L_p -ESTIMATES OF THE SOLUTION OF A LINEAR PROBLEM ARISING IN MAGNETOHYDRODYNAMICS

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Dedicated to the memory of M. Sh. Birman

ABSTRACT. Coercive estimates in anisotropic Sobolev spaces $W_p^{2,1}(Q_T)$ are established for solutions of a linearized problem of magnetohydrodynamics for the magnetic field. The result can help analyze nonlinear problems of magnetohydrodynamics, in particular, free boundary problems.

§1. INTRODUCTION

Let Ω_1 be a bounded domain in $\mathbb{R}^3$ with boundary S_1 that is a strictly interior subdomain of a domain Ω with boundary S , and let $\Omega_2 = \Omega \setminus \overline{\Omega}_1$. In the present paper we consider the linear problem

$$\begin{aligned} \mu_1 \vec{H}_t + \alpha^{-1} \operatorname{curl} \operatorname{curl} \vec{H} &= \vec{G}(x, t), \quad \nabla \cdot \vec{H}(x, t) = 0, \quad x \in \Omega_1, \quad t > 0, \\ \operatorname{curl} \vec{H} &= 0, \quad \nabla \cdot \vec{H}(x, t) = 0, \quad x \in \Omega_2, \\ (1.1) \quad [\mu H_n] &= 0, \quad [\vec{H}_\tau] = 0, \quad x \in S_1, \\ H_n &= 0, \quad x \in S, \\ \vec{H}(x, 0) &= \vec{H}_0(x), \quad x \in \Omega_1 \cup \Omega_2, \end{aligned}$$

where $\vec{H}(x, t) = (H_1, H_2, H_3)$ is unknown, and $\vec{G}$ and $\vec{H}_0$ are two given solenoidal vector fields. By $[u]$ we mean the jump of the function $u(x, t)$, $x \in \Omega_1 \cup \Omega_2$, on the surface S_1 : $[u] = u^{(1)}(x, t) - u^{(2)}(x, t)$, $u^{(i)} = u(x, t)|_{x \in \overline{\Omega}_i}$; $H_n = \vec{H} \cdot \vec{n}$ and $\vec{H}_\tau = \vec{H} - \vec{n} H_n$ are the normal and the tangential components of the vector $\vec{H}$ on S and S_1 ; $\alpha = \text{const} > 0$; and μ is a piecewise constant function equal to μ_i in Ω_i , $i = 1, 2$, $\mu_i > 0$.

Problem (1.1) arises in the analysis of problems of magnetohydrodynamics. In such problems, Ω_1 is a domain filled with a viscous incompressible electrically conducting fluid, Ω_2 is a vacuum region surrounding Ω , S is a perfectly conducting surface, $\vec{H}$ is a magnetic field, μ_1 and μ_2 are the magnetic permeability of the fluid and of vacuum, and α is the conductivity of the fluid. The velocity and pressure of the fluid are subject to the Navier–Stokes equations; we do not dwell on this in the present paper. Relations (1.1) are none other than the linearized Maxwell equations from which the displacement current is excluded, and the standard boundary conditions for the magnetic field. We assume that the domains Ω_1 and Ω are simply connected. Then the equations $\operatorname{curl} \vec{H} = 0$, $\nabla \cdot \vec{H} = 0$ in Ω_2 imply $\vec{H}^{(2)}(x, t) = \nabla \varphi(x, t)$, where $\varphi(x, t)$ is the solution of the Neumann

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problem

$$(1.2) \quad \begin{aligned} \nabla^2 \varphi(x, t) &= 0, \quad x \in \Omega_2, \quad \frac{\partial \varphi}{\partial n} \Big|_{x \in S} = 0, \\ \mu_2 \frac{\partial \varphi}{\partial n} \Big|_{x \in S_1} &= \mu_1 \vec{H}^{(1)} \cdot \vec{n} \Big|_{x \in S_1}. \end{aligned}$$

Hence, problem (1.1) can be written in the form

$$(1.3) \quad \begin{aligned} \mu_1 \vec{H}_t^{(1)} + \alpha^{-1} \operatorname{curl} \operatorname{curl} \vec{H}^{(1)} &= \vec{G}(x, t), \quad \nabla \cdot \vec{H}^{(1)}(x, t) = 0, \quad x \in \Omega_1, \quad t > 0, \\ \operatorname{curl} \vec{H}^{(2)} &= 0, \quad \nabla \cdot \vec{H}^{(2)}(x, t) = 0, \quad x \in \Omega_2, \\ \nabla^2 \varphi(x, t) &= 0, \quad x \in \Omega_2, \quad \frac{\partial \varphi}{\partial n} \Big|_{x \in S} = 0, \\ \mu_2 \frac{\partial \varphi}{\partial n} - \mu_1 \vec{H}^{(1)} \cdot \vec{n} \Big|_{x \in S_1} &= 0, \\ \vec{H}_\tau^{(1)} &= \nabla_\tau \varphi(x, t), \quad x \in S_1, \\ \vec{H}^{(1)}(x, 0) &= \vec{H}_0^{(1)}. \end{aligned}$$

The vector field $\vec{H}_0^{(2)}(x)$ is also equal to the gradient of a harmonic function: $\vec{H}_0^{(2)}(x) = \nabla \varphi_0(x)$,

$$(1.4) \quad \begin{aligned} \nabla^2 \varphi_0(x) &= 0, \quad x \in \Omega_2, \quad \frac{\partial \varphi_0}{\partial n} \Big|_{x \in S} = 0, \\ \mu_2 \frac{\partial \varphi_0}{\partial n} - \mu_1 \vec{H}_0^{(1)} \cdot \vec{n} \Big|_{x \in S_1} &= 0, \\ \vec{H}_{0\tau}^{(1)} &= \nabla_\tau \varphi_0(x) \Big|_{x \in S_1}. \end{aligned}$$

Hence, $\vec{H}^{(2)}$ is completely determined by $\vec{H}^{(1)} \cdot \vec{n} \Big|_{x \in S_1}$. The boundary conditions for $\vec{H}$ on S_1 can be written in a short form as $\vec{H}_\tau = \mathcal{B}(\vec{H} \cdot \vec{n})$, where $\mathcal{B}$ is a nonlocal linear operator.

Problems of magnetohydrodynamics were studied, in particular, in the papers [1, 2] for simply connected domains Ω and Ω_1 and in [3, 4, 5] for multi-connected Ω and Ω_1 . For solutions of nonlinear problems, estimates in $W_2^{2,1}(Q_T^i)$, $Q_T^i = \Omega_i \times (0, T)$, were obtained; in particular, the following estimate for the solution of problem (1.1) was proved (see [2, 5]):

$$\|\vec{H}^{(1)}\|_{W_2^{2,1}(Q_T^1)} \leq c(\|\vec{G}\|_{L_2(Q_T^1)} + \|\vec{H}_0\|_{W_2^1(\Omega_1)}).$$

In the present paper, this estimate is extended to the spaces $W_p^{2,1}(Q_T^1)$, $p > 1$, with the norm

$$\|\vec{H}^{(1)}\|_{W_p^{2,1}(Q_T^1)}^p = \sum_{2k+|j| \leq 2} \|D_t^k D_x^j \vec{H}^{(1)}\|_{L_p(Q_T^1)}^p.$$

Theorem 1. Assume that $\vec{G} \in L_p(Q_T^1)$, $\vec{H}_0^{(1)} \in W_p^{2-2/p}(\Omega_1)$ and that the conditions

$$(1.5) \quad \begin{aligned} \nabla \cdot \vec{G}(x, t) &= 0, \quad \nabla \cdot \vec{H}_0^{(1)}(x) = 0, \quad x \in \Omega_1, \\ \vec{H}_{0\tau}^{(1)} &= \mathcal{B}(\vec{H}_0^{(1)} \cdot \vec{n}) \end{aligned}$$

are satisfied (possibly, in the distributional sense). Then problem (1.1) has a unique solution $\vec{H}^{(1)} \in W_p^{2,1}(Q_T^1)$, and this solution satisfies the inequality

$$(1.6) \quad \|\vec{H}^{(1)}\|_{W_p^{2,1}(Q_T^1)} \leq c(\|\vec{G}\|_{L_p(Q_T^1)} + \|\vec{H}_0\|_{W_p^{2-2/p}(\Omega_1)}).$$

We recall that $W_p^r(\Omega_1)$, where $r = [r] + \mu$ and $\mu \in (0, 1)$, is the space with the norm

$$\|v\|_{W_p^r(\Omega_1)} = \left(\sum_{0 \leq |j| \leq [r]} \|D^j v\|_{L_p(\Omega_1)}^p + \sum_{|j|=[r]} \int_{\Omega_1} \int_{\Omega_1} |D^j v(x) - D^j v(y)|^p \frac{dx dy}{|x - y|^{3+p\mu}} \right)^{1/p}.$$

It is easy to verify that (1.6) implies the same estimate for $\vec{H}^{(2)}$. Indeed, the solution of (1.2) satisfies the inequality

$$(1.7) \quad \|\nabla \varphi\|_{W_p^2(\Omega_2)} \leq c \|\vec{H}^{(1)} \cdot \vec{n}\|_{W_p^{2-1/p}(S_1)} \leq c \|\vec{H}^{(1)}\|_{W_p^2(\Omega_1)}.$$

Moreover, since

$$\mu_2 \int_{\Omega_2} \nabla \varphi \cdot \nabla \eta dx = - \int_{S_1} \mu_1 \vec{H}^{(1)} \cdot \vec{n} \eta dS = -\mu_1 \int_{\Omega_1} \vec{H}^{(1)} \cdot \nabla \eta dx$$

for arbitrary $\eta \in W_{p'}^1(\Omega_2)$, $1/p' = 1 - 1/p$, we have

$$\|\nabla \varphi\|_{L_p(\Omega_2)} \leq c \|\vec{H}^{(1)}\|_{L_p(\Omega_1)}$$

and

$$(1.8) \quad \|\nabla \varphi_t\|_{L_p(\Omega_2)} \leq c \|\vec{H}_t^{(1)}\|_{L_p(\Omega_1)}.$$

From (1.7) and (1.8) it follows that

$$(1.9) \quad \|\vec{H}^{(2)}\|_{W_p^{2,1}(Q_T^2)} \leq c \|\vec{H}^{(1)}\|_{W_p^{2,1}(Q_T^1)}.$$

For $t = 0$ we also have $\vec{H}_0^{(2)} = \nabla \varphi_0(x)$, where φ_0 is a solution of the Neumann problem

$$(1.10) \quad \nabla^2 \varphi_0 = 0, \quad \frac{\partial \varphi_0}{\partial n} \Big|_{x \in S} = 0, \quad \mu_2 \frac{\partial \varphi_0}{\partial n} \Big|_{x \in S_1} - \mu_1 \vec{H}_0^{(1)} \cdot \vec{n} \Big|_{x \in S_1} = 0;$$

in other words, φ_0 satisfies the integral identity

$$(1.11) \quad \mu_2 \int_{\Omega_2} \nabla \varphi_0 \cdot \nabla \eta dx + \int_{S_1} \mu_1 \vec{H}^{(1)} \cdot \vec{n} \eta dS = 0$$

for arbitrary smooth η . We have

$$\|\vec{H}_0^{(2)}\|_{W_p^{2-2/p}(\Omega_2)} \leq c \|\nabla \varphi_0\|_{W_p^{2-2/p}(\Omega_2)} \leq c \|\vec{H}^{(1)}\|_{W_p^{2-2/p}(\Omega_1)}.$$

As usual, the condition of being solenoidal (for instance, $\nabla \cdot \vec{G} = 0$) is understood as $\int_{\Omega_1} \vec{G} \cdot \nabla \eta dx = 0$ for arbitrary smooth η vanishing on S_1 . For $p > \frac{3}{2}$, the boundary condition (1.4) is understood as the coincidence of the traces of $\vec{H}$ and $\nabla \varphi_0$ on S_1 : $\vec{H}_{0\tau}^{(2)} = \nabla_\tau \varphi_0 = \vec{H}_{0\tau}^{(1)} \in W_p^{2-3/p}(S_1)$. For $p < 3/2$ this becomes meaningless, and for $p = 3/2$ condition (1.5) is understood as the boundedness of the integral

$$\int_{\Omega_2} (\vec{\mathfrak{H}}_0 - \vec{H}_0^{(2)} - \vec{n}^* \vec{n}^* \cdot (\vec{\mathfrak{H}}_0 - \vec{H}_0^{(2)})) \rho^{-1}(x) dx,$$

where $\rho(x)$ is a smooth function equal to $\text{dist}(x, S_1)$ near S_1 , $\vec{n}^*$ is a smooth extension of the normal $\vec{n}$ to the interior of Ω_2 , and $\vec{\mathfrak{H}}_0 \in W_{3/2}^{2/3}(\Omega_2)$ is the extension of the vector field $\vec{H}_0^{(1)} \in W_{3/2}^{2/3}(\Omega_1)$ with preservation of the class.

For applications to problems of magnetohydrodynamics, the case where $p > 3/2$ is most interesting.

§2. MODEL PROBLEM

In this section, the following model problem in $\mathbb{R}_+^3 \cup \mathbb{R}_-^3$ ($\mathbb{R}_+^3 = \{x_3 > 0\}$, $\mathbb{R}_-^3 = \{x_3 < 0\}$) is considered:

$$\begin{aligned}
 (2.1) \quad & \mu_1 \vec{h}_t - \alpha^{-1} \nabla^2 \vec{h} = \vec{g}(x, t), \quad x_3 > 0, \quad t > 0, \\
 & \nabla^2 \varphi(x, t) = f(x, t), \quad x_3 < 0, \quad t > 0, \\
 & \nabla \cdot \vec{h} = d(x', t), \quad x_3 = 0, \quad x' = (x_1, x_2), \\
 & h_\alpha = \frac{\partial \varphi}{\partial x_\alpha} + b_\alpha(x', t), \quad \alpha = 1, 2, \quad \mu_1 h_3 = \mu_2 \frac{\partial \varphi}{\partial x_3} + b_3, \quad x_3 = 0, \\
 & \vec{h}(x, 0) = 0;
 \end{aligned}$$

here $\vec{g}$, f , d , b_i are given functions having compact supports and vanishing for $t = 0$. Our aim is to construct and estimate the solution $(\vec{h}, \varphi)$. Suppose $0 \leq T \leq \infty$, $G \subset \mathbb{R}^n$, and $Q_T = G \times (0, T)$. We introduce the following seminorms:

$$\begin{aligned}
 \|u\|_{\dot{W}_p^{2,1}(Q_T)} &= \left(\sum_{|j|=2} \|D_x^j u\|_{L_p(Q_T)}^p + \|u_t\|_{L_p(Q_T)}^p \right)^{1/p}, \\
 \|u\|_{\dot{W}_p^r(G)} &= \left(\sum_{|j|=r} \|D_x^j u\|_{L_p(G)}^p \right)^{1/p}
 \end{aligned}$$

if r is an integer,

$$\|u\|_{\dot{W}_p^r(G)} = \left(\sum_{|j|=[r]} \int_G \int_G |D_x^j u(x) - D_y^j u(y)|^p \frac{dx dy}{|x - y|^{n+p\mu}} \right)^{1/p}$$

if $r = [r] + \mu$, $0 < \mu < 1$, and

$$\|u\|_{\dot{W}_p^{r,r/2}(G \times (-\infty, T))} = \left(\int_0^T \|u\|_{\dot{W}_p^r(G)}^p dt + \int_G \|u\|_{\dot{W}_p^{r/2}(-\infty, T)}^p dx \right)^{1/p}.$$

Theorem 2.1. *Let*

$$f(x, t) = \nabla \cdot \vec{F}(x, t) + F_0(x, t),$$

where $\vec{F}$, F_0 are functions with compact support, $f(x, 0) = 0$, $d(x', 0) = 0$, $\vec{b}(x', 0) = 0$. Problem (2.1) has a unique solution $\vec{h} \in W_p^{2,1}(\mathbb{R}_+^3 \times (0, T))$, $\nabla \varphi \in W_p^{2,1}(\mathbb{R}_-^3 \times (0, T))$, and this solution is subject to the inequality

$$\begin{aligned}
 (2.2) \quad & \|\vec{h}\|_{\dot{W}_p^{2,1}(\mathbb{R}_+^3 \times (0, T))} + \|\nabla \varphi\|_{\dot{W}_p^{2,1}(\mathbb{R}_-^3 \times (0, T))} \\
 & \leq c \left(\|\vec{g}\|_{L_p(\mathbb{R}_+^3 \times (0, T))} + \|\nabla f\|_{L_p(\mathbb{R}_-^3 \times (0, T))} + \|F_t\|_{L_p(\mathbb{R}_-^3 \times (0, T))} + \|F_0 t\|_{L_p(\mathbb{R}_-^3 \times (0, T))} \right. \\
 & \quad \left. + \|\vec{b}\|_{\dot{W}_p^{2-1/p, 1-1/2p}(R^2 \times (-\infty, T))} + \|\vec{d}\|_{\dot{W}_p^{1-1/p, 1/2-1/2p}(R^2 \times (-\infty, T))} \right),
 \end{aligned}$$

where $\vec{b}(x', 0) = 0$, $d(x', 0) = 0$ for $t \leq 0$.

Proof. We set $\vec{g} = 0$, $f = 0$, $\vec{b} = 0$, $d = 0$ for $t < 0$ and extend these functions with preservation of the class to the domain $t > T$ (for instance, as even functions of $t - T$).

Consider the problem

$$\begin{aligned}
 (2.3) \quad & \mu_1 \vec{h}_t - \alpha^{-1} \nabla^2 \vec{h} = \vec{g}(x, t), \quad x_3 > 0, \quad -\infty < t < \infty, \\
 & \nabla^2 \varphi(x, t) = f(x, t), \quad x_3 < 0, \quad -\infty < t < \infty, \\
 & \nabla \cdot \vec{h} = \vec{d}(x', t), \quad x_3 = 0, \quad x' = (x_1, x_2), \\
 & h_\alpha = \frac{\partial \varphi}{\partial x_\alpha} + b_\alpha(x', t), \quad \alpha = 1, 2, \\
 & \mu_1 h_3 = \mu_2 \frac{\partial \varphi}{\partial x_3} + b_3, \quad x_3 = 0, \quad -\infty < t < \infty.
 \end{aligned}$$

The Fourier transformation with respect to $x' = (x_1, x_2)$ and the Laplace transformation with respect to t (it is defined by the standard formula

$$FLu = \tilde{u}(\xi, s) = \int_0^\infty e^{-st} dt \int_{\mathbb{R}^2} e^{-ix' \cdot \xi} dx',$$

where $s = \sigma + i\xi_0$, $\sigma \geq 0$) converts (2.3) to a boundary value problem for a system of ordinary differential equations:

$$\begin{aligned}
 (2.4) \quad & r^2(\xi, s) \vec{h}(\xi, s, x_3) - \frac{d^2 \vec{h}}{dx_3^2} = \alpha \vec{g}(\xi, s, x_3), \quad x_3 > 0, \\
 & |\xi|^2 \tilde{\varphi} - \frac{d^2 \tilde{\varphi}}{dx_3^2} = \tilde{f}(\xi, s, x_3), \quad x_3 > 0, \\
 & \frac{d\tilde{h}_3}{dx_3} + \sum_{\beta=1}^2 i\xi_\beta \tilde{h}_\beta = \tilde{d}(\xi, s), \quad x_3 = 0, \\
 & \tilde{h}_\alpha = i\xi_\alpha \tilde{\varphi} + \tilde{b}_\alpha, \quad \alpha = 1, 2, \\
 & \mu_1 \tilde{h}_3 = \mu_2 \frac{d\tilde{\varphi}}{dx_3} + \tilde{b}_3, \quad x_3 = 0, \\
 & \vec{h}(\xi, s, x_3) \xrightarrow{x_3 \rightarrow +\infty} 0, \quad \tilde{\varphi}(\xi, s, x_3) \xrightarrow{x_3 \rightarrow -\infty} 0,
 \end{aligned}$$

where $r^2 = \mu_1 \alpha s + |\xi|^2$, $-\pi < \arg r < \pi$.

Let $\vec{h}_0(\xi, s, x_3)$ and $\tilde{\varphi}_0(\xi, s, x_3)$ be solutions of the problems

$$\begin{aligned}
 & r^2 \vec{h}_0 - \frac{d^2 \vec{h}_0}{dx_3^2} = \alpha \vec{g}, \quad x_3 > 0, \quad \vec{h}_0 \xrightarrow{x_3 \rightarrow +\infty} 0, \\
 & \tilde{h}_{01} = \tilde{h}_{02} = 0, \quad \frac{d\tilde{h}_{03}}{dx_3} = 0, \quad x_3 = 0, \\
 & |\xi|^2 \tilde{\varphi}_0 - \frac{d^2 \tilde{\varphi}_0}{dx_3^2} = \tilde{f}(\xi, s, x_3), \quad x_3 < 0, \quad \tilde{\varphi}_0 \xrightarrow{x_3 \rightarrow -\infty} 0, \\
 & \tilde{\varphi}_0 = 0, \quad x_3 = 0.
 \end{aligned}$$

The functions $\vec{h}' = \vec{h} - \vec{h}_0$, $\vec{\varphi}' = \vec{\varphi} - \vec{\varphi}_0$ satisfy the relations

$$\begin{aligned}
 (2.5) \quad & r^2 \vec{h}' - \frac{d^2 \vec{h}'}{dx_3^2} = 0, \quad x_3 > 0, \quad \vec{h}'_0 \xrightarrow{x_3 \rightarrow -\infty} 0, \\
 & |\xi|^2 \vec{\varphi}'_0 - \frac{d^2 \vec{\varphi}'_0}{dx_3^2} = 0, \quad x_3 < 0, \quad \vec{\varphi}'_0 \xrightarrow{x_3 \rightarrow -\infty} 0, \\
 & \frac{d\vec{h}'_3}{dx_3} + \sum_{\beta=1}^2 i\xi_\beta \vec{h}'_\beta = \vec{d}', \\
 & \vec{h}'_\alpha = i\xi_\alpha \vec{\varphi}' + \vec{b}'_\alpha, \quad \alpha = 1, 2, \\
 & \mu_1 \vec{h}'_3 = \mu_2 \frac{d\vec{\varphi}'}{dx_3} + \vec{b}'_3, \quad x_3 = 0,
 \end{aligned}$$

where $\vec{b}'_\alpha = \vec{b}_\alpha$, $\vec{d}' = \vec{d}$, and

$$(2.6) \quad \vec{b}'_3 = \vec{b}_3 - \mu_1 \vec{h}_{03} + \mu_2 \frac{d\vec{\varphi}_0}{dx_3} \Big|_{x_3=0}.$$

Problem (2.5) can be solved easily. We have

$$\begin{aligned}
 \vec{h}'(\xi, s, x_3) &= \vec{h}'(\xi, s, 0)e^{-rx_3}, \quad \vec{\varphi}'(\xi, s, x_3) = \vec{\varphi}'(\xi, s, 0)e^{|\xi|x_3}, \\
 \vec{h}'_\alpha(\xi, s, 0) &= i\xi_\alpha \vec{\varphi}'(\xi, s, 0) + \vec{b}'_\alpha, \quad \alpha = 1, 2, \\
 \mu_1 \vec{h}'_3(\xi, s, 0) &= \mu_2 |\xi| \vec{\varphi}'(\xi, s, 0) + \vec{b}'_3 = -\mu_2 \sum_{\beta=1}^2 \frac{i\xi_\beta}{|\xi|} i\xi_\beta \vec{\varphi}'(\xi, s, 0) + \vec{b}'_3, \\
 &\quad - r\vec{h}'_3(\xi, s, 0) + \sum_{\beta=1}^2 i\xi_\beta \vec{h}'_\beta = \vec{d}',
 \end{aligned}$$

which implies

$$\begin{aligned}
 (2.7) \quad & \vec{h}'_\alpha(\xi, s, 0) = \vec{b}'_\alpha - \frac{i\xi_\alpha}{|\xi|} \frac{1}{mr + |\xi|} \left(\vec{d}' - \sum_{\beta=1}^2 i\xi_\beta \vec{b}'_\beta + \mu_1^{-1} r \vec{b}'_3 \right), \quad \alpha = 1, 2, \\
 & \vec{h}'_3(\xi, s, 0) = \mu_1^{-1} \vec{b}'_3 - \frac{m}{mr + |\xi|} \left(\vec{d}' - \sum_{\beta=1}^2 i\xi_\beta \vec{b}'_\beta + \mu_1^{-1} r \vec{b}'_3 \right), \\
 & \vec{\varphi}'(\xi, s, 0) = -\frac{1}{|\xi|(mr + |\xi|)} \left(\vec{d}' - \sum_{\beta=1}^2 i\xi_\beta \vec{b}'_\beta + \mu_1^{-1} r \vec{b}'_3 \right), \quad m = \mu_2 \mu_1^{-1}.
 \end{aligned}$$

Now we pass to estimates. Since $h_{0i} = (FL)^{-1} \tilde{h}_{0i}$ and $\varphi_0 = (FL)^{-1} \tilde{\varphi}_0$ solve the problems

$$\begin{aligned}
 & \mu_1 \frac{\partial}{\partial t} h_{0i} - \alpha^{-1} \nabla^2 h_{0i} = g_i(x, t), \quad i = 1, 2, 3, \quad x_3 > 0, \\
 & h_{0\alpha}|_{x_3=0} = 0, \quad \alpha = 1, 2, \quad \frac{\partial h_{03}}{\partial x_3} \Big|_{x_3=0} = 0, \\
 & h_{0i}(x, 0) = 0, \quad \lim_{x_3 \rightarrow \infty} h_{0i}(x, t) = 0, \\
 & \nabla^2 \varphi_0(x, t) = f(x, t), \quad x_3 < 0, \quad \varphi_0|_{x_3=0} = 0, \\
 & \lim_{x_3 \rightarrow -\infty} \varphi_0(x, t) = 0,
 \end{aligned}$$

the following estimates hold true:

$$(2.8) \quad \|\vec{h}_0\|_{\dot{W}_p^{2,1}(\mathbb{R}^3 \times (0,T))} \leq c\|g\|_{L_p(\mathbb{R}^3 \times (0,T))}, \quad T > 0,$$

$$(2.9) \quad \|\nabla \varphi_0\|_{\dot{W}_p^2(\mathbb{R}_-^3)} \leq c\|\nabla f\|_{L_2(\mathbb{R}_-^3)};$$

moreover, since

$$\int_{\mathbb{R}_-^3} \nabla \varphi \cdot \nabla \eta \, dx = - \int_{\mathbb{R}^3} f \eta \, dx = \int_{\mathbb{R}^3} (\vec{F} \cdot \nabla \eta - F_0 \eta) \, dx$$

for arbitrary smooth η with $\eta|_{x_3=0} = 0$, we have

$$(2.10) \quad \|\nabla \varphi_0\|_{L_p(\mathbb{R}_-^3)} \leq c(\|\vec{F}_t\|_{L_p(\mathbb{R}_-^3)} + \|F_{0t}\|_{L_p(\mathbb{R}_-^3)}).$$

From (2.9) and (2.10) it follows that

$$(2.11) \quad \begin{aligned} \|\nabla \varphi_0\|_{\dot{W}_p^{2,1}(\mathbb{R}_-^3 \times (0,T))} &\leq c(\|\nabla f\|_{L_p(\mathbb{R}_-^3 \times (0,T))} + \|\vec{F}_t\|_{L_p(\mathbb{R}_-^3 \times (0,T))} \\ &\quad + \|F_{0t}\|_{L_p(\mathbb{R}_-^3 \times (0,T))}). \end{aligned}$$

Now we estimate $\vec{h}'|_{x_3=0}$ by using (2.7). The expression $i\xi_\alpha/|\xi|$ is the symbol of the Riesz operator, which is bounded in $L_p(\mathbb{R}^2)$, and

$$\widetilde{M}_0 = \frac{r(\xi, s)}{mr(\xi, s) + |\xi|}, \quad \widetilde{M}_\alpha = \frac{i\xi_\alpha}{mr + |\xi|}$$

satisfy the well-known Marcinkiewicz–Mikhlin–Lizorkin condition

$$\begin{aligned} |\widetilde{M}(\xi, \sigma + i\xi_0)| + \sum_{\alpha_0}^2 \left| \xi_j \frac{\partial \widetilde{M}(\xi, \sigma + i\xi_0)}{\partial \xi_j} \right| + \sum_{j \neq k} \left| \xi_j \xi_k \frac{\partial^2 \widetilde{M}(\xi, \sigma + i\xi_0)}{\partial \xi_j \partial \xi_k} \right| \\ + \left| \xi_0 \xi_1 \xi_2 \frac{\partial^3 \widetilde{M}(\xi, \sigma + i\xi_0)}{\partial \xi_0 \partial \xi_1 \partial \xi_2} \right| \leq c, \quad \sigma \geq 0. \end{aligned}$$

Moreover, the $\widetilde{M}_i$ are analytic in s for $\operatorname{Re} s > 0$, whence $M_i(x', t) = (FL)^{-1} \widetilde{M}_i$ vanishes for $t < 0$ and we have

$$\|\vec{h}'(\cdot, 0)\|_{L_p(\mathbb{R}^2 \times (-\infty, T))} \leq c(\|\vec{b}'(\cdot, 0)\|_{L_p(\mathbb{R}^2 \times (-\infty, T))} + \|D(\cdot, 0)\|_{L_p(\mathbb{R}^2 \times (-\infty, T))}),$$

where $D = (FL)^{-1} r^{-1} \widetilde{d}'$. Similar estimates are true for the finite differences

$$\Delta_j^2(h_j) \vec{h}'(x', t, 0) = \vec{h}'(x' + 2e_j h_j, t, 0) - 2\vec{h}'(x' + e_j h_j, t, 0) + \vec{h}'(x', t, 0)$$

and $\Delta_t(-h_0) = \vec{h}'(x', t - h_0, 0) - \vec{h}'(x', t, 0)$, where $e_1 = (1, 0)$, $e_2 = (0, 1)$. These estimates and Golovkin's theorem on equivalent norms in the spaces W_p^l [6] allow us to conclude that

$$\begin{aligned} \|\vec{h}'(\cdot, 0)\|_{\dot{W}_p^{2-1/p, 1-1/2p}(\mathbb{R}^2 \times (-\infty, T))} \\ \leq c(\|\vec{b}'(\cdot, 0)\|_{\dot{W}_p^{2-1/p, 1-1/2p}(\mathbb{R}^2 \times (-\infty, T))} + \|\vec{D}\|_{\dot{W}_p^{1-1/p, 1/2-1/2p}(\mathbb{R}^2 \times (-\infty, T))}). \end{aligned}$$

Finally, we have $D(x', t) = (FL)^{-1} r^{-1} \widetilde{d} e^{-rx_3}|_{x_3=0} = w(x, t)|_{x_3=0}$, where $w(x, t)$ is a solution of the problem

$$\mu_1 w_t - \alpha^{-1} \nabla^2 w = 0, \quad x_3 > 0, \quad \frac{\partial w}{\partial x_3} \Big|_{x_3=0} = -d(x', t), \quad w|_{t=0} = 0, \quad \lim_{x_3 \rightarrow \infty} w(x, t) = 0,$$

and, as a consequence,

$$\|\vec{D}\|_{\dot{W}_p^{2-1/p, 1-1/2p}(\mathbb{R}^2 \times (-\infty, T))} \leq c\|w\|_{\dot{W}_p^{2,1}(\mathbb{R}_+^3 \times (0,T))} \leq c\|\vec{d}\|_{\dot{W}_p^{1-1/p, 1/2-1/2p}(\mathbb{R}^2 \times (-\infty, T))}.$$

Thus,

$$(2.12) \quad \begin{aligned} \|\vec{h}'\|_{\dot{W}_p^{2,1}(\mathbb{R}_+^3 \times (0,T))} &\leq c(\|\vec{h}'(\cdot, 0)\|_{\dot{W}_p^{2-1/p, 1-1/2p}(\mathbb{R}^2 \times (-\infty, T))}) \\ &\leq c(\|\vec{b}'\|_{\dot{W}_p^{2-1/p, 1-1/2p}(\mathbb{R}^2 \times (-\infty, T))} + \|\vec{d}'(\cdot, 0)\|_{\dot{W}_p^{1-1/p, 1/2-1/2p}(\mathbb{R}^2 \times (-\infty, T))}). \end{aligned}$$

Since $\vec{h}_0 = 0$ and $\varphi_0 = 0$ for $t < 0$, we can use (2.6) and the trace theorem for the space $W_p^{2,1}(\mathbb{R}_+^3 \times (0, T))$ to obtain

$$(2.13) \quad \begin{aligned} \|b'_3\|_{\dot{W}_p^{2-1/p, 1-1/2p}(\mathbb{R}^2 \times (-\infty, T))} &\leq \|b_3\|_{\dot{W}_p^{2-1/p, 1-1/2p}(\mathbb{R}^2 \times (-\infty, T))} \\ &\quad + c(\|h_{03}\|_{\dot{W}_p^{2,1}(\mathbb{R}_+^3 \times (0, T))} + \|\nabla \varphi_0\|_{\dot{W}_p^{2,1}(\mathbb{R}_+^3 \times (0, T))}). \end{aligned}$$

Combining (2.8), (2.11), (2.12), and (2.13), we arrive at inequality (2.2). $\square$

§3. PROOF OF INEQUALITY (1.6) FOR $\vec{H}_0 = 0$

First, we obtain yet another estimate of the solution of (1.2).

Proposition 3.1. *The solution of the problem (1.2) normalized by $\int_{\Omega_2} \varphi \, dx = 0$ satisfies the inequality*

$$(3.1) \quad \|\varphi_t\|_{L_p(\Omega_2)} \leq c(\|\vec{G}\|_{L_p(\Omega_2)} + \|\operatorname{curl} \vec{H}^{(1)}\|_{L_p(S_1)}).$$

Proof. Let η be an arbitrary function of class $L_{p'}(\Omega_2)$, $1/p = 1 - 1/p'$, such that $\int_{\Omega_2} \eta \, dx = 0$, and let X be the solution of the problem

$$\nabla^2 X = \eta, \quad x \in \Omega_2, \quad \frac{\partial X}{\partial n} \Big|_S = 0, \quad \frac{\partial X}{\partial n} \Big|_{S_1} = 0.$$

By the Green formula,

$$\begin{aligned} \int_{\Omega_2} \varphi_t \eta \, dx &= \int_{\Omega_2} \varphi_t \nabla^2 X \, dx = - \int_{\Omega_2} \nabla \varphi_t \cdot \nabla X \, dx \\ &= \int_{S_1} \frac{\mu_1}{\mu_2} \vec{H}_t^{(1)} \cdot \vec{n} X \, dS = \mu_2^{-1} \int_{S_1} (\vec{G} \cdot \vec{n} - \alpha^{-1} \operatorname{curl}_n \operatorname{curl} \vec{H}^{(1)}) X \, dS \\ &= \mu_2^{-1} \int_{S_1} \vec{G} \cdot \vec{n} X \, dS + (\mu_2 \alpha)^{-1} \int_{S_1} \vec{n} \cdot (\nabla X \times \operatorname{curl} \vec{H}^{(1)}) \, dS, \end{aligned}$$

whence

$$\begin{aligned} \left| \int_{\Omega_2} \varphi_t \eta \, dx \right| &\leq \mu_2^{-1} (\|\vec{G} \cdot \vec{n}\|_{W_p^{-1/p}(S_1)} \|X\|_{W_p^{1/p}(S_1)} + (\mu_2 \alpha)^{-1} \|\operatorname{curl} \vec{H}^{(1)}\|_{L_p(S_1)} \|\nabla X\|_{L_{p'}(S_1)}) \\ &\leq c(\|\vec{G}\|_{L_p(\Omega_1)} + \|\operatorname{curl} \vec{H}^{(1)}\|_{L_p(S_1)}) \|\eta\|_{L_{p'}(\Omega_2)}. \end{aligned}$$

This implies (3.1), because η is arbitrary.

Now we proceed to the proof of (1.6).

Assume that $\vec{H} \in W_2^{2,1}(Q_T^1) \cap W_p^{2,1}(Q_T^1)$ is a solution of (1.1) with $\vec{H}_0 = 0$. It is easily seen that $\vec{H}^{(1)}$ is also a solution of

$$\begin{aligned} \mu_1 \vec{H}_t - \alpha^{-1} \nabla^2 \vec{H} &= \vec{G}(x, t), \quad x \in \Omega_1, \quad t \in (0, T), \\ \nabla^2 \varphi(x, t) &= 0, \quad x \in \Omega_2, \quad \frac{\partial \varphi}{\partial n} \Big|_{x \in S} = 0, \\ \mu_1 \vec{H} \cdot \vec{n} &= \mu_2 \frac{\partial \varphi}{\partial n} = 0, \quad \vec{H}_\tau = \nabla_\tau \varphi(x, t), \quad x \in S_1, \\ \nabla \cdot \vec{H} \Big|_{x \in S_1} &= 0, \quad \vec{H}(x, 0) = 0. \end{aligned}$$

We use the localization method and, first, estimate $\vec{H}(x, t)$ near S_1 .

Let x_0 be an arbitrary point of S_1 . Without loss of generality, we may assume that $x_0 = 0$ and that the x_3 -axis is directed along the interior normal $-\vec{n}(x_0)$. In the neighborhood of the point $x_0 = 0$, the surface S_1 can be defined by an equation of the form

$$x_3 = \Phi(x'), \quad x' = (x_1, x_2), \quad |x'| \leq d_0,$$

where Φ is a smooth function such that $\Phi(0) = 0$, and $\nabla\Phi(0) = 0$. Let $\zeta(x)$ be a smooth function of $|x|$ equal to 1 for $|x| \leq d_1/2$ and to 0 for $|x| \geq d_1$, where $d_1 \leq d$, and satisfying $\frac{\partial\zeta}{\partial n}\Big|_{S_1} = 0$. The functions $\vec{h} = \vec{H}\zeta$ and $\psi = \varphi\zeta$ satisfy the relations

$$\begin{aligned} (3.2) \quad & \mu_1 \vec{h}_t - \alpha^{-1} \nabla^2 \vec{h} = \vec{G}\zeta - \alpha^{-1} l(\vec{H}), \quad x \in \omega_1, \quad t \in (0, T), \\ & \nabla^2 \psi(x, t) = l(\varphi), \quad x \in \omega_2, \\ & \vec{h}_\tau - \nabla_\tau \psi = -\varphi \nabla_\tau \zeta, \\ & \mu_1 \vec{h} \cdot \vec{n} - \mu_2 \frac{\partial\psi}{\partial n} = 0, \\ & \nabla \cdot \vec{h} = -\vec{H} \cdot \nabla \zeta, \quad x \in \sigma, \\ & \vec{h}(x, 0) = 0, \end{aligned}$$

where $\omega_i = \{x \in \Omega_i : |x| \leq d_1\}$, $\sigma = \{x \in S_1 : |x| \leq d_1\}$, $l(\varphi) = 2(\nabla\zeta \cdot \nabla)\varphi + \varphi \nabla^2 \zeta$, and $l(\vec{H}) = 2(\nabla\zeta \cdot \nabla)\vec{H} + \vec{H} \nabla^2 \zeta$.

In ω_i we introduce the new variables

$$y' = x', \quad y_3 = x_3 - \Phi(x').$$

Under this transformation, the operator $\nabla_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ converts to $\tilde{\nabla} = \nabla - \nabla\Phi \frac{\partial}{\partial y_3}$, where $\nabla = \nabla_y$, $\nabla\Phi = (\Phi_1, \Phi_2, 0)$, $\Phi_\alpha = \frac{\partial\Phi}{\partial y_\alpha}$, $\alpha = 1, 2$, and relations (3.2) take the form (2.1) with

$$\begin{aligned} (3.3) \quad & \vec{g} = \zeta \vec{G} - \alpha^{-1} l(\vec{H}) - \alpha^{-1} (\nabla \vec{h} - \tilde{\nabla}^2 \vec{h}), \\ & f = l(\varphi) + \nabla^2 \psi - \tilde{\nabla}^2 \psi, \\ & b_\beta = -n_\beta \frac{\partial\psi}{\partial n} + n_\beta \vec{h} \cdot \vec{n} - \varphi \tilde{\nabla}_\beta \zeta, \quad \beta = 1, 2, \\ & b_3 = \mu_1 (h_3 + \vec{h} \cdot \vec{n}) + \mu_2 \left(-\frac{\partial\psi}{\partial y_3} - \frac{\partial\psi}{\partial n} \right), \\ & d = -\vec{H} \cdot \tilde{\nabla} \zeta - (\tilde{\nabla} - \nabla) \cdot \vec{h}. \end{aligned}$$

Since

$$\vec{n} = \left(\frac{\Phi_1}{\sqrt{1 + |\nabla\Phi|^2}}, \frac{\Phi_2}{\sqrt{1 + |\nabla\Phi|^2}}, -\frac{1}{\sqrt{1 + |\nabla\Phi|^2}} \right)$$

and $\tilde{\nabla} - \nabla = -\nabla\Phi \frac{\partial}{\partial y_3}$, all differential expressions in (3.3) have small leading coefficients with respect to $\vec{h}$ and ψ (they are proportional to the derivatives $\Phi_\alpha(y) = \frac{\partial\Phi}{\partial y_\alpha}$). Moreover, we have

$$\begin{aligned} f &= 2\tilde{\nabla} \cdot (\varphi \tilde{\nabla}) - \varphi \tilde{\nabla}^2 \zeta + \frac{\partial}{\partial y_3} \left(2 \sum_{\alpha=1}^2 \Phi_\alpha \frac{\partial\psi}{\partial y_\alpha} + \sum_{\alpha=1}^2 \left(\frac{\partial^2 \Phi}{\partial y_\alpha^2} \psi - \Phi_\alpha^2 \frac{\partial\psi}{\partial y_3} \right) \right) \\ &= \nabla \cdot \vec{F}_1 + \frac{\partial}{\partial y_3} F_2 + F_0, \end{aligned}$$

where

$$F_0 = -\varphi \tilde{\nabla}^2 \zeta, \quad F_2 = \frac{\partial}{\partial y_3} \left(2 \sum_{\alpha=1}^2 \Phi_\alpha \frac{\partial \psi}{\partial y_\alpha} + \sum_{\alpha=1}^2 \left(\frac{\partial^2 \Phi}{\partial y_\alpha^2} \psi - \Phi_\alpha^2 \frac{\partial \psi}{\partial y_3} \right) \right),$$

and $\vec{F}_1$ is determined by the equation $\nabla \cdot \vec{F}_1 = 2\tilde{\nabla} \cdot (\varphi \nabla \tilde{\zeta})$, i.e., $\vec{F}_1 = 2\varphi \tilde{\nabla} \zeta - 2e_3 \nabla \Phi \cdot (\varphi \tilde{\nabla} \zeta)$, $e_3 = (0, 0, 1)$.

We apply estimate (2.2) and the theorem on the traces of functions of class $W_p^{2,1}(\mathbb{R}_+^3 \times (0, T))$. For d_1 sufficiently small, we obtain

$$(3.4) \quad \begin{aligned} \|\vec{h}\|_{\dot{W}_p^{2,1}(\mathbb{R}_+^3 \times (0, T))}^p &\leq c \left(\|\zeta \vec{G}\|_{L_p(\mathbb{R}_+^3 \times (0, T))} + \|\vec{H}\|_{W_p^{1,0}(\omega_1 \times (0, T))} \right. \\ &\quad + \|\tilde{\nabla} \zeta \cdot \vec{H}\|_{\dot{W}_p^{1-1/p, 1/2-1/2p}(\mathbb{R}^2 \times (-\infty, T))}^p + d_1 \|\nabla \psi\|_{\dot{W}_p^{2,1}(\omega_2 \times (0, T))}^p \\ &\quad \left. + \|\varphi\|_{W_p^{2,1}(\omega_2 \times (0, T))} \right). \end{aligned}$$

Inequalities of this type can be obtained for arbitrary $x_0 \in S$. On the other hand, in the neighborhood $\omega(x_0)$ of an arbitrary interior point $x_0 \in \Omega_1$ whose distance from S does not exceed $d_2 \leq d_1$, we can get the estimate

$$(3.5) \quad \|\vec{h}\|_{\dot{W}_p^{2,1}(\mathbb{R}^3 \times (0, T))}^p \leq c \left(\|\zeta \vec{G}\|_{L_p(\mathbb{R}^3 \times (0, T))} + \|\vec{H}\|_{W_p^{1,0}(\omega \times (0, T))} \right),$$

where $\vec{h} = \vec{H}\zeta$, ζ is a cutoff function in $\omega(x_0)$ similar to that defined above, and

$$\|\vec{H}\|_{W_p^{1,0}(\omega \times (0, T))} = \left(\int_0^T \|\vec{H}\|_{W_p^1(\omega)}^p dt \right)^{1/p}.$$

We take a finite net of points x_{0k} such that the neighborhoods $\omega_1(x_{0k})$ and $\omega(x_{0k})$ cover Ω_1 and take the sum of the corresponding inequalities (3.4), (3.5). As a result, we obtain

$$\begin{aligned} \|\vec{H}\|_{W_p^{2,1}(Q_T^1)}^p &\leq c \left(\|\vec{G}\|_{L_p(Q_T^1)}^p + \|\vec{H}\|_{W_p^{1,0}(Q_T^1)}^p + \|\vec{H}^{(1)}\|_{W_p^{0,1/2-1/2p}(S_1 \times (-\infty, T))}^p \right. \\ &\quad \left. + d_1^p \|\nabla \varphi\|_{W_p^{2,1}(Q_T^2)}^p + \|\varphi\|_{W_p^{2,1}(Q_T^2)}^p \right). \end{aligned}$$

The last term can be estimated with the help of (1.7), (1.8), (3.1), and the inequality

$$\|\varphi\|_{W_p^2(\Omega_2)} \leq c \|\vec{H}^{(1)} \cdot \vec{n}\|_{W_p^{1-1/p}(S_1)} \leq c \|\vec{H}^{(1)}\|_{W_p^1(\Omega_1)}.$$

Thus,

$$\|\varphi\|_{W_p^{2,1}(Q_T^2)}^p \leq c \left(\|\vec{H}\|_{W_p^{1,0}(Q_T^1)}^p + \|\vec{G}\|_{L_p(Q_T^1)}^p + \|\operatorname{curl} \vec{H}^{(1)}\|_{L_p(S_1 \times (0, T))}^p \right).$$

In order to estimate the intermediate norms of $\vec{H}$, we use the interpolation inequality

$$\begin{aligned} \|\vec{H}\|_{W_p^{1,0}(Q_T^1)}^p + \|\operatorname{curl} \vec{H}^{(1)}\|_{L_p(S_1 \times (0, T))}^p + \|\vec{H}^{(1)}\|_{W_p^{0,1/2-1/2p}(S_1 \times (-\infty, T))}^p \\ \leq \epsilon \|\vec{H}\|_{W_p^{2,1}(Q_T^1)}^p + c(\epsilon) \|\vec{H}\|_{L_p(Q_T^1)}^p \end{aligned}$$

with arbitrarily small $\epsilon > 0$. Taking d_1 sufficiently small, we arrive at the estimate

$$(3.6) \quad \|\vec{H}\|_{W_p^{2,1}(Q_T^1)} \leq c \left(\|\vec{G}\|_{L_p(Q_T^1)} + \|\vec{H}\|_{L_p(Q_T^1)} \right)$$

with a constant independent of T .

Since T is arbitrary, relation (3.6) and the Gronwall inequality imply the estimate

$$(3.7) \quad \|\vec{H}\|_{W_p^{2,1}(Q_T^1)} \leq c(T) \|\vec{G}\|_{L_p(Q_T^1)}.$$

In fact, the constant c can be taken independent of T . The proof of this is based on an estimate for the solution of the parameter-dependent problem

$$(3.8) \quad \begin{aligned} \lambda \mu_1 \vec{h} + \alpha^{-1} \operatorname{curl} \operatorname{curl} \vec{h} &= g, \quad \nabla \cdot \vec{h} = 0, \quad x \in \Omega_1, \\ \operatorname{curl} \vec{h} &= 0, \quad \nabla \cdot \vec{h} = 0, \quad x \in \Omega_2, \\ [\mu \vec{h} \cdot \vec{n}]|_{S_1} &= 0, \quad [\vec{h}_\tau]|_{S_1} = 0, \quad \vec{h} \cdot \vec{n}|_S = 0, \end{aligned}$$

Arguing in the same way as it was done in [7] for the Stokes problem, from (3.6) we can deduce the estimate

$$(3.9) \quad (1 + |\lambda|) \|\vec{h}\|_{L_p(\Omega_1)} + \|\vec{h}\|_{W_p^2(\Omega_1)} \leq c \|\vec{g}\|_{L_p(\Omega_1)},$$

provided λ is contained in the set

$$\operatorname{Re} \lambda + \kappa |\operatorname{Im} \lambda| \geq 0, \quad |\lambda| \geq \rho,$$

with some $\kappa, \rho > 0$. Moreover, since the spectrum of problem (3.8) is positive (see [2, 9]), condition (3.9) is also fulfilled if

$$\operatorname{Re} \lambda + \kappa |\operatorname{Im} \lambda| \geq 0, \quad \operatorname{Re} \lambda \geq -\delta, \quad \delta > 0.$$

Under these assumptions, the semigroup theory ensures the estimate

$$\|\vec{H}\|_{L_p(Q_T^1)} \leq c \|\vec{G}\|_{L_p(Q_T^1)}$$

and moreover, a weighted estimate (3.7) with the exponential weight e^{at} , $a \in (0, \delta)$, and with a constant independent of T .

For the proof that (1.3) admits a solution $\vec{H} \in W_p^{2,1}(Q_T^1)$, $\nabla \varphi \in W_p^{2,1}(Q_T^2)$ (assuming that $p > 2$), we use a trick from [8] and approximate $\vec{G}(x, t)$ by the vector fields

$$\vec{G}_\epsilon(x, t) = \int_{-\epsilon}^\epsilon \omega_\epsilon(\tau) \vec{G}(x, t - \tau - 2\epsilon) d\tau,$$

where ω_ϵ is a standard mollifying kernel and $\vec{G}(x, t) = 0$ for $t < 0$. Let $\vec{h} \in W_2^2(Q_T^1)$, $\nabla \varphi \in W_2^{2,1}(Q_T^2)$ be a solution of problem (1.3) (the existence of this was proved in [2]). Every $\vec{G}_\epsilon$ gives rise to the solution

$$\vec{H}_\epsilon(x, t) = \int_{-\epsilon}^\epsilon \omega_\epsilon(\tau) \vec{H}(x, t - \tau - 2\epsilon) d\tau, \quad \nabla \varphi_\epsilon(x, t) = \int_{-\epsilon}^\epsilon \omega_\epsilon(\tau) \nabla \varphi(x, t - \tau - 2\epsilon) d\tau,$$

vanishing for $t < \epsilon$ and infinitely differentiable with respect to t . The imbedding theorems show that $\vec{H}_\epsilon$, $\vec{H}_{\epsilon t}$, $\nabla \varphi_\epsilon$, $\nabla \varphi_{\epsilon t}$ are bounded, and from the equation $\operatorname{curl} \operatorname{curl} \vec{H}_\epsilon = -\alpha \mu_1 \vec{H}_{\epsilon t} + \alpha \vec{G}_\epsilon$ it follows that $\vec{H}_\epsilon \in W_p^{2,1}(Q_T^1)$ (the proof is the same as in [2] for $p = 2$; we omit this). Using (3.7) and letting ϵ go to zero, we obtain a solution $\vec{H} \in W_p^2(Q_T^1)$, $\nabla \varphi \in W_p^{2,1}(Q_T^2)$ of problem (1.3). $\square$

§4. PROBLEM (1.1) IN MULTI-CONNECTED DOMAINS Ω_1 AND Ω

The results obtained above are extended (with some modifications) to the case of multi-connected domains Ω_1 and Ω . Let h_1 and h be the first Betti numbers of these domains; suppose that at least one of them is different from zero. In this case in Ω_2 we have $h + h_1$ smooth and linearly independent “Neumann vector fields” $\vec{u}_k(x)$ satisfying

$$\operatorname{curl} \vec{u}_k(x) = 0, \quad \nabla \cdot \vec{u}_k(x) = 0, \quad x \in \Omega_2, \quad \vec{u}_k \cdot \vec{n}|_{S \cup S_1} = 0.$$

Moreover, if $h \neq 0$, then in Ω there exist h “modified Neumann vector fields” $\vec{U}_q(x)$, $q = 1, \dots, h$, such that

$$\begin{aligned} \operatorname{curl} \vec{U}_q(x) &= 0, \quad \nabla \cdot \vec{U}_q(x) = 0, \quad x \in \Omega_1 \cup \Omega_2, \\ \vec{U}_q \cdot \vec{n}|_S &= 0, \quad [\mu \vec{U}_q \cdot \vec{n}]_{S_1} = 0, \quad [\vec{U}_{q\tau}]_{S_1} = 0. \end{aligned}$$

As for the structure of the $\vec{u}_k$ and $\vec{U}_q$, we refer the reader to [5, 9]. We denote by $\mathcal{H}_1(\Omega)$ the linear set of solenoidal vector fields $\vec{H}$ such that $\operatorname{curl} \vec{H} = 0$ in Ω_2 and $\int_{\Omega} \mu \vec{H} \cdot \vec{U}_q dx = 0$, $q = 1, \dots, h$.

The equations $\operatorname{curl} \vec{H}^{(2)} = 0$, $\nabla \cdot \vec{H}^{(2)} = 0$ imply that

$$(4.1) \quad \vec{H}^{(2)} = \nabla \varphi + \vec{u}(x, t), \quad \vec{u}(x, t) = \sum_{j=1}^{h+h_1} k_j(t) \vec{u}_j(x),$$

where φ is a solution of (1.2). The functions $k_j(t)$ satisfy the relations

$$(4.2) \quad \sum_{j=1}^{h+h_1} k_j(t) C_{mj} = \int_{\Omega_2} \vec{H}^{(2)}(x, t) \cdot \vec{u}_m(x) dx, \quad m = 1, \dots, h + h_1,$$

where the $C_{mj} = \int_{\Omega_2} \vec{u}_m(x) \cdot \vec{u}_j(x) dx$ are entries of a positive definite matrix.

We pass to consideration of problem (1.1). In the case of multi-connected domains, it is convenient to write it in the form

$$(4.3) \quad \begin{aligned} \mu \vec{H}_t + \operatorname{curl} \vec{E} &= 0, \quad \nabla \cdot \vec{H} = 0, \quad x \in \Omega_1 \cup \Omega_2, \\ \operatorname{curl} \vec{H} &= \alpha \vec{E} + \vec{j}(x, t), \quad x \in \Omega_1, \\ \operatorname{curl} \vec{H} &= 0, \quad \nabla \cdot \vec{E} = 0, \quad x \in \Omega_2, \\ [\mu \vec{H} \cdot \vec{n}] &= 0, \quad [\vec{H}_\tau] = 0, \quad [\vec{E}_\tau] = 0, \quad x \in S_1, \\ \vec{H} \cdot \vec{n} &= 0, \quad \vec{E}_\tau = 0, \quad x \in S, \\ \vec{H}(x, 0) &= \vec{H}_0(x), \quad x \in \Omega_1 \cup \Omega_2, \end{aligned}$$

where $\vec{j}(x, t)$ is a given vector field and $\vec{E}$ is an additional unknown. It is clear that $\vec{E}$ can be removed easily and (4.3) implies (1.1) with $\vec{G} = \alpha^{-1} \operatorname{curl} \vec{j}$.

Thus, $\vec{H}^{(1)}(x, t)$ satisfies the relations

$$(4.4) \quad \mu \vec{H}_t^{(1)} + \alpha^{-1} \operatorname{curl} \operatorname{curl} \vec{H}^{(1)} = \alpha^{-1} \operatorname{curl} \vec{j}, \quad \nabla \cdot \vec{H}^{(1)} = 0, \quad x \in \Omega_1,$$

$$(4.5) \quad \mu_1 \vec{H}^{(1)} \cdot \vec{n} = \mu_2 \frac{\partial \varphi}{\partial n}, \quad \vec{H}_\tau^{(1)} = \nabla_\tau \varphi + \vec{u}_\tau(x, t), \quad x \in S_1, \quad \vec{H}^{(1)}(x, 0) = 0$$

(if $\vec{H}_0 = 0$). As above, the function φ is a solution of (1.2). Moreover, it is easy to check that $\vec{H}$ satisfies the integral identity

$$(4.6) \quad \int_{\Omega} \mu \vec{H}_t \cdot \vec{\psi} dx + \alpha^{-1} \int_{\Omega_1} \operatorname{curl} \vec{H} \cdot \overrightarrow{\operatorname{curl}} \psi dx = \alpha^{-1} \int_{\Omega_1} \vec{j} \cdot \operatorname{curl} \vec{\psi} dx,$$

where ψ is an arbitrary vector field of class $W_2^1(\Omega_1) \cap W_2^1(\Omega_2)$ with $\operatorname{curl} \psi = 0$ in Ω_2 and with a continuous tangential component on S_1 . Let $\vec{u}_m^*$ be a smooth divergence-free extension of $\vec{u}_m$ to the domain Ω . Taking $\psi = \vec{u}_m^*$ in (4.6), we obtain

$$\begin{aligned} \int_{\Omega} \mu \vec{H}_t \cdot \vec{u}_m^* dx &= -\alpha^{-1} \int_{\Omega_1} \operatorname{curl} \operatorname{curl} \vec{H}^{(1)} \cdot \vec{u}_m^* dx + \alpha^{-1} \int_{\Omega_1} \operatorname{curl} \vec{j} \cdot \vec{u}_m^* dx \\ &\quad - \alpha^{-1} \int_{S_1} (\operatorname{curl} \vec{H}^{(1)} - \vec{j}) \cdot (\vec{n} \times \vec{u}_m) dS, \end{aligned}$$

which reduces to

$$(4.7) \quad \mu_2 \sum_{j=1}^{h+h_1} C_{mj} k'_j(t) = -\alpha^{-1} \int_{S_1} (\operatorname{curl} \vec{H}^{(1)} - \vec{j}) \cdot (\vec{n} \times \vec{u}_m) dS,$$

by (4.4), (4.1). For $t = 0$ we have

$$(4.8) \quad k_j(0) = 0, \quad j = 1, \dots, h + h_1.$$

We show that estimation of $\vec{H}$ reduces to that of $\vec{H}^{(1)}$ and k_j satisfying (4.7), (4.8), (4.2). Problem (4.4), (4.5) differs from (1.3) only in the nonhomogeneity in the boundary condition. Precisely as above, we establish the inequality

$$\begin{aligned} \|\vec{H}^{(1)}\|_{W_p^{2,1}(Q_T^1)} &\leq c(\|\operatorname{curl} \vec{j}\|_{L_p(Q_T^1)} + \|\vec{u}\|_{W_p^{2-2/p, 1-1/2p}(S_1 \times (-\infty, T))} + \|\vec{H}^{(1)}\|_{L_p(Q_T^1)}) \\ &\leq c(\|\operatorname{curl} \vec{j}\|_{L_p(Q_T^1)} + \|\vec{k}\|_{W_p^{1-1/2p}((0, T))} + \|\vec{H}^{(1)}\|_{L_p(Q_T^1)}) \end{aligned}$$

with $\vec{k} = (k_1(t), \dots, k_{h+h_1}(t))$. Moreover, we have estimate (1.9) for φ :

$$\|\nabla \varphi\|_{W_p^{2,1}(Q_T^1)} \leq c\|\vec{H}^{(1)}\|_{W_p^{2,1}(Q_T^1)},$$

and, as a consequence,

$$\|\vec{H}^{(2)}\|_{W_p^{2,1}(Q_T^2)} \leq c(\|\vec{H}^{(1)}\|_{W_p^{2,1}(Q_T^1)} + \|\vec{k}\|_{W_p^1(0, T)}).$$

The norm of $\vec{k}'_t$ is estimated with the help of (4.7). Since

$$\int_{S_1} \vec{j} \cdot (\vec{n} \times \vec{u}_m) dS = \int_{\Omega_1} (\vec{j} \cdot \operatorname{curl} \vec{u}_m^* - \operatorname{curl} \vec{j} \cdot \vec{u}_m^*) dx,$$

(4.7) implies

$$\|\vec{k}'_t\|_{L_p(0, T)} \leq c(\|\operatorname{curl} \vec{H}^{(1)}\|_{L_p(S_1 \times (0, T))} + \|\vec{j}\|_{L_p(Q_T^1)} + \|\operatorname{curl} \vec{j}\|_{L_p(Q_T^1)}).$$

Now we use the interpolation inequalities

$$\begin{aligned} \|\operatorname{curl} \vec{H}^{(1)}\|_{L_p(S_1 \times (0, T))} &\leq \epsilon \|D^2 \vec{H}^{(1)}\|_{L_p(Q_T^1)} + c(\epsilon) \|\vec{H}^{(1)}\|_{L_p(Q_T^1)}, \\ \|\vec{k}\|_{W_p^{1-1/p}(-\infty, T)} &\leq \epsilon \|\vec{k}'_t\|_{L_p(0, T)} + c_1(\epsilon) \|\vec{k}\|_{L_p(0, T)} \end{aligned}$$

with sufficiently small $\epsilon > 0$ and the inequality

$$\|\vec{k}\|_{L_p(0, T)} \leq c\|\vec{H}^{(2)}\|_{L_p(Q_T^2)},$$

which is a consequence of (4.2). Uniting the above estimates, we arrive at an analog of (3.6):

$$(4.9) \quad \sum_{i=1}^2 \|\vec{H}^{(i)}\|_{W_p^{2,1}(Q_T^1)} \leq c \left(\|\vec{j}\|_{L_p(Q_T^1)} + \|\operatorname{curl} \vec{j}\|_{L_p(Q_T^1)} + \sum_{i=1}^2 \|\vec{H}^{(i)}\|_{L_p(Q_T^i)} \right).$$

Since T is arbitrary, (4.9) implies

$$(4.10) \quad \sum_{i=1}^2 \|\vec{H}^{(i)}\|_{W_p^{2,1}(Q_T^1)} \leq c(\|\vec{j}\|_{L_p(Q_T^1)} + \|\operatorname{curl} \vec{j}\|_{L_p(Q_T^1)}).$$

Since

$$\int_{\Omega} \mu \vec{H} \cdot \vec{U}_q dx = - \int_{\Omega_1} \operatorname{curl} E \cdot \vec{U}_q dx = 0,$$

it is natural to consider problem (4.3) (at least for $\vec{H}_0 = 0$) on the linear space $\mathcal{H}_{\perp}(\Omega)$ of vector fields satisfying the orthogonality condition

$$(4.11) \quad \int_{\Omega} \mu \vec{H} \cdot \vec{U}_q dx = 0, \quad q = 1, \dots, h.$$

In the paper [5] it was shown that problem (4.3), (4.11) can be written in the form

$$(4.12) \quad \vec{H}_t + A\vec{H} = \vec{J}, \quad \vec{H}|_{t=0} = \vec{H}_0,$$

where A is a positive definite selfadjoint operator defined on the space $D(A)$ of divergence free vector fields of class $W_2^2(\Omega_1) \cap W_2^2(\Omega_2) \cap \mathcal{H}_\perp(\Omega)$ satisfying the conditions

$$\operatorname{curl} \vec{H}^{(2)} = 0, \quad [\mu \vec{H} \cdot \vec{n}]|_{x \in S_1} = 0, \quad [\vec{H}_\tau]|_{x \in S_1} = 0, \quad \vec{H} \cdot \vec{n}|_{x \in S} = 0.$$

Let $\mathcal{H}_\perp^0$ be the closure of $D(A)$ in $L_2(\Omega)$ with the scalar product $\int_\Omega \mu \vec{v} \cdot \vec{u} dx$. The operator A is defined by

$$A\vec{H} = P_{\mathcal{H}_\perp^0} \alpha^{-1} \mu^{-1} \operatorname{curl} \mathcal{E} \operatorname{curl} \vec{H},$$

where $P_{\mathcal{H}_\perp^0}$ is the orthogonal projection onto $\mathcal{H}_\perp^0$, and $\mathcal{E}$ is the extension operator that takes a vector field $\vec{B} = \operatorname{curl} \vec{H}$ to its extension to Ω such that $(\mathcal{E}\vec{B})_\tau|_S = 0$ and $\|\mathcal{E}\vec{B}\|_{W_2^1(\Omega)} \leq c\|\vec{B}\|_{W_2^1(\Omega_1)}$. Moreover, in (4.12) we have $\vec{H}_0 \in D(A^{1/2})$ and $\vec{J} = \alpha^{-1} \mu^{-1} \operatorname{curl} P_{\mathcal{H}_\perp^0} \mathcal{E} \vec{J}^*$.

The positivity of the spectrum of A makes it possible to show that

$$\sum_{l=1}^2 \|\vec{H}^{(i)}\|_{L_p(Q_T^i)} \leq c\|\vec{J}\|_{L_p(Q_T)}, \quad Q_T = \Omega \times (0, T);$$

hence, the constant in (4.10) is independent of T .

The vector field $\vec{E}$ is recovered in the following way (see [1, 5]). Since $\mu \vec{H}_t \cdot \vec{n}|_S = 0$, $[\mu \vec{H}_t \cdot \vec{n}]|_{S_1} = 0$, and $\vec{H}_t \in \mathcal{H}_\perp$, there exists $\vec{E}_0 \in W_2^{1,0}(Q_T)$ such that

$$\begin{aligned} \mu \vec{H}_t &= -\operatorname{curl} \vec{E}_0, \quad \nabla \cdot \vec{E}_0 = 0, \quad x \in \Omega, \\ \vec{E}_0|_{x \in S} &= 0, \\ \|\vec{E}_0\|_{W_p^{1,0}(Q_T)} &\leq c\|\mu \vec{H}_t\|_{L_p(Q_T)}. \end{aligned}$$

Consequently, (4.6) implies

$$\int_{\Omega_1} (-\vec{E}_0 + \alpha^{-1} \operatorname{curl} \vec{H} - \alpha \vec{J}) \cdot \operatorname{curl} \vec{\psi} dx = 0.$$

Therefore

$$-\vec{E}_0 + \alpha^{-1} \operatorname{curl} \vec{H} - \alpha \vec{J} = \nabla \omega_1, \quad x \in \Omega_1,$$

and $\vec{E} = \vec{E}_0 + \nabla \omega_1(x, t)$, $x \in \Omega_1$. Since $\operatorname{curl}(\vec{E} - \vec{E}_0) = 0$ and $\nabla \cdot (\vec{E} - \vec{E}_0) = 0$ in Ω_2 , we can set $\vec{E} = \vec{E}_0 + \nabla \omega_2$, $x \in \Omega_2$, where ω_2 is a solution of the problem

$$\nabla^2 \omega_2 = 0, \quad x \in \Omega_2, \quad \omega_2 - \omega_1|_{x \in S_1} = 0, \quad \omega_2|_{x \in S} = 0.$$

However, this determines $\vec{E}^{(2)}$ only up to a linear combination $\sum_{j=1}^l c_j(t) \vec{v}_j(x)$, where the $\vec{v}_j(x)$ are ‘‘Dirichlet vector fields’’. They satisfy

$$\operatorname{curl} \vec{v}_j(x) = 0, \quad \nabla \cdot \vec{v}_j(x) = 0, \quad x \in \Omega_2, \quad \vec{v}_j|_{x \in S_k} = 0,$$

and are the gradients $\nabla \chi_j$ of harmonic functions satisfying the conditions

$$\chi_j(x)|_{x \in S_k} = \delta_{jk}, \quad j, k = 1, \dots, l,$$

where the S_k are all connected components of the boundary of Ω_2 except for one of them (e.g., except for S or S_1). Clearly, in Ω_2 there exists at least one Dirichlet vector field such that

$$\chi_1(x)|_{x \in S_1} = 1, \quad \chi_1(x)|_{x \in S} = 0.$$

To determine $\vec{E}^{(2)}$ in a unique way, it suffices to impose the condition

$$(4.13) \quad 0 = \int_{\Omega_2} \vec{v}_j(x) \cdot \vec{E}(x, t) dx = \int_{S_j} \vec{E} \cdot \vec{n} dS, \quad j = 1, \dots, l.$$

For $l = 1$ this reduces to $\int_S \vec{E}^{(2)} \cdot \vec{n} dS = \int_{S_1} \vec{E} \cdot \vec{n} dS = 0$. The vector field $\vec{E}$ defined in this way satisfies the inequalities

$$\|\vec{E}^{(i)}\|_{W_p^{1,0}(Q_T^i)} \leq c(\|\vec{E}_0\|_{W_p^{1,0}(Q_T^i)} + \|\nabla \omega_i\|_{W_p^{1,0}(Q_T^i)}), \quad i = 1, 2,$$

which imply

$$(4.14) \quad \begin{aligned} \sum_{i=1}^2 \|\vec{E}^{(i)}\|_{W_p^{1,0}(Q_T^i)} &\leq c(\|\mu \vec{H}_t\|_{L_p(Q_T)} + \|\operatorname{curl} \vec{H}\|_{W_p^{1,0}(Q_T^1)} + \|\vec{j}\|_{W_p^{1,0}(Q_T^1)}) \\ &\leq c\left(\sum_{i=1}^2 \|\vec{H}^{(i)}\|_{W_p^{2,1}(Q_T^i)} + \|\vec{j}\|_{W_p^{1,0}(Q_T^1)}\right). \end{aligned}$$

§5. THE CASE OF NONHOMOGENEOUS INITIAL CONDITIONS

Now we consider problem (4.3), (4.11) with $\vec{H}_0 \in W_p^{2-2/p}(\Omega_1) \cap W_p^{2-2/p}(\Omega_2) \cap \mathcal{H}_\perp$ different from zero. This means that $\vec{H}_0^{(i)}$ are divergence free vector fields of class $W_p^{2-2/p}(\Omega_i)$, $i = 1, 2$, $\vec{H}_0^{(2)}(x) = \nabla \varphi_0(x) + \sum_{i=1}^{h+h_1} \varkappa_i \vec{u}_i(x)$, the function $\varphi_0 \in W_p^{3-2/p}(\Omega_2)$ is defined (up to a constant) by (1.10), (1.11); also, in the case where $p \geq 3/2$, condition $\vec{H}_{0\tau}^{(1)} = \nabla_\tau \varphi_0 \sum_{i=0}^{h+h_1} \varkappa_i \vec{u}_{i\tau}(x)$ is satisfied (see the comments on Theorem 1.1). Finally, $\vec{H}_0$ should satisfy (4.11).

We construct an auxiliary vector field $\vec{B}(x, t)$ such that $\vec{B}^{(i)} \in W_p^{2,1}(Q_T^i)$, $\vec{B} \in \mathcal{H}_\perp$, and moreover,

$$(5.1) \quad \begin{aligned} \vec{B}^{(i)}(x, t) &= 0 \quad \text{for } t > T_0, \quad \vec{B}(x, 0) = \vec{H}_0(x), \\ \nabla \cdot \vec{B}^{(i)}(x, t) &= 0, \quad i = 1, 2, \quad \vec{B}^{(2)}(x, t) = \nabla \Phi(x, t) + \sum_{i=1}^{h+h_1} k_i(t) \vec{u}_i(x), \\ \nabla^2 \Phi &= 0, \quad x \in \Omega_2, \quad \frac{\partial \Phi}{\partial n} \Big|_{x \in S} = 0, \quad \mu_2 \frac{\partial \Phi}{\partial n} - \mu_1 \vec{B}^{(1)} \cdot \vec{n} \Big|_{x \in S_1} = 0, \\ \vec{B}_\tau^{(1)}(x, t) &= \vec{B}_\tau^{(2)}(x, t), \quad x \in S_1, \\ k_i(0) &= \varkappa_i, \quad i = 1, \dots, h + h_1, \end{aligned}$$

We outline the construction of such a field. In the case of simply connected domains Ω_1 and Ω , this construction was given in [10] for $p = 2$.

First, we construct a divergence free vector field $\vec{B}^{(1)}(x, t)$, $x \in \Omega_1$ such that $\vec{B}^{(1)} \Big|_{t=0} = \vec{H}_0^{(1)}$,

$$\|\vec{B}_1^{(1)}\|_{W_p^{2,1}(Q_\infty^1)} \leq c\|\vec{H}_0^{(1)}\|_{W_p^{2-2/p}(\Omega_1)},$$

and $\vec{B}_1^{(1)}(x, t) = 0$ for $t > T_0$ (the last property is achieved by multiplication by an appropriate cutoff function $\zeta(t)$). Then we find $\Phi(x, t)$ as a solution of problem (1.2) with $\vec{H}^{(1)} = \vec{B}_1^{(1)}$; it satisfies the inequality

$$\|\nabla \Phi\|_{W_p^{2,1}(Q_\infty^2)} \leq c\|\vec{B}_1^{(1)}\|_{W_p^{2,1}(Q_\infty^1)}$$

and also vanishes for $t > T_0$. Also, we find $k_i(t)$ such that $k_i(0) = \varkappa_i$, $k_i(t) = 0$ for $t > T_0$, and

$$\|\vec{k}\|_{W_p^1(0,T)} \leq c|\vec{\varkappa}|.$$

Then we set $\vec{B}_1^{(2)} = \nabla \Phi + \sum_{i=1}^{h+h_1} k_i(t) \vec{u}_i(x)$. It is easily seen that $\mu_2 \vec{B}_1^{(2)} \cdot \vec{n}|_{x \in S_1} = \mu_1 \vec{B}_1^{(1)} \cdot \vec{n}|_{x \in S_1}$, but the tangential components of $\vec{B}_1^{(1)}$ and $\vec{B}_1^{(2)}$ on S_1 do not coincide. Let $\vec{b} = [\vec{B}_1]|_{x \in S_1}$, and let $\vec{B}_2^{(1)}$ be a vector field (it may fail to be solenoidal) such that $\vec{B}_2^{(1)}(x, t)|_{x \in S_1} = -\vec{b}(x, t)$, $\vec{B}_2^{(1)}(x, t) = 0$ for $t < 0$ and $t > T_0$, and

$$\|\vec{B}_2^{(1)}\|_{W_p^{2,1}(Q_\infty)} \leq c \|\vec{b}\|_{W_p^{2-1/p, 1-1/2p}(S_1 \times \mathbb{R})} \leq c \sum_{i=1}^2 \|\vec{B}_1^{(i)}\|_{W_p^{2,1}(Q_\infty)}.$$

Finally, we define $\vec{B}_3^{(1)}$ as a vector field satisfying the relations

$$\begin{aligned} \nabla \cdot \vec{B}_3^{(1)}(x, t) &= -\nabla \cdot \vec{B}_2^{(1)}(x, t), \quad x \in \Omega_1, \\ \vec{B}_3^{(1)}(x, t)|_{x \in S_1} &= 0 \end{aligned}$$

and the inequalities

$$\|\vec{B}_3^{(1)}\|_{W_p^2(\Omega_1)} \leq c \|\vec{B}_2^{(1)}\|_{W_p^2(\Omega_1)}, \quad \|\vec{B}_{3,t}^{(1)}\|_{L_p(\Omega_1)} \leq c \|\vec{B}_{2,t}^{(1)}\|_{L_p(\Omega_1)},$$

which imply that

$$\|\vec{B}_3^{(1)}\|_{W_p^{2,1}(Q_T^1)} \leq c \|\vec{B}_2^{(1)}\|_{W_p^{2,1}(Q_T^1)}.$$

The existence of such a vector field follows from Bogovskii's result [11]. Clearly, we have $\vec{B}_3^{(1)}(x, t) = 0$ for $t > T_0$, $t < 0$.

We set $\vec{B}_2^{(2)} = \vec{B}_3^{(2)} = 0$.

The sum $\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3$ satisfies conditions (5.1) and

$$(5.2) \quad \sum_{i=1}^2 \|\vec{B}^{(i)}\|_{W_p^{2,1}(Q_\infty^{(i)})} \leq c \sum_{i=1}^2 \|\vec{H}_0^{(i)}\|_{W_p^{2-2/p}(\Omega_i)},$$

but, in general, $\vec{B} \notin \mathcal{H}_\perp$. Therefore, we introduce $\vec{B}_4(x, t) = \sum_{q=1}^h C_q(t) \vec{U}_q$, where the C_q are solutions of the system $\sum_{q=1}^h C_q(t) \int_\Omega \mu \vec{U}_q \cdot \vec{U}_m dx = -\int_\Omega \mu \vec{B} \cdot \vec{U}_m dx$, $m = 1, \dots, h$. The vector field $\vec{B}_4$ is such that $\vec{B}_4(x, t) = 0$ for $t < 0$, $t > T_0$ and

$$\sum_{i=1}^2 \|\vec{B}_4^{(i)}\|_{W_p^{2,1}(Q_\infty^{(i)})} \leq c \|\vec{C}\|_{W_p^1(0, \infty)} \leq c (\|\vec{B}\|_{L_p(Q_\infty)} + \|\vec{B}_t\|_{L_p(Q_\infty)}).$$

The sum $\vec{D} = \vec{B} + \vec{B}_4$ belongs to $\mathcal{H}_\perp$ and satisfies the initial condition $\vec{D}|_{t=0} = \vec{H}_0$, and inequality (5.2). For the difference $\vec{\mathfrak{H}} = \vec{H} - \vec{D}$ we obtain the problem

$$\begin{aligned} \mu \vec{\mathfrak{H}}_t + \text{curl } \vec{E} &= -\mu \vec{D}_t, \quad \nabla \cdot \vec{\mathfrak{H}} = 0, \quad x \in \Omega \cup \Omega_2, \\ \text{curl } \vec{\mathfrak{H}} &= \alpha \vec{E} + \vec{j}(x, t) - \text{curl } \vec{D}, \quad x \in \Omega_1, \\ \text{curl } \vec{\mathfrak{H}} &= 0, \quad \nabla \cdot \vec{E} = 0, \quad x \in \Omega_2, \\ [\mu \vec{\mathfrak{H}} \cdot \vec{n}] &= 0, \quad [\vec{\mathfrak{H}}_\tau] = 0, \quad [\vec{E}_\tau] = 0, \quad x \in S_1, \\ \vec{\mathfrak{H}} \cdot \vec{n} &= 0, \quad \vec{E}_\tau = 0, \quad x \in S, \\ \int_{S_j} \vec{E}_j \cdot \vec{n} dS &= 0, \quad j = 1, \dots, l, \\ \vec{\mathfrak{H}}(x, 0) &= 0, \quad x \in \Omega_1 \cup \Omega_2. \end{aligned}$$

The solution $\vec{\mathfrak{H}}, \vec{E}$ satisfies the inequality

$$\sum_{i=1}^2 \|\vec{\mathfrak{H}}\|_{W_p^{2,1}(Q_\infty^{(i)})} + \sum_{i=1}^2 \|\vec{E}\|_{W_p^{1,0}(Q_\infty^{(i)})} \leq c \left(\|\vec{j}\|_{L_p(Q_T^1)} + \|\text{curl } \vec{j}\|_{L_p(Q_T^1)} + \sum_{i=1}^2 \|\vec{D}\|_{W_p^{2,1}(Q_\infty^{(i)})} \right),$$

the proof of which does not differ from the proof of (4.10), (4.14). Hence,

$$(5.3) \quad \sum_{i=1}^2 \|\vec{H}\|_{W_p^{2,1}(Q_\infty^{(i)})} + \sum_{i=1}^2 \|\vec{E}\|_{W_p^{1,0}(Q_\infty^{(i)})} \\ \leq c \left(\|\vec{j}\|_{L_p(Q_T^1)} + \|\operatorname{curl} \vec{j}\|_{L_p(Q_T^1)} + \sum_{i=1}^2 \|\vec{H}_0^{(i)}\|_{W_p^{2-2/p}(\Omega_i)} \right).$$

Estimate (1.6) serves as an analog of (4.14) for problem (1.1).

REFERENCES

- [1] O. A. Ladyzhenskaya and V. A. Solonnikov, *Solution of some nonstationary problems of magnetohydrodynamics for a viscous incompressible fluid*, Trudy Mat. Inst. Steklov. **59** (1960), 115–173. (Russian) MR0170130 (30:370)
- [2] ———, *The linearization principle and invariant manifolds for problems of magnetohydrodynamics*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **38** (1973), 46–93; English transl., J. Soviet Math. **8** (1977), no. 4, 384–422. MR0377310 (51:13482)
- [3] G. Ströhmer, *About an initial-boundary value problem from magnetohydrodynamics*, Math. Z. **209** (1992), 345–362. MR1152262 (93k:76126)
- [4] ———, *About the equations of non-stationary magnetohydrodynamics*, Nonlinear Anal. **52** (2003), 1249–1273. MR1941256 (2004h:76169)
- [5] S. J. N. Mosconi and V. A. Solonnikov, *On a problem of magnetohydrodynamics in a multi-connected domain*, Nonlinear Anal. **74** (2011), no. 2, 462–478. MR2733223 (2011g: 35320)
- [6] K. K. Golovkin, *On equivalent normalizations of fractional spaces*, Trudy Mat. Inst. Steklov. **66** (1962), 364–383. (Russian) MR0154028 (27:3988)
- [7] V. A. Solonnikov, *Estimates for solutions of the nonstationary Stokes problem in anisotropic Sobolev spaces and estimates for the resolvent of the Stokes operator*, Uspekhi Math. Nauk **58** (2003), no. 2, 123–156; English transl., Russian Math. Surveys **58** (2003), no. 2, 331–365. MR1992567 (2004h:35184)
- [8] ———, *The solvability of the second initial-boundary value problem for a linear nonstationary system of Navier–Stokes equations*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **69** (1977), 200–218; English transl., J. Soviet Math. **10** (1978), no. 1, 141–155. MR0609731 (58:29438)
- [9] È. B. Byhovskii and N. V. Smirnov, *Orthogonal decomposition of the space of vector functions square-summable on a given domain, and the operators of vector analysis*, Trudy Mat. Inst. Steklov **59** (1960), 5–36. (Russian) MR0121641 (22:12375)
- [10] M. Padula and V. A. Solonnikov, *On the free boundary problem of magnetohydrodynamics*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **385** (2010), 135–186. MR2749373
- [11] M. E. Bogovskii, *Solutions of some problems of vector analysis, associated with the operators div and grad*, Trudy Sem. S. L. Soboleva **1980**, no. 1, 5–40. (Russian) MR0631691 (82m:26014)

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