha^2}} \right\}$$

consists of isolated points

$$\frac{1}{\gamma} \left(2m + \frac{\alpha\sqrt{1 + \alpha^2}}{\sqrt{1 + \alpha^2} \pm 1} i \right)$$

on the upper and lower boundaries of the strip $S_{\alpha,\gamma}$, and that the distance from any such point to the set of zeros $\{z_k\}$ is equal to $1/\sqrt{1+2\alpha^2}$.

Lemma 5. *The Blaschke product $B = \prod_{n=0}^{\infty} B_{\alpha,\beta^n\rho}$ converges for all α, β, ρ such that $\alpha > 0$, $0 < \beta < 1$, $\rho > 0$. If*

$$\beta = \frac{\sqrt{1+\alpha^2} - 1}{\sqrt{1+\alpha^2} + 1},$$

then the half-plane

$$\Pi_{\alpha,\rho} = \left\{ z : \operatorname{Im} z \geq \frac{\alpha\sqrt{1+\alpha^2}}{\rho(\sqrt{1+\alpha^2} + 1)} \right\}$$

lies in the set

$$\bigcup_{\lambda \in \sigma(B)} \{z : |b_\lambda(z)| < \varepsilon\}$$

for $\varepsilon > 1/\sqrt{1+2\alpha^2}$.

FIGURE 1

(Figure 1 illustrates zeros of B and four circles $|b_\lambda(z)| = \frac{1}{\sqrt{1+2\alpha^2}}$ for the zeros $\lambda = \frac{\pm 1 + i\alpha}{\rho}$ and $\lambda = \frac{\pm 1 + i\alpha}{\beta\rho}$.)

Proof. The following estimate implies that B converges:

$$1 - B_{\alpha,\beta^n\rho}(i) = 1 - \frac{e^{-\pi\beta^n\rho} + e^{-\pi\alpha}}{1 + e^{-\pi(\alpha+\beta^n\rho)}} = \frac{(1 - e^{-\pi\beta^n\rho})(1 - e^{-\pi\alpha})}{1 + e^{-\pi(\alpha+\beta^n\rho)}} \leq (1 - e^{-\pi\alpha})\pi\beta^n\rho.$$

It remains to note that for $\beta = (\sqrt{1+\alpha^2} - 1)/(\sqrt{1+\alpha^2} + 1)$ and $\gamma_n = \beta^n\rho$, the upper boundary of the strip of Lemma 4 for $\gamma = \gamma_{n-1}$ coincides with the lower boundary of the strip for $\gamma = \gamma_n$. Therefore, the union of these strips coincides with the required half-plane. $\square$

Remark 2. The set of points

$$v_n = \frac{1}{\rho\beta^n} \frac{\alpha\sqrt{1+\alpha^2}}{\sqrt{1+\alpha^2} + 1} i$$

on the imaginary axis (the points of intersection of four corresponding circles, as in Figure 1) is included into

$$\Pi_{\alpha,\rho} \setminus \bigcup_{\lambda \in \sigma(B)} \left\{ z : |b_\lambda(z)| < \frac{1}{\sqrt{1+2\alpha^2}} \right\}.$$

Every point v_n has four nearest zeros of B , namely, $(i\alpha \pm 1)/(\rho\beta^n)$ and $(i\alpha \pm 1)/(\rho\beta^{n-1})$ with the pseudohyperbolic distance from each of them equal precisely to $1/\sqrt{1+2\alpha^2}$.

Recall that the pseudohyperbolic distance between two points $z, w \in \mathbb{C}_+$ is defined by

$$|b_w(z)| = \left| \frac{z-w}{z-\bar{w}} \right|,$$

while between two points $z, w \in \mathbb{D}$ by

$$|b_w(z)| = \left| \frac{z-w}{1-\bar{w}z} \right|.$$

Lemma 6. *For the Blaschke product $B = \prod_{n=0}^{\infty} B_{\alpha,\beta^n\rho}$ we have*

$$|B(x+iy)| \geq \exp \left\{ -\frac{(1+e^{-\pi\alpha})\pi\rho y}{(e^{-\pi\rho y} - e^{-\pi\alpha})(1-\beta)} \right\}$$

in the strip $0 < y < \frac{\alpha}{\rho}$. In the complementary half-plane $y > \frac{\alpha}{\rho}$, we have the following upper estimate:

$$|B(x+iy)| \leq \exp \left\{ -\frac{\log(\cosh \pi\alpha) \cdot \log \frac{\rho y}{\alpha}}{\log \frac{1}{\beta}} \right\}.$$

Proof. For the product $B_{\alpha,\gamma}$, we have

$$|B_{\alpha,\gamma}(x+iy)|^2 = \frac{e^{-2\pi\gamma y} + e^{-2\pi\alpha} + 2e^{-\pi(\gamma y+\alpha)} \cos \pi\gamma x}{1 + e^{-2\pi(\gamma y+\alpha)} + 2e^{-\pi(\gamma y+\alpha)} \cos \pi\gamma x},$$

whence

$$\left| \frac{e^{-\pi\gamma y} - e^{-\pi\alpha}}{1 - e^{-\pi(\gamma y+\alpha)}} \right| \leq |B_{\alpha,\gamma}(x+iy)| \leq \frac{e^{-\pi\gamma y} + e^{-\pi\alpha}}{1 + e^{-\pi(\gamma y+\alpha)}}.$$

Now, we deduce an estimate from below, assuming $0 < y < \frac{\alpha}{\rho}$:

$$\begin{aligned} \log \frac{1}{|B(x+iy)|} &= \sum_{n=0}^{\infty} \log \frac{1}{|B_{\alpha,\beta^n\rho}(x+iy)|} \leq \sum_{n=0}^{\infty} \log \frac{1 + e^{-\pi(\beta^n\rho y+\alpha)}}{e^{-\pi\beta^n\rho y} - e^{-\pi\alpha}} \\ &\leq \sum_{n=0}^{\infty} \left(\frac{1 - e^{-\pi(\beta^n\rho y+\alpha)}}{e^{-\pi\beta^n\rho y} - e^{-\pi\alpha}} - 1 \right) = \sum_{n=0}^{\infty} \frac{(1 + e^{-\pi\alpha})(1 - e^{-\pi\beta^n\rho y})}{e^{-\pi\beta^n\rho y} - e^{-\pi\alpha}} \\ &\leq (1 + e^{-\pi\alpha}) \sum_{n=0}^{\infty} \frac{\pi\beta^n\rho y}{e^{-\pi\beta^n\rho y} - e^{-\pi\alpha}} = \frac{(1 + e^{-\pi\alpha})\pi\rho y}{(e^{-\pi\rho y} - e^{-\pi\alpha})(1-\beta)}, \end{aligned}$$

as claimed. To estimate $|B(x+iy)|$ from above, we replace B by a finite product $\prod_{0 \leq n \leq N} B_{\alpha,\beta^n\rho}$, where

$$N \stackrel{\text{def}}{=} \frac{\log \frac{\rho y}{\alpha}}{\log \frac{1}{\beta}}.$$

The number of such indices n is $[N] + 1 > N$. Since for these n we have

$$\beta^n\rho y \geq \alpha,$$

for each factor we get the estimate

$$|B_{\alpha,\beta^n\rho}(x+iy)| \leq \frac{e^{-\pi\beta^n\rho y} + e^{-\pi\alpha}}{1 + e^{-\pi(\beta^n\rho y+\alpha)}} \leq \frac{2e^{-\pi\alpha}}{1 + e^{-2\pi\alpha}} = \frac{1}{\cosh \pi\alpha}.$$

Therefore, the entire product satisfies

$$|B(x + iy)| \leq (\cosh \pi \alpha)^{-N}. \quad \square$$

From now on, we fix

$$\beta = \frac{\sqrt{1 + \alpha^2} - 1}{\sqrt{1 + \alpha^2} + 1},$$

and consider $B = \prod_{n=0}^{\infty} B_{\alpha, \beta^n \rho}$ corresponding to this β .

Theorem 7. *We have*

$$\delta_1(H^\infty / BH^\infty) = \frac{1}{\sqrt{1 + 2\alpha^2}}.$$

Moreover, there exists an absolute constant c and another constant $C = C(\delta_1)$ such that

$$c_1(\delta) \leq \max \left\{ \frac{c}{(\delta - \delta_1)^2} \log \frac{1}{\delta - \delta_1}, C \right\}$$

for every δ with $\delta_1 < \delta \leq 1$.

The proof of this theorem is contained in two following lemmas, where δ_1 denotes the number $\frac{1}{\sqrt{1 + 2\alpha^2}}$. After proving these lemmas, we can conclude that $\delta_1 = \delta_1(H^\infty / BH^\infty)$.

Lemma 8. *Let $\delta > \delta_1$. Then*

$$c_1(\delta) \leq \max \left\{ \frac{c}{(\delta - \delta_1)^2} \log \frac{1}{\delta - \delta_1}, C \right\}$$

for an absolute constant c and another constant $C = C(\delta_1)$.

Proof. First we check that the function $|f(z)| + |B(z)|$ can be separated away from zero by some constant η depending on α and δ only. By Lemma 6, in the strip

$$0 < y\rho \leq \frac{\alpha\sqrt{1 + \alpha^2}}{\sqrt{1 + \alpha^2} + 1}$$

we have the estimate

$$|B(x + iy)| \geq \exp \left\{ -\frac{\alpha\sqrt{1 + \alpha^2} (e^{\pi\alpha} + 1)}{2 \left(e^{\frac{\pi\alpha}{\sqrt{1 + \alpha^2} + 1}} + 1 \right)} \right\}.$$

Now we check that $|f(z)|$ is separated away from zero in the half-plane

$$y\rho \geq \frac{\alpha\sqrt{1 + \alpha^2}}{\sqrt{1 + \alpha^2} + 1}.$$

Fix any $\varepsilon, \delta > \varepsilon > \delta_1$. By Lemma 5,

$$\left\{ z : \operatorname{Im} z \geq \frac{\alpha\sqrt{1 + \alpha^2}}{\rho(\sqrt{1 + \alpha^2} + 1)} \right\} \subset \bigcup_{\lambda \in \sigma(B)} \{z : |b_\lambda(z)| < \varepsilon\},$$

so it suffices to verify that f is separated away from zero on each disk $\{z : |b_\lambda(z)| < \varepsilon\}$, $\lambda \in \sigma(B)$, uniformly with respect to λ .

By the Schwarz lemma, we have

$$\left| \frac{f(z) - f(\lambda)}{1 - \overline{f(\lambda)}f(z)} \right| \leq |b_\lambda(z)|,$$

i.e., for z in the disk $\{z : |b_\lambda(z)| < \varepsilon\}$ we have

$$\left| \frac{f(z) - f(\lambda)}{1 - \overline{f(\lambda)}f(z)} \right| < \varepsilon.$$

Rewriting the inequality

$$\left| \frac{a+b}{1+\bar{a}b} \right| \leq \frac{|a|+|b|}{1+|a||b|}$$

(in fact, this is the triangle inequality for the hyperbolic metric for the points a , b , and 0) in the form

$$|b| \geq \frac{|a| - \left| \frac{a+b}{1+\bar{a}b} \right|}{1 - |a| \left| \frac{a+b}{1+\bar{a}b} \right|}$$

with $a = f(\lambda)$ and $b = -f(z)$, we get

$$|f(z)| \geq \frac{\delta - \varepsilon}{1 - \delta\varepsilon} > \frac{\delta - \delta_1}{1 - \delta\delta_1}.$$

Therefore, in the entire half-plane we have

$$|f(z)| + |B(z)| \geq \eta,$$

where

$$(2.1) \quad \eta = \min \left\{ \exp \left[- \frac{\alpha\sqrt{1+\alpha^2}(e^{\pi\alpha}+1)}{2(e^{\frac{\pi\alpha}{\sqrt{1+\alpha^2}+1}}+1)} \right], \frac{\delta - \delta_1}{1 - \delta\delta_1} \right\}.$$

Finally, from the Carleson corona theorem (see, e.g., [Nik2]) we know that the Bezout equation $fh + Bg = 1$ admits a solution h with the norm estimate

$$\|h\|_\infty \leq \frac{c}{\eta^2} \log \frac{1}{\eta},$$

which means that

$$c_1(\delta) \leq \frac{c}{\eta^2} \log \frac{1}{\eta}.$$

Recall that $\delta_1 = \frac{1}{\sqrt{1+2\alpha^2}}$. If the first term in (2.1) is less than the second, then

$$\eta = \eta(\delta_1) = \exp \left[- \frac{\alpha\sqrt{1+\alpha^2}(e^{\pi\alpha}+1)}{2(e^{\frac{\pi\alpha}{\sqrt{1+\alpha^2}+1}}+1)} \right],$$

and we can put

$$C(\delta_1) = \frac{c}{\eta^2(\delta_1)} \log \frac{1}{\eta(\delta_1)}.$$

If the second term is smaller, we have

$$c_1(\delta) \leq \frac{c}{(\delta - \delta_1)^2} \log \frac{1}{\delta - \delta_1}.$$

□

Lemma 9. *Let $\delta \leq \delta_1$. Then $c_1(\delta) = +\infty$.*

Proof. Consider a sequence of points v_n as in Remark 2 and put $f_n = b_{v_n}$. In Remark 2 it was mentioned that

$$|b_{v_n}(\lambda)| \geq \frac{1}{\sqrt{1+2\alpha^2}} = \delta_1 \geq \delta, \quad \lambda \in \sigma(B).$$

We would like to estimate from below the H^∞ -norm of a solution g_n of the Bezout equation $g_nf_n + Bh_n = 1$. Since

$$\|g_n\|_\infty = \|1 - Bh_n\|_\infty = \|h_n - \bar{B}\|_\infty \geq \|h\|_\infty - 1$$

and

$$\|h_n\|_\infty \geq |h_n(v_n)| = \frac{1}{|B(v_n)|},$$

we can use the estimate of Lemma 6 to obtain

$$\|g_n\|_\infty \rightarrow \infty,$$

which yields $c_1(\delta) = +\infty$. $\square$

Remark 3. If we take an arbitrary δ , $\delta < \delta_1$, and use the above construction, it will be easy to construct a function f with the properties $\|f\|_\infty \leq 1$, $|f(\lambda)| \geq \delta$ for every $\lambda \in \sigma$ that is not invertible in H^∞/BH^∞ , so that the Bezout equation $gf + Bh = 1$ admits no bounded solution g, h . Indeed, for f it suffices to take a product of the factors b_{v_n} with sufficiently rare subsequence of zeros v_n to ensure the condition $|f(\lambda)| \geq \delta$. However, for the Blaschke product B from Theorem 7, we do not know whether such a function exists in the case where $\delta = \delta_1$. In order to guarantee this property, i.e., to have a noninvertible element f of the algebra H^∞/BH^∞ with $\delta_1 \leq f(\lambda) \leq \|f\|_\infty \leq 1$ ($\lambda \in \sigma(B)$), we need a Blaschke product B with a more sophisticated zero set, which will be exhibited in the following theorem.

Theorem 10. *For an arbitrary fixed number δ_1 in $(0, 1)$, there exists a Blaschke product B such that*

- 1) $c_1(\delta, H^\infty/BH^\infty) < \infty$ for every δ with $\delta_1 < \delta \leq 1$;
- 2) there exists a function f such that $\delta_1 \leq |f(\lambda)| \leq \|f\|_\infty \leq 1$ for $\lambda \in \sigma(B)$, but $\frac{1}{f} \notin H^\infty/BH^\infty$.

Proof. Step 1. We start with an arbitrary bounded monotone increasing sequence of positive numbers α_n with $\alpha = \lim \alpha_n$, $\delta_1 \stackrel{\text{def}}{=} \frac{1}{\sqrt{1+2\alpha^2}}$. Our Blaschke product B will be of the form

$$B(z) = \prod_{n=1}^{\infty} \prod_{m=0}^{m_n-1} B_{\alpha_n, \beta_n^m \rho_n}(z),$$

where

$$\beta_n = \frac{\sqrt{1 + \alpha_n^2} - 1}{\sqrt{1 + \alpha_n^2} + 1}$$

and

$$\rho_{n+1} = \rho_n \beta_n^{m_n} \frac{\sqrt{1 + \alpha_n^2} + 1}{\alpha_n \sqrt{1 + \alpha_n^2}} \cdot \frac{\alpha_{n+1} \sqrt{1 + \alpha_{n+1}^2}}{\sqrt{1 + \alpha_{n+1}^2} + 1}.$$

The initial value $\rho_1 = \rho$ can be taken arbitrarily. The required noninvertible function f will be the Blaschke product

$$f(z) = \prod_{n=0}^{\infty} b_{v_n}(z),$$

where

$$v_n = \frac{1}{\rho_{n+1}} \cdot \frac{\alpha_{n+1} \sqrt{1 + \alpha_{n+1}^2}}{\sqrt{1 + \alpha_{n+1}^2} + 1} i = \frac{1}{\rho_n \beta_n^{m_n-1}} \cdot \frac{\alpha_n \sqrt{1 + \alpha_n^2}}{\sqrt{1 + \alpha_n^2} - 1} i,$$

i.e., we put the root v_n on the common boundary of the last strip determined by α_n and the first strip determined by α_{n+1} . So, the only parameters at our disposal are the numbers m_n of strips of equal hyperbolic width, or, in other words, the distances between the neighboring roots v_n . We subordinate these distances to the following condition:

$$(2.2) \quad \left| \frac{v_l - v_k}{v_l + v_k} \right| \geq \left(\delta \sqrt{1 + 2\alpha_{l+1}^2} \right)^{2^{-k}}.$$

If we take any zero λ of the Blaschke product B with $\operatorname{Im} v_{n-1} < \operatorname{Im} \lambda < \operatorname{Im} v_n$, then

$$\begin{aligned}
 |f(\lambda)| &= \prod_{k=0}^{\infty} |b_{v_k}(\lambda)| = \prod_{k=0}^{n-2} |b_{v_k}(\lambda)| \cdot |b_{v_{n-1}}(\lambda)b_{v_n}(\lambda)| \cdot \prod_{k=n+1}^{\infty} |b_{v_k}(\lambda)| \\
 &\geq \prod_{k=0}^{n-2} |b_{v_k}(v_{n-1})| \cdot |b_{v_{n-1}}(\lambda)b_{v_n}(\lambda)| \cdot \prod_{k=n+1}^{\infty} |b_{v_k}(v_n)| \\
 &\geq \prod_{k=0}^{n-2} \left(\delta \sqrt{1 + 2\alpha_n^2} \right)^{2^{-k}} \cdot |b_{v_{n-1}}(\lambda)b_{v_n}(\lambda)| \cdot \prod_{k=n+1}^{\infty} \left(\delta \sqrt{1 + 2\alpha_{n+1}^2} \right)^{2^{-k}} \\
 &\geq \left(\delta \sqrt{1 + 2\alpha_n^2} \right)^{1-3 \cdot 2^{-n}} \cdot |b_{v_{n-1}}(\lambda)b_{v_n}(\lambda)|.
 \end{aligned}$$

Thus, if we guarantee the estimate

$$(2.3) \quad |b_{v_{n-1}}(\lambda)b_{v_n}(\lambda)| \geq \frac{(\delta \sqrt{1 + 2\alpha_n^2})^{3 \cdot 2^{-n}}}{\sqrt{1 + 2\alpha_n^2}}$$

for any root λ in the strip between v_{n-1} and v_n , we immediately obtain the required estimate for f : $|f(\lambda)| \geq \delta$.

Step 2. We shall construct the roots v_n by induction. Assume that all v_k with $k < n$ are already fixed and we need to choose v_n . First, we need to take v_n sufficiently far from the preceding roots in order to satisfy (2.2) for $k = n$ and all $l < n$ as well as for $l = n$ and all $k < n$.

Note that it suffices to check condition (2.3) only for the roots λ of B that have positive real part and are nearest to the imaginary axis, i.e., for $\lambda = (1 + i\alpha_n)/\rho_n \beta_n^m$, because the hyperbolic distance between any other λ with positive real part and any v_k is strictly larger, and the consideration of λ with negative real part can be omitted due to symmetry.

Now, we reduce the problem to the case of two roots of B only, namely, to the roots nearest to one of the zeroes of f , either v_{n-1} or v_n , i.e., for $m = 0$ and $m = m_n - 1$.

For the root $\lambda = (1 + i\alpha_n)/\rho_n$, we have

$$|b_{v_{n-1}}(\lambda)| = \frac{1}{\sqrt{1 + 2\alpha_n^2}},$$

and hence, (2.3) turns into

$$(2.4) \quad |b_{v_n}(\lambda)| \geq (\delta \sqrt{1 + 2\alpha_n^2})^{3 \cdot 2^{-n}}.$$

For the root $\lambda = (1 + i\alpha_n)/\rho_n \beta_n^{m_n-1}$, we have

$$|b_{v_n}(\lambda)| = \frac{1}{\sqrt{1 + 2\alpha_n^2}},$$

and (2.3) turns into

$$(2.5) \quad |b_{v_{n-1}}(\lambda)| \geq (\delta \sqrt{1 + 2\alpha_n^2})^{3 \cdot 2^{-n}}.$$

In fact, estimates (2.4) and (2.5) follow from (2.2), but we do not want to enter into additional calculations and simply add (2.4)–(2.5) to the list of requirements for the inductive choice of m_n . For this, we check that (2.4)–(2.5) are fulfilled, as well as (2.2), provided m_n is sufficiently large.

Now, we consider the behavior of the following function $\phi_a(t)$:

$$\phi_a(t) \stackrel{\text{def}}{=} |b_{ia}((1 + i\alpha_n)t)|^2 = \frac{(a - \alpha_n t)^2 + t^2}{(a + \alpha_n t)^2 + t^2}.$$

Since

$$\frac{\phi'_a(t)}{\phi_a(t)} = \frac{4\alpha_n a[t^2(1 + \alpha_n^2) - a^2]}{[t^2(1 + \alpha_n^2) + a^2]^2 - 4\alpha_n^2 a^2 t^2},$$

the function ϕ_a decreases monotonously from 1 to its minimal value when t changes from 0 to $a/\sqrt{1 + \alpha_n^2}$, and then increases tending again to 1 as $t \rightarrow \infty$. Consider the product of two such functions $\phi_a(t)\phi_b(t)$ with sufficiently large b/a ; clearly, this product has two local minima: first of them tends to $a/\sqrt{1 + \alpha_n^2}$ decreasing monotonously as $b \rightarrow \infty$, and the second tends to $b/\sqrt{1 + \alpha_n^2}$ increasing monotonously as $a \rightarrow 0$.

We shall apply these arguments to our requirement (2.3). To this end, we set

$$a = |v_{n-1}| = \frac{\alpha_n \sqrt{1 + \alpha_n^2}}{\rho_n(\sqrt{1 + \alpha_n^2} + 1)}, \quad b = |v_n| = \frac{\alpha_n \sqrt{1 + \alpha_n^2}}{\rho_n \beta_n^{m_n}(\sqrt{1 + \alpha_n^2} + 1)},$$

and

$$\lambda = \frac{1 + i\alpha_n}{\rho_n \beta_n^m},$$

and compare the values of our function at the points $t = t_m = 1/\rho_n \beta_n^m$, $m = 0, \dots, m_n - 1$, in order to guarantee that the minimal value is attained either for $m = 0$ or for $m = m_n - 1$, where, by our assumption, either (2.4) or (2.5) is fulfilled. For this, we note that

$$t_0 = \frac{1}{\rho_n} > \frac{\alpha_n}{\rho_n(\sqrt{1 + \alpha_n^2} + 1)} = \frac{a}{\sqrt{1 + \alpha_n^2}};$$

therefore, if m_n is sufficiently large, then the point of minimum is located to the left of t_0 and the function $\phi_a(t)\phi_b(t)$ is monotone increasing at t_0 . Symmetrically, we have

$$t_{m_n-1} = \frac{1}{\rho_n \beta_n^{m_n-1}} < \frac{\alpha_n}{\rho_n \beta_n^{m_n-1}(\sqrt{1 + \alpha_n^2} - 1)} = \frac{b}{\sqrt{1 + \alpha_n^2}};$$

therefore, if m_n is sufficiently large, then the point of minimum is located to the right of t_{m_n-1} and the function $\phi_a(t)\phi_b(t)$ is monotone decreasing at t_{m_n-1} . It follows that condition (2.3) is fulfilled for all sufficiently large m_n .

Thus, we have proved that conditions (2.2) and (2.3) are fulfilled if the sequence m_n increases sufficiently fast. Therefore, the construction of the Blaschke product B and a function f such that $\|f\|_\infty = 1$ and $|f(\lambda)| \geq \delta_1$ for all λ , $\lambda \in \sigma(B)$, is complete. The function f represents a noninvertible element of H^∞/BH^∞ . Indeed, assuming that f is invertible in H^∞/BH^∞ , i.e., that there exist two H^∞ -functions g and h such that $fg + Bh = 1$, we arrive at a contradiction, because

$$\lim_{n \rightarrow \infty} (f(v_n)g(v_n) + B(v_n)h(v_n)) = 0.$$

So, we have finished the proof of the second statement of the theorem.

Step 3. To complete the proof, we need to check that any other function f satisfying the conditions $\|f\|_\infty = 1$ and $|f(\lambda)| \geq \delta$ for all λ , $\lambda \in \sigma(B)$, and for arbitrary δ , $\delta > \delta_1$, represents an invertible element of H^∞/BH^∞ .

Fix such a δ and such a function f . We must check that

$$(2.6) \quad \inf_{\operatorname{Im} z > 0} (|f(z)| + |B(z)|) > 0.$$

As in the proof of Theorem 7, we take an arbitrary ε , $\delta_1 < \varepsilon < \delta$, and split the upper half-plane into two parts: the set

$$\Pi_\varepsilon \stackrel{\text{def}}{=} \bigcup_{\lambda \in \sigma(B)} \{z : |b_\lambda(z)| \leq \varepsilon\}$$

and its complement Π_ε^c . We shall check that f is bounded away from zero on Π_ε and so is B on Π_ε^c .

If $|b_\lambda(z)| \leq \varepsilon$, then, by the Schwarz lemma,

$$\left| \frac{f(z) - f(\lambda)}{1 - \overline{f(\lambda)}f(z)} \right| \leq |b_\lambda(z)| \leq \varepsilon.$$

Now, as in the proof of Lemma 8, we can use the triangle inequality in the form

$$|b| \geq \frac{|a| - \left| \frac{a+b}{1+\bar{a}b} \right|}{1 - |a| \left| \frac{a+b}{1+\bar{a}b} \right|}$$

with $a = f(\lambda)$ and $b = -f(z)$ to get

$$|f(z)| \geq \frac{\delta - \varepsilon}{1 - \delta\varepsilon}$$

for arbitrary z in Π_ε .

On the complement Π_ε^c , we estimate $|B(z)|$ by splitting the product B into two sub-products $B = B'B''$. Namely, we fix a number N such that $\delta\sqrt{1+2\alpha_n^2} > 1$ for $n \geq N$ and put

$$B' = \prod_{n=1}^{N-1} \prod_{m=0}^{m_n-1} B_{\alpha_n, \beta_n^m \rho_n}, \quad B'' = \prod_{n=N}^{\infty} \prod_{m=0}^{m_n-1} B_{\alpha_n, \beta_n^m \rho_n}.$$

Note that the first product is an interpolating Blaschke product. Indeed, all $B_{\alpha, \gamma}$ are interpolating, because the relation

$$\left(\frac{B}{b_\lambda} \right)(\lambda) = 2i \cdot \operatorname{Im} \lambda \cdot \frac{dB}{dz}(\lambda)$$

shows that

$$B_k(z_k) = \frac{\pi\alpha}{\sinh \pi\alpha},$$

where $B_k = B_{\alpha, \gamma}/b_{z_k}$, $z_k = (2k+1+i\alpha)/\gamma$. Therefore, the zeros of B' form a finite union of interpolating sets. Since these sets are uniformly separated, the entire product B' is also interpolating. Using the general form of the Carleson condition (see, e.g., [Nik1, Nik2])

$$|B'(z)| \geq c \inf_{\lambda \in \sigma(B')} |b_\lambda(z)|,$$

we get $|B'(z)| \geq c\varepsilon$ in Π_ε^c . As to the second product B'' , we can use the estimate of Lemma 6 for $\rho = \rho_N$. The estimate of Lemma 6 was obtained for the strips of equal hyperbolic width, but in our situation the width of the strips decreases, because α_n is monotone increasing. This means that the hyperbolic distance from any point below the first strip to the corresponding zero of $B_{\alpha_n, \gamma_n, m}$ is strictly larger than a similar distance in the case where all α_n are equal to α . Therefore, below the first strip, each factor $|B_{\alpha_n, \gamma_n, m}|$ is strictly larger than in the equidistant case. The entire half-plane

$$\operatorname{Im} z > \operatorname{Im} v_{N-1} = \frac{1}{\rho_N} \cdot \frac{\alpha_N \sqrt{1 + \alpha_N^2}}{\sqrt{1 + \alpha_N^2} + 1}$$

is in the set Π_ε by Lemma 5; therefore, B'' is separated away from zero on the set Π_ε^c , whence we see that B is separated away from zero on Π_ε^c . So, condition (2.6) is fulfilled, which means that the f is invertible in the algebra H^∞/BH^∞ , i.e., $c(\delta) < \infty$ for any δ with $\delta > 1/\sqrt{1+2\alpha^2}$. $\square$

§3. A VERSION OF THE RESTRICTED INVERTIBILITY CONJECTURE

3.1. Bourgain–Tzafriri’s restricted invertibility theorem. The following statement is known as *Bourgain–Tzafriri’s restricted invertibility theorem*.

Theorem 11 ([BTz]). *Whatever be a bounded operator T on a Hilbert space H , for any orthogonal basis $\{e_j\}_{j \in \mathbb{N}}$ satisfying $\inf_j \frac{\|Te_j\|}{\|e_j\|} > 0$ there exists a subset $I \subset \mathbb{N}$ of positive upper density,*

$$0 < \overline{\text{dens}}(I) \stackrel{\text{def}}{=} \limsup_{n \rightarrow \infty} \frac{|I \cap \{1, 2, \dots, n\}|}{n},$$

such that the restriction $T|_{H_I}$ is left invertible:

$$\inf\{\|Tx\| : x \in H_I, \|x\| = 1\} > 0,$$

where $H_I = \text{span}\{e_j : j \in I\}$.

See also [SS] for a generalization and a simpler proof of a matrix version of the Bourgain–Tzafriri theorem.

The following conjecture is often quoted as Bourgain–Tzafriri’s *restricted invertibility conjecture* (**RIC**) (it seems although that these authors never actually stated this as a conjecture). It is known that the famous Kadison–Singer conjecture on pure states on C^* -algebras (see [KS, CT]) implies **RIC**: in [CCLV] it was proved that if the Kadison–Singer conjecture is valid, then so is **RIC**. For more details about these conjectures, we refer the reader to the papers mentioned above, as well as to the WEB page [ARCC]. Both conjectures are still open (June 2010).

3.2. Restricted invertibility conjecture (RIC).

Conjecture. *For every bounded operator T on a Hilbert space H and every orthogonal basis $\{e_j\}_{j \in \mathbb{N}}$ satisfying $\inf_j \frac{\|Te_j\|}{\|e_j\|} > 0$, there exists a finite partition $\bigcup_{s=1}^r I_s = \mathbb{N}$ such that all restrictions $T|_{H_{I_s}}$ are left invertible.*

It is easily seen that the **RIC** is equivalent to saying that the same quality partitions exist for every bounded T and every *unconditional* basis in H (in place of an orthogonal one). Having watched no much progress concerning this conjecture during the last 20 years, we can try to approach the truth by treating some stronger conjectures first. Namely, we may replace an “unconditional basis” by a “Schauder basis”, and even by a “summation basis”. We denote the corresponding conjectures by **B-RIC** and **SB-RIC**, respectively.

Recall that a *summation basis* relative to a (triangular) matrix $V = \{v_{nj}\}$ of scalars v_{nj} is a sequence $\{e_j\}_{j \in \mathbb{N}}$ in H such that for every $x \in H$ there exists a unique sequence of scalars $\{a_j\}_{j \in \mathbb{N}}$ satisfying $x = (V) \sum_{j \geq 1} a_j e_j$, which means the following:

- $v_{nj} = 0$ for $j > n$;
- $x = \lim_{n \rightarrow \infty} \sum_{j=1}^n v_{nj} a_j e_j = x$ (norm convergence).

Clearly, **SB-RIC** $\implies$ **B-RIC** $\implies$ **RIC**. Here, we present a counterexample to the **SB-RIC**.

3.3. Counterexample.

Theorem 12. *Given δ , $0 < \delta < 1$, in a Hilbert space H there exists a sequence $\{e_j\}_{j \in \mathbb{N}}$ with the following properties.*

- 1) $\{e_j\}_{j \in \mathbb{N}}$ is a summation basis (relative to a triangular matrix).
- 2) $\{e_j\}_{j \in \mathbb{N}}$ is block orthogonal: there exists a monotone increasing sequence of integers n_s such that $H_{[n_s, n_{s+1})} \perp H_{[n_t, n_{t+1})}$ for every $s \neq t$, where $H_{[n_s, n_{s+1})} = \text{span}\{e_j : n_s \leq j < n_{s+1}\}$.

- 3) *There exists a bounded linear operator $A : H \rightarrow H$ satisfying $\|A\| \leq 1$, $Ae_j = \lambda_j(A)e_j$, $\delta \leq |\lambda_j(A)| = \frac{\|Ae_j\|}{\|e_j\|} \leq \|A\| \leq 1$ ($j \in \mathbb{N}$), and such that for every finite partition $\bigcup_{s=1}^r I_s = \mathbb{N}$ there is a restriction $A|_{H_{I_s}}$ ($1 \leq s \leq r$) that is NOT left invertible.*
- 4) *Every bounded operator $T : H \rightarrow H$ satisfying $Te_j = \lambda_j(T)e_j$ ($j \in \mathbb{N}$) and $1 \geq \|T\| \geq \inf_j |\lambda_j(T)| > \delta$ is invertible.*

Proof. We use our main construction from Theorem 10, replacing the upper half-plane by the unit disk. Namely, given δ (δ_1 in Theorem 10), $0 < \delta < 1$, there exists a Blaschke sequence $\sigma = \{z_j\}$ of distinct points in $\mathbb{D}$ such that

- a) for every $f \in H^\infty$ with $\delta < \inf_j |f(z_j)|$ and $\|f\|_\infty \leq 1$, we have $\frac{1}{f} \in H^\infty|\sigma$;
- b) there is an H^∞ -function g such that $\delta \leq |g(z_j)| \leq \|g\|_\infty \leq 1$ but $\frac{1}{g} \notin H^\infty|\sigma$.

Now, we interpret a) and b) in terms of the model operator M_B^* and the reproducing kernels $x_j = \frac{(1-|z_j|^2)^{1/2}}{1-\bar{z}_j z}$.

First, by (a scalar version of) the commutant lifting theorem, the operator T occurring in statement 4) of the theorem is of the form $T = f(M_B)^*$, where f possesses all properties listed in a): $f \in H^\infty$, $\delta < \inf_j |f(z_j)|$, and $\|f\|_\infty \leq 1$. Hence, $\frac{1}{f} \in H^\infty|\sigma$, which means that T is invertible (and proves statement 4)).

Second, in order to verify statements 1)–3), we restate b) in terms of the same model operator. Namely, for an operator

$$T = g(M_B)^*$$

with a function g as in b) we have $\|T\| \leq 1$, $Tx_j = \lambda_j(T)x_j$, $\delta \leq |\lambda_j(T)| = |g(z_j)| \leq \|T\| \leq 1$ ($j \in \mathbb{N}$), and $\inf\{\|Tx\| : x \in K_B, \|x\| = 1\} = 0$.

Notice that should we restrict ourselves to properties 1) and 3) only, we would simply set $A = T = f(M_B)^*$. Property 1) follows from the fact that the sequence $\{x_j\}_{j \in \mathbb{N}}$ corresponding to a Blaschke sequence $\{z_j\}_{j \in \mathbb{N}}$ is a summation basis, see [Nik1, p. 194]. In order to check 3), suppose that there exists a partition $\bigcup_{s=1}^r \sigma_s = \mathbb{N}$ such that all restrictions $T|_{H_{\sigma_s}}$ are left invertible:

$$0 < \inf\{\|Tx\| : x \in H_{\sigma_s}, \|x\| = 1\}$$

for every s , $1 \leq s \leq r$. We show that this leads to a contradiction as follows. Let B_s be the Blaschke product whose zero sequence is σ_s . Since the restriction $T|_{H_{\sigma_s}} = g(M_{B_s})^*$ is, in fact, invertible, there exist functions $f_s, h_s \in H^\infty$ such that $gf_s + B_sh_s = 1$. Hence, $B \cdot \prod_{s=1}^r h_s = \prod_{s=1}^r (1 - gf_s) = 1 - gF$, where $F \in H^\infty$. This implies that the operator $T = g(M_B)^*$ is invertible, which contradicts the construction of T . Therefore, a counterexample satisfying 1), 3), and 4) of the theorem is constructed.

In order to satisfy (2), we modify the above construction in the following way. Let $N \in \mathbb{N}$, and let $T_N = T|_{H_N}$ be the restriction of T to

$$H_N = \text{span}\{x_j : 1 \leq j \leq N\}.$$

Then

$$\begin{aligned} \|T_N\| &\leq 1, \quad T_N x_j = \lambda_j(T_N) x_j, \\ \delta &\leq |\lambda_j(T_N)| \leq \|T\| \leq 1 \quad (1 \leq j \leq N), \end{aligned}$$

and

$$\lim_{N \rightarrow \infty} \inf\{\|T_N x\| : x \in H_N, \|x\| = 1\} = 0.$$

Now, we set

$$A = \sum_{N \geq 1} \oplus T_N,$$

defining A coordinatewise on the l^2 -orthogonal sum

$$H = \sum_{N \geq 1} \oplus H_N.$$

In particular, this means that the point spectrum of A is $\{\lambda_j(T)\}_{j \geq 1}$, but each eigenvalue is repeated infinitely many times.

Next, we denote by $\{e_j\}_{j \in \mathbb{N}}$ the sequence of eigenvectors of A ordered naturally: if $f_k = (\delta_{N,k})_{N \geq 1} \in l^2$, then

$$\{e_j\}_{j \in \mathbb{N}} = (x_1 f_1, x_1 f_2, x_2 f_2, \dots, x_1 f_N, x_2 f_N, \dots, x_N f_N, x_1 f_{N+1}, \dots),$$

or, more formally,

$$e_j = x_m f_N, \quad \text{where } N = \left\lceil \sqrt{2j} + \frac{1}{2} \right\rceil, \quad m = j - \frac{N(N-1)}{2}.$$

We show that $\{e_j\}_{j \in \mathbb{N}}$ has properties 1) and 2), and A enjoys all requirements of 3).

Indeed, properties 1) and 2) for $\{e_j\}$ easily follow from property 1) for $\{x_j\}$ and the block orthogonal nature of $\{e_j\}$.

In order to prove 3), suppose the contrary; then there exists a finite partition $\bigcup_{s=1}^r I_s = \mathbb{N}$ such that all restrictions $A|_{H_{I_s}}$ ($1 \leq s \leq r$) are left invertible. Taking the intersection of $\bigcup_{s=1}^r I_s = \mathbb{N}$ with the N th group of indices corresponding to the eigenfunctions $\{x_m f_N\}_{1 \leq m \leq N}$, we obtain a partition $\bigcup_{s=1}^r I_{s,N} = I^N$ of the set $I^N = \{1, 2, \dots, N\}$ where the index m runs. Arguing by induction, assume we have an infinite subsequence $\{N_i\}$ of $\mathbb{N}$ such that for a given N all partitions $\bigcup_{s=1}^r (I_{s,N_i} \cap I^N) = I^N$, $i \geq 1$, are the same. Since there are only finitely many partitions of I^{N+1} , we can choose an infinite subsequence of $\{N_i\}$, say $\{N'_l\}$, such that all partitions $\bigcup_{s=1}^r (I_{s,N'_l} \cap I^{N+1}) = I^{N+1}$, $l \geq 1$, are the same. Applying a diagonal process to this table of sequences, we obtain a growing sequence of integers $\{M_i\}_{i \geq 1}$ such that all partitions $\bigcup_{s=1}^r (I_{s,M_i} \cap I^N) = I^N$, $i \geq 1$, are the same, for all $N = 1, 2, \dots$. This means that we have a partition $\bigcup_{s=1}^r \sigma_s = \mathbb{N}$, $I_{s,M_i} \cap I^N = \sigma_s \cap I^N$. Next, observe that, for every s with $1 \leq s \leq r$, we have

$$0 < \delta := \inf\{\|Ax\| : x \in H_{I_s}, \|x\| = 1\} \leq \inf\{\|T_N f\| : f \in H_{I_{s,M_i} \cap I^N}, \|f\| = 1\}$$

for every $N \geq 1$. Letting $N \rightarrow \infty$, we get

$$0 < \delta \leq \inf\{\|Tf\| : f \in H_{\sigma_s}, \|f\| = 1\}$$

for every s , $1 \leq s \leq r$.

But we saw above that this is impossible. □

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