TRAPPED MODES IN AN ELASTIC PLATE WITH A HOLE

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In the memory of our dear teacher M. Sh. Birman

ABSTRACT. For an infinite linear elastic plate with stress-free boundary, the trapped modes arising around holes in the plate are investigated. These are L^2 -eigenvalues of the elastostatic operator in the punched plate subject to Neumann type stress-free boundary conditions at the surface of the hole. It is proved that the perturbation gives rise to infinitely many eigenvalues embedded into the essential spectrum. The eigenvalues accumulate to a positive threshold. An estimate of the accumulation rate is given.

§1. INTRODUCTION

1.1. We consider a homogeneous and isotropic linear elastic medium in the domain $G = \mathbb{R}^2 \times J$ with $J = (-\frac{\pi}{2}, \frac{\pi}{2})$. For functions $u, v \in H^1(G; \mathbb{C}^3)$, we set

$$\epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \quad \text{and} \quad \langle \epsilon(u), \epsilon(v) \rangle_{\mathbb{C}^{3 \times 3}} = \sum_{i,j=1}^3 \epsilon_{ij}(u) \overline{\epsilon_{ji}(v)}.$$

Then the quadratic form

$$(1.1) \quad a[u] = 2 \int_G \langle \epsilon(u), \epsilon(u) \rangle_{\mathbb{C}^{3 \times 3}} dx$$

induces the elastostatic operator A for materials with zero Poisson ratio; A corresponds to the differential expression

$$(1.2) \quad -(\Delta + \operatorname{grad} \operatorname{div})$$

with stress-free (Neumann-type) boundary conditions.¹ The operator A has purely absolutely continuous spectrum filling the nonnegative half-line.

Now, consider a punched plate $\Omega^c = G \setminus \overline{\Omega}$ with a hole $\Omega = \Omega_0 \times J$, where $\Omega_0 \subset \mathbb{R}^2$ is a bounded Lipschitz domain. Let A_{Ω^c} be the elastostatic operator corresponding to the differential expression (1.2) on the outer domain Ω^c subject to stress-free (Neumann-type) boundary conditions. This geometric perturbation does not change the location of the essential spectrum, but it gives rise to a somewhat unexpected trapping effect. As the main result of this paper we prove the existence of infinitely many eigenvalues $\{\nu_k\}_{k \in \mathbb{N}}$ of A_{Ω^c} , embedded into the essential spectrum, which accumulate at a certain threshold $\Lambda > 0$, and we compute the accumulation rate of these trapped modes. We establish the formula

$$(1.3) \quad \ln(\Lambda - \nu_k) = -2k \ln k + o(k \ln k) \quad \text{as} \quad k \rightarrow \infty.$$

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¹Here we have chosen a suitable set of units such that Young's modulus E fulfills $E = 2$. Moreover, we point out that the special choice of zero Poisson ratio is essential for the results of this paper.

This paper is related to an extensive series of works on trapped modes in perturbed quantum [1]–[5], acoustic [6]–[10], and elastic waveguides [11]–[15].

The existence of eigenvalues near a lower threshold of a suitable spectral branch of the operator can be shown by reducing the problem to the corresponding scattering channel via separation of variables. The spectral structure of the model near the threshold is then given by the behavior of an eigenvalue branch of a parameter dependent Sturm–Liouville problem. In the acoustic and the quantum case, this usually turns out to be a nondegenerate quadratic minimum. Therefore, one can carry out a weak-coupling analysis in the spirit of [16] for an effective Laplacian.

However, elastic waveguides are given by systems of partial differential equations. The structure of the spectral minimum of a scattering channel is described by the behavior of an eigenvalue branch of a certain effective matrix-valued Sturm–Liouville problem. It turns out that for the scattering channel of interest this minimum is strongly degenerate, what leads to a weak-coupling analysis of a pseudodifferential operator with degenerate symbol in the spirit of [17]. In particular, the minimum will be attained on the circle $\{\xi \in \mathbb{R}^2 : |\xi| = \varkappa\}$ for some $\varkappa > 0$. This means that there are infinitely many wave functions in the spectral minimum, which may turn into eigenfunctions in the case of perturbation. A corresponding analysis for additive perturbations was carried out in [15]. Here we consider the case of geometric perturbations by Neumann-type boundary conditions.

First, we transform the problem with boundary perturbation into a problem with additive perturbation. For this, we pass to the resolvents of the elasticity operators. Then the additive perturbation is described by suitable operators of elastic potential theory. The main task is now to analyze the spectral behavior of this perturbation operator.

As in [15], we need to deal with super-polynomial accumulation rates of the eigenvalues. Note that such accumulation rates have recently been obtained also in other physical settings; see, e.g., [18]–[20]. In general, they rest on the combination of compactly supported perturbations applied to strongly degenerate operators.

The paper is structured as follows. In §2 we state the problem in detail. In §3 we describe the spatial and internal symmetries of the problem that allow us to distinguish embedded eigenvalues from the surrounding essential spectrum. Considering the elasticity problem for a suitable symmetry case, we give a short description of its degenerate spectral minimum. See [14, 15] for a detailed investigation.

In §4 we state our main results. There we also give a proof of the existence result for holes, which is based on the results of [15]. In §5 we provide the formulation of the perturbation problem in terms of the inverse operators. For the investigation of the accumulation rate we need further auxiliary material, which is presented in §6. There we introduce general classes of operators that allow us to handle super-polynomial eigenvalue asymptotics beyond the well-known Σ_p -classes for compact operators. Furthermore, we state a special type of Birman–Schwinger principle appropriate to our problem. Finally, we introduce the calculus of [17] and describe the behavior of the Birman–Schwinger operator along the spectral minimum of the unperturbed elasticity operator.

The main proof of the accumulation rate asymptotics can be found in §7. The general idea of the proof is to obtain properties of the spectrum of the unbounded elasticity operator by investigating the compact Birman–Schwinger operator. This is done by extracting the part of the Birman–Schwinger operator that is responsible for the leading term in the eigenvalue asymptotics, calculating its properties, and estimating the remainder terms.

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§2. THE LINEAR ELASTOSTATIC OPERATOR FOR ZERO POISSON COEFFICIENT

2.1. The unperturbed operator. We recall that $G = \mathbb{R}^2 \times J$ with $J = (-\frac{\pi}{2}, \frac{\pi}{2})$. In general, the quadratic form describing linear elastostatics with stress-free boundary conditions is given by

$$(2.1) \quad a[u] = \int_G \langle \sigma(u), \epsilon(u) \rangle_{\mathbb{C}^{3 \times 3}} dx$$

on all functions $u \in D[a] = H^1(G; \mathbb{C}^3)$. Here

$$\epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \quad \text{and} \quad \sigma(u) = \lambda(\text{Tr } \epsilon(u))\mathbf{I} + 2\mu\epsilon(u)$$

denote the strain and stress matrix with the Lamé constants

$$\mu = \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)},$$

where ν is Poisson's ratio and E is Young's modulus.

In this paper we focus on the special case where Poisson's ratio is zero. For a suitable set of units such that $E = 2$, the form (2.1) turns into

$$(2.2) \quad a[u] = 2 \int_G \langle \epsilon(u), \epsilon(u) \rangle_{\mathbb{C}^{3 \times 3}} dx \quad \text{on} \quad D[a] = H^1(G; \mathbb{C}^3).$$

This form is well defined because, obviously,

$$(2.3) \quad a[u] \leq 2\|u\|_{H^1(G; \mathbb{C}^3)}^2, \quad u \in H^1(G; \mathbb{C}^3).$$

On the other hand, Korn's inequality implies also the reverse estimate

$$(2.4) \quad a[u] + \|u\|_{L^2(G; \mathbb{C}^3)}^2 \geq c\|u\|_{H^1(G; \mathbb{C}^3)}^2, \quad u \in H^1(G; \mathbb{C}^3),$$

for some $c > 0$, see [22]. Thus, the nonnegative form a is closed on the maximal domain $D[a] = H^1(G; \mathbb{C}^3)$. It is associated with the positive selfadjoint operator

$$(2.5) \quad A = -(\Delta + \text{grad div})$$

on the domain

$$D(A) = \{u \in H^2(G; \mathbb{C}^3) : \partial_j u_3 + \partial_3 u_j = 0, j = 1, 2, 3 \text{ on } \partial G\}.$$

2.2. Perturbation by a hole. Let $\Omega_0 \subset \mathbb{R}^2$ be a nonempty, connected, bounded Lipschitz domain. Put $\Omega := \Omega_0 \times J$ and $\Omega^c = G \setminus \overline{\Omega}$. We shall study the two quadratic forms

$$(2.6) \quad a_{\Omega^c}[u] = 2 \int_{\Omega^c} \langle \epsilon(u), \epsilon(u) \rangle_{\mathbb{C}^{3 \times 3}} dx, \quad u \in D[a_{\Omega^c}] := H^1(\Omega^c; \mathbb{C}^3),$$

$$(2.7) \quad a_{\Omega}[u] = 2 \int_{\Omega} \langle \epsilon(u), \epsilon(u) \rangle_{\mathbb{C}^{3 \times 3}} dx, \quad u \in D[a_{\Omega}] := H^1(\Omega; \mathbb{C}^3).$$

As before, we have

$$(2.8) \quad a_{\Omega^c}[u] \leq 2\|u\|_{H^1(\Omega^c; \mathbb{C}^3)}^2 \quad \text{and} \quad a_{\Omega}[u] \leq 2\|u\|_{H^1(\Omega; \mathbb{C}^3)}^2,$$

and Korn's inequality implies that

$$(2.9) \quad a_{\Omega^c}[u] + \|u\|_{L^2(\Omega^c; \mathbb{C}^3)}^2 \geq c(\Omega^c) \|u\|_{H^1(\Omega^c; \mathbb{C}^3)}^2, \quad u \in H^1(\Omega^c; \mathbb{C}^3),$$

$$(2.10) \quad a_{\Omega}[u] + \|u\|_{L^2(\Omega; \mathbb{C}^3)}^2 \geq c(\Omega) \|u\|_{H^1(\Omega; \mathbb{C}^3)}^2, \quad u \in H^1(\Omega; \mathbb{C}^3).$$

Hence, the two forms a_{Ω^c} and a_{Ω} are closed on the maximal domains $D[a_{\Omega^c}] = H^1(\Omega^c; \mathbb{C}^3)$ and $D[a_{\Omega}] = H^1(\Omega; \mathbb{C}^3)$, respectively. They are associated with the selfadjoint operators A_{Ω^c} in $L^2(\Omega^c; \mathbb{C}^3)$ and A_{Ω} in $L^2(\Omega; \mathbb{C}^3)$, subject to the stress-free boundary conditions

$$(\nabla u + (\nabla u)^T) \mathbf{n}_{\partial\Omega^c} = 0, \quad (\nabla u + (\nabla u)^T) \mathbf{n}_{\partial\Omega} = 0,$$

where $\mathbf{n}_{\partial\Omega^c}$ and $\mathbf{n}_{\partial\Omega}$ denote the outward unit normals for $\partial\Omega^c$ and $\partial\Omega$, respectively. The operator A_{Ω} describes the interior problem in Ω and A_{Ω^c} describes the exterior problem in Ω^c . The orthogonal sum of these two operators gives

$$(2.11) \quad A_{\Gamma} := A_{\Omega^c} \oplus A_{\Omega} \quad \text{in} \quad L^2(G; \mathbb{C}^3) = L^2(\Omega^c; \mathbb{C}^3) \oplus L^2(\Omega; \mathbb{C}^3).$$

Here we use the notation

$$(2.12) \quad \Gamma = \Gamma_0 \times J \quad \text{with} \quad \Gamma_0 = \partial\Omega_0.$$

Observe that the operator A_{Γ} acts in the same Hilbert space $L^2(G; \mathbb{C}^3)$ as A . It is associated with the quadratic form

$$(2.13) \quad a_{\Gamma}[u] = 2 \int_{G \setminus \Gamma} \langle \epsilon(u), \epsilon(u) \rangle_{\mathbb{C}^3 \times 3} dx, \quad u \in D[a_{\Gamma}] := H^1(G \setminus \Gamma; \mathbb{C}^3).$$

We point out that the operator A_{Γ} corresponds to a perturbation of A by Neumann-type boundary conditions at Γ . We have

$$\begin{aligned} D(A_{\Gamma}) &= \{u \in H^2(G \setminus \Gamma; \mathbb{C}^3) : (\nabla u + (\nabla u)^T) \mathbf{n}_{\partial G} = 0 \text{ on } \partial G, \\ &\quad (\nabla u + (\nabla u)^T) \mathbf{n}_{\Gamma} = 0 \text{ on } \Gamma\}, \end{aligned}$$

where $\mathbf{n}_{\partial G}$ and $\mathbf{n}_{\Gamma}$ denote the outward unit normals for ∂G and Γ , respectively.

§3. SYMMETRY DECOMPOSITION AND INITIAL SPECTRAL ANALYSIS

3.1. Spatial and internal symmetries. As in [15], we define the subspaces $\mathcal{H}_j$, $j = 1, 2$, of $\mathcal{H} = L^2(G; \mathbb{C}^3)$ by

$$\begin{aligned} \mathcal{H}_j &:= \{u \in \mathcal{H} : u_l(\cdot, \cdot, -x_3) = (-1)^{j-1} u_l(\cdot, \cdot, x_3), \quad l = 1, 2, \\ &\quad u_3(\cdot, \cdot, -x_3) = (-1)^j u_3(\cdot, \cdot, x_3), \quad x_3 \in J\}. \end{aligned}$$

Furthermore, we set

$$\mathcal{H}_3 := \{u \in \mathcal{H} : u(x) = (u_1(x_1, x_2), u_2(x_1, x_2), 0), \quad x \in G\}$$

and denote by $\mathcal{H}_4$ the orthogonal complement of $\mathcal{H}_3$ in $\mathcal{H}_1$, which consists of all functions $u \in \mathcal{H}_1$ such that

$$\int_J u_i(x_1, x_2, x_3) dx_3 = 0 \quad \text{for almost every } (x_1, x_2) \in \mathbb{R}^2, \quad i = 1, 2.$$

By P_j we denote the orthogonal projections onto $\mathcal{H}_j$, $j = 1, \dots, 4$. Then $P_j P_1 = P_1 P_j = P_j$ for $j = 3, 4$. Defining

$$(3.1) \quad D[a^{(j)}] := P_j D[a] \subset D[a], \quad j = 1, \dots, 4,$$

we see that

$$(3.2) \quad a[u, v] = 0 \quad \text{for all } u \in D[a^{(l)}], \quad v \in D[a^{(j)}]$$

with $l, j = 2, 3, 4$ and $l \neq j$. Hence, these subspaces are reducing² for the operator A , and $A = A^{(3)} \oplus A^{(4)} \oplus A^{(2)}$ on

$$(3.3) \quad \mathcal{H} = \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_2,$$

where the operators $A^{(j)}$ are the restrictions of A to $D(A^{(j)}) = D(A) \cap \mathcal{H}_j$ and correspond to the closed forms $a^{(j)}$ given by the differential expression (1.1) on $D[a^{(j)}]$, $j = 2, 3, 4$.

Since the boundary Γ is parallel to the x_3 -direction, a simple calculation shows that the orthogonal decomposition (3.3) is also reducing for the perturbed operator $A_\Gamma = A_\Gamma^{(3)} \oplus A_\Gamma^{(4)} \oplus A_\Gamma^{(2)}$. In the case of a Lipschitz domain $\Omega = \Omega_0 \times J$, we have a similar symmetry decomposition for

$$A_\Omega = A_\Omega^{(3)} \oplus A_\Omega^{(4)} \oplus A_\Omega^{(2)} \quad \text{and} \quad A_{\Omega^c} = A_{\Omega^c}^{(3)} \oplus A_{\Omega^c}^{(4)} \oplus A_{\Omega^c}^{(2)}.$$

3.2. Separation of variables for the operator A . Applying the unitary Fourier transformation Φ in (x_1, x_2) -direction and its inverse Φ^* , we see that $\Phi A \Phi^*$ admits the orthogonal expansion

$$\Phi A \Phi^* = \int_{\mathbb{R}^2}^\oplus A(\xi) \, d\xi \quad \text{on} \quad \mathcal{H} = \int_{\mathbb{R}^2}^\oplus h \, d\xi, \quad h = L^2(J, \mathbb{C}^3).$$

The selfadjoint operators $A(\xi)$ are given by the differential expressions

$$(3.4) \quad A(\xi) = \begin{pmatrix} -\partial_3^2 + |\xi|^2 + \xi_1^2 & \xi_1 \xi_2 & -i\xi_1 \partial_3 \\ \xi_1 \xi_2 & -\partial_3^2 + |\xi|^2 + \xi_2^2 & -i\xi_2 \partial_3 \\ -i\xi_1 \partial_3 & -i\xi_2 \partial_3 & -2\partial_3^2 + |\xi|^2 \end{pmatrix}$$

on the domains

$$(3.5) \quad D(A(\xi)) = \left\{ u \in H^2(J; \mathbb{C}^3) : \begin{aligned} \partial_3 u_3(\pm \frac{\pi}{2}) &= \partial_3 u_1(\pm \frac{\pi}{2}) + i\xi_1 u_3(\pm \frac{\pi}{2}) = 0, \\ \partial_3 u_2(\pm \frac{\pi}{2}) &+ i\xi_2 u_3(\pm \frac{\pi}{2}) = 0 \end{aligned} \right\}.$$

The symmetry (3.3) extends to the operators $A(\xi)$. Indeed, if for $j = 1, 2$ we put

$$h_j := \{ u \in h : u_l(t) = (-1)^{j-1} u_l(-t), \, l = 1, 2; \, u_3(t) = (-1)^j u_3(-t) \}$$

and

$$h_3 := \text{span}\{(1, 0, 0), (0, 1, 0)\}, \quad h_4 := \left\{ u \in h_1 : \int_J u_i(t) \, dt = 0, \, i = 1, 2 \right\},$$

then

$$(3.6) \quad \Phi A^{(j)} \Phi^* = \int_{\mathbb{R}^2}^\oplus A^{(j)}(\xi) \, d\xi \quad \text{on} \quad \mathcal{H}_j = \int_{\mathbb{R}^2}^\oplus h_j \, d\xi, \quad j = 1, \dots, 4,$$

where the operators $A^{(j)}(\xi)$ are the restrictions of $A(\xi)$ to $D(A^{(j)}(\xi)) = D(A(\xi)) \cap h_j$. Moreover,

$$(3.7) \quad \begin{aligned} A(\xi) &= A^{(1)}(\xi) \oplus A^{(2)}(\xi) \quad \text{on} \quad h = h_1 \oplus h_2, \\ A(\xi) &= A^{(3)}(\xi) \oplus A^{(4)}(\xi) \oplus A^{(2)}(\xi) \quad \text{on} \quad h = h_3 \oplus h_4 \oplus h_2. \end{aligned}$$

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²For $\mathcal{H}_3$ and $\mathcal{H}_4$ to be reducing it is crucial to work in the case of the zero Poisson coefficient.

3.3. Spectral analysis for the operator $A^{(4)}(\xi)$. Throughout this paper, the spectral resolution of the operator $A^{(4)}$ will be of particular interest. Because of the expansion (3.6), in fact we need to carry out spectral analysis for the operators $A^{(4)}(\xi)$.

Being the restrictions of the nonnegative second order Sturm–Liouville systems (3.4) to $D(A(\xi)) \cap h_4$, the operators $A^{(4)}(\xi)$ have a nonnegative discrete spectrum, which accumulates to infinity only. We denote the eigenvalues by $\lambda_j(\xi)$, $j \in \mathbb{N}$, where the numbering is in the nondecreasing order, including multiplicities. Note that the eigenvalues depend on the absolute value of ξ only. Thus, sometimes we use the notation $\lambda_k = \lambda_k(r)$ for $r := |\xi| \geq 0$.

The following results on the lowest branch of eigenvalues were obtained in [14, 15]. They are crucial for our further investigations. We have

$$(3.8) \quad \Lambda := \min_{\xi \in \mathbb{R}^2} \lambda_1(\xi) > 0,$$

where the minimum is attained on the set $\{\xi \in \mathbb{R}^2 : |\xi| = \varkappa\}$ for some $\varkappa > 0$. Furthermore, in a suitable neighborhood of this set we have $\lambda_2(\xi) > \lambda_1(\xi)$, and $\lambda_1 = \lambda_1(r)$ is real analytic. The behavior of $\lambda_1(\cdot)$ near its minimum is given by

$$(3.9) \quad \lambda_1(\varkappa + \varepsilon) = \Lambda + q^2 \varepsilon^2 + O(\varepsilon^3) \quad \text{as } \varepsilon \rightarrow 0$$

with a certain $q > 0$. As stated in [15], a numerical evaluation gives

$$(3.10) \quad \begin{aligned} \varkappa &= 0.632138 \pm 10^{-6}, \\ \Lambda &= 1.887837 \pm 10^{-6}, \\ q &= 0.849748 \pm 10^{-6}. \end{aligned}$$

Furthermore, for $\xi \in \mathbb{R}^2$ in a neighborhood of $|\xi| = \varkappa$ the eigenfunction corresponding to $\lambda_1(\xi)$ is given by $\psi_1 = \tilde{\psi}_1 / \|\tilde{\psi}_1\|_{L^2(J; \mathbb{C}^3)}$, where

$$(3.11) \quad \tilde{\psi}_1(\xi, x_3) := (i\xi_1 d_1(|\xi|, x_3), i\xi_2 d_1(|\xi|, x_3), d_2(|\xi|, x_3))$$

and

$$(3.12) \quad \begin{aligned} d_1(r, t) &:= r\beta \cos\left(\frac{\pi}{2}\beta\right) \cos(\gamma t) + \frac{\gamma^2 \beta}{r} \cos\left(\frac{\pi}{2}\gamma\right) \cos(\beta t), \\ d_2(r, t) &:= -r\beta\gamma \cos\left(\frac{\pi}{2}\beta\right) \sin(\gamma t) + r\gamma^2 \cos\left(\frac{\pi}{2}\gamma\right) \sin(\beta t) \end{aligned}$$

with $\beta = \sqrt{\lambda_1(r) - r^2}$ and $\gamma = \sqrt{\frac{\lambda_1(r)}{2} - r^2}$. For ξ with $|\xi| = \varkappa$, the functions

$$(3.13) \quad w_\xi(x) = \psi_1(\xi, x_3) e^{i\xi \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$$

satisfy

$$(3.14) \quad -(\Delta + \text{grad div})w_\xi(x) = \Lambda w_\xi(x) \quad \text{for } x \in G.$$

3.4. The spectrum of A_Ω and $A_\Omega^{(4)}$. For a bounded Lipschitz domain Ω_0 and the respective hole $\Omega = \Omega_0 \times J$, the bounds (2.8) and (2.10) show that the quadratic form $a_\Omega[u] + \|u\|_{L^2(\Omega; \mathbb{C}^3)}^2$ is equivalent to the square of the norm on $H^1(\Omega; \mathbb{C}^3)$. Moreover, under these conditions the embedding of $H^1(\Omega; \mathbb{C}^3)$ into $L^2(\Omega; \mathbb{C}^3)$ is compact. Hence, the spectrum of A_Ω is discrete and accumulates to infinity only. The same holds true for the reduced component $A_\Omega^{(4)}$.

3.5. The essential spectrum. By (3.6), the spectrum of $A^{(4)}$ is given by the union of the eigenvalues $\lambda_j(\xi)$ of $A^{(4)}(\xi)$ over all $\xi \in \mathbb{R}^2$ and $j \in \mathbb{N}$. The eigenvalues are analytic with respect to $|\xi|$. Moreover, Korn's inequality shows that $\lambda_j(\xi) + d \geq c|\xi|^2$ for suitable constants $c, d > 0$. Therefore, the spectrum of $A^{(4)}$ is pure absolutely continuous. Since

$$\min \sigma_e(A^{(4)}) = \min_{\xi, j} \lambda_j(\xi) = \min_{\xi} \lambda_1(\xi) = \Lambda,$$

the essential spectrum $\sigma_e(A^{(4)})$ coincides with $[\Lambda, \infty)$.

As in [23], it can be shown that the essential spectrum of the locally perturbed operator $A_\Gamma^{(4)}$ coincides with the essential spectrum of $A^{(4)}$. Moreover, recall that $A_\Gamma = A_{\Omega^c} \oplus A_\Omega$, so that $A_\Gamma^{(4)} = A_{\Omega^c}^{(4)} \oplus A_\Omega^{(4)}$. Since the spectrum of $A_\Omega^{(4)}$ is purely discrete, the essential spectrum of $A_{\Omega^c}^{(4)}$ coincides with the essential spectrum of $A_\Gamma^{(4)}$, and we conclude finally that

$$(3.15) \quad \sigma_e(A_{\Omega^c}^{(4)}) = \sigma_e(A_\Gamma^{(4)}) = \sigma_e(A^{(4)}) = [\Lambda, \infty).$$

Moreover, we shall see that the perturbed operators $A_{\Omega^c}^{(4)}$ and $A_\Gamma^{(4)}$ have infinite discrete spectrum on $[0, \Lambda)$. Since a similar analysis for the full operators yields

$$(3.16) \quad \sigma_e(A_{\Omega^c}) = \sigma_e(A_\Gamma) = \sigma_e(A) = [0, \infty),$$

this discrete spectrum gives rise to embedded eigenvalues of A_{Ω^c} and A_Γ .

The reduction of the problem to that on the subspace $\mathcal{H}_4$ is a crucial step, which “uncovers” these eigenvalues and opens them for a detailed analysis via variational techniques.

§4. STATEMENT OF THE MAIN RESULTS

4.1. Existence of infinitely many eigenvalues. In the following, we state our main results on the existence and accumulation behavior of the eigenvalues of $A_{\Omega^c}^{(4)}$ and $A_\Gamma^{(4)}$.

Theorem 4.1. *The operators $A_{\Omega^c}^{(4)}$ and $A_\Gamma^{(4)}$ have infinitely many eigenvalues in $[0, \Lambda)$.*

Since the spectrum of $A_{\Omega^c}^{(4)}$ and $A_\Gamma^{(4)}$ in $[0, \Lambda)$ is discrete, these eigenvalues will accumulate to the threshold Λ only.

Here we provide a short proof, which involves a result in [15].

Proof. Let $\Gamma = \partial\Omega_0 \times J$ be the Lipschitz boundary of the hole $\Omega = \Omega_0 \times J$. Also, let $A_\alpha^{(4)}$ be the operator associated with the quadratic form

$$a_\alpha^{(4)}[u] = 2 \int_G (1 - \alpha\chi_\Omega) \langle \epsilon(u), \epsilon(u) \rangle_{\mathbb{C}^{3 \times 3}} dx, \quad u \in D[a_\alpha^{(4)}] := H^1(G; \mathbb{C}^3) \cap \mathcal{H}_4,$$

and let $A_{\alpha, \Gamma}^{(4)}$ be the operator associated with the form

$$a_{\alpha, \Gamma}^{(4)}[u] = 2 \int_{G \setminus \Gamma} (1 - \alpha\chi_\Omega) \langle \epsilon(u), \epsilon(u) \rangle_{\mathbb{C}^{3 \times 3}} dx, \quad u \in D[a_{\alpha, \Gamma}^{(4)}],$$

where $D[a_{\alpha, \Gamma}^{(4)}] := H^1(G \setminus \Gamma; \mathbb{C}^3) \cap \mathcal{H}_4$. Note that $A_{\alpha, \Gamma}^{(4)}$ can be viewed as a perturbation of $A_\alpha^{(4)}$ by Neumann-type boundary conditions on Γ . Thus, we have $A_{\alpha, \Gamma}^{(4)} < A_\alpha^{(4)}$. Moreover, by analogy with [23], the essential spectra of the two operators coincide.

It was shown in Theorem 4.1 in [15] that $A_\alpha^{(4)}$ has infinitely many eigenvalues in $[0, \Lambda)$ for any $\alpha \in (0, 1)$. By a standard variational argument, the operator $A_{\alpha, \Gamma}^{(4)}$ must have infinitely many eigenvalues in $[0, \Lambda)$ as well. Note that

$$A_{\alpha, \Gamma}^{(4)} = A_{\Omega^c}^{(4)} \oplus (1 - \alpha)A_\Omega^{(4)}.$$

Since $A_\Omega^{(4)}$ has purely discrete spectrum, the spectra of the operators $A_{\Omega^c}^{(4)}$ and $A_{\alpha,\Gamma}^{(4)}$ below Λ differ only in finitely many eigenvalues. Thus, also $A_{\Omega^c}^{(4)}$ has infinitely many eigenvalues in $[0, \Lambda)$. $\square$

4.2. Accumulation rate of the eigenvalues near Λ . For asymptotic estimates on the accumulation of eigenvalues at Λ , we set

$$(4.1) \quad w(t) := t^{-2t}, \quad t > 1.$$

For formal convenience we put $\omega(t) = t^{-1}$ for $0 < t \leq 1$, extending w to a continuous bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$. Furthermore, for a selfadjoint operator T we denote by $n_-(s, T)$ the number of eigenvalues smaller than $s \in \mathbb{R}$ (including multiplicities).

Theorem 4.2. *Consider the plate with a hole $\Omega = \Omega_0 \times J$, and let $\Gamma = \partial\Omega_0 \times J$. Then*

$$(4.2) \quad \lim_{\tau \rightarrow 0+} \frac{n_-(\Lambda - \tau, A_\Gamma^{(4)})}{w^{-1}(\tau)} = \lim_{\tau \rightarrow 0+} \frac{n_-(\Lambda - \tau, A_{\Omega^c}^{(4)})}{w^{-1}(\tau)} = 1.$$

Our result is similar to the asymptotics obtained in [15] for the elastic plate with changes of material properties.

Asymptotical estimates on the eigenvalues can be derived easily from these expressions. Let $\{\nu_k\}_{k \in \mathbb{N}}$ denote the eigenvalues of $A_{\Omega^c}^{(4)}$ and $\{\tilde{\nu}_k\}_{k \in \mathbb{N}}$ the eigenvalues of $A_\Gamma^{(4)}$ that lie below Λ , numbered in the nondecreasing order including multiplicities.

Corollary 4.3. *Consider the plate with a hole $\Omega = \Omega_0 \times J$. Then for every $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that*

$$(4.3) \quad w((1 + \varepsilon)k) \leq \Lambda - \nu_k \leq w((1 - \varepsilon)k) \quad \text{for } k > N_\varepsilon.$$

Moreover,

$$(4.4) \quad \ln(\Lambda - \nu_k) = -2k \ln k + o(k \ln k) \quad \text{as } k \rightarrow \infty.$$

In the remaining part of this paper we prove Theorem 4.2. Note that for the hole $\Omega = \Omega_0 \times J$ with boundary $\Gamma = \partial\Omega_0 \times J$ we have

$$(4.5) \quad A_\Gamma^{(4)} = A_{\Omega^c}^{(4)} \oplus A_\Omega^{(4)}.$$

Since $A_\Omega^{(4)}$ has purely discrete spectrum, the spectra of $A_\Gamma^{(4)}$ and $A_{\Omega^c}^{(4)}$ below Λ differ only in finitely many eigenvalues. Thus, it suffices to prove Theorem 4.2 for $A_\Gamma^{(4)}$ only.

§5. PASSAGE TO THE RESOLVENTS

5.1. Inverse operators. It is advantageous to study the geometric perturbation of $A^{(4)}$ to $A_\Gamma^{(4)}$ in terms of an additive perturbation of the inverse operators. Since $\sigma(A^{(4)}) = [\Lambda, \infty)$, the operator $A^{(4)}$ has a bounded inverse $R := (A^{(4)})^{-1}$. Put $R_\Gamma := (A_\Gamma^{(4)})^{-1}$.

Lemma 5.1. *The operator R_Γ is bounded on $L^2(G; \mathbb{C}^3)$.*

Proof. In view of (3.15), it remains to show that 0 is not an eigenvalue of $A_\Gamma^{(4)}$. Suppose the contrary. Then the corresponding eigenfunction u satisfies

$$(5.1) \quad 0 = a_\Gamma[u] = \int_{G \setminus \Gamma} 2 \sum_i |\partial_i u_i|^2 + \sum_{i < j} |\partial_i u_j + \partial_j u_i|^2 dx.$$

In particular, $\partial_3 u_3 = 0$ in $G \setminus \Gamma$. Since $u \in \mathcal{H}_4$, u_3 is antisymmetric, whence $u_3 = 0$. Inserting this into (5.1), we see that $\partial_3 u_1 = \partial_3 u_2 = 0$ in $G \setminus \Gamma$. Again, the condition $u \in \mathcal{H}_4$ implies $u_1 = u_2 = 0$. Thus, $u = 0$, which proves that zero is not an eigenvalue of $A_\Gamma^{(4)}$. $\square$

Both R and R_Γ are positive operators, and we denote the associated quadratic forms by r and r_Γ , respectively. Since $D[a^{(4)}] \subset D[a_\Gamma^{(4)}]$, it follows that $A_\Gamma^{(4)} < A^{(4)}$ in the quadratic form sense. Therefore, the operator

$$(5.2) \quad B_\Gamma := R_\Gamma - R$$

is bounded and nonnegative. In fact, B_Γ is a compact operator; this can be shown by using the standard method of [23]. In the following we give an initial characterization of B_Γ . A more detailed analysis of B_Γ is provided in Subsection 6.4.

5.2. First analysis of B_Γ . Since Γ_0 is Lipschitz, it has a surface measure S_{Γ_0} and a unit normal $\mathbf{n}_{\Gamma_0}$, which exists S_{Γ_0} -almost everywhere on Γ_0 ; see [22, p. 96]. For closed curves Γ_0 we choose $\mathbf{n}_{\Gamma_0}$ to point into the unbounded part of $\mathbb{R}^2$. For $\Gamma = \Gamma_0 \times J$, the corresponding unit normal is given by

$$(5.3) \quad \mathbf{n}_\Gamma = \begin{pmatrix} \mathbf{n}_{\Gamma_0} \\ 0 \end{pmatrix}.$$

Assume the normal vector exists at a point $x \in \Gamma$. We can choose some sufficiently small neighborhood $\mathcal{O} \subset G$ of x such that $\mathcal{O} \setminus \Gamma$ consists of two disjoint sets. We denote them by $\mathcal{O}^+$ and $\mathcal{O}^-$ so that $x \pm \varepsilon \mathbf{n}_\Gamma \in \mathcal{O}^\pm$ for sufficiently small $\varepsilon > 0$. Then

$$(5.4) \quad \gamma_D^+ u(x) := \lim_{\substack{y \rightarrow x \\ y \in \mathcal{O}^+}} u(y) \quad \text{and} \quad \gamma_D^- u(x) := \lim_{\substack{y \rightarrow x \\ y \in \mathcal{O}^-}} u(y)$$

are the trace operators for $u \in C_b(G \setminus \Gamma)$ with respect to Γ . The operators $\gamma_D^\pm$ admit extensions to bounded operators

$$(5.5) \quad \gamma_D^\pm : H^s(G \setminus \Gamma) \rightarrow H^{s-1/2}(\Gamma), \quad 1/2 < s \leq 1,$$

see [22, p. 100].

The application of $\gamma_D^\pm$ to a vector-valued or matrix-valued function means an individual application to each vector or matrix component. If for a given u we have $\gamma_D^+ u = \gamma_D^- u$, then we use also the shorthand notation $\gamma_D u := \gamma_D^\pm u$.

For functions $u \in C_b^1(G \setminus \Gamma; \mathbb{C}^3)$, we define the conormal derivative with respect to the operator A at Γ by

$$(5.6) \quad \gamma_N^\pm u := 2^{-\frac{1}{2}} (\gamma_D^\pm (\nabla u + (\nabla u)^T)) \mathbf{n}_\Gamma \quad \text{on } \Gamma.$$

Again we set $\gamma_N u := \gamma_N^+ u$ if $\gamma_N^+ u = \gamma_N^- u$. The boundedness of the operators (5.5) implies that the $\gamma_N^\pm$ extend to bounded operators

$$\gamma_N^\pm : H^s(G \setminus \Gamma; \mathbb{C}^3) \rightarrow H^{s-3/2}(\Gamma; \mathbb{C}^3), \quad 3/2 < s \leq 2.$$

Lemma 5.2. *For any $v \in \mathcal{H}_4$, the function $B_\Gamma v$ is a unique solution of the boundary-value problem*

$$(5.7) \quad -(\Delta + \text{grad div})u = 0 \quad \text{in } G \setminus \Gamma, \quad \gamma_N u = -\gamma_N Rv \quad \text{on } \Gamma,$$

for $u \in \mathcal{H}_4$ with $(\nabla u + (\nabla u)^T) \mathbf{n}_{\partial G} = 0$ on ∂G .

Proof. By the definition of B_Γ in (5.2), we have

$$B_\Gamma v = R_\Gamma v - Rv \in D(A_\Gamma^{(4)}) - D(A^{(4)}) \subset \mathcal{H}_4.$$

Furthermore,

$$-(\Delta + \text{grad div})B_\Gamma v = v - v = 0 \quad \text{in } G \setminus \Gamma$$

and

$$\gamma_N B_\Gamma v = \gamma_N R_\Gamma v - \gamma_N Rv = -\gamma_N Rv.$$

It remains to show that the solution of the problem (5.7) is unique. Suppose that $u \in \mathcal{H}_4$ with $(\nabla u + (\nabla u)^T) \mathbf{n}_{\partial G} = 0$ on ∂G satisfies (5.7). Put $w := u + Rv$. Then

$w \in P_4 H^2(G \setminus \Gamma; \mathbb{C}^3)$, $(\nabla w + (\nabla w)^T) \mathbf{n}_{\partial G} = 0$ on ∂G and $\gamma_N w = 0$ on Γ . Thus, $w \in D(A_\Gamma^{(4)})$. Since

$$A_\Gamma^{(4)} w = -(\Delta + \text{grad div})w = 0 = (\Delta + \text{grad div})Rv = v,$$

we see that $w = R_\Gamma v$. This proves that $u = B_\Gamma v$. $\square$

§6. AUXILIARY MATERIAL

6.1. The classes Σ_π and $\mathcal{S}^\pi$. In order to describe the accumulation rate for the eigenvalues of $A_\Gamma^{(4)}$ at Λ , we need to generalize the well-known weak Neumann–Schatten classes Σ_p ; see [24, 11.6], and the classes $\mathcal{S}^p$ introduced by Safronov [25, 17]. This is done as in [15].

Let $\pi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function with $\lim_{s \rightarrow 0+} \pi(s) = 0$. For suitable compact operators T , we define the functionals

$$\Delta_\pi(T) = \limsup_{s \rightarrow 0+} \pi(s)n(s, T), \quad \delta_\pi(T) = \liminf_{s \rightarrow 0+} \pi(s)n(s, T).$$

Here $n(s, T)$ denotes the number of singular values of T strictly larger than s (including multiplicities). We denote by Σ_π the class of compact operators T such that $\Delta_\pi(T) < \infty$. Similarly, for suitable operator-valued functions $\mathcal{T} : \mathbb{R}_+ \rightarrow \mathfrak{S}_\infty$ we define the functionals

$$\Delta_\pi(\mathcal{T}) = \limsup_{s \rightarrow 0+} \pi(s)n(1, \mathcal{T}(s)), \quad \delta_\pi(\mathcal{T}) = \liminf_{s \rightarrow 0+} \pi(s)n(1, \mathcal{T}(s)).$$

We denote by $\mathcal{S}^\pi$ the class of operator-valued functions $\mathcal{T}$ such that $\Delta_\pi(\beta\mathcal{T}) < \infty$ for all $\beta > 0$. Furthermore, $\mathcal{S}_0^\pi$ is the class of all $\mathcal{T} \in \mathcal{S}^\pi$ such that $\Delta_\pi(\beta\mathcal{T}) = 0$ for all $\beta > 0$. We will need the following two lemmas, which were proved in [15]. See also [17] for the polynomial case.

Lemma 6.1. *For $\mathcal{T} \in \mathcal{S}^\pi$ and $\mathcal{T}_0 \in \mathcal{S}_0^\pi$,*

$$(6.1) \quad \lim_{\beta \rightarrow 1} \Delta_\pi(\beta\mathcal{T}) = \Delta_\pi(\mathcal{T}) \quad \text{implies} \quad \Delta_\pi(\mathcal{T} + \mathcal{T}_0) = \Delta_\pi(\mathcal{T}), \quad \text{and}$$

$$(6.2) \quad \lim_{\beta \rightarrow 1} \delta_\pi(\beta\mathcal{T}) = \delta_\pi(\mathcal{T}) \quad \text{implies} \quad \delta_\pi(\mathcal{T} + \mathcal{T}_0) = \delta_\pi(\mathcal{T}).$$

Lemma 6.2. *Let $\mathcal{T} : \mathbb{R}_+ \rightarrow \mathfrak{S}_\infty$, and let $\mathcal{T}(s) \rightarrow T$ in $\mathfrak{S}_\infty$ as $s \rightarrow 0$. Then $\mathcal{T} \in \mathcal{S}_0^\pi$ for all $\pi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\lim_{s \rightarrow 0+} \pi(s) = 0$.*

6.2. A modified Birman–Schwinger principle. Using the definitions in §5, we define the Birman–Schwinger operator

$$(6.3) \quad \mathcal{Y}(\tau) := \left(\frac{1}{\Lambda - \tau} - R \right)^{-\frac{1}{2}} B_\Gamma \left(\frac{1}{\Lambda - \tau} - R \right)^{-\frac{1}{2}}$$

in $\mathcal{H}_4$ for $\tau \in (0, \Lambda)$.

For an operator T , we denote by $n_+(s, T)$ and $n_-(s, T)$ the number of eigenvalues strictly larger and strictly smaller than $s > 0$, respectively (counting multiplicities).

We shall need the following version of the Birman–Schwinger principle [28, 29].

Lemma 6.3. *Let $\tau \in (0, \Lambda)$. Then*

$$(6.4) \quad n_- \left(\Lambda - \tau, A_\Gamma^{(4)} \right) = n_+ \left(1, \mathcal{Y}(\tau) \right).$$

Proof. Using Glazman's lemma [24, 10.2.2], we obtain

$$\begin{aligned}
n_-(\Lambda - \tau, A_\Gamma^{(4)}) &= n_+((\Lambda - \tau)^{-1}, R + B_\Gamma) \\
&= \sup_L \{ \dim L : (\Lambda - \tau)^{-1} \|u\|^2 < r[u] + \|B_\Gamma^{\frac{1}{2}} u\|^2, \quad 0 \neq u \in L \} \\
&= \sup_L \{ \dim L : \|v\|^2 < \|B_\Gamma^{\frac{1}{2}}((\Lambda - \tau)^{-1} - R)^{-\frac{1}{2}} v\|^2, \quad 0 \neq v \in L \} \\
&= n_+(1, \mathcal{V}(\tau)),
\end{aligned}$$

where L runs through subspaces of $\mathcal{H}_4$. $\square$

6.3. Reduction to the spectral minimum of $A^{(4)}$. The reduction of $A^{(4)}$ to its spectral minimum was already described in [15]. Nevertheless, here we present it again because of its importance for the paper. As in Subsection 3.2, $\lambda_1(\xi)$ denotes the lowest branch of eigenvalues of $A^{(4)}(\xi)$. The corresponding normed eigenfunction in $L^2(J; \mathbb{C}^3)$ is $\psi_1(\xi, \cdot)$. We set

$$\lambda_0(\xi) := \lambda_1(\xi) - \Lambda \quad \text{for } \xi \in \Xi,$$

where $\Xi := \lambda_1^{-1}([\Lambda, \Lambda + \delta))$ for suitable $\delta > 0$. The spectral projection of $A^{(4)}$ onto $[\Lambda, \Lambda + \delta)$ is given by

$$(\Pi_c u)(x) := \Phi_{\xi \rightarrow (x_1)}^* \left(\chi_\Xi(\xi) \langle (\Phi u)(\xi, \cdot), \psi_1(\xi, \cdot) \rangle_{L^2(J; \mathbb{C}^3)} \psi_1(\xi, x_3) \right), \quad u \in \mathcal{H}_4.$$

Then the unitary operator

$$\Pi_0 : \Pi_c \mathcal{H}_4 \rightarrow L^2(\Xi), \quad (\Pi_0 u)(\xi) := \langle (\Phi u)(\xi, \cdot), \psi_1(\xi, \cdot) \rangle_{L^2(J; \mathbb{C}^3)}$$

maps $\Pi_c \mathcal{H}_4$ into scalar-valued functions.

Next, we define

$$P(\xi) := \begin{cases} +\sqrt{\lambda_0(\xi)} & \text{if } |\xi| \geq \varkappa, \\ -\sqrt{\lambda_0(\xi)} & \text{if } |\xi| \leq \varkappa. \end{cases}$$

The properties of $\lambda_1(\xi)$ collected in Subsection 3.2 show that $P \in C^1(\Xi)$ for sufficiently small $\delta > 0$. Furthermore, P depends on $|\xi|$ only. We have

$$|\nabla P(\xi)| = q \quad \text{for } |\xi| = \varkappa,$$

whence $|\nabla P(\xi)| > 0$ for $\xi \in \Xi$ if $\delta > 0$ is sufficiently small. Now we are able to describe $A^{(4)}$ by a suitable direct integral. Let

$$M_\lambda = \{\xi \in \mathbb{R}^2 : P(\xi) = \lambda\} \quad \text{for } \lambda \in \Theta := (-\sqrt{\delta}, \sqrt{\delta}).$$

In particular, $M_0 = \{\xi \in \mathbb{R}^2 : |\xi| = \varkappa\}$ is the set where $\lambda_1(\cdot)$ attains its minimum. We denote by dM_λ the measure on M_λ induced by $d\xi$ on $\mathbb{R}^2$. Furthermore, we set $d\mu_\lambda := |\nabla P|^{-1} dM_\lambda$ and

$$G(\lambda) := L^2(M_\lambda, d\mu_\lambda) \quad \text{for } \lambda \in \Theta.$$

The measures $d\xi$ and $d\mu_\lambda d\lambda$ coincide on Ξ . With these definitions, we have

$$\lambda_0 = \int_\Theta^\oplus \lambda^2 d\lambda \quad \text{on } L^2(\Xi) = \int_\Theta^\oplus G(\lambda) d\lambda,$$

where λ_0 denotes the operator of multiplication by $\lambda_0(\cdot)$ in $L^2(\Xi)$ and λ^2 is the operator of multiplication by λ^2 in $G(\lambda)$. The spaces $G(\lambda)$ are unitarily equivalent. We denote by $U_\lambda : G(\lambda) \rightarrow G(0)$ the corresponding unitary operator. It follows that the space $L^2(\Xi)$ is equivalent to

$$G_\Theta := L^2(\Theta) \otimes G(0),$$

where the corresponding unitary operator $\Upsilon : L^2(\Xi) \rightarrow G_\Theta$ is given by

$$(\Upsilon u)(\lambda) = U_\lambda(u|_{M_\lambda}) \quad \text{for } u \in L^2(\Xi) \quad \text{and almost every } \lambda \in \Theta.$$

Finally, we define the isometry

$$(6.5) \quad F := \Upsilon \Pi_0$$

between $\Pi_c \mathcal{H}_4$ and G_Θ , which reduces the operator $A^{(4)} - \Lambda$ to its essential behavior near the spectral minimum. Namely, if we define $J_\lambda u := (Fu)(\lambda)$ for $u \in \Pi_c \mathcal{H}_4$ and almost every $\lambda \in \Theta$, then

$$(6.6) \quad J_\lambda(A^{(4)} - \Lambda)u = \lambda^2 J_\lambda u.$$

6.4. Reduction of $\mathcal{Y}(\tau)$ to the spectral minimum. We reduce the Birman–Schwinger operator by considering its restriction to the wave functions at the spectral minimum of $A^{(4)}$. The main task is to describe the behavior of B_Γ on the wave functions. Since a direct calculation is only possible for rotation invariant perturbations, we restrict ourselves to the case of a circular boundary perturbation

$$(6.7) \quad \Gamma := \Gamma^\beta = \{x \in \mathbb{R}^2 : |x| = \beta\} \times J, \quad \beta > 0.$$

Note that the decomposition of $L^2(G; \mathbb{C}^3)$ into symmetry spaces obtained in Subsection 3.1 carries over to $L^2(\Gamma^\beta; \mathbb{C}^3)$. In the following, we denote by P_4 not only the orthogonal projection onto $\mathcal{H}_4$ in $L^2(G; \mathbb{C}^3)$ but also the projection onto the corresponding symmetry space in $L^2(\Gamma^\beta; \mathbb{C}^3)$.

The reduction procedure for $\mathcal{Y}(\tau)$ involves operators from elastic potential theory. These operators shall be introduced as follows. First, we mention that

$$\gamma_N R : P_4 L^2(G; \mathbb{C}^3) \rightarrow P_4 L^2(\Gamma^\beta; \mathbb{C}^3)$$

is a bounded operator. This follows immediately from the boundedness of the operators $\gamma_N : H^2(G; \mathbb{C}^3) \rightarrow H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)$ and $R : P_4 L^2(G; \mathbb{C}^3) \rightarrow P_4 H^2(G; \mathbb{C}^3)$. The latter follows from the boundedness of R in $P_4 L^2(G; \mathbb{C}^3)$ and from the estimate

$$(6.8) \quad \|Au\|^2 + \|u\|^2 \geq c \|u\|_{H^2(G; \mathbb{C}^3)}^2, \quad u \in D(A),$$

for some $c > 0$. Note that (6.8) is a consequence of local regularity estimates going back to Nirenberg [30]; see also [22, p. 133]. We conclude that the adjoint operator $(\gamma_N R)^* : P_4 L^2(\Gamma^\beta; \mathbb{C}^3) \rightarrow P_4 L^2(G; \mathbb{C}^3)$ is well defined and bounded. This operator is known as the double-layer potential operator. The following properties of $(\gamma_N R)^*$ will be of importance for us. We refer to [22, Chapter 6] for proofs and further references.

Lemma 6.4. *The double-layer operator $(\gamma_N R)^*$ gives rise to bounded linear operators*

$$(6.9) \quad \begin{aligned} \gamma_D^\pm (\gamma_N R)^* &: P_4 H^{s+1/2}(\Gamma^\beta; \mathbb{C}^3) \rightarrow P_4 H^{s+1/2}(\Gamma^\beta; \mathbb{C}^3), \\ \gamma_N (\gamma_N R)^* &: P_4 H^{s+1/2}(\Gamma^\beta; \mathbb{C}^3) \rightarrow P_4 H^{s-1/2}(\Gamma^\beta; \mathbb{C}^3) \end{aligned}$$

for $-\frac{1}{2} \leq s \leq \frac{1}{2}$. Moreover, let $\varphi \in P_4 L^2(\Gamma^\beta; \mathbb{C}^3)$, and let $u := -(\gamma_N R)^* \varphi$. Then

$$(6.10) \quad -(\Delta + \text{grad div})u = 0 \quad \text{on } G \setminus \Gamma^\beta$$

and

$$(6.11) \quad 0 = \gamma_N^- u - \gamma_N^+ u, \quad \varphi = \gamma_D^- u - \gamma_D^+ u \quad \text{on } \Gamma^\beta.$$

The above results allow us to define the operator

$$(6.12) \quad D\varphi := -\gamma_N (\gamma_N R)^* \varphi$$

for $\varphi \in P_4 H^1(\Gamma^\beta; \mathbb{C}^3)$. The operator D is often called a hypersingular operator, which refers to its representation as an integral operator. The next lemma shows that D is a positive definite selfadjoint operator in $P_4 L^2(\Gamma^\beta; \mathbb{C}^3)$.

Lemma 6.5. *We have*

$$(6.13) \quad c^{-1} \|\varphi\|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2 \leq \langle D\varphi, \varphi \rangle_{L^2(\Gamma^\beta; \mathbb{C}^3)} \leq c \|\varphi\|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2, \quad \varphi \in P_4 H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3),$$

for some $c > 0$.

Proof. Looking at (6.9) in Lemma 6.4, we immediately deduce that

$$\langle D\varphi, \varphi \rangle_{L^2(\Gamma^\beta; \mathbb{C}^3)} \leq \|D\varphi\|_{H^{-\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)} \|\varphi\|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)} \leq c \|\varphi\|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2$$

for all $\varphi \in P_4 H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)$ and some $c > 0$. This proves the upper estimate. For the lower estimate, we follow the arguments in [31, 6.6.2]. Let $\varphi \in P_4 H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)$, and let $u = -(\gamma_N R)^* \varphi$. Then Lemma 6.4 gives

$$\begin{aligned} \langle D\varphi, \varphi \rangle_{L^2(\Gamma^\beta; \mathbb{C}^3)} &= \langle \gamma_N u, \gamma_D^- u - \gamma_D^+ u \rangle_{L^2(\Gamma^\beta; \mathbb{C}^3)} \\ &= \langle \gamma_N^- u, \gamma_D^- u \rangle_{L^2(\Gamma^\beta; \mathbb{C}^3)} - \langle \gamma_N^+ u, \gamma_D^+ u \rangle_{L^2(\Gamma^\beta; \mathbb{C}^3)} \\ &= 2 \int_{G \setminus \Gamma^\beta} |\epsilon(u)|_{\mathbb{C}^3 \times 3}^2 dx. \end{aligned}$$

On the other hand, using Lemma 6.4 again, and recalling the embedding properties of the trace operator (5.5) and Korn's inequality, we see that

$$\begin{aligned} \|\varphi\|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2 &= \|\gamma_D^- u - \gamma_D^+ u\|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2 \\ &\leq 2\|\gamma_D^- u\|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2 + 2\|\gamma_D^+ u\|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2 \\ &\leq c_1 \|u\|_{H^1(G \setminus \Gamma^\beta; \mathbb{C}^3)}^2 \\ &\leq c_2 \int_{G \setminus \Gamma^\beta} |\epsilon(u)|_{\mathbb{C}^3 \times 3}^2 dx \end{aligned}$$

for $\varphi \in P_4 H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)$ and suitable $c_1, c_2 > 0$. This proves the lower estimate. $\square$

Now we are able to give the second description of B_Γ .

Lemma 6.6. *We have*

$$(6.14) \quad B_\Gamma = (\gamma_N R)^* D^{-1} (\gamma_N R).$$

Proof. Note that D^{-1} is bounded in $P_4 L^2(\Gamma^\beta; \mathbb{C}^3)$ because D is positive definite. We make use of the characterization of B_Γ given in (5.7). Let $u \in P_4 L^2(G; \mathbb{C}^3)$. Then

$$-(\Delta + \text{grad div})(\gamma_N R)^* D^{-1} (\gamma_N R) u = 0 \quad \text{on } G \setminus \Gamma^\beta,$$

which follows from (6.10). Furthermore,

$$\gamma_N (\gamma_N R)^* (-\gamma_N (\gamma_N R)^*)^{-1} (\gamma_N R) u = -(\gamma_N R) u \quad \text{on } \Gamma^\beta.$$

Therefore, $(\gamma_N R)^* D^{-1} (\gamma_N R) u$ is a solution of (5.7). Since the solution is unique, it equals $B_\Gamma u$. $\square$

In the following we describe the behavior of B_Γ on wave functions corresponding to the spectral minimum of $A^{(4)}$. This is done with the help of the results of Subsection 6.3.

We define $W_\lambda : P_4 L^2(\Gamma^\beta; \mathbb{C}^3) \rightarrow G(0)$ for almost every $\lambda \in \Theta$ by

$$(6.15) \quad W_\lambda u := J_\lambda \Pi_c (\gamma_N R)^* D^{-\frac{1}{2}} u = U_\lambda \left[\Pi_0 \Pi_c (\gamma_N R)^* D^{-\frac{1}{2}} u \right]_{M_\lambda}$$

for $u \in P_4 L^2(\Gamma^\beta; \mathbb{C}^3)$. Recall that $w_\xi(x) = \psi_1(\xi, x_3) e^{i\xi \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$, as stated in (3.13).

Lemma 6.7. *For $u \in P_4 L^2(\Gamma^\beta; \mathbb{C}^3)$ we have*

$$(6.16) \quad W_\lambda u = U_\lambda \left[(2\pi\lambda_1(\xi))^{-1} \left\langle D^{-\frac{1}{2}} u, \gamma_N w_\xi \right\rangle_{L^2(\Gamma^\beta; \mathbb{C}^3)} \right]_{\xi \in M_\lambda}.$$

Proof. The proof is accomplished by showing that

$$\left(\Pi_0 \Pi_c (\gamma_N R)^* D^{-\frac{1}{2}} u \right) (\xi) = (2\pi\lambda_1(\xi))^{-1} \left\langle D^{-\frac{1}{2}} u, \gamma_N w_\xi \right\rangle_{L^2(\Gamma^\beta; \mathbb{C}^3)}, \quad \xi \in \Xi,$$

in the $L^2(\Xi)$ -sense. Note that

$$\left(\Pi_0 \Pi_c (\gamma_N R)^* D^{-\frac{1}{2}} u \right) (\xi) = \left\langle \Phi_{\left(\frac{x_1}{x_2}\right) \rightarrow \xi} (\gamma_N R)^* D^{-\frac{1}{2}} u, \psi_1(\xi, \cdot) \right\rangle_{L^2(J; \mathbb{C}^3)}.$$

We apply an approximation argument which makes use of the following cutoff function.

Choose $\tilde{\zeta} \in C^\infty(\mathbb{R})$ with $\tilde{\zeta}(t) = 1$ for $t < 1$, $\tilde{\zeta}(t) = 0$ for $t > 2$, and $0 \leq \tilde{\zeta}(t) \leq 1$ for $t \in (1, 2)$ and set

$$(6.17) \quad \zeta_\varepsilon(x) := \tilde{\zeta}(\varepsilon \ln |x|) \quad \text{for } x \in \mathbb{R}^2, \varepsilon > 0.$$

Then $\zeta_\varepsilon \in C_c^\infty(\mathbb{R}^2)$ and

$$(6.18) \quad \int_{\mathbb{R}^2} |\nabla^k \zeta_\varepsilon|^2 dx \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad k = 1, 2.$$

We have

$$\begin{aligned} & \left\| \langle \Phi(1 - \zeta_\varepsilon) (\gamma_N R)^* D^{-\frac{1}{2}} u, \psi_1 \rangle_{L^2(J; \mathbb{C}^3)} \right\|_{L^2(\Xi)}^2 \\ & \leq \int_{\mathbb{R}^2} \|\Phi(1 - \zeta_\varepsilon) (\gamma_N R)^* D^{-\frac{1}{2}} u\|_{L^2(J; \mathbb{C}^3)}^2 \|\psi_1\|_{L^2(J; \mathbb{C}^3)}^2 d\xi \\ & = \left\| (1 - \zeta_\varepsilon) (\gamma_N R)^* D^{-\frac{1}{2}} u \right\|_{L^2(G; \mathbb{C}^3)}^2. \end{aligned}$$

Since $(\gamma_N R)^* D^{-\frac{1}{2}} u \in L^2(G; \mathbb{C}^3)$, the last expression tends to zero as $\varepsilon \rightarrow 0$. Therefore,

$$\langle \Phi \zeta_\varepsilon (\gamma_N R)^* D^{-\frac{1}{2}} u, \psi_1 \rangle_{L^2(J; \mathbb{C}^3)} \rightarrow \langle \Phi (\gamma_N R)^* D^{-\frac{1}{2}} u, \psi_1 \rangle_{L^2(J; \mathbb{C}^3)}$$

in $L^2(\Xi)$ as $\varepsilon \rightarrow 0$. On the other hand,

$$\begin{aligned} \langle \Phi \zeta_\varepsilon (\gamma_N R)^* D^{-\frac{1}{2}} u, \psi_1 \rangle_{L^2(J; \mathbb{C}^3)} &= (2\pi)^{-1} \int_G \left\langle (\gamma_N R)^* D^{-\frac{1}{2}} u, \zeta_\varepsilon w_\xi \right\rangle_{\mathbb{C}^3} dx \\ &= (2\pi)^{-1} \left\langle D^{-\frac{1}{2}} u, \gamma_N R (\zeta_\varepsilon w_\xi) \right\rangle_{L^2(\Gamma^\beta; \mathbb{C}^3)}. \end{aligned}$$

Therefore, it remains to show that

$$\left\langle D^{-\frac{1}{2}} u, \gamma_N R (\zeta_\varepsilon w_\xi) \right\rangle_{L^2(\Gamma^\beta; \mathbb{C}^3)} \rightarrow (\lambda_1(\xi))^{-1} \left\langle D^{-\frac{1}{2}} u, \gamma_N w_\xi \right\rangle_{L^2(\Gamma^\beta; \mathbb{C}^3)}$$

in $L^2(\Xi)$ as $\varepsilon \rightarrow 0$. This is obtained by showing that

$$(6.19) \quad \|R(\zeta_\varepsilon w_\xi) - \lambda_1(\xi)^{-1} w_\xi\|_{H^2(M; \mathbb{C}^3)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

uniformly in $\xi \in \Xi$ for a suitable neighborhood M of Γ^β in G . Note that

$$A^{(4)}(w_\xi \zeta_\varepsilon) = \lambda_1(\xi) w_\xi \zeta_\varepsilon + \sum_{k=1}^2 a_k \partial_k \zeta_\varepsilon + \sum_{k,l=1}^2 b_{kl} \partial_k \partial_l \zeta_\varepsilon$$

for some functions $a_k, b_{kl} : G \rightarrow \mathbb{C}^3$, $k, l = 1, 2$, which are uniformly bounded with respect to $\xi \in \Xi$. Let $\varepsilon > 0$ be so small that $\zeta_\varepsilon = 1$ on M . Then by (6.8) and by the boundedness of R we obtain

$$\begin{aligned} \|R(w_\xi \zeta_\varepsilon) - \lambda_1(\xi)^{-1} w_\xi\|_{H^2(M; \mathbb{C}^3)} &\leq \left\| \lambda_1(\xi)^{-1} R \left(\sum_k a_k \partial_k \zeta_\varepsilon + \sum_{k,l} b_{kl} \partial_k \partial_l \zeta_\varepsilon \right) \right\|_{H^2(G; \mathbb{C}^3)} \\ &\leq C \int_{\mathbb{R}^2} (|\nabla \zeta_\varepsilon|^2 + |\nabla^2 \zeta_\varepsilon|^2) dx \end{aligned}$$

for some $C > 0$ independent of $\xi \in \Xi$. Since the latter expression tends to zero as $\varepsilon \rightarrow 0$, this proves (6.19). $\square$

Lemma 6.7 shows that the operator $X : P_4 L^2(\Gamma^\beta; \mathbb{C}^3) \rightarrow L^2(M_0, d\mu_0)$ given by

$$(6.20) \quad Xu(\xi) := W_0 u(\xi) = \frac{1}{2\pi\Lambda} \left\langle D^{-\frac{1}{2}} u, \gamma_N w_\xi \right\rangle_{L^2(\Gamma^\beta; \mathbb{C}^3)}, \quad \xi \in M_0,$$

for $u \in P_4 L^2(\Gamma^\beta; \mathbb{C}^3)$ is well defined. Moreover, there exists a constant $C > 0$ such that

$$(6.21) \quad \|Xu - W_\lambda u\|_{G(0)}^2 \leq C\lambda^2 \|u\|_{L^2(\Gamma^\beta; \mathbb{C}^3)}^2 \quad \text{for } u \in P_4 L^2(\Gamma^\beta; \mathbb{C}^3), \lambda \in \Theta.$$

The essential spectral properties of the Birman–Schwinger operator (6.3) can be described in terms of the operator

$$(6.22) \quad \mathcal{L}(\tau) := F^* \mathcal{K}(\tau) \mathcal{K}^*(\tau) F \oplus \mathbb{O} \quad \text{on} \quad \Pi_c \mathcal{H}_4 \oplus (\mathbf{I} - \Pi_c) \mathcal{H}_4,$$

where $\mathcal{K}(\tau)$ is given by

$$(6.23) \quad \mathcal{K}(\tau) : P_4 L^2(\Gamma^\beta; \mathbb{C}^3) \rightarrow G_\Theta : u \mapsto \eta_\tau \otimes Xu.$$

The function

$$(6.24) \quad \eta_\tau(\lambda) = \left(\frac{(\lambda^2 + \Lambda)(\Lambda - \tau)}{\lambda^2 + \tau} \right)^{\frac{1}{2}} = \left(\frac{\tau}{\Lambda(\Lambda - \tau)} + \frac{1}{\Lambda} - \frac{1}{\lambda^2 + \Lambda} \right)^{-\frac{1}{2}}, \quad \lambda \in \Theta,$$

corresponds to the outer terms of the Birman–Schwinger operator (6.3). The spectrum of $\mathcal{L}(\tau)$ is basically determined by the spectrum of

$$(6.25) \quad K = XX^*.$$

Note that this is a compact selfadjoint integral operator in $L^2(M_0, d\mu_0)$.

§7. PROOF OF THE MAIN ESTIMATE

In this section we prove Theorem 4.2.

7.1. Proof for circular boundaries. At the first step, we prove Theorem 4.2 for circular boundary perturbations $\Gamma^\beta = \{x \in \mathbb{R}^2 : |x| = \beta\} \times J$ for $\beta > 0$.

In the sequel, we use the notation

$$(7.1) \quad \varrho(\tau) := \frac{1}{w^{-1}(\tau)}, \quad \pi(\tau) := \frac{1}{v^{-1}(\tau)},$$

where $w(t) = t^{-2t}$ and $v(t) = t^{-t}$ for $t > 1$, while for $0 < t \leq 1$ we put $w(t) = t^{-1}$ and $v(t) = t^{-1/2}$. Note that $\pi(\tau) = \varrho(\tau^2)$. Furthermore, in [15] it was shown that

$$(7.2) \quad \lim_{\tau \rightarrow 0+} \frac{\varrho(\tau)}{\varrho(c\tau)} = 1, \quad c > 0.$$

In §8 below we shall prove that the operator K defined in (6.25) satisfies

$$(7.3) \quad \Delta_\pi(K) = 1, \quad \delta_\pi(K) = 1,$$

with the functionals Δ_π and δ_π as in §6. Let $\{\lambda_j(K)\}_{j \in \mathbb{N}}$ denote the eigenvalues of K , in the nonincreasing order including multiplicities. Using the minimax principle [24, 9.2.4], from (6.22) we see that the j th eigenvalue of $\mathcal{L}(\tau)$ fulfills

$$\lambda_j(\mathcal{L}(\tau)) = \|\eta_\tau\|_{L^2(\Theta)}^2 \lambda_j(K).$$

Note that

$$\zeta_\tau := \sqrt{\tau} \|\eta_\tau\|_{L^2(\Theta)}^2 \rightarrow \pi \Lambda^2 \quad \text{as } \tau \rightarrow 0$$

and

$$n(1, \alpha \mathcal{L}(\tau)) = n((\alpha \zeta_\tau)^{-1} \sqrt{\tau}, K), \quad \alpha > 0.$$

Here $n(s, T)$ denotes the number of singular values of T strictly larger than s (including multiplicities). Thus, using (7.2), we get

$$\begin{aligned} \Delta_\varrho(\alpha \mathcal{L}) &= \limsup_{\tau \rightarrow 0+} \varrho(\tau) n(1, \alpha \mathcal{L}(\tau)) \\ &= \limsup_{\tau \rightarrow 0+} \frac{\varrho(\tau)}{\varrho((\alpha \zeta_\tau)^{-2} \tau)} \varrho([(\alpha \zeta_\tau)^{-1} \sqrt{\tau}]^2) n((\alpha \zeta_\tau)^{-1} \sqrt{\tau}, K) = \Delta_\pi(K) = 1. \end{aligned}$$

In the same way we show that $\delta_\varrho(\alpha \mathcal{L}) = 1$. From the definition of $\mathcal{L}$ in (6.22), it follows that $n(1, \alpha \mathcal{L}(\tau)) = n(1, \sqrt{\alpha} F^* \mathcal{K}(\tau))$. Therefore,

$$(7.4) \quad \Delta_\varrho(\alpha F^* \mathcal{K}) = 1, \quad \delta_\varrho(\alpha F^* \mathcal{K}) = 1 \quad \text{for all } \alpha > 0.$$

We introduce

$$\mathcal{Z}(\tau) := D^{-\frac{1}{2}}(\gamma_N R) \left(\frac{\tau}{\Lambda(\Lambda - \tau)} + \frac{1}{\Lambda} - R \right)^{-\frac{1}{2}}.$$

Then the Birman–Schwinger operator (6.3) equals $\mathcal{Y}(\tau) = \mathcal{Z}^*(\tau) \mathcal{Z}(\tau)$. As in [15, Lemma 9.1], it can be checked that $\mathcal{Z}^*(\tau) - F^* \mathcal{K}(\tau)$ converges in $\mathfrak{S}_\infty$ to a compact operator as $\tau \rightarrow 0$. Therefore, we can apply Lemma 6.2 for this operator family to show that

$$(\mathcal{Z}^*(\tau) - F^* \mathcal{K}(\tau)) \in \mathcal{S}_0^g.$$

By (7.4), we have $\alpha F^* \mathcal{K} \in \mathcal{S}^g$ and $\Delta_\varrho(\alpha F^* \mathcal{K}) = 1$ for all $\alpha > 0$. Hence, Lemma 6.1 is applicable, which yields

$$(7.5) \quad \Delta_\varrho(\mathcal{Y}) = \Delta_\varrho(\mathcal{Z}^*) = \Delta_\varrho(F^* \mathcal{K}) = 1.$$

Similarly, $\delta_\varrho(\mathcal{Y}) = 1$. The proof is completed by applying the Birman–Schwinger principle (6.4). $\square$

7.2. Proof for arbitrary boundaries. We use the result for circular boundaries to derive the asymptotics for general boundary perturbations.

Since the problem is invariant under translations of the first two variables, the proof of Subsection 7.1 shows that Theorem 4.2 holds true for boundary perturbations of the type $\Gamma^\beta(y) = \{x \in \mathbb{R}^2 : |x - y| = \beta\} \times J$ with $\beta > 0$ and $y \in \mathbb{R}^2$. In order to prove Theorem 4.2 for general boundaries $\Gamma = \Gamma_0 \times J$, first we show that

$$\frac{n_-(\Lambda - \tau, A_\Gamma^{(4)})}{w^{-1}(\tau)} \leq \lim_{\tau \rightarrow 0+} \frac{n_-(\Lambda - \tau, A_{\Gamma_e}^{(4)})}{w^{-1}(\tau)} = 1,$$

where $\Gamma_e = \Gamma^\beta(0)$ with $\beta > \sup\{|x| : x \in \Gamma_0\}$.

Consider the operator $A_{\Gamma \cup \Gamma_e}^{(4)}$ associated with the quadratic form

$$a_{\Gamma \cup \Gamma_e}^{(4)}[u] = 2 \int_{G \setminus (\Gamma \cup \Gamma_e)} |\epsilon(u)|_{\mathbb{C}^{3 \times 3}}^2 dx, \quad u \in D[a_{\Gamma \cup \Gamma_e}^{(4)}],$$

where $D[a_{\Gamma \cup \Gamma_e}^{(4)}] := P_4 H^1(G \setminus (\Gamma \cup \Gamma_e); \mathbb{C}^3)$. Arguing as in [23], we can check that the essential spectrum of $A_{\Gamma \cup \Gamma_e}^{(4)}$ coincides with that of $A_\Gamma^{(4)}$. Since $D[a_\Gamma^{(4)}] \subset D[a_{\Gamma \cup \Gamma_e}^{(4)}]$ and $a_\Gamma^{(4)} = a_{\Gamma \cup \Gamma_e}^{(4)}$ on $D[a_\Gamma^{(4)}]$, we have $A_{\Gamma \cup \Gamma_e}^{(4)} < A_\Gamma^{(4)}$. Thus,

$$n_-(\Lambda - \tau, A_\Gamma^{(4)}) \leq n_-(\Lambda - \tau, A_{\Gamma \cup \Gamma_e}^{(4)}), \quad \tau \in (0, \Lambda).$$

On the other hand, the discrete spectra of $A_{\Gamma \cup \Gamma_e}^{(4)}$ and $A_{\Gamma_e}^{(4)}$ below Λ differ only by finitely many eigenvalues. To see this, we observe that both operators admit orthogonal expansions into operators acting on $G \setminus \overline{B_\beta(0)}$ and $B_\beta(0)$, respectively. Here $B_\beta(y) = \{x \in \mathbb{R}^2 : |x - y| < \beta\} \times J$. In both cases, the parts on $B_\beta(0)$ have discrete spectrum accumulating at infinity only and therefore having a finite number of eigenvalues below Λ . But the parts on the unbounded domain coincide. Thus,

$$\lim_{\tau \rightarrow 0+} \frac{n_-(\Lambda - \tau, A_\Gamma^{(4)})}{w^{-1}(\tau)} \leq \lim_{\tau \rightarrow 0+} \frac{n_-(\Lambda - \tau, A_{\Gamma \cup \Gamma_e}^{(4)})}{w^{-1}(\tau)} = \lim_{\tau \rightarrow 0+} \frac{n_-(\Lambda - \tau, A_{\Gamma_e}^{(4)})}{w^{-1}(\tau)} = 1.$$

The corresponding lower estimate can be verified much as the upper estimate by reduction of the problem to some circular boundary $\Gamma_\beta(y)$. Here $\beta > 0$ and $y \in \mathbb{R}^2$ are chosen so that $\Gamma_\beta(y)$ lies in the bounded part of G bordered by Γ . $\square$

§8. TREATMENT OF THE OPERATOR K

In this section, we estimate the eigenvalues of the operator K given in (6.25) for the case of a circular boundary perturbation $\Gamma^\beta = S_\beta \times J$ with $S_\beta = \{x \in \mathbb{R}^2 : |x| = \beta\}$. From (6.20) we obtain

$$(8.1) \quad K = \tilde{X} D^{-1} \tilde{X}^*,$$

where

$$(8.2) \quad \tilde{X}u(\xi) := (2\pi\Lambda)^{-1} \langle u, \gamma_N w_\xi \rangle_{L^2(\Gamma^\beta; \mathbb{C}^3)}, \quad \xi \in M_0,$$

for $u \in P_4 L^2(\Gamma^\beta; \mathbb{C}^3)$. We estimate the operator D with the help of simple model operators T_1 and T_2 , which reflect the main spectral properties of D . Let

$$(8.3) \quad \varphi_n(x) := (2\pi^2\beta)^{-\frac{1}{2}} e^{in \cdot \tau^{-1}(x)} = (2\pi^2\beta)^{-\frac{1}{2}} e^{in_1 \mu^{-1}(x_1, x_2)} e^{in_2(2x_3 + \pi)}$$

for $x \in \Gamma^\beta = S_\beta \times J$ and $n \in \mathbb{Z}^2$, where the mappings $\tau : [0, 2\pi]^2 \rightarrow \Gamma^\beta$ and $\mu : [0, 2\pi] \rightarrow S_\beta$ are given by

$$\tau(s, t) = (\beta \cos s, \beta \sin s, (t - \pi)/2)^T \quad \text{and} \quad \mu(s) = (\beta \cos s, \beta \sin s).$$

The functions φ_n , $n \in \mathbb{Z}^2$, constitute an orthonormal basis of $L^2(\Gamma^\beta)$. We define two quadratic forms t_1 and t_2 by

$$(8.4) \quad t_j[u] := \sum_{n \in \mathbb{Z}^2} (1 + |n_1| + \delta_{j2}|n_2|) \left| \int_{\Gamma^\beta} u(x) \overline{\varphi_n(x)} dS_{\Gamma^\beta}(x) \right|_{\mathbb{C}^3}^2$$

for suitable $u \in P_4 L^2(\Gamma^\beta; \mathbb{C}^3)$. It is easily seen that t_1 and t_2 are well defined on the spaces

$$D_1 := P_4 H^1(\Gamma^\beta; \mathbb{C}^3),$$

$$D_2 := \{u \in P_4 H^1(\Gamma^\beta; \mathbb{C}^3) : u(\cdot, \cdot, -\pi/2) = u(\cdot, \cdot, \pi/2)\}.$$

Denote by $D[t_j]$ for $j = 1, 2$ the completion of D_j with respect to the norm $u \mapsto \sqrt{t_j[u]}$. Then t_j admits a unique extension up to a closed positive definite form on $D[t_j]$, which

is again denoted by t_j . It is associated with a positive definite selfadjoint operator T_j in $P_4 L^2(\Gamma_\beta; \mathbb{C}^3)$ given by

$$(T_j u)(x) = \sum_{n \in \mathbb{Z}^2} (1 + |n_1| + \delta_{j2}|n_2|) \varphi_n(x) \int_{\Gamma^\beta} u(y) \overline{\varphi_n(y)} dS_{\Gamma^\beta}(y), \quad x \in \Gamma^\beta,$$

for suitable $u \in P_4 L^2(\Gamma^\beta; \mathbb{C}^3)$. Note that the inverse of T_j is given by

$$(8.5) \quad (T_j^{-1} u)(x) = \sum_{n \in \mathbb{Z}^2} (1 + |n_1| + \delta_{j2}|n_2|)^{-1} \varphi_n(x) \int_{\Gamma^\beta} u(y) \overline{\varphi_n(y)} dS_{\Gamma^\beta}(y), \quad x \in \Gamma^\beta.$$

Lemma 8.1. *We have*

$$(8.6) \quad \tilde{c}^{-1} T_1 \leq D \leq \tilde{c} T_2$$

in the quadratic form sense for some $\tilde{c} > 0$.

The proof of this lemma is rather technical; we present it in the Appendix.

Set $\tilde{K}_j := \tilde{X} T_j^{-1} \tilde{X}^*$ for $j = 1, 2$ and recall that $K = \tilde{X} D^{-1} \tilde{X}^*$. The above lemma shows that

$$(8.7) \quad \tilde{c}^{-1} \tilde{K}_2 \leq K \leq \tilde{c} \tilde{K}_1.$$

This allows us to study the operators $\tilde{K}_j$ instead of K .

8.1. Calculation of the eigenvalues. Note that each $\tilde{K}_j$, $j = 1, 2$, is an integral operator in $L^2(M_0, d\mu_0)$ with the kernel

$$(8.8) \quad \tilde{k}_j(\eta, \xi) = (2\pi\Lambda)^{-2} \langle T_j^{-1} \gamma_N w_\xi, \gamma_N w_\eta \rangle_{L^2(\Gamma^\beta; \mathbb{C}^3)}, \quad \eta, \xi \in M_0.$$

Since $\mathbf{n}_{\Gamma^\beta}(x) = \beta^{-1}(x_1, x_2, 0)$ for $x \in \Gamma^\beta$, from the definition of w_ξ in (3.13) and of γ_N in (5.6) we obtain

$$\gamma_N w_\xi(x) = (2p\beta^2)^{-\frac{1}{2}} \xi \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} e^{i\xi \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} \begin{pmatrix} -2\xi d_1(x_3) \\ i(d'_1(x_3) + d_2(x_3)) \end{pmatrix}, \quad x \in \Gamma^\beta,$$

where $p = \varkappa^2 \|d_1\|_{L^2(J)}^2 + \|d_2\|_{L^2(J)}^2$ and $d_j(t) = d_j(\varkappa, t)$ for $j = 1, 2$. Thus, (8.5) and passage to the cylindrical coordinates $(s, t) \in (0, 2\pi)^2$ for integration over $x \in \Gamma^\beta$ show that

$$(8.9) \quad \tilde{k}_j(\eta, \xi) = (64\pi^4 \Lambda^2 p \beta)^{-1} \sum_{n \in \mathbb{Z}^2} (1 + |n_1| + \delta_{j2}|n_2|)^{-1} I_{n_1}(\xi) \overline{I_{n_1}(\eta)} J_{n_2}(\xi) \cdot \overline{J_{n_2}(\eta)}$$

with

$$(8.10) \quad I_k(\xi) = \int_0^{2\pi} e^{-iks} i\beta(\xi_1 \cos s + \xi_2 \sin s) e^{i\beta(\xi_1 \cos s + \xi_2 \sin s)} ds$$

and

$$J_k(\xi) = \int_0^{2\pi} e^{-ikt} \begin{pmatrix} -2\xi d_1(\frac{t-\pi}{2}) \\ i(d'_1(\frac{t-\pi}{2}) + d_2(\frac{t-\pi}{2})) \end{pmatrix} dt.$$

With the substitutions

$$(8.11) \quad \eta = \varkappa \begin{pmatrix} \cos v \\ \sin v \end{pmatrix}, \quad \xi = \varkappa \begin{pmatrix} \cos w \\ \sin w \end{pmatrix}, \quad d\mu_0 = \frac{\varkappa}{|\nabla P|} dv = \frac{\varkappa}{q} dv,$$

we can regard the operator $\tilde{K}_j$ also as an operator in $L^2((0, 2\pi))$. Using the identity $\eta \cdot \xi = \varkappa^2 \cos(w - v)$, we see that

$$J_k(\xi) \cdot \overline{J_k(\eta)} = \sum_{m=-1}^1 p_{|m|}(k) e^{im(w-v)},$$

where

$$(8.12) \quad p_0(k) = \left| \int_{-\pi}^{\pi} e^{-ikt} (d'_1(\tfrac{t}{2}) + d_2(\tfrac{t}{2})) dt \right|^2, \quad p_1(k) = 2\kappa^2 \left| \int_{-\pi}^{\pi} e^{-ikt} d_1(\tfrac{t}{2}) dt \right|^2.$$

Furthermore, if we expand the second exponential function in (8.10) in a Taylor series and apply (8.11), we get

$$I_k(\xi) = I_k(w) = e^{-ikw} 2\pi \sum_{l=\delta_{0k}}^{\infty} \left(\frac{i\beta\kappa}{2} \right)^{2l+|k|} \frac{2l+|k|}{l!(l+|k|)!}.$$

Therefore, the kernel of $\tilde{K}_j$ as an operator in $L^2((0, 2\pi))$ is given by

$$(8.13) \quad \tilde{k}_j(v, w) = \sum_{n_1=-\infty}^{\infty} \mu_{|n_1|}^j \frac{e^{in_1(w-v)}}{2\pi},$$

where

$$(8.14) \quad \mu_k^j = \frac{\kappa}{8\pi\Lambda^2 pq\beta} \sum_{m=-1}^1 \left(\sum_{n_2 \in \mathbb{Z}} \frac{p_{|m|}(n_2)}{1 + |k-m| + \delta_{j2}|n_2|} \right) \times \left(\sum_{l=\delta_{0,k-m}}^{\infty} \left(\frac{\beta\kappa}{2} \right)^{2l+|k-m|} \frac{(-1)^l (2l+|k-m|)}{l!(l+|k-m|)!} \right)^2$$

for $k \in \mathbb{N}_0$. This means that $\tilde{K}_j$ has exactly the eigenfunctions $v \mapsto e^{\pm ikv}$ corresponding to the eigenvalues μ_k^j for $k \in \mathbb{N}_0$.³

8.2. Estimates for the eigenvalues. A detailed analysis of the expression (8.14) (to be given in the Appendix) shows that we can find $c > 0$ and $N \in \mathbb{N}$ such that

$$(8.15) \quad c^{-1}(2k)^2 \left(\frac{\beta\kappa e}{2k} \right)^{2k} \leq \mu_k^2 \leq \mu_k^1 \leq c(2k)^2 \left(\frac{\beta\kappa e}{2k} \right)^{2k} \quad \text{for all } k > N.$$

By (8.13) the eigenspaces of the eigenvalues μ_k^j have dimension 2 for $k \geq 1$ and dimension 1 for $k = 0$. We denote by $\lambda_n(K)$, $n \in \mathbb{N}$, the eigenvalues of the operator K in the nonincreasing order including multiplicities. Then from (8.7) and (8.15) we see that there exists a constant $c > 0$ such that

$$(8.16) \quad c^{-1}n^2 \left(\frac{\beta\kappa e}{n} \right)^n \leq \lambda_{n+1}(K) \leq \lambda_n(K) \leq cn^2 \left(\frac{\beta\kappa e}{n} \right)^n$$

for all sufficiently large $n \in 2\mathbb{N}$.

8.3. Estimates for the number of eigenvalues. From (8.16), it is easily seen that

$$(8.17) \quad \left(\frac{1}{cn} \right)^n < \lambda_n(K) < \left(\frac{c}{n} \right)^n, \quad n \in \mathbb{N},$$

for some constant $c > 0$. Proceeding as in [15], from (8.17) we obtain the desired asymptotic estimate (7.3) on the counting function for eigenvalues of K .

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³We point out that similar types of eigenvalue asymptotics appeared, e.g., in [19]–[21].

§9. APPENDIX

9.1. Proof of (8.15). By the definition (8.12), we have

$$\sum_{n_2=-\infty}^{\infty} p_m(n_2) < \infty \quad \text{for } m = 0, 1.$$

Setting

$$C := \frac{\varkappa}{8\pi\Lambda^2 pq\beta} \sum_{m=-1}^1 \left(\frac{\beta\varkappa}{2}\right)^{-2m} \sum_{n_2=-\infty}^{\infty} p_{|m|}(n_2),$$

from (8.14) for $k \geq 2$ we get

$$(9.1) \quad \mu_k^1 \leq C \left(\frac{\beta\varkappa}{2}\right)^{2k} \frac{1}{k} \left(\sum_{l=0}^{\infty} \left(\frac{\beta\varkappa}{2}\right)^{2l} \frac{2l+k+1}{l!(l+k-1)!} \right)^2.$$

Estimating the right bracket and using Stirling's formula, we arrive at the inequality

$$\mu_k^1 \leq 9C \left(\frac{\beta\varkappa}{2}\right)^{2k} \frac{k^3}{(k!)^2} \left(\sum_{l=0}^{\infty} \left(\frac{\beta\varkappa}{2}\right)^{2l} \frac{1}{l!} \right)^2 \leq \frac{9Ce^{(\beta\varkappa)^2/2}}{8\pi} (2k)^2 \left(\frac{\beta\varkappa e}{2k}\right)^{2k}.$$

This proves the upper estimate in (8.15). In order to check the lower, we set

$$C := \frac{\varkappa}{16\pi\Lambda^2 pq\beta} \left(\frac{\beta\varkappa}{2}\right)^{-2} \sum_{n_2 \in \mathbb{Z}} \frac{p_1(n_2)}{1+|n_2|}.$$

Then

$$\begin{aligned} \mu_k^2 &\geq C \left(\frac{\beta\varkappa}{2}\right)^{2k} \frac{1}{k-1} \left(\sum_{l=0}^{\infty} \left(\frac{\beta\varkappa}{2}\right)^{2l} \frac{(-1)^l (2l+k-1)}{l!(l+k-1)!} \right)^2 \\ &\geq \frac{C}{2} \left(\frac{\beta\varkappa}{2}\right)^{2k} \frac{k^3}{(k!)^2} \left| 1 - \frac{1}{k} \left(\frac{\beta\varkappa}{2}\right)^2 \sum_{l=0}^{\infty} \left(\frac{\beta\varkappa}{2}\right)^{2l} \frac{(2l+k+1)k!}{(k-1)(l+1)!(l+k)!} \right|_+^2, \end{aligned}$$

where $|x|_+ = \frac{1}{2}(|x| + x)$ for $x \in \mathbb{R}$. Using the inequality $\frac{2l+k+1}{(k-1)(l+1)} \leq 3$ for $k \geq 2$ and Stirling's formula, we obtain

$$\mu_k^2 \geq \frac{C}{16\pi e^{\frac{1}{6n}}} \left| 1 - \frac{3}{k} \left(\frac{\beta\varkappa}{2}\right)^2 e^{(\beta\varkappa)^2/4} \right|_+^2 (2k)^2 \left(\frac{\beta\varkappa e}{2k}\right)^{2k}.$$

This proves the lower bound in (8.15) for sufficiently large $k \in \mathbb{N}$.

9.2. Proof of Lemma 8.1. As stated in Lemma 6.5, we have

$$c^{-1} \|u\|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2 \leq \langle Du, u \rangle_{L^2(\Gamma^\beta; \mathbb{C}^3)} \leq c \|u\|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2, \quad u \in P_4 H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3),$$

for some $c > 0$. Therefore, it suffices to prove that

$$(9.2) \quad t_1[u] \leq c_1 \|u\|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2, \quad u \in D_1,$$

and

$$(9.3) \quad t_2[u] \geq c_2 \|u\|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2, \quad u \in D_2,$$

for some constants $c_1, c_2 > 0$. We start with (9.2). Note that among several equivalent norms we may choose

$$(9.4) \quad \|u\|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)} = \left(\|u\|_{L^2(\Gamma^\beta; \mathbb{C}^3)}^2 + |u|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2 \right)^{\frac{1}{2}},$$

where

$$(9.5) \quad |u|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2 = \int_{\Gamma^\beta} \int_{\Gamma^\beta} \frac{|u(x) - u(y)|^2}{|x - y|^3} dS_{\Gamma^\beta}(x) dS_{\Gamma^\beta}(y)$$

denotes the Sobolev–Slobodetskii seminorm of order $1/2$; see [31]. Recall that $\Gamma_\beta = S_\beta \times J$ with $S_\beta = \{x \in \mathbb{R}^2 : |x| = \beta\}$. We define a new seminorm $|\cdot|_w$ on $H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)$ by

$$(9.6) \quad |u|_w^2 := \int_J \int_{S_\beta} \int_{S_\beta} \frac{|u(x', t) - u(y', t)|^2}{|x' - y'|^2} dS_{S_\beta}(x') dS_{S_\beta}(y') dt.$$

Roughly speaking, $|\cdot|_w$ measures smoothness of order $1/2$ only along S_β , whereas $|\cdot|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}$ measures smoothness also along the third coordinate. Thus, it is reasonable to expect that

$$(9.7) \quad \|u\|_{L^2(\Gamma^\beta; \mathbb{C}^3)}^2 + |u|_w^2 \leq c \|u\|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2, \quad u \in H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3),$$

for some $c > 0$. This is true indeed. We give an outline of the proof after we have finished the present proof.

Recall the definition of φ_n given in (8.3). Applying the expansion

$$u = \sum_{n \in \mathbb{Z}^2} u_n \varphi_n, \quad u_n = \int_{\Gamma^\beta} u(x) \overline{\varphi_n(x)} dS_{\Gamma^\beta}(x)$$

for $u \in P_4 H^{\frac{1}{2}}(\Gamma_\beta; \mathbb{C}^3)$, we see that

$$\begin{aligned} |u|_w^2 &= \int_J \int_{S_\beta} \int_{S_\beta} \frac{|\sum_{n \in \mathbb{Z}^2} u_n (\varphi_n(x', t) - \varphi_n(y', t))|^2}{|x' - y'|^2} dS_{S_\beta}(x') dS_{S_\beta}(y') dt \\ &= \frac{1}{2\pi\beta} \sum_{n_2 \in \mathbb{Z}} \int_{S_\beta} \int_{S_\beta} \frac{|\sum_{n_1 \in \mathbb{Z}} u_n (e^{in_1 \mu^{-1}(x')} - e^{in_1 \mu^{-1}(y')})|^2}{|x' - y'|^2} dS_{S_\beta}(x') dS_{S_\beta}(y'). \end{aligned}$$

In order to prove (9.3), we note that the substitutions $x = \tau(s)$ and $y = \tau(t)$ with $s, t \in (0, 2\pi)^2$ yield

$$\begin{aligned} |u|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2 &= \int_{\Gamma^\beta} \int_{\Gamma^\beta} \frac{|\sum_{n \in \mathbb{Z}^2} u_n (\varphi_n(x) - \varphi_n(y))|^2}{(|x_1 - y_1|^2 + |x_2 - y_2|^2 + |x_3 - y_3|^2)^{\frac{3}{2}}} dS_{\Gamma^\beta}(x) dS_{\Gamma^\beta}(y) \\ &= \frac{\beta}{8\pi^2} \int_{(0, 2\pi)^2} \int_{(0, 2\pi)^2} \frac{|\sum_{n \in \mathbb{Z}^2} u_n (e^{in \cdot s} - e^{in \cdot t})|^2}{(4\beta^2 \sin^2(\frac{s_1 - t_1}{2}) + 4^{-1}|s_2 - t_2|^2)^{\frac{3}{2}}} ds dt. \end{aligned}$$

Using the change of variables $s = t + h$ and denoting $I = (-t_1, 2\pi - t_1) \times (-t_2, 2\pi - t_2)$, we arrive at

$$\begin{aligned} |u|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2 &\leq \frac{\beta}{8\pi^2} \int_{(0, 2\pi)^2} \int_I \frac{|\sum_{n \in \mathbb{Z}^2} u_n (e^{in \cdot h} - 1) e^{in \cdot t}|^2}{(4\beta^2 \sin^2 \frac{h_1}{2} + 4^{-1}|h_2|^2)^{\frac{3}{2}}} dh dt \\ &\leq \frac{\beta}{2} \sum_{n \in \mathbb{Z}^2} |u_n|^2 \int_{(-\pi, \pi)^2} \frac{|e^{in \cdot h} - 1|^2}{(4\beta^2 \sin^2 \frac{h_1}{2} + 4^{-1}|h_2|^2)^{\frac{3}{2}}} dh. \end{aligned}$$

Note that

$$2\beta^{-1}(4\beta^2 \sin^2 \frac{h_1}{2} + 4^{-1}|h_2|^2) > c^{-1}|h|^2, \quad h \in (-\pi, \pi)^2,$$

for some $c > 0$. Therefore, the substitution $h = (r \cos \phi, r \sin \phi)$ gives

$$\begin{aligned} |u|_{H^{\frac{1}{2}}(\Gamma^\beta; \mathbb{C}^3)}^2 &\leq c \sum_{n \in \mathbb{Z}^2} |u_n|^2 \int_{|h| < \sqrt{2}\pi} \frac{|e^{in \cdot h} - 1|^2}{|h|^3} dh \\ &= c \sum_{n \in \mathbb{Z}^2} |u_n|^2 \int_0^{\sqrt{2}\pi} \int_0^{2\pi} \frac{|e^{i|n|r \cos \phi} - 1|^2}{r^2} d\phi dr \\ &\leq c \sum_{n \in \mathbb{Z}^2} |n| |u_n|^2 \int_0^\infty \int_0^{2\pi} \frac{|e^{it \cos \phi} - 1|^2}{t^2} d\phi dt. \end{aligned}$$

Since the integral on the right-hand side is finite and $|n| \leq |n_1| + |n_2|$, this proves (9.3).

9.3. Outline of the proof of (9.7). First, we consider the case of the entire space $\mathbb{R}^n$ for $n \geq 2$. Let $s \in (0, 1)$. In [22, Lemma 3.15], it was proved that

$$|u|_{H^s(\mathbb{R}^n)}^2 = a_s \int_{\mathbb{R}^n} |\xi|^{2s} |\Phi u(\xi)|^2 d\xi$$

for some constant $a_s > 0$, where Φ denotes the unitary Fourier transformation in $\mathbb{R}^n$. Similarly,

$$|u|_{w,s}^2 = a'_s \int_{\mathbb{R}^n} |\xi'|^{2s} |\Phi u(\xi)|^2 d\xi$$

for some $a'_s > 0$, where $\xi' = (\xi_1, \dots, \xi_{n-1})$, and $|u|_{w,s}^2$ is given by (cf. (9.6))

$$|u|_{w,s}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|u(x', t) - u(y', t)|^2}{|x' - y'|^{n-1+2s}} dx' dy' dt.$$

Therefore,

$$|u|_{w,s}^2 \leq a'_s a_s^{-1} |u|_{H^s(\mathbb{R}^n)}^2, \quad u \in H^s(\mathbb{R}^d).$$

The second step is to carry over this result to domains $\Omega = \Omega_0 \times I$, where $\Omega_0 \subset \mathbb{R}^{n-1}$ is a Lipschitz domain and $I \subset \mathbb{R}$ is an interval. This is done by using a continuous extension operator $E : H^s(\Omega) \rightarrow H^s(\mathbb{R}^n)$. The existence of such an object was proved in [22]. Note that our definition of $\|\cdot\|_{H^s}$ corresponds to $\|\cdot\|_{W^s}$ in [22]. Put

$$|u|_{w,s,\Omega}^2 = \int_I \int_{\Omega_0} \int_{\Omega_0} \frac{|u(x', t) - u(y', t)|^2}{|x' - y'|^{n-1+2s}} dx' dy' dt.$$

Then

$$(9.8) \quad \|u\|_{L^2(\Omega)}^2 + |u|_{w,s,\Omega}^2 \leq \|Eu\|_{L^2(\mathbb{R}^n)}^2 + |Eu|_{w,s}^2 \leq c_1 \|Eu\|_{H^s(\mathbb{R}^n)}^2 \leq c_2 \|u\|_{H^s(\Omega)}^2$$

for some $c_1, c_2 > 0$ and $u \in H^s(\Omega)$. Recalling the standard definition of the Sobolev spaces on a boundary via the Sobolev spaces on the preimages of its parametrization mappings [31, 2.5], we immediately obtain an analog of (9.8) for the standard Sobolev norm on a boundary. Finally, (9.7) is obtained by using the equivalence of the standard Sobolev norm and the norm (9.4), which carries over to the corresponding “weaker” norms.

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