THE POISSON MODEL OF THE FOCK SPACE AND REPRESENTATIONS OF CURRENT GROUPS

A. M. VERSHIK AND M. I. GRAEV

ABSTRACT. The quasi-Poisson measures are considered, i.e., the σ -finite measures given by a density with respect to a Poisson measure. Representations of current groups are constructed in Hilbert spaces of functionals integrable with respect to a quasi-Poisson measure. For the groups $O(n, 1)$ and $U(n, 1)$, these models give new, more convenient, realizations of the representations in Fock spaces constructed in the previous papers by the authors. A crucial role in considerations is played by spaces of configurations and an analogy between quasi-Poisson and stable measures.

§1. INTRODUCTION

1.1. The problem of defining a continuous tensor product of Hilbert spaces. The functor EXP . Since the 1930s, the concept of a continuous tensor product of Hilbert spaces has been associated with Fock's construction. In mathematical physics, after V. Fock's work this construction was called *second quantization*; later, it took the form of the *Hilbert exponential* $\text{EXP}(H)$. It is natural to view this operation EXP of exponentiation as a functor in the category of Hilbert spaces Hilb . With an arbitrary¹ Hilbert space H , this functor associates the new Hilbert space

$$\text{EXP } H = \bigoplus_{n=0}^{\infty} H^{\text{sym} \otimes n}$$

(the Hilbert sum of the symmetric tensor powers of H); this space has a number of important properties and, in particular, a factorization structure (see below).

The most important property of the spaces $\text{EXP}(H)$, or, more precisely, of the functor EXP , is its *multiplicativity*: every decomposition of H into a finite direct sum of mutually orthogonal subspaces gives rise to a decomposition of $\text{EXP}(H)$ into a finite tensor product of the corresponding EXP -spaces:

$$\text{EXP} \left(\bigoplus_{i=1}^n H_i \right) = \bigotimes_{i=1}^n \text{EXP}(H_i).$$

It is this property that makes it possible to define a *factorization structure*. Another important element of the EXP construction is as follows: with every vector $h \in H$, it associates the so-called decomposable (or product) vector $\exp(h) \in \text{EXP}(H)$. In the functional model, $\text{EXP}(H)$ can be realized as the Wiener–Itô space, i.e., the space of all square-integrable functionals with respect to the standard Gaussian measure (“white noise”) on the canonical extension $\tilde{H}$ of H .

2010 *Mathematics Subject Classification.* Primary 81R10.

Key words and phrases. Current group, integral model, Fock representation, canonical representation, special representation, infinite-dimensional Lebesgue measure.

Supported by the RFBR grants 08-01-00379a, 09-01-12175-ofi-m (the first author) and 10-01-00041a (the second author).

¹We shall consider only separable Hilbert spaces over the field of complex numbers $\mathbb{C}$.

In the most important case, which we consider below and which is the only one needed for applications to field theory, the space H to be “second-quantized” is itself not merely a Hilbert space, but the space $L_m^2(X; K)$ of square-integrable functions taking values in an auxiliary Hilbert space K and defined on a phase space X that is endowed with a probability (discrete or continuous) measure m . (In quantum field theory, X is the phase space on which observables are defined.) From the more abstract operator point of view, this means that we consider the functor EXP not merely in the category of Hilbert spaces, but in the category of Hilbert spaces H with a fixed Abelian W^* -subalgebra in the algebra of all linear operators. This is exactly the case where H has the form $L_m^2(X; K)$, with m being a σ -finite continuous measure and K an auxiliary Hilbert space of “spins”; functions defined on X take values in the set of operators on K . In a natural sense, such Hilbert spaces form a subcategory in the category of spaces with EXP structure.

It is this special case that is used in the free field construction; as we shall see below, the space $\text{EXP}(L_m^2(X; K))$ has the *structure of a continuous (continual) tensor product of Hilbert spaces*. The choice of a parametrization of this continuum, i.e., the base X of the product, is irrelevant; for instance, it can be the Minkowski space, an arbitrary manifold, simply the interval $[0, 1]$ with the Lebesgue measure, etc. Note that instead of $L_m^2(X; K)$ it is more consistent to consider the isomorphic space of square-integrable (with respect to the measure m) sections of the fiber bundle over X with fiber K ; in what follows, we adopt this point of view.

Should the base X be finite or countable, the required (Hilbert) space of functionals would be the (Hilbert) tensor product of spaces K_x of functionals at different points x :

$$\bigotimes_{x \in X} K_x.$$

The problem is to realize how this product should be understood in the case of continual X .²

The functor EXP allows us to solve this problem as follows. Extrapolating the multiplicative property of EXP to the continual case, we obtain the conjectural formula

$$(1.1) \quad \text{EXP} \left(\int_X^{\oplus} K_x dm(x) \right) = \int_X^{\otimes} \text{EXP}(K_x) dm(x).$$

The right-hand side here is still to be defined, but the left-hand side is well defined, because so is the Hilbert direct integral under the EXP sign. Namely, if we assume that the spaces K_x of values of functions at all points x are identified, i.e., $K_x \cong K$, then

$$\int_X^{\oplus} K_x dm(x) = L_m^2(X; K).$$

However, it is better not to identify the spaces K_x and regard the direct integral as the space of square-integrable sections of the “locally trivial” fiber bundle with fiber K . As usual, one should consider classes of sections that coincide mod 0 rather than individual sections. Thus, a continual direct integral of (isomorphic) Hilbert spaces can be identified with the space of square-integrable sections of a fiber bundle or the space of square-integrable functions with values in a typical fiber.

Thus, (1.1) can be written in more detail as follows:

$$\text{EXP} \left(\int_X^{\oplus} K_x dm(x) \right) \equiv \text{EXP}(L_m^2(X; K)) = \int_X^{\otimes} \text{EXP}(K_x) dm(x) \equiv \int_X^{\otimes} L^2(K_x) dm(x);$$

²To define separable (“small”) countable tensor products, one should choose a stabilizing vector in the subspace K or a sequence of such vectors, one in each subspace K_x ; if the factors have the structure of a ring or a unital algebra, then this vector is the identity.

the last identity identifies $\text{EXP}(K_x)$ with the L^2 space with respect to the Gaussian measure on K_x , as above. In all these formulas, the spaces K_x can be replaced with the space K isomorphic to them.

In other words, there is a good reason to regard the left-hand side of (1.1) as the definition of the right-hand side, i.e., the definition, in terms of the functor EXP , of the continuous tensor product of Hilbert spaces corresponding to a direct integral of Hilbert spaces over a measure space. Therefore, the restriction of the functor EXP to the category of Hilbert spaces of the form $\text{EXP}(L_m^2(X; K))$ can be viewed as the continuous tensor product functor defined on the category of measurable fiber bundles $E(X, m, \{K_x\})$ with base (X, m) and fibers K_x , $x \in X$, which sends such a fiber bundle to the Hilbert space

$$\text{EXP}(L_m^2(X; K)) = \int_X^{\otimes} L^2(K_x) dm(x).$$

Thus, we have formally justified the claim that the second quantization of a direct integral is the integral of tensor products of the quantized factors.

This definition, with various modifications, is the generally accepted definition of a continuous tensor product, and there are hundreds of papers studying this notion and its applications. What is the disadvantage of this method, and what is the novelty of the new model of a continuous tensor product suggested in this paper and based on the “configurational” approach to second quantization?

We begin with explaining why the EXP notion does not allow one to give a direct meaning to a continuous tensor product, i.e., to the right-hand side of (1.1). The reason is that for the white noise, which is a generalized random process, the values at points are not defined, so that in the framework of *this* construction (the functor EXP and the Gaussian measure), all spaces $K_x \cong K$, $x \in X$, are only infinitesimal objects: we cannot define a measure on them and, consequently, cannot define the exponential of this infinitesimal space directly.

More exactly: the definition of the direct integral of Hilbert space in the sense of von Neumann (or the space of square-integrable sections of a vector bundle, see below) does not rely on the existence of values at points, so that the left-hand side of (1.1) can be defined directly via the EXP functor, and also approximatively, as the limit of a sequence of finite tensor products with respect to a sequence of finer and finer finite partitions of X (this definition is completely similar to that of the Lebesgue integral for classes of integrable functions coinciding mod 0). In probabilistic terms, the left-hand side is the space of square-integrable functionals with respect to the corresponding white noise on $L^2(X; K)$.

At the same time, the right-hand side, which is a continuous tensor product of continually many spaces, each being the exponential of an infinitesimal subspace, has no definition for its own and is identified with the left-hand side by convention.

1.2. Factorizations. An immediate cause for the introduction of a new, integral, model of the Fock space and the functor INT was the fact that the Fock model is not always convenient for constructing representations of current groups (for more details, see below). Obviously, every representation of a current group relies on the existence of a factorization in the representation space. Up to now, representation theory has only addressed Fock factorizations;³ we shall briefly describe them. Let (X, m) be a space with a standard continuous probability measure. A factorization of a Hilbert space H over the base (X, m) is a map Φ that assigns a W^* -subalgebra W_A in the algebra $\text{END}(H)$ of all linear operators in H to every class of coinciding (mod 0) measurable sets $A \subset X$

---

³Factorizations that are not isomorphic to Fock ones were discovered in [37] (see also the Appendix to [14]), but they have not yet found applications in representation theory.

so that the following conditions are satisfied: monotonicity (if $A \subset B$, then $W_A \subset W_B$); monotone continuity in the natural sense; and independence: a finite or countable partition of X into measurable subsets $A_1, A_2, \dots$ gives rise to a decomposition of $\text{END}(H)$ into mutually commuting subalgebras W_{A_i} whose pairwise intersections consist only of scalars. The trivial partition (into two sets, the empty one and the entire space) leads to the algebra of scalars, and the partition into separate points leads to the algebra of all operators (see [37]). In a somewhat loose sense, with any finite measurable partition of the space (X, m) , a factorization associates the decomposition of the Hilbert space H into the tensor product of factors corresponding to the elements of the partition. The unitary equivalence of factorizations is defined in a natural way.

A canonical factorization in the Fock space $\text{EXP}(L^2(X; K))$ is defined as follows: with a partition of the base (X, m) into finitely many measurable sets (classes of sets coinciding mod 0) A_i , $i = 1, \dots, n$, we associate the decomposition into the tensor product $\bigotimes_{i=1}^n \text{EXP}(L^2(A_i; K))$; one can check easily that this correspondence determines a factorization in the above sense, which is called a Fock factorization. In the language of Gaussian measures, it can be described as follows: a measurable set A in the base is taken to the subspace of all square-integrable functionals of the white noise that depend only on the behavior of the trajectory on the set A . A highly nontrivial theorem in [24] asserts that the only unitary invariant of a Fock factorization is the (finite or infinite) dimension of the Hilbert space K . It coincides with the dimension of the space of values of the white noise in the Gaussian model.

In what follows, we shall be interested in infinite-dimensional Fock factorizations. We emphasize that a unitary equivalence is not at all an isomorphism of EXP space structures; in other words, the group of automorphisms of a Fock factorization is substantially wider than the group of automorphisms of the Fock space structure in $\text{EXP}(L^2(X; K))$, which is merely the group of projective automorphisms of a space with Gaussian measure.

The definition of the processes with independent values (Lévy processes) shows that such a process generates a factorization of the L^2 space of functionals in accordance with the same pattern as described above. It turns out that all these factorizations are unitarily equivalent to Fock factorizations, and the corresponding dimension is equal to the cardinality of the support of the Lévy measure; in particular, for the classical white noise and the standard Poisson process, this dimension is equal to one. But, of course, an isomorphism of such factorizations is not an isomorphism of the corresponding processes (more precisely, of the corresponding measure spaces). Hence, a particular Fock factorization should not be identified with the canonical factorization in the Fock space $\text{EXP } H$.

Studying representations of current groups with coefficients in the semisimple groups $O(n, 1)$, $U(n, 1)$, $n = 1, 2, \dots$, the authors discovered that it is not convenient to diagonalize the unipotent subgroups of parabolic groups in the Gaussian model, i.e., in the usual Fock space $\text{EXP}(L^2(X; K))$. It is much more convenient to employ other measures instead of the Gaussian one and, consequently, other models of factorizations. In particular, it turned out that a remarkable (non-Gaussian) σ -finite (i.e., nonprobability) measure on the space of Schwartz distributions, which will be called the stable measure and has a large group of linear symmetries, allows one to simplify constructions and proofs of the irreducibility of representations of current groups. It is the analysis of this measure that has led the authors to the definition of the functor INT. In the introduction, we give a brief definition of this functor; for a detailed description, see §1.

1.3. The stable measure as a base for a discrete model of the Fock space. As was mentioned above, a continuous factorization is defined naturally not only in the space $\text{EXP}(L^2(X; K))$, i.e., for a Gaussian process (white noise), but also in the Hilbert space

of square-integrable functionals with respect to the measure of an arbitrary Lévy process (a process with independent values). Hence, every Lévy process induces its own second quantization, i.e., its own EXP-type functor. But it seems that among all Lévy processes, there are two distinguished ones having remarkable properties, the most important of which is that their structure is quite convenient for describing representations of current groups. Along with the white noise (i.e., the functor EXP), this is the so-called *gamma process* and the related *stable (Lebesgue)* measure on the space of Schwartz distributions. This fact was first observed in [29], but without reference to the gamma process, its properties, and the stable measure.

The main feature of both measures, the Gaussian measure and the stable one, is the existence of an infinite-dimensional group of linear symmetries (the orthogonal group for the Gaussian measure and the group of multipliers, or the Cartan group, for the stable measure).⁴ Precisely these symmetries allow one to use the corresponding spaces in representation theory, primarily for constructing representations of current groups. A special interest to the gamma process is due to a large number of applications it has in combinatorics, number theory, statistical physics, etc. The group of linear transformations preserving the stable measure and the properties of this measure were studied in detail in [34, 3, 36]. This measure was originally called the Lebesgue measure (because its properties are similar to those of the finite-dimensional Lebesgue measure), but in this paper it is called the *stable measure*. The stable measure $\mathcal{L}$ is concentrated on the cone $\mathcal{K}$ of nonnegative discrete finite measures on the base X :

$$\begin{aligned}\mathcal{K}(X) &\equiv l_+^1(X) \\ &\equiv \left\{ \xi = \sum_k c_k \delta_{x_k} : x_k \in X; c_k \geq 0, k = 1, 2, \dots; c_1 \geq c_2 \geq \dots; \sum_k c_k < \infty \right\};\end{aligned}$$

its Laplace transform is given by the formula

$$\int_{\mathcal{K}} e^{\langle f, \xi \rangle} d\mathcal{L} = \exp \left\{ - \int_X \log f(x) dx \right\}, \quad f \geq 0.$$

Clearly, this measure is invariant under the group of multipliers M_a that act in accordance with the formula

$$\xi = \sum_k c_k \delta_{x_k} \longrightarrow M_a \xi = \sum_k c_k a(x_k) \delta_{x_k},$$

where $a(\cdot)$ is a nonnegative measurable function such that $\int \log a(x) dx = 0$.

The stable measure allows one to construct a new (isomorphic) model of the Fock space and define a new functor INT on the category of Hilbert spaces, similar to the functor EXP.

While with a measure space (X, m) and a Hilbert space of coefficients K the functor EXP associates the Fock space $\text{EXP}(L_m^2(X; K))$, the functor INT associates with this pair the Hilbert space

$$\text{INT}(l_+^1(X); K) \equiv L_{\mathcal{L}}^2(\mathcal{K}(X); K) = \int_{l_+^1(X)} \bigotimes_k K_{x_k} d\xi,$$

where $\mathcal{K} = l_+^1(X)$ is the cone of finite nonnegative countably supported measures ξ on X (see above) endowed with the stable measure $\mathcal{L}$, and an element of the space $\text{INT}(l_+^1(X); K)$ is a square-integrable functional with respect to $\mathcal{L}$ that sends a countable measure ξ to an element of the countable tensor product $\bigotimes_k K$ (for the details, see [11] and §4).

⁴The Gaussian measure has also the infinite-dimensional commutative group of translations by smooth vectors that preserve the type of the measure.

The isomorphism between the spaces $\text{EXP}(L_m^2(X; K))$ and

$$\text{INT}(\mathcal{K}(X); K) \equiv L_{\mathcal{L}}^2(l_+^1(X); K)$$

is an isometry preserving the factorizations.

Since the definition of the INT functor involves only countable tensor products, we may say that the definition of a continuous tensor product is reduced to a direct integral of countable tensor products:

$$\int_X^{\otimes} K_x dm(x) = L_{\mathcal{L}}^2(l_+^1(X); K).$$

Thus, we have obtained a new formula for a continuous tensor factorization. This model has a clear interpretation, namely, *a continuous tensor product is a direct integral of countable tensor products over the trajectories of a certain random process (the gamma process)*.

In this connection it should be mentioned that Lévy processes, or processes with independent values, appeared in approximately the same time as the Fock space and works on second quantization; however, a very close relationship between these objects, and even a link to the Wiener process and the Gaussian measure were discovered much later.

For more information on the introduction and study of the stable (Lebesgue) measure, see [29, 8, 34, 14, 3, 35, 36]. The papers [10, 11] contain a survey of applications of this measure to the representation theory of current groups for Lie groups of rank 1; in particular, instead of the EXP functor corresponding to the white noise, the functor INT is introduced that corresponds to the σ -finite stable (Lebesgue) measure. The isomorphism between the Fock space and the space of functionals on the set of trajectories of a general Lévy process is discussed from various viewpoints in [14].

1.4. The configurational approach to the Fock space. In this paper, we make the next step, whose principal goal is to describe a new, configurational (or, as we call it, quasi-Poisson), construction of a continuous tensor product and its applications to the representation theory of current groups. The point is that most Lévy processes (in particular, the gamma process) can be obtained by the following general construction (see, e.g., [18]).

Consider the half-strip $Y = X \times \mathbb{R}_+$ and endow it with the measure $\mu = dm \times P$, where dm is a measure on X and P is the so-called Lévy measure on $\mathbb{R}_+$, occurring in the Lévy–Khinchin formula for the logarithm of the characteristic functional of an infinitely divisible distribution on the real line. Assume that P satisfies the condition $\int_{\mathbb{R}_+} \min[1, x] dP < \infty$ (a more familiar form of this condition is $P[[1, \infty]] < \infty$ and $\int_0^1 x dP < \infty$). Then (see [18]) the corresponding Lévy process can be obtained as follows.

Starting with the measure μ , construct a Poisson field on locally finite configurations in the strip, and let ν be the measure on the space of configurations corresponding to this field (in this case, the Poisson field ν is said to have intensity measure μ). With every configuration

$$\omega = \bigcup_{i=1}^n (x_i, r_i), \quad x_i \in X, \quad r_i \in \mathbb{R}_+, \quad i = 1, \dots, n,$$

in the strip Y we associate its projection $\xi_\omega = \sum_i r_i \delta x_i$ to the space of distributions on X . Then the image of the measure on configurations under this map is the measure (on the cone $\mathcal{K}$, see above) corresponding to the Lévy process with Lévy measure P . For instance, the gamma process on the half-line $[0, \infty]$ corresponds to the Poisson field

in the strip $Y = X \times \mathbb{R}_+$ with intensity measure $dx \times \frac{e^{-r} dr}{r}$, and all properties of the gamma process can be reformulated tautologically in terms of this Poisson field. The stable measure has density $e^{\sum r_i}$ with respect to the law of the gamma process. Hence, all considerations related to the stable measure can also be carried over to the space of configurations endowed with a *quasi-Poisson measure*, which differs from the Poisson measure only by the density $\frac{dq}{d\nu} \omega = e^{\sum \omega r_i}$. This allows us to give another model of the Fock space, which involves the space of functionals of configurations. This model is even more geometric than the INT model.

It turns out that many steps of the construction do not require that the space of configurations be considered in the strip; we may take an arbitrary topological space Y , the set of configurations $\text{Conf}(Y)$, and Poisson fields on $\text{Conf}(Y)$. This extends the class of examples.⁵ In particular, we may consider generalized strips of the form $X \times S$, where S is a group (or even a semigroup), which allows us to study representations of a wider class of current groups of semidirect products.

To summarize, we may say that the model of the Fock space with the stable measure (or the functor INT) and the configurational model (with a quasi-Poisson measure) studied in this paper differ from the classical construction of a continuous tensor factorization in that a continual tensor product is interpreted as a direct integral over countable tensor products; in other words, we replace continuous integrals with integrals over countable trajectories. Of course, here a crucial role is played by an appropriate choice of measures (the stable measure and the corresponding quasi-Poisson measure). The fact that these measures are infinite (σ -finite) makes it possible to simplify formulas for representations of current groups. Certainly, we could use more familiar probability measures equivalent to the original ones; for the stable measure, this is the law of the gamma process and its generalizations to the vector case. However, in our opinion, passing to σ -finite measures leads to an important change of the viewpoint on integration in infinite-dimensional spaces. It should be emphasized once again that we heavily rely on the following fact: the parabolic subgroups of $O(n, 1)$ and $U(n, 1)$ are semidirect products, and irreducible representations of the corresponding current group extend to irreducible representations of the entire current group in the same spaces.

It is in pertinent to recall that the theorem on isomorphism between the Fock and Poisson factorizations, i.e., the result on isomorphism between the L^2 spaces over the white noise and the Poisson process on the line preserving the factorizations, was obtained in [6] as a consequence of the fact that the representations of the group of diffeomorphisms in these spaces are equivalent; in its turn, this equivalence followed from the coincidence of the spherical functions in both representations and from their irreducibility. (For other approaches, see [21, 14].) Though this proof did not provide a simple explicit formula for the isomorphism in question, now it becomes clear (see also [14, 11]) how one can write it explicitly without involving the above representations; namely, one should use the correspondences between the elements $\exp v$, $\text{int } v$, and $\text{qps } v$ (see § 4).

1.5. Summary of the paper. The twelve sections of the paper are divided naturally into three groups. In § 2–6, we define and study Poisson and quasi-Poisson measures, the structure of the corresponding spaces of functions, and groups of operators in them. Apparently, the notion of a quasi-Poisson is considered and analyzed for the first time. In § 7–9, we describe constructions of representations of current groups for crossed products in quasi-Poisson spaces; here the main examples are the parabolic subgroups of rank 1 groups. Finally, in § 10–12 the constructed representations of the current groups of parabolic subgroups are extended to the current groups of semisimple groups of rank 1,

⁵Note that for constructing representations of current groups, we only need the case of the strip $Y = X \times \mathbb{R}_+$.

$O(n, 1)$ and $U(n, 1)$. We emphasize that these representations themselves have been known since the 1970s and were studied in our previous papers (see [4, 5] and others). The novelties are in the realization of these representations in the spaces introduced in the paper and the simple proof of their irreducibility based on this realization. The key role in this approach is played by the maximal parabolic subgroups and their representations.

Now we describe the content of the paper in more detail. We consider the space of configurations $\mathcal{E}(Y)$ on a space Y with an infinite σ -finite measure. The *quasi-Poisson measures* on $\mathcal{E}(Y)$ are infinite σ -finite measures $d\sigma(\omega)$ on $\mathcal{E}(Y)$ related to Poisson measures $d\nu(\omega)$ on $\mathcal{E}(Y)$ by the formula $d\sigma(\omega) = \pi(\omega) d\nu(\omega)$.

With every triple (Y, μ, u) , where μ is an infinite σ -finite measure on Y and u is a nonnegative function ("action") on Y satisfying the additional conditions (2.5) (see the main text), we associate the Poisson measure $d\nu(\omega)$ on $\mathcal{E}(Y)$ with intensity measure $e^{-u(y)} d\mu(y)$ and the quasi-Poisson measure $d\sigma(\omega) = \pi(\omega) d\nu(\omega)$, where $\pi(\omega) = \exp(\sum_{y \in \omega} u(y))$. The *quasi-Poisson space* associated with (Y, μ, u) and a pair (K, h) , where K is a separable Hilbert space and h is a fixed unit vector in K , is the Hilbert space

$$\text{QPS}(\mathcal{E}(Y), \sigma; K) = L^2(\mathcal{E}(Y), \sigma; K_\omega^\otimes),$$

i.e., the Hilbert space of measurable square-integrable sections of the fiber bundle whose base is the space of configurations $\mathcal{E}(Y)$ and the fiber over a point $\omega \in \mathcal{E}(Y)$ is the countable tensor product $K_\omega^\otimes = \bigotimes_{y \in \omega} K_y$, $K_y = K$.

In the special case where $(Y, \mu) = (\mathbb{R}_+^*, d^*r) \times (X, m)$ and $u(r, x) = r$, the projection $P : Y = \mathbb{R}_+^* \times X \rightarrow X$ induces a map from the space of configurations $\mathcal{E}(Y)$ to the space of configurations of points in X marked by numbers $r \in \mathbb{R}_+^*$, i.e., to the cone of discrete (nonnegative) measures on X . Under this map, the quasi-Poisson measure σ on $\mathcal{E}(Y)$ goes to the stable measure $\mathcal{L}$ on the cone $l_+^1(X)$, and there is a natural isomorphism of spaces

$$\text{QPS}(\mathcal{E}(\mathbb{R}_+^* \times X), \sigma; K) \cong \text{INT}(X; K).$$

Our main goal in this paper is to present a construction of irreducible unitary representations of current groups in quasi-Poisson spaces. This can be done by using the classical models of these representations in Fock spaces and the isomorphism between quasi-Poisson and Fock spaces:

$$\text{QPS}(\mathcal{E}(Y), \sigma; K) \cong \text{EXP}(L^2(Y, \mu; K)).$$

For the class of groups G that can be presented as semidirect products $G = S \ltimes G_0$, we suggest a new model of representations of the corresponding current groups in quasi-Poisson spaces. For this, we introduce the group $\mathfrak{U}(H)$ of all unitary operators on $H = L^2(Y, \mu; K)$ and the subgroup

$$G(Y; K) = \text{Aut}(Y, \mu) \ltimes B(Y; K)$$

of $\mathfrak{U}(H)$, where $B(Y; K)$ is the group of operators preserving the fibers K of the fiber bundle over Y and $\text{Aut}(Y, \mu)$ is the group of μ -preserving transformations of the space Y realized as the group of unitary operators permuting the fibers K of this fiber bundle. With every nonnegative function u on Y we associate a subgroup

$$G_u(Y; K) = \text{Aut}_u(Y, \mu) \ltimes B_u(Y; K),$$

which has a nontrivial 1-cocycle.

There exists a natural representation of the group $G_u(Y; K)$ in the quasi-Poisson space

$$\text{QPS}_u(\mathcal{E}(Y), \sigma; K) = \int_{\mathcal{E}(Y)} K_\omega^\otimes d\sigma(\omega).$$

In this representation, the operators corresponding to elements of the subgroup $B_u(Y; K)$ act in the fibers $K_\omega^\otimes$ of the fiber bundle over $\mathcal{E}(Y)$, and the operators $U(a)$ corresponding to elements of the subgroup $\text{Aut}_u(Y, \mu)$ permute these fibers:

$$U(a)\varphi(\omega) = e^{-\frac{1}{2} \text{Log } a} \varphi(a^{-1}\omega),$$

where $\text{Log } a$ is a functional on $\text{Aut}_u(Y; \nu)$ that depends only on u . The irreducibility of this representation follows from the ergodicity of the quasi-Poisson measure σ . Note that this construction, in contrast to the Fock model, does not involve the existence of a nontrivial 1-cocycle explicitly.

To construct a representation of an arbitrary current group G^X , where $G = S \ltimes G_0$, in a quasi-Poisson space $\text{QPS}_u(\mathcal{E}(Y), \sigma; K)$, it suffices to describe the embeddings of groups $S^X \rightarrow \text{Aut}_u(Y, \mu)$ and $G_0^X \rightarrow B_u(Y, \mu)$. This construction applies, in particular, to the groups P^X , where P is the maximal parabolic subgroup of the semisimple Lie group $O(n, 1)$ or $U(n, 1)$, because P can be presented as a semidirect product $P = \mathbb{R}_+^* \ltimes G_0$ (see §8 and 9). It does not apply directly to the current groups $U(n, 1)^X$ and $O(n, 1)^X$; however, representations of the groups $U(n, 1)^X$ and $O(n, 1)^X$ in quasi-Poisson spaces can be obtained as extensions of the constructed representations of the current groups P^X . These representations of $U(n, 1)^X$ and $O(n, 1)^X$ are described in §11 and 12.

The main difference of the approach suggested in this paper from that adopted in [12] and the earlier papers devoted to the study of the integral model of the Fock space and the corresponding representations of current groups is that we work with the space of configurations rather than with the space of discrete measures. In other words, we view a discrete measure on X with values in $\mathbb{R}_+$ as a measure on the space of configurations on $\mathbb{R}_+ \times X$. This approach is widely used in the theory of Lévy processes. Another novelty, which seems to have never been used before, is that instead of the Poisson measure on the space of configurations we consider the so-called quasi-Poisson measure, a σ -finite measure defined by a special density with respect to the Poisson measure (a kind of “external field”). This allows us to simplify the statements. The old approach is a special case of the new one, namely, the case where the space with a Poisson or quasi-Poisson measure is $\mathcal{E}(\mathbb{R}_+ \times X)$.

§2. POISSON AND QUASI-POISSON MEASURES

2.1. The Poisson measure ν . Let Y be a standard Borel space with a σ -finite infinite continuous (nonatomic) Borel measure μ .⁶

Configurations on Y are finite or countable discrete submultisets in Y , i.e., collections ω of points of Y that contain every point y with a finite multiplicity.

Denote by $\mathcal{E}(Y)$ the set of all configurations on Y . Consider a basis of the measure space (Y, μ) , i.e., an arbitrary countable family $\mathcal{A} = \{A\}$ of measurable subsets that is dense in the σ -algebra of all measurable subsets of Y (for example, if Y is a topological space, we can take a countable base of open sets), and denote by $\mathfrak{B}$ the σ -algebra generated by this family. Using this σ -algebra, we introduce the σ -algebra $\mathcal{B}$ of subsets in $\mathcal{E}(Y)$ generated by the subsets of the form

$$\mathcal{E}(A, n) = \{\omega \in \mathcal{E}(Y) : \#(\omega \cap A) = n\}, \quad \text{where } A \in \mathfrak{B} \quad \text{and} \quad n = 0, 1, 2, \dots$$

---

⁶Without entering into measure-theoretic details, we only note that the space (Y, μ) is assumed to be isomorphic, in the sense of measure theory, to the real line $\mathbb{R}$ with Lebesgue measure; thus, it may be (up to a set of zero measure) an arbitrary separable complete metric space with a σ -finite continuous measure.

The *Poisson measure* on $\mathcal{E}(Y)$ with intensity measure μ is the probability measure $\nu_\mu = \nu$ defined on the σ -algebra $\mathcal{B}$ by the formulas

$$\nu(\mathcal{E}(A, n)) = \frac{\mu(A)^n}{n!} e^{-\mu(A)} \quad \text{for all } \mathcal{E}(A, n).$$

Up to isomorphism, the measure on $\mathcal{E}(Y)$ defined in this way does not depend on the choice of a basis $\mathcal{A}$, but depends substantially on the intensity measure μ ; we shall omit the subscript indicating the dependence on μ if μ is fixed.

One usually says that Y is the *state space*, μ is the *intensity measure* of the Poisson measure ν , and $(\mathcal{E}(Y), \nu)$ is the *Poisson pair* associated with the pair (Y, μ) . Note that we do not impose any conditions on the intensity measure apart from continuity and σ -finiteness.

The existence of the Poisson measure with a given intensity measure and its basic properties can easily be derived from the definition; this material can be found in textbooks (see, e.g., [18]). Now we state some properties of the Poisson measure ν .

2.1.1. For every subset $A \in \mathfrak{B}$ of finite measure, the set $\{\omega \in \mathcal{E}(Y) : \#(\omega \cap A) = \infty\}$ has zero measure.

2.1.2. The measure ν is concentrated on the set of infinite configurations (i.e., configurations ω such that $\#\omega = \infty$) without multiple points.

2.1.3. Infinite divisibility. If $Y = Y_1 \cup \dots \cup Y_n$ is an arbitrary finite partition of the space Y into pairwise disjoint measurable subsets, $\mathcal{E}(Y) = \mathcal{E}(Y_1) \times \dots \times \mathcal{E}(Y_n)$ is the corresponding direct product decomposition of the space $\mathcal{E}(Y)$, and ν_i is the projection of the Poisson measure ν to $\mathcal{E}(Y_i)$, $i = 1, \dots, n$, then $\nu = \nu_1 \times \dots \times \nu_n$.

2.1.4. Campbell's theorem.

Theorem 2.1. *The series*

$$F(\omega) = \sum_{y \in \omega} f(y), \quad \omega \in \mathcal{E}(Y),$$

where f is a measurable function on Y , converges absolutely on a subset of full measure if and only if

$$\int_Y \min[1, |f(y)|] d\mu(y) < \infty.$$

In this case,

$$(2.1) \quad \int_{\mathcal{E}(Y)} e^{\theta F(\omega)} d\nu(\omega) = \exp \left(\int_Y (e^{\theta f(y)} - 1) d\mu(y) \right)$$

for every $\theta \in \mathbb{C}$ provided that the integral on the right-hand side converges (in particular, if θ is purely imaginary).

2.1.5. Characteristic functional. Campbell's theorem implies the following result.

Proposition 2.1. *For every pair (Y, μ) and every nonnegative measurable function f on Y such that $\int_Y \min[c, f(y)] d\mu(y) < \infty$ for every $c > 0$, the series $F(\omega) = \sum_{y \in \omega} f(y)$ converges on a subset of full Poisson measure ν in the space of configurations $\omega \in \mathcal{E}(Y)$, and*

$$(2.2) \quad \int_{\mathcal{E}(Y)} e^{-F(\omega)} d\nu(\omega) = \exp \left(\int_Y (e^{-f(y)} - 1) d\mu(y) \right).$$

The integral on the left-hand side of (2.2) is called the *characteristic functional*, or the *Laplace transform*, of the Poisson measure ν .

Remark. Formula (2.2) can be taken as the definition of the Poisson measure. It is also convenient to replace $e^{-f(y)}$ with $1 + \varphi(y)$ and write (2.2) in the form

$$(2.3) \quad \int_{\mathcal{E}(Y)} \left(\prod_{y \in \omega} (1 + \varphi(y)) \right) d\nu(\omega) = \exp \left(\int_Y \varphi(y) d\mu(y) \right)$$

provided that both sides converge.

2.1.6. The action of the transformation group. Denote by $G(Y, \mu)$ the set of measurable transformations $g : y \mapsto gy$ of the space Y leaving the measure μ quasiinvariant and satisfying the following condition on the Radon–Nikodym derivative $\varphi_g(y) = \frac{d\mu(gy)}{d\mu(y)}$:

$$\int_Y |\varphi_g(y) - 1| d\mu(y) < \infty.$$

Such transformations form a group and induce transformations $\omega \mapsto g\omega$ in the space of configurations $\mathcal{E}(Y)$, which will be denoted by the same letter.

Theorem 2.2. *The measure ν on the space $\mathcal{E}(Y)$ is quasiinvariant under the transformations $\omega \mapsto g\omega$, $g \in G(Y)$, and the corresponding Radon–Nikodym derivative is equal to*

$$(2.4) \quad e^{-\int_Y (\varphi(y)-1) d\mu(y)} \left(\prod_{y \in \omega} \varphi(y) \right).$$

Corollary. *The Poisson measure ν with intensity measure μ is invariant under the subgroup $\text{Aut}(Y; \mu)$ of all μ -preserving transformations $g \in G(Y, \mu)$.*

Theorem 2.3. *The Poisson measure ν is ergodic with respect to the subgroup $\text{Aut}(Y, \mu)$ of all μ -preserving transformations $g : y \mapsto gy$.*

2.2. An important special case. In what follows, we are especially interested in the case where

$$Y = S \times X, \quad d\mu(y) = ds dm(x).$$

Here S is a space with an action of a locally compact group of transformations J , ds is a J -invariant measure on S (i.e., $d(as) = ds$ for every $a \in J$), and (X, m) is a space with a continuous probability measure m . In this case, on Y we have an action of the group J^X of maps $X \rightarrow J$, which is defined by the formula

$$(s, x) \mapsto a(\cdot)(s, x) = (a(x)s, x),$$

and this action induces an action of the group J^X on the space $\mathcal{E}(Y)$ of configurations $\omega = \{s_k, x_k\}$:

$$\omega \mapsto a(\cdot)\omega, \quad \text{where} \quad a(\cdot)\{s_k, x_k\} = \{a(x_k)s_k, x_k\}.$$

Theorem 2.2 implies the following result.

Theorem 2.4. *The Poisson measure ν associated with the state space $Y = S \times X$ and the intensity measure $d\mu(y) = ds \times dm(x)$, is invariant under the action of the group J^X , i.e.,*

$$d\nu(a(\cdot)\omega) = d\nu(\omega) \quad \text{for all} \quad a(\cdot) \in J^X.$$

2.3. Definition of a quasi-Poisson measure. A *quasi-Poisson measure* on the space of configurations $\mathcal{E}(Y)$ is an infinite σ -finite measure on $\mathcal{E}(Y)$ of the form $d\sigma(\omega) = \pi(\omega)d\nu(\omega)$, where ν is the Poisson measure.

Let (Y, μ) be a space with an infinite σ -finite measure μ . Fix a nonnegative function $u(y)$ on Y satisfying the conditions

$$(2.5) \quad \int_Y e^{-u(y)} d\mu(y) = \infty \quad \text{and} \quad \int_Y \min[1, u(y)] e^{-u(y)} d\mu(y) < \infty.$$

We shall denote by $(\mathcal{E}(Y), \nu)$ the Poisson pair in which $\nu = \nu_{\mu, u}$ is the Poisson measure with intensity measure $e^{-u(y)} d\mu(y)$, i.e., the characteristic functional of ν is given by

$$\int_{\mathcal{E}(Y)} \exp \left(- \sum_{y \in \omega} f(y) \right) d\nu(\omega) = \exp \left(\int_Y (e^{-f(y)} - 1) e^{-u(y)} d\mu(y) \right).$$

The conditions on u and Campbell's theorem show that the series $\pi(\omega) = \sum_{y \in \omega} u(y)$, $\omega \in \mathcal{E}(Y)$, converges absolutely on a subset of full measure ν in the space $\mathcal{E}(Y)$ of all configurations. Thus, we may define a measure σ on $\mathcal{E}(Y)$ by the formula

$$(2.6) \quad d\sigma(\omega) = \pi(\omega) d\nu(\omega), \quad \pi(\omega) = e^{\sum_{y \in \omega} u(y)}.$$

Theorem 2.5. The characteristic functional of the measure σ on $\mathcal{E}(Y)$ is equal to

$$(2.7) \quad \int_{\mathcal{E}(Y)} \exp \left(- \sum_{y \in \omega} f(y) \right) d\sigma(\omega) = \exp \left(\int_Y (e^{-f(y)} - e^{-u(y)}) d\mu(y) \right),$$

where f is an arbitrary positive measurable function on Y such that the integral on the right-hand side converges absolutely.

Indeed,

$$\int_{\mathcal{E}(Y)} \exp \left(- \sum_{y \in \omega} f(y) \right) d\sigma(\omega) = \int_{\mathcal{E}(Y)} \exp \left(- \sum_{y \in \omega} f(y) - u(y) \right) d\nu(\omega).$$

Hence, by Campbell's formula for the Poisson measure $d\nu$, we have

$$\int_{\mathcal{E}(Y)} \exp \left(- \sum_{y \in \omega} f(y) \right) d\sigma(\omega) = \exp \left(\int_Y (e^{u(y)-f(y)} - 1) e^{-u(y)} d\mu(y) \right),$$

which is equivalent to (2.7).

Corollary. The measure $d\sigma$ on the space $\mathcal{E}(Y)$ is infinite, i.e., it is a quasi-Poisson measure, if and only if

$$(2.8) \quad \int_Y (1 - e^{-u(y)}) d\mu(y) = \infty.$$

Remark. If $s(u) \equiv \int_Y (1 - e^{-u(y)}) d\mu(y) < \infty$, then $d\sigma = e^{s(u)} d\nu$, where $\nu = \nu_\mu$ is the Poisson measure with intensity measure $\mu(y)$.

Definition 2.1. Given a triple (Y, μ, u) , where μ is a measure on Y and u is a nonnegative function on Y satisfying conditions (2.5) and (2.8), the associated quasi-Poisson measure is the infinite σ -finite measure $d\sigma(\omega)$ defined by (2.6) on the space of configurations $\mathcal{E}(Y)$, i.e., the measure with the characteristic functional (2.7).

Remark. Formula (2.7) can be taken as the definition of the quasi-Poisson measure σ .

The quasi-Poisson measure σ and the Poisson measure ν have the same sets of full measure. Hence, σ is concentrated on the set of infinite configurations ω without multiple points; one can easily check that this measure is infinitely divisible.

2.4. The group $\text{Aut}_u(Y, \mu)$, the logarithm of a transformation, and the projective invariance property.

Definition 2.2. Denote by $\text{Aut}(Y, \mu)$ the group of all μ -preserving (measurable) transformations $T : y \mapsto Ty$ of the space Y . Given a function u on Y satisfying conditions (2.5), we also denote by $\text{Aut}_u(Y, \mu)$ the subgroup of transformations $T \in \text{Aut}(Y, \mu)$ such that the integral

$$(2.9) \quad \text{Log } T \equiv \int_Y (e^{-u(T^{-1}y)} - e^{-u(y)}) d\mu(y)$$

converges absolutely.

One can easily check that the functional $\text{Log } T$ on the group $\text{Aut}_u(Y, \mu)$ has the logarithm property:

$$\text{Log}(T_1 T_2) = \text{Log } T_1 + \text{Log } T_2 \quad \text{for any } T_1, T_2 \in \text{Aut}_u; \quad \text{Log } T^{-1} = -\text{Log } T.$$

The integrand in (2.9) is a nontrivial 1-cocycle of the group $\text{Aut}_u(Y, \mu)$ with values in the space $L^1(Y, \mu)$ (because the coboundary does not lie in $L^1(Y, \mu)$; cf. [14]).

The action of $\text{Aut}_u(Y, \mu)$ on Y induces an action $\omega \mapsto T\omega$ of this group on the space of configurations $\mathcal{E}(Y)$.

Theorem 2.6. *The quasi-Poisson measure σ on the space $\mathcal{E}(Y)$ associated with a triple (Y, μ, u) is projectively invariant under the action of the group $\text{Aut}_u(Y, \mu)$ on $\mathcal{E}(Y)$:*

$$d\sigma(T\omega) = e^{\text{Log } T} d\sigma(\omega) \quad \text{for every } T \in \text{Aut}_u(Y, \mu).$$

In particular, σ is invariant under all transformations $T \in \text{Aut}_u(Y, \mu)$ that preserve the function $\pi(\omega)$ (i.e., satisfy $\pi(T\omega) \equiv \pi(\omega)$); in other words, under all transformations that leave the function u invariant.

Proof. It suffices to verify that

$$\int_{\mathcal{E}(Y)} \exp\left(-\sum_{y \in \omega} f(y)\right) d\sigma(T\omega) = e^{\text{Log } T} \int_{\mathcal{E}(Y)} \exp\left(-\sum_{y \in \omega} f(y)\right) d\sigma(\omega)$$

for every $T \in \text{Aut}_u(Y, \mu)$. We have

$$\int_{\mathcal{E}(Y)} \exp\left(-\sum_{y \in \omega} f(y)\right) d\sigma(T\omega) = \int_{\mathcal{E}(Y)} \exp\left(-\sum_{y \in \omega} f(T^{-1}y)\right) d\sigma(\omega).$$

Along with the formula for the characteristic functional of σ , this implies the relation

$$\int_{\mathcal{E}(Y)} \exp\left(-\sum_{y \in \omega} f(y)\right) d\sigma(T\omega) = e^{\text{Log } T} \exp\left(\int_Y (e^{-f(T^{-1}y)} - e^{-u(T^{-1}y)}) d\mu(y)\right).$$

From the invariance of $d\mu(y)$ it follows that the second factor on the right-hand side equals $\exp(\int_Y (e^{-f(y)} - e^{-u(y)}) d\mu(y))$ and, consequently, coincides with the original integral $\int_{\mathcal{E}(Y)} \exp(-\sum_{y \in \omega} f(y)) d\sigma(\omega)$, as required. $\square$

Theorem 2.7. *The quasi-Poisson measure σ is ergodic relative to the group $\text{Aut}_u(Y, \mu)$.*

Proof. The claim follows from the ergodicity of the Poisson measure ν with respect to the action of $\text{Aut}_u(Y, \mu)$, because these two measures are equivalent.

The ergodicity of the Poisson measure is a consequence of the following general fact. Let (Y, μ) be a space with a finite or σ -finite measure μ and an action of a group G ; if μ is invariant and ergodic with respect to this action, then the corresponding Poisson measure on the space of configurations $\mathcal{E}(Y)$ is also invariant and ergodic with respect to the induced action of G . Since all spaces (Y, μ) with σ -finite continuous standard measures

are metrically isomorphic (each of them is a countable union of Lebesgue spaces), it suffices to consider the real line $\mathbb{R}$ and the one-parameter group of translations, whose action on $\mathbb{R}$ is transitive (and hence ergodic); thus, our assertion is equivalent to the classical theorem on the ergodicity of a stationary Poisson process on $\mathbb{R}$. $\square$

2.5. An example. Let

$$(Y, \mu) = (\mathbb{R}_+^* \times X, d^*r \, dm(x)),$$

where (X, m) is a standard space with a probability measure and $d^*r = r^{-1}dr$ is the Haar measure on $\mathbb{R}_+^*$.

In this example, $\text{Aut}_u(Y, \mu)$ is the group of transformations $T : (r, x) \mapsto T(r, x)$ of the space $Y = \mathbb{R}_+^* \times X$ preserving the measure $d\mu(y) = d^*r \, dm(x)$ and such that the integral

$$\text{Log } T = \int_X \int_0^\infty (e^{-u(T^{-1}(r, x))} - e^{-u(r, x)}) d^*r \, dm(x)$$

converges. In the special case where u is a function depending only on r (e.g., $u(r) = r$), the group $\text{Aut}_u(Y, \mu)$ is the semidirect product

$$\text{Aut}_u(Y, \mu) = \Sigma \ltimes (\mathbb{R}_+^*)^X,$$

where Σ is the group of m -preserving transformations of X and $(\mathbb{R}_+^*)^X$ is the subgroup of transformations $(r, x) \mapsto (a(x)r, x)$ such that the integral

$$\text{Log } a = \int_X \int_0^\infty (e^{-u(a(x)r)} - e^{-u(r)}) d^*r \, dm(x)$$

converges. An element of $\text{Aut}_u(Y, \mu)$ is a pair $T = (\tau, a(\cdot))$, where $\tau \in \Sigma$, $a(\cdot) \in (\mathbb{R}_+^*)^X$, the multiplication law is given by the formula

$$(\tau_1, a_1(\cdot))(\tau_2, a_2(\cdot)) = (\tau_1\tau_2, a_1^{\tau_2}(\cdot)a_2(\cdot)), \quad \text{where } a^\tau(x) = a(x^\tau),$$

and the action of $\text{Aut}_u(Y, \mu)$ on Y is given by the formula

$$T(r, x) = (a(x)r, x^\tau) \quad \text{for } T = (\tau, a(\cdot)).$$

Proposition 2.2. *By integrating with respect to r , the formula for $\text{Log } a$ can be simplified to*

$$\text{Log } a = \int_X \log a(x) \, dm(x).$$

Indeed, this follows from the relation

$$(2.10) \quad \int_0^\infty (e^{-u(ar)} - e^{-u(br)}) d^*r = \log(b/a) \quad \text{for any } a, b > 0.$$

§3. THE FIELD OF POISSON AND QUASI-POISSON SPACES (THE SCALAR CASE)

3.1. The Poisson–Hilbert space $\text{PS}(\mathcal{E}(Y), \nu)$. Let Y be a space with an infinite continuous measure μ , and let $(\mathcal{E}(Y), \nu)$ be the Poisson pair associated with (Y, μ) .

Definition 3.1. The Poisson space (or the Poisson field of configurations) associated with the pair (Y, μ) is the Hilbert space

$$\text{PS}(\mathcal{E}(Y), \nu) = L^2(\mathcal{E}(Y), \nu),$$

i.e., the Hilbert space of measurable functions $f(\omega)$ on the space of configurations $\mathcal{E}(Y)$ with the norm

$$\|f\|^2 = \int_{\mathcal{E}(Y)} |f(\omega)|^2 d\nu(\omega) < \infty.$$

Observe three fundamental properties of the space $\text{PS}(\mathcal{E}(Y), \nu)$.

The first is the *factorization property*: if $Y = Y_1 \cup \dots \cup Y_n$ is an arbitrary finite partition of Y into pairwise disjoint measurable subsets and μ_i is the restriction of μ to Y_i , then

$$\text{PS}(\mathcal{E}(Y), \nu) \approx \bigotimes_{i=1}^n \text{PS}(\mathcal{E}(Y_i), \nu_i),$$

where $\mathcal{E}(Y_i, \nu_i)$ is the Poisson pair associated with (Y_i, μ_i) .

The second property is as follows: *in the space $\text{PS}(\mathcal{E}(Y), \nu)$, a unitary representation of the group $\text{Aut}(Y, \mu)$ of μ -preserving transformations $a : y \mapsto ay$ of Y is defined by the formula*

$$(U(a)f)(\omega) = f(a^{-1}\omega).$$

The third property: *there exists a continuous map $\text{ps} : L^2(Y, \mu) \rightarrow \text{PS}(\mathcal{E}(Y), \nu)$ onto a total subset in $\text{PS}(\mathcal{E}(Y), \nu)$ satisfying the relation*

$$(3.1) \quad \langle \text{ps } v_1, \text{ps } v_2 \rangle = e^{\langle v_1, v_2 \rangle},$$

where the angle brackets on the left- and right-hand sides stand for the inner products in $\text{PS}(\mathcal{E}(Y), \nu)$ and $L^2(Y, \mu)$, respectively.

To construct this map, we introduce the dense subspace $L(Y, \mu) \subset L^2(Y, \mu)$ of measurable functions v that take finitely many values and vanish outside a subset of finite measure in Y . Given $v \in L(Y, \mu)$, we define the vector $\text{ps } v$ by the formula

$$(3.2) \quad (\text{ps } v)(\omega) = e^{-\int_Y v(y) d\mu(y)} \prod_{y \in \omega} (1 + v(y)), \quad \omega \in \mathcal{E}(Y);$$

note that the product in (3.2) contains only finitely many factors different from 1.

The definition of $\text{ps } v$ and formula (2.3) show that

$$(3.3) \quad \int_{\mathcal{E}(Y)} (\text{ps } v)(\omega) = 1 \quad \text{for } v \in L(Y, \mu).$$

Formula (3.1) for $v \in L(Y, \mu)$ is an easy consequence of (3.3). Next, from (3.1) it follows that the map $\text{ps} : L(Y, \mu) \rightarrow \text{PS}(\mathcal{E}(Y), \nu)$ defined by this formula is continuous. Hence, it extends by continuity to a map $\text{ps} : L^2(Y, \mu) \rightarrow \text{PS}(\mathcal{E}(Y), \nu)$.

Remark. For an arbitrary vector $v \in L^2(Y, \mu)$, the infinite product in (3.2) may fail to converge; however, for every function $v \in L^1(Y, \mu)$, it converges on a set of full measure in $\mathcal{E}(Y)$.

The map $v \mapsto \text{ps } v$ induces a continuous unitary representation in $\text{PS}(\mathcal{E}(Y), \nu)$ of the group $\mathfrak{U}$ of all unitary operators of the space $L^2(Y, \mu)$. Namely, by (3.1), the operators $T : \text{ps } v \mapsto \text{ps}(Tv)$, $T \in \mathfrak{U}$, preserve the inner product on the set of all vectors $\text{ps } v$. Since this set is total, they can be extended to unitary operators on the entire space $\text{PS}(\mathcal{E}(Y), \nu)$.

3.2. The quasi-Poisson space $\text{QPS}(\mathcal{E}(Y), \sigma)$. Let σ be the quasi-Poisson measure on $\mathcal{E}(Y)$ associated with a triple (Y, μ, u) , where (Y, μ) is a space with an infinite continuous measure and $u(y)$ is a continuous nonnegative function on Y satisfying (2.5). By Definition 2.1, the measure σ is given by the formula

$$d\sigma(\omega) = \pi(\omega) d\nu(\omega),$$

where $\pi(\omega) = e^{\sum_{y \in \omega} u(y)}$ and ν is the Poisson measure with intensity measure $e^{-u(y)} d\mu(y)$, and its characteristic functional equals

$$\int_{\mathcal{E}(Y)} \exp \left(- \sum_{y \in \omega} f(y) \right) d\sigma(\omega) = \exp \left(\int_Y (e^{-f(y)} - e^{-u(y)}) d\mu(y) \right).$$

Definition 3.2. The quasi-Poisson Hilbert space associated with (Y, μ, u) is the Hilbert space

$$\text{QPS}(\mathcal{E}(Y), \sigma) = L^2(\mathcal{E}(Y), \sigma)$$

of measurable functions f on $\mathcal{E}(Y)$ with the norm

$$\|f\|^2 = \int_{\mathcal{E}(Y)} |f(\omega)|^2 d\sigma(\omega).$$

Now we consider the main properties of the space $\text{QPS}(\mathcal{E}(Y), \sigma)$.

3.2.1. Relationship with the Poisson space $\text{PS}(\mathcal{E}(Y), \nu)$. The formula $d\sigma(\omega) = e^{\sum_{y \in \omega} u(y)} d\nu(\omega)$, where ν is the Poisson measure on $\mathcal{E}(Y)$ with the intensity measure $e^{-u(y)} d\mu(y)$, implies the following result.

Proposition 3.1. *The map*

$$\tau_u : \varphi(\omega) \mapsto \pi^{1/2}(\omega) \varphi(\omega), \quad \text{where } \pi(\omega) = e^{\sum_{y \in \omega} u(y)},$$

is an isomorphism of Hilbert spaces $\text{PS}(\mathcal{E}(Y), \nu) \rightarrow \text{QPS}(\mathcal{E}(Y), \sigma)$, *where* ν *is the Poisson measure with the intensity measure* $e^{-u(y)} d\mu(y)$.

3.2.2. The action of the group $\text{Aut}_u(Y, \mu)$. By Theorem 2.6, the quasi-Poisson measure σ is projectively invariant under the group $\text{Aut}_u(Y, \mu)$ of transformations $a : y \mapsto ay$ of the space Y preserving the measure $d\mu(y)$ and such that the integral $\text{Log } a = \int_Y (e^{-u(a^{-1}y)} - e^{-u(y)}) d\mu(y)$ converges. Namely, $d\sigma(a\omega) = e^{\text{Log } a} d\sigma(\omega)$. This implies the following theorem.

Theorem 3.1. *The operators*

$$(U(a)f)(\omega) = e^{-\frac{1}{2} \text{Log } a} f(a^{-1}\omega), \quad a \in \text{Aut}_u(Y, \mu),$$

on the space $\text{QPS}(\mathcal{E}(Y), \sigma)$ *form a unitary representation of the group* $\text{Aut}_u(Y, \mu)$ *in this space.*

3.2.3. The embedding $L^2(Y, \mu) \rightarrow \text{QPS}(\mathcal{E}(Y), \sigma)$.

Definition 3.3. We define a map

$$\text{qps} : L^2(Y, \mu) \rightarrow \text{QPS}(\mathcal{E}(Y), \sigma)$$

as the composition of three maps: the isomorphism $\lambda_u : L^2(Y, \mu(y)) \rightarrow L^2(Y, e^{-u(y)} \mu(y))$ defined by the formula $(\lambda_u v)(y) = e^{-\frac{1}{2} u(y)} v(y)$, the map $\text{ps} : L^2(Y, e^{-u(y)} \mu(y)) \rightarrow \text{PS}(\mathcal{E}(Y), \nu)$, and the map $\tau_u : \text{PS}(\mathcal{E}(Y), \nu) \rightarrow \text{QPS}(\mathcal{E}(Y), \sigma)$.

In accordance with this definition,

$$(3.4) \quad (\text{qps } v(y))(\omega) = \pi^{-1/2}(\omega) (\text{ps}(e^{\frac{1}{2} u(y)} v(y)))(\omega).$$

Combined with the definition of ps , this implies the following theorem.

Theorem 3.2.

- 1) *The map* $\text{qps} : L^2(Y, \mu) \rightarrow \text{QPS}(\mathcal{E}(Y), \sigma)$ *is continuous.*
- 2) *The vectors* $\text{qps } v$, $v \in L^2(Y, \mu)$, *form a total subset in the space* $\text{QPS}(\mathcal{E}(Y), \sigma)$ *and are related by the formula*

$$(3.5) \quad \langle \text{qps } v_1, \text{qps } v_2 \rangle = e^{\langle v_1, v_2 \rangle} \quad \text{for any } v_1, v_2 \in L^2(Y, \mu).$$

- 3) *On the subspace* $L(Y, \mu) \subset L^2(Y, \mu)$ *of piecewise constant functions, the vectors* $\text{qps } v$ *take the form*

$$(3.6) \quad (\text{qps}_u v)(\omega) = e^{-\int_Y v(y) e^{-\frac{1}{2} u(y)} d\mu(y)} \bigotimes_{y \in \omega} (e^{-\frac{1}{2} u(y)} + v(y)).$$

Remark. For $u(y) \equiv 0$, formula (3.6) turns into the similar formula (3.2) for $\text{ps } v$.

Precisely as in the case of the Poisson space, these properties of the vectors $\text{qps } v$ imply the following result.

Proposition 3.2. *The map $v \mapsto \text{qps } v$ generates a continuous unitary representation in the space $\text{QPS}(\mathcal{E}(Y), \nu)$ of the group $\mathfrak{U}$ of all unitary operators of $L^2(Y, \mu)$.*

§4. THE FIELD OF POISSON AND QUASI-POISSON SPACES (THE VECTOR CASE)

4.1. Countable tensor products. We begin with the definition of a countable tensor product of Hilbert spaces.

A *punctured Hilbert space* is a pair (K, h) , where K is a Hilbert space and $h \in K$ is a fixed unit vector. We assume that the one-dimensional spaces are punctured automatically. An isomorphism of punctured Hilbert spaces is defined in a natural way.

Definition 4.1. The countable tensor product $\bigotimes_{n=1}^{\infty} (K_n, h_n)$ of punctured Hilbert spaces (K_n, h_n) , $n = 1, 2, \dots$, is the punctured Hilbert space

$$(\tilde{K}, \tilde{h}) = \left(\bigotimes_{n=1}^{\infty} K_n, \bigotimes_{n=1}^{\infty} h_n \right),$$

where $\bigotimes_{n=1}^{\infty} K_n$ is the norm completion of the inductive limit of the finite tensor products $\bigotimes_{n=1}^k K_n$ with respect to the embeddings $\bigotimes_{n=1}^k K_n \ni f \mapsto f \otimes h_{k+1} \in \bigotimes_{n=1}^{k+1} K_n$.

In particular, if all (K_n, h_n) are identified with a fixed punctured space (K, h) , we write $\bigotimes_{n=1}^{\infty} (K, h) = (\bigotimes_{n=1}^{\infty} K, \bigotimes_{n=1}^{\infty} h)$ and call this punctured space the *countable tensor power* of (K, h) .

It is natural to write elements $\bigotimes_{m=1}^n f_k$ of the subspaces $\bigotimes_{m=1}^n K_m$ as infinite products that stabilize at the $(n+1)$ st step:

$$(4.1) \quad y_n = f_1 \otimes f_2 \otimes \cdots \otimes f_n \otimes h_{n+1} \otimes h_{n+2} \otimes \cdots, \quad \text{where } f_m \in K_m.$$

These vectors form a total subset in $\bigotimes_{n=1}^{\infty} K_n$, i.e., their linear hull is dense in $\bigotimes_{n=1}^{\infty} K_n$.

Note that if in each punctured space (K_n, h_n) there is a fixed orthonormal basis $\{e_{ni} : i = 0, 1, \dots\}$ with $e_{n0} = h_n$, then the vectors $\bigotimes_{n=1}^{\infty} e_{n, i_n}$, where $i_n = 0$ for sufficiently large n , form an orthonormal basis in the countable tensor product $\bigotimes_{n=1}^{\infty} (K_n, h_n)$.

Proposition 4.1. *For every sequence $f_n \in K_n$ such that*

$$\sum_{n=1}^{\infty} \|f_n\| < \infty,$$

the sequence $y_n = \bigotimes_{m=1}^n (h_m + f_m) \in \bigotimes_{m=1}^n K_m$ converges in $\bigotimes_{n=1}^{\infty} K_n$.

The limit of such a sequence will be written as an infinite product: $y = \bigotimes_{m=1}^{\infty} (h_m + f_m)$. In particular, $y \in \bigotimes_{m=1}^n K_m$ if $f_m = 0$ for $m > n$.

In what follows, it is convenient to identify tensor products that differ only by the order of factors, and write $\bigotimes_{y \in \omega} (K_y, h_y)$, where ω is a finite or countable (unordered) set, instead of $\bigotimes_{m=1}^n (K_m, h_m)$ or $\bigotimes_{m=1}^{\infty} (K_m, h_m)$. Accordingly, instead of $\bigotimes_{m=1}^n v_m$ and $\bigotimes_{m=1}^{\infty} v_m$, where $v_m \in K_m$, we shall write $\bigotimes_{y \in \omega} v_y$ or $\prod_{y \in \omega} v_y$, where $v_y \in K_y$.

The inner product of two such monomials equals

$$\left\langle \bigotimes_{y \in \omega} h_y, \bigotimes_{y \in \omega} h'_y \right\rangle = \prod_{y \in \omega} \langle h_y, h'_y \rangle.$$

In the special case where all spaces (K_y, h_y) , $y \in \omega$, are identified with a fixed space (K, h) , we write $K_{\omega}^{\otimes} = (K, h)^{\otimes}$ instead of $\bigotimes_{y \in \omega} (K_y, h_y)$.

By definition, the tensor product of two symmetrized tensor products $\bigotimes_{y \in \omega_1} K_y$ and $\bigotimes_{y \in \omega_2} K_y$, where ω_1 and ω_2 are disjoint subsets, is the symmetrized tensor product $\bigotimes_{y \in \omega_1 \cup \omega_2} K_y$.

4.2. The Poisson–Hilbert space $\text{PS}(\mathcal{E}(Y), \nu; K)$. Let $(\mathcal{E}(y), \nu)$ be the Poisson pair associated with a state space Y and intensity measure $\mu(y)$, and let (K, h) be a punctured finite-dimensional or separable Hilbert space.

Definition 4.2. With every configuration $\omega \in \mathcal{E}(Y)$ we associate the countable tensor product $K_\omega^\otimes$ of punctured Hilbert spaces:

$$K_\omega^\otimes = \bigotimes_{y \in \omega} (K_y, h_y), \quad (K_y, h_y) = (K, h);$$

the Poisson–Hilbert space associated with the punctured Hilbert space (K, h) and the pair (Y, μ) is the Hilbert space

$$\text{PS}(\mathcal{E}(Y), \nu; K) = L^2(\mathcal{E}(Y), \nu; K_\omega^\otimes),$$

i.e., the space of measurable sections $F : \omega \mapsto F(\omega)$ of the fiber bundle over $\mathcal{E}(Y)$ with fibers $K_\omega^\otimes = \bigotimes_{y \in \omega} (K_y, h_y)$, $(K_y, h_y) = (K, h)$, endowed with the norm

$$\|F\|^2 = \int_{\mathcal{E}(Y)} \|F(\omega)\|_{K_\omega^\otimes}^2 d\nu(\omega).$$

In other words, $\text{PS}(\mathcal{E}(Y), \nu; K)$ is the direct integral with respect to the Poisson measure ν on $\mathcal{E}(Y)$ of the countable tensor products $K_\omega^\otimes$, i.e.,

$$(4.2) \quad \text{PS}(\mathcal{E}(Y), \nu; K) = \int_{\mathcal{E}(Y)}^\oplus K_\omega^\otimes d\nu(\omega).$$

In the special case where $\dim K = 1$, the spaces $K_\omega^\otimes$ are also one-dimensional, and hence $\text{PS}(\mathcal{E}(Y), \nu; K)$ coincides with the space $\text{PS}(\mathcal{E}(Y), \nu)$ introduced above.

In the space of functions $v(y)$ on Y there is a natural action of the additive group $A = \mathbb{R}^Y$ of real-valued, compactly supported, piecewise constant functions $a(y)$ on Y : $v(y) \mapsto e^{ia(y)}v(y)$. This action induces a unitary representation of the group A in the space $\text{PS}(\mathcal{E}(Y), \nu; K)$:

$$(T(a)f)(\omega) = e^{i \sum_{y \in \omega} a(y)} f(\omega).$$

Obviously, formula (4.2) gives rise to a decomposition of the space $\text{PS}(\mathcal{E}(Y), \nu; K)$ into a direct integral of pairwise disjoint eigenspaces of the group A .

4.3. The map $\text{ps} : L^2(Y, K) \rightarrow \text{PS}(\mathcal{E}(Y), \nu; K)$. Denote by $L^2(Y, \mu; K)$ (or, in short, by $L^2(Y; K)$) the Hilbert space of measurable maps $v : Y \rightarrow K$ with the norm

$$\|v\|^2 = \int_Y \|v(y)\|^2 d\mu(y) < \infty.$$

Here it is reasonable to regard $L^2(Y; K)$ not as the space of K -valued square integrable functions on Y , but rather as the space of sections of a certain measurable fiber bundle. More precisely, we consider the measurable fiber bundle $E \rightarrow Y$ over Y with fibers $K_y = K$, $y \in Y$, $K_y \simeq K$.⁷ The space of square integrable sections is defined as the following direct integral of Hilbert spaces:

$$L^2(Y; K) = \int_Y K_y d\mu(y), \quad K_y = K.$$

⁷We shall not dwell on the routine definitions of a measurable locally trivial fiber bundle of vector spaces over a measure space and related notions; local triviality is understood as triviality over every set of positive measure, isomorphisms are measurable maps, etc.

In what follows, we adhere to this definition of the space $L^2(Y; K)$.

Observe the following simple fact.

Proposition 4.2. *If $(Y, \mu) = (Y_1, \mu_1) \times (Y_2, \mu_2)$, then*

$$L^2(Y, \mu; K) = L^2(Y_1, \mu_1; K^{Y_2}), \quad \text{where } K^{Y_2} = L^2(Y_2, \mu_2, K).$$

Definition 4.3. Denote by $L(Y; K)$ the subspace $L(Y; K) \subset L^2(Y; K)$ of measurable functions v that take finitely many values and vanish outside a subset of finite measure in Y . Obviously, $L(Y; K)$ is dense in $L^2(Y; K)$. Given $v \in L(Y; K)$ and $\omega \in \mathcal{E}(Y)$, set

$$(4.3) \quad (\text{ps } v)(\omega) = \exp\left(-\int_Y \langle v(y), h_y \rangle_{K_y} d\mu(y)\right) \bigotimes_{y \in \omega} (h_y + v(y)).$$

Proposition 4.3. *The vectors $\text{ps } v$ defined by (4.3) lie in $\text{PS}(\mathcal{E}(Y), \nu; K)$; for any $v_1, v_2 \in L(Y; K)$, we have*

$$(4.4) \quad \langle \text{ps}(v_1), \text{ps}(v_2) \rangle = e^{\langle v_1, v_2 \rangle}.$$

Proof. The definition of $L(Y; K)$ shows that the tensor product on the right-hand side of (4.3) contains only finitely many factors different from h_y . Hence, it converges and lies in $K_\omega^\otimes$.

Next, from (4.3) it follows that

$$\langle (\text{ps } v_1)(\omega), (\text{ps } v_2)(\omega) \rangle_{K_\omega^\otimes} = \exp\left(-\int_Y (\langle v_1(y), h_y \rangle_{K_y} + \overline{\langle v_2(y), h_y \rangle_{K_y}}) d\mu(y)\right) \prod_{y \in \omega} (1 + \varphi(y)),$$

where

$$\varphi(y) = \langle v_1(y), v_2(y) \rangle_{K_y} + \langle v_1(y), h_y \rangle_{K_y} + \langle h_y, v_2(y) \rangle_{K_y}.$$

Since $\varphi(y) \in L(Y, \mu)$, formulas (3.2) and (3.3) imply (4.4). $\square$

Theorem 4.1. *The map $\text{ps} : L(Y; K) \rightarrow \text{PS}(\mathcal{E}(Y), \nu; K)$ defined by (4.3) extends by continuity to a continuous map*

$$\text{ps} : L^2(Y; K) \rightarrow \text{PS}(\mathcal{E}(Y), \nu; K)$$

satisfying (4.4) for any $v_1, v_2 \in L^2(Y; K)$.

From (4.4) it follows that the map $\text{ps} : L(Y; K) \rightarrow \text{PS}(\mathcal{E}(Y), \nu; K)$ is continuous. Hence, since $L(Y; K)$ is dense in $L^2(Y; K)$, it can be extended by continuity to a continuous map $\text{ps} : L^2(Y; K) \rightarrow \text{PS}(\mathcal{E}(Y), \nu; K)$.

Proposition 4.4. *For every fixed $\omega \in \mathcal{E}(Y)$, the set of vectors $(\text{ps } v)(\omega) \in K_\omega^\otimes$, $v \in L(Y, K)$, is total in $K_\omega^\otimes$.*

Proof. We use the fact that the set of vectors of the form

$$\Phi(\omega) = \bigotimes_{y \in \omega} (h_y + f_y), \quad f_y \in K_y,$$

where $f_y = 0$ for all but finitely many $y \in \omega$, is total in $K_\omega^\otimes$. Hence, it suffices to show that for every such vector $\Phi(\omega)$, where $\omega = \{y_k\}$ and $f_{y_k} = 0$ for $k > n$, there exists a vector $v \in L(Y, K)$ such that $(\text{ps } v)(\omega) = c\Phi(\omega)$, i.e., $v(y_k) = cf_k$ for all $k = 1, 2, \dots$.

In Y we fix pairwise disjoint neighborhoods $U_1, \dots, U_n$ of the points $y_1, \dots, y_n$ that do not contain the points y_k , $k > n$; then we define a function $v \in L(Y, K)$ by the formulas

$$v(y) = f_k \quad \text{for } y \in U_k, \quad k = 1, \dots, n,$$

and $v(y) = 0$ on the complement of $\bigcup_{k=1}^n U_k$. Obviously, the vector $v \in L(Y, K)$ defined in this way satisfies the required property. $\square$

Theorem 4.2. *The set of vectors $\text{ps } v$, $v \in L(Y, K)$, is total in $\text{PS}(\mathcal{E}(Y), \nu; K)$.*

Proof. Denote by P_0 the norm completion in $\text{PS}(\mathcal{E}(Y), \nu; K)$ of the linear hull of the vectors $\text{ps } v$, $v \in L(Y, K)$. We show that $P_0 = \text{PS}(\mathcal{E}(Y), \nu; K)$. The definition of $\text{ps } v$ implies that P_0 is invariant under the unitary representation of the group A in the space $\text{PS}(\mathcal{E}(Y), \nu; K)$. Hence, since $\text{PS}(\mathcal{E}(Y), \nu; K)$ is the direct integral with respect to the Poisson measure ν of pairwise disjoint eigenspaces $K_\omega^\otimes$ of A , the space P_0 itself can be presented in a similar form:

$$P_0 = \int_{\mathcal{E}(Y)} \tilde{K}_\omega^\otimes d\nu(\omega), \quad \text{where} \quad \tilde{K}_\omega^\otimes \subset K_\omega^\otimes.$$

On the other hand, Proposition 4.4 shows that $\tilde{K}_\omega^\otimes = K_\omega^\otimes$ on a subset of full measure in $\mathcal{E}(Y)$. Hence, $P_0 = \text{PS}(\mathcal{E}(Y), \nu; K)$. $\square$

Theorem 4.2 and the continuity of the map $v \mapsto \text{ps } v$ imply the following corollary.

Corollary. *For every linear subspace $\tilde{L}(Y, K)$ dense in $L(Y, K)$, the set of vectors $\text{ps } v$, $v \in \tilde{L}(Y, K)$, is also total in $\text{PS}(\mathcal{E}(Y), \nu; K)$.*

Note that the structure of the space $\text{PS}(\mathcal{E}(Y), \nu; K)$ endowed with the canonical factorization over the base Y is completely determined by the total subset of vectors $\text{ps } v$.

By analogy with the scalar case, we have the following result.

Proposition 4.5. *The map $v \mapsto \text{ps } v$ generates a continuous unitary representation in the space $\text{PS}(\mathcal{E}(Y), \nu; K)$ of the group $\mathfrak{U}(H)$ of all unitary operators in the space $H = L^2(Y, \mu; K)$.*

4.4. The quasi-Poisson space $\text{QPS}(\mathcal{E}(Y), \sigma; K)$. Let σ be the quasi-Poisson measure on $\mathcal{E}(Y)$ associated with a triple (Y, μ, u) (see Definition 2.1).

Definition 4.4. The quasi-Poisson space associated with a triple (Y, μ, u) and a punctured Hilbert space (K, h) is the Hilbert space

$$\text{QPS}(\mathcal{E}(Y), \sigma; K) = L^2(\mathcal{E}(Y), \sigma; K_\omega^\otimes),$$

i.e., the space of measurable sections $F : \omega \mapsto F(\omega)$ of the fiber bundle over $\mathcal{E}(Y)$ with fibers $K_\omega^\otimes = \bigotimes_{y \in \omega} (K_y, h_y)$, $(K_y, h_y) = (K, h)$, endowed with the norm

$$\|F\|^2 = \int_{\mathcal{E}(Y)} \|F(\omega)\|_{K_\omega^\otimes}^2 d\sigma_u(\omega).$$

In other words, $\text{QPS}(\mathcal{E}(Y), \sigma; K, h)$ is the direct integral with respect to the quasi-Poisson measure σ on $\mathcal{E}(Y)$ of the countable tensor products $K_\omega^\otimes$:

$$(4.5) \quad \text{QPS}(\mathcal{E}(Y), \sigma; K) = \int_{\mathcal{E}(Y)}^\oplus K_\omega^\otimes d\sigma_u(\omega).$$

In the special case where $\dim K = 1$, the spaces $K_\omega^\otimes$ are also one-dimensional and we write $\text{QPS}(\mathcal{E}(Y), \sigma)$ instead of $\text{QPS}(\mathcal{E}(Y), \sigma; K)$.

As in the case of $\text{PS}(\mathcal{E}(Y), \nu; K)$, the action on Y of the group $A = \mathbb{R}^Y$ of real-valued, compactly supported, piecewise constant functions $a(y)$, $v(y) \mapsto e^{ia(y)}v(y)$, induces a unitary representation of A in the space $\text{QPS}(\mathcal{E}(Y), K)$:

$$(T(a)f)(\omega) = e^{i \sum_{y \in \omega} a(y)} f(\omega).$$

Obviously, formula (4.5) gives rise to a decomposition of the space $\text{QPS}(\mathcal{E}(Y), \sigma; K)$ into a direct product of pairwise disjoint eigenspaces of A .

The relation $d\sigma(\omega) = e^{\sum_{y \in \omega} u(y)} d\nu(\omega)$, where $d\nu$ is the Poisson measure associated with the pair $(Y, e^{-u(y)} d\mu(y))$, implies the following result.

Proposition 4.6. *The map*

$$\tau : \varphi(\omega) \mapsto \pi^{1/2}(\omega)\varphi(\omega), \quad \pi(\omega) = e^{\sum_{y \in \omega} u(y)},$$

is an isomorphism of Hilbert spaces $\text{PS}(\mathcal{E}(Y), \nu; K) \rightarrow \text{QPS}(\mathcal{E}(Y), \sigma; K)$, where ν is the Poisson measure with the intensity measure $e^{-u(y)} d\mu(y)$.

Definition 4.5. We define a map

$$\text{qps} : L^2(Y, \mu; K) \rightarrow \text{QPS}(\mathcal{E}(Y), \sigma; K)$$

as the composition of three maps: the isomorphism

$$\lambda : L^2(Y, \mu(y); K) \rightarrow L^2(Y, e^{-u(y)} \mu(y); K)$$

defined by the formula $(\lambda v)(y) = e^{-\frac{1}{2}u(y)} v(y)$, the map $\text{ps} : L^2(Y, e^{-u(y)} \mu(y); K) \rightarrow \text{PS}(\mathcal{E}(Y), \nu; K)$, and the map $\tau : \text{PS}(Y, e^{-u(y)} \mu(y); K) \rightarrow \text{QPS}(\mathcal{E}(Y), \sigma; K)$.

In accordance with this definition,

$$(4.6) \quad (\text{qps } v(y))(\omega) = \pi^{-1/2}(\omega)(\text{ps}(e^{\frac{1}{2}u(y)} v(y)))(\omega).$$

Combined with the properties of the map ps , this implies the following theorem.

Theorem 4.3.

- 1) *The map $\text{qps} : L^2(Y, \mu; K) \rightarrow \text{QPS}(\mathcal{E}(Y), \sigma; K)$ is continuous.*
- 2) *The vectors $\text{qps } v$ with $v \in L^2(Y, \mu; K)$ constitute a total subset in the space $\text{QPS}(\mathcal{E}(Y), \sigma; K)$ and are related by the formula*

$$(4.7) \quad \langle \text{qps } v_1, \text{qps } v_2 \rangle = e^{\langle v_1, v_2 \rangle} \quad \text{for any } v_1, v_2 \in L^2(Y, \mu; K).$$

- 3) *On the subspace $L(Y, \mu; K) \subset L^2(Y, \mu; K)$ of compactly supported piecewise constant functions, the vectors $\text{qps } v$ have the form*

$$(4.8) \quad (\text{qps } v)(\omega) = e^{-\int_Y \langle v(y), f_u(y) \rangle_{K_y} d\mu(y)} \bigotimes_{y \in \omega} (f_u(y) + v(y)), \quad \text{where } f_u(y) = e^{-\frac{1}{2}u(y)} h_y.$$

Corollary. *The map $\text{ps } v \mapsto \text{qps } v$, $v \in L^2(Y, \mu; K)$, where $\text{ps } v \in \text{PS}(\mathcal{E}(Y), \nu; K)$, $\text{qps } v \in \text{QPS}(\mathcal{E}(Y), \sigma; K)$, generates an isomorphism of Hilbert spaces*

$$\text{PS}(\mathcal{E}(Y), \nu; K) \rightarrow \text{QPS}(\mathcal{E}(Y), \sigma; K).$$

Indeed, the subsets of vectors $\text{ps } v$ and $\text{qps } v$ are total in the corresponding subspaces, and the map $\text{ps } v \mapsto \text{qps } v$ preserves the inner product.

Remark. It is natural to interpret PS and QPS as functors on the subcategory of Hilbert spaces $L^2(Y, \mu; K)$. Note that in the categories of spaces $\text{PS}(\mathcal{E}(Y), \nu; K)$ and $\text{QPS}(\mathcal{E}(Y), \sigma; K)$ there are two canonical factorizations, over the base Y and over the fiber K , so that every partition of the base (Y, μ) into finitely many disjoint measurable subsets and every decomposition of the fiber K into a finite direct sum of subspaces give rise to a decomposition of the spaces $\text{PS}(Y; K) = \text{PS}(\mathcal{E}(Y), \nu; K)$ and $\text{QPS}(Y; K) = \text{QPS}(\mathcal{E}(Y), \sigma; K)$ into a finite tensor product:

$$\begin{aligned} \text{PS} \left(\bigcup_{i=1}^n Y_i; K \right) &\approx \bigotimes_{i=1}^n \text{PS}(Y_i; K); & \text{PS} \left(Y; \bigoplus_{i=1}^n K_i \right) &\approx \bigotimes_{i=1}^n \text{PS}(Y; K_i); \\ \text{QPS}_u \left(\bigcup_{i=1}^n Y_i; K \right) &\approx \bigotimes_{i=1}^n \text{QPS}_u(Y_i; K); & \text{QPS}_u \left(Y; \bigoplus_{i=1}^n K_i \right) &\approx \bigotimes_{i=1}^n \text{QPS}_u(Y; K_i). \end{aligned}$$

4.5. The stable measure $\mathcal{L}$ and the associated Hilbert space. In the special case where $(Y, \mu) = (\mathbb{R}_+^*, d^*r) \times (X, m)$, the projection $P : Y = \mathbb{R}_+^* \times X \rightarrow X$ induces a map from the space of configurations $\mathcal{E}(Y)$ onto the space of configurations of points in X marked by numbers $r \in \mathbb{R}_+^*$, i.e., onto the cone of discrete (nonnegative) measures on X :

$$l_+^1(X) = \left\{ \xi = \sum_{n=1}^{\infty} r_n \delta_{x_n} : r_n > 0, \sum r_n < \infty, x_n \in X \right\}.$$

Elements of $l_+^1(X)$ will be written as $\xi = \{r_k, x_k\}_{k=1}^{\infty}$ or, in short, $\xi = \{r_k, x_k\}$.

Under this map, the quasi-Poisson measure σ on $\mathcal{E}(Y)$ goes to a measure $P\sigma = \mathcal{L}$ on $l_+^1(X)$. This measure $\mathcal{L}$ is called the *stable measure*; it was first introduced in [29] and then studied in detail in [3, 35, 36].

There is a natural isomorphism of Hilbert spaces

$$\text{QPS}(X; K) \cong \text{INT}(X; K),$$

where $\text{QPS}(X; K) = \text{QPS}_u(\mathcal{E}(\mathbb{R}_+^* \times X), \sigma; K)$, $u(r) = r$, and

$$\text{INT}(X; K) = L^2(l_+^1(X), \mathcal{L}; K_{\xi}^{\otimes}),$$

i.e., $\text{INT}(X; K)$ is isomorphic to the space of functions $\varphi(\xi)$ on $l_+^1(X)$ with values in the tensor products

$$K_{\xi}^{\otimes} = \bigotimes_{(r,s) \in \xi} K_{r,s}, \quad K_{r,s} = K,$$

endowed with the norm

$$\|\varphi\|^2 = \int \|\varphi(\xi)\|_{K_{\xi}^{\otimes}}^2 d\mathcal{L}(\xi).$$

Under this isomorphism, the vectors $\text{qps } v$, $v \in L(\mathbb{R}_+^* \times X, d^*r dm(x); K)$, of the total subset in $\text{QPS}_u(\mathcal{E}(\mathbb{R}_+^* \times X), \sigma; K)$, given by the formula

$$(\text{qps } v)(\omega) = e^{-\int_{\mathbb{R}_+^* \times X} \langle v(r,x), f(r,x) \rangle_{K_{r,x}} d^*r dm(x)} \bigotimes_{(r,x) \in \omega} (f(r,x) + v(r,x)),$$

where $f(r,x) = e^{-\frac{r}{2}} h_{r,x}$, go to vectors of a similar total set in $L^2(l_+^1(X), \mathcal{L}; K_{\xi}^{\otimes})$:

$$(\text{int } v)(\xi) = e^{-\int_{\mathbb{R}_+^* \times X} \langle v(r,x), f(r,x) \rangle_{K_{r,x}} d^*r dm(x)} \bigotimes_{(r,x) \in \xi} (f(r,x) + v(r,x)).$$

For completeness, we give an independent definition and describe the basic properties of the stable measure $\mathcal{L}$.

On the cone $l_+^1(X)$, there is a natural action of the group of multipliers M_a , $a \in A$, where A is the group of nonnegative measurable functions on (X, m) such that the integral $\int_X \log a(x) dm(x)$ converges:

$$M_a : \xi \mapsto a(\cdot)\xi, \quad \text{where} \quad a(\cdot)\xi = \{a(x_k)r_k, x_k\} \quad \text{for} \quad \xi = \{r_k, x_k\}.$$

Theorem 4.4 (see [3, 35]). *There exists a σ -finite (infinite) measure $\mathcal{L}$ on the cone $l_+^1(X)$ that is uniquely determined by its characteristic functional*

$$F(a) \equiv \int_{l_+^1(X)} \exp \left(- \sum_k r_k a(x_k) \right) d\mathcal{L}(\xi) = \exp \left(- \int_X \log a(x) dm(x) \right) \quad \text{for every } a \in A$$

and possesses the following fundamental properties:

1) $\mathcal{L}$ is projectively invariant under the group of multipliers M_a ,

$$(4.9) \quad d\mathcal{L}(a(\cdot)\xi) = \exp \left(\int_X \log a(x) dm(x) \right) d\mathcal{L}(\xi),$$

and ergodic with respect to this group;

2) $\mathcal{L}$ is invariant and ergodic with respect to the group of all m -preserving transformations of (X, m) .

We call $\mathcal{L}$ the *stable measure* on $l_+^1(X)$.⁸ Now we state yet two other fundamental properties of $\mathcal{L}$.

Theorem 4.5. *For every function $v \in L^1(\mathbb{R}_+^* \times X, d^*r dm(x))$, we have*

$$(4.10) \quad \int_{l_+^1(X)} \left(\prod_{(r,x) \in \xi} (e^{-r} + v(r, x)) \right) d\mathcal{L}(\xi) = \exp \left(\int_{\mathbb{R}_+^* \times X} v(r, x) d^*r dm(x) \right).$$

Formula (4.10) can be taken as the definition of $\mathcal{L}$.

Theorem 4.6. *The measure $\mathcal{L}$ is related to the Poisson measure by the formula*

$$(4.11) \quad e^{-\sum_{(r,x) \in \xi} r} d\mathcal{L}(\xi) = d\nu(\xi),$$

where ν is the Poisson measure on $\mathcal{E}(X \times \mathbb{R}_+^*)$ with the intensity measure $d\mu(r, x) = e^{-r} d^*r dm(x)$.

§5. RELATIONSHIP WITH FOCK SPACES

5.1. The Fock spaces $\text{EXP}(\mathcal{K})$. By definition, for an arbitrary fixed Hilbert space $\mathcal{K}$, the Fock space $\text{EXP}(\mathcal{K})$ is the direct sum of the symmetrized tensor powers $S^n \mathcal{K}$ of $\mathcal{K}$:

$$\text{EXP}(\mathcal{K}) = \bigoplus_{n=0}^{\infty} (S^n \mathcal{K}).$$

Here $S^0 \mathcal{K} = \mathbb{C}$. Note that the spaces $\text{EXP}(\mathcal{K})$ are canonically punctured: each of them contains a fixed unit vector from $S^0 \mathcal{K} = \mathbb{C}$, which is denoted by $\mathbb{1}$ and called the *vacuum* vector.

It is natural to interpret the symbol EXP as a functor in the category of Hilbert spaces. The most important property of this functor is multiplicativity: every decomposition $\mathcal{K} = \bigoplus_{i=1}^n K_i$ of $\mathcal{K}$ into a finite direct sum of mutually orthogonal subspaces gives rise to a decomposition of the space $\text{EXP}(\mathcal{K})$ into a finite tensor product of the corresponding EXP -spaces:

$$(5.1) \quad \text{EXP} \left(\bigoplus_{i=1}^n K_i \right) = \bigotimes_{i=1}^n \text{EXP}(K_i).$$

This property defines a *Fock factorization* structure on the spaces $\text{EXP}(\mathcal{K})$.

Another important element of the EXP construction is the continuous map $\mathcal{K} \rightarrow \text{EXP}(\mathcal{K})$ that sends every vector $h \in \mathcal{K}$ to its exponential $\exp h \in \text{EXP} \mathcal{K}$:

$$\exp h = \mathbb{1} \oplus h \oplus \frac{h \otimes h}{\sqrt{2!}} \oplus \cdots \oplus \frac{\otimes^n h}{\sqrt{n!}} \oplus \cdots.$$

In particular, $\exp 0 = \mathbb{1}$.

The vectors $\exp h$ defined in this way are linearly independent, form a total subset in $\text{EXP}(\mathcal{K})$, and satisfy the relation

$$(5.2) \quad \langle \exp h_1, \exp h_2 \rangle = e^{\langle h_1, h_2 \rangle} \quad \text{for any } h_1, h_2 \in \mathcal{K},$$

where the angle brackets on the left- and right-hand side stand for the inner products in the spaces $\text{EXP}(\mathcal{K})$ and $\mathcal{K}$, respectively.

⁸In the papers [3, 35, 36], $\mathcal{L}$ was called the Lebesgue measure.

5.2. The spaces $\text{EXP}(Y; K)$. In the most important case, which is the only one needed for applications to field theory, the initial Hilbert spaces $\mathcal{K}$ are the spaces

$$\mathcal{K} = K^Y = L^2(Y, \mu; K)$$

introduced above, where Y is a space with a continuous (nonatomic) measure μ and K is an arbitrary finite-dimensional or separable Hilbert space.

Definition 5.1. In what follows, we shall only consider the restriction of the functor EXP to the subcategory of spaces of the form $L^2(Y, \mu; K)$ and write $\text{EXP}(Y, \mu; K)$ or, in short, $\text{EXP}(Y, K)$ instead of $\text{EXP}(L^2(Y, \mu; K))$. By Fock spaces we mean only spaces of the form $\text{EXP}(Y, K)$.

Every finite partition $Y = \bigcup_{i=1}^n Y_i$ of the base Y into pairwise disjoint subsets and every decomposition $K = \bigoplus_{i=1}^n K_i$ of the space K into a finite direct sum of subspaces give rise to a decomposition of the space $L^2(Y, K)$ into a direct sum of subspaces:

$$L^2(Y, K) = \bigoplus_{i=1}^n L^2(Y_i, K) = \bigoplus_{i=1}^n L^2(Y, K_i).$$

Hence, the Fock factorization induces two canonical factorizations of the category of spaces $\text{EXP}(Y, K)$, over the base Y and over the fiber K :

$$\text{EXP}\left(\bigcup_{i=1}^n Y_i; K\right) \approx \bigotimes_{i=1}^n \text{EXP}(Y_i; K); \quad \text{EXP}\left(Y; \bigoplus_{i=1}^n K_i\right) \approx \bigotimes_{i=1}^n \text{EXP}(Y; K_i).$$

The structure of the space $\text{EXP}(Y; K)$ endowed with the canonical factorization over the base Y is completely determined by the total subset of vectors $\exp v$. We emphasize that the group of automorphisms of the Hilbert space $\text{EXP}(Y; K)$ and the group of automorphisms of the same space preserving the usual Fock factorization are much wider than the group of automorphisms preserving the canonical factorization.

5.3. Relationship with the spaces $\text{PS}(\mathcal{E}(Y), \nu; K)$ and $\text{QPS}(\mathcal{E}(Y), \sigma; K)$. Comparing the formulas for the inner products of the vectors $\exp v$, $\text{ps } v$, and $\text{qps } v$ in the spaces $\text{EXP}(Y; K)$, $\text{PS}(\mathcal{E}(Y), \nu; K)$, and $\text{QPS}(\mathcal{E}(Y), \sigma; K)$, respectively, we see that they coincide. This implies the following theorem.

Theorem 5.1. *The maps $\exp v \mapsto \text{ps } v$ and $\exp v \mapsto \text{qps } v$, $v \in L^2(Y, \mu; K)$, generate factorization-preserving isomorphisms of Hilbert spaces*

$$\text{EXP}(Y, \mu; K) \cong \text{PS}(\mathcal{E}(Y), \nu; K) \quad \text{and} \quad \text{QPS}(\mathcal{E}(Y), \sigma; K).$$

Corollary. *In the special case where $Y = \mathbb{R}_+^* \times X$, the map $\exp v \mapsto \text{int } v$ induces an isomorphism of Hilbert spaces $\text{EXP}(X; K) \cong \text{INT}(X; K)$ (see Subsection 4.5).*

§6. THE GROUPS $G_u(Y; K)$ AND THEIR UNITARY REPRESENTATIONS IN QUASI-POISSON SPACES

In this and the next sections, we describe representations of the current groups G^X , where X is a space with a probability measure m , in quasi-Poisson spaces. The classical models of representations of G^X are representations in Fock spaces. In view of the isomorphism between Fock and quasi-Poisson spaces, they give rise to representations of G^X in quasi-Poisson spaces.

In this paper, we construct representations of current groups G^X in quasi-Poisson spaces without involving Fock models and using only remarkable properties of these spaces.

We begin with a brief description of representations of current groups G^X in Fock spaces. For this, we first introduce the “large group” $\text{Aff}(H)$ of all motions in a Hilbert

space H and construct a unitary projective (in general) representation of this group in the Fock space $\text{EXP}(H)$. To construct a Fock representation of the group G^X , it suffices to describe an embedding of G^X into $\text{Aff}(H)$.

In this paper, a similar approach is used to describe unitary representations of a certain class of current groups in quasi-Poisson spaces. Namely, we introduce a family of “large” groups $G_u(Y; K)$ and construct unitary representations of these groups in quasi-Poisson spaces. To construct a unitary representation of a current group in a quasi-Poisson space, it suffices to describe an embedding of this group into one of these “large” groups.

6.1. The Fock model of representations of current groups G^X . The current group G^X , where X is a space with a probability measure m and G is a locally compact group, is the group of measurable maps $X \rightarrow G$ with pointwise multiplication. It is convenient to assume that the images of these maps are contained in compact subsets of G .

A Fock representation of a current group G^X is determined by a pair (τ, b) , where τ is a unitary representation of the group G in a Hilbert space $\mathcal{K}$ and b is a 1-cocycle of this representation, i.e., a map $b : G \rightarrow \mathcal{K}$ satisfying

$$b(g_1 g_2) = b(g_1) + \tau(g_1) b(g_2) \quad \text{for any } g_1, g_2 \in G.$$

This representation is realized in the Fock space $\text{EXP}(H, \mathcal{K}) = \text{EXP}(H)$, where $H = L^2(X, m, \mathcal{K})$, i.e., in the Hilbert space of sections $x \mapsto v(x) \in K_x$ of the fiber bundle over X with the fibers $K_x = K$, endowed with the norm $\|v\|^2 = \int_X \|v(x)\|^2 dm(x)$.

To construct this representation, first we introduce the “large” group $\text{Aff}(H)$ of all motions $h \mapsto Th + v$ on H , where T is a unitary operator and $v \in H$, i.e., the group of pairs (T, v) , $T \in \mathfrak{U}(H)$, $v \in H$, where $\mathfrak{U}(H)$ is the group of all unitary operators on H , with the multiplication law

$$(T_1, v_1)(T_2, v_2) = (T_1 T_2, v_1 + T_1 v_2).$$

There exists an irreducible unitary representation of the group $\text{Aff}(H)$ in the space $\text{EXP}(H)$. The operators $U_{T,v}$ of this representation are defined on the total subset of vectors $\exp h$, $h \in H$, by the formula

$$(6.1) \quad U_{T,v}(\exp h) = e^{-\frac{1}{2}\|v\|^2 - (Th, v)} \exp(Th + v).$$

They satisfy the relation

$$U_{g_1 g_2} = e^{i \text{Im} \langle v_1, T_1 v_2 \rangle} U_{g_1} U_{g_2} \quad \text{for all } g_1 = (T_1, v_1), g_2 = (T_2, v_2) \in \text{Aff}(H).$$

Thus, in general, the representation U of the group $\text{Aff}(H)$ is projective.

Every pair (τ, b) , where τ is a unitary representation of a group G in a Hilbert space $\mathcal{K}$ and b is a 1-cocycle of this representation, gives rise to a natural embedding $\pi : G^X \rightarrow \text{Aff}(H)$ of G^X into $\text{Aff}(H)$:

$$\pi(g(\cdot)) = (T(g(\cdot)), v(g(\cdot))) \in \text{Aff}(H),$$

where $T(g(\cdot)) \in \mathfrak{U}(H)$ and $v(g(\cdot)) \in H$ are given by the formulas

$$(6.2) \quad [T(g(\cdot))h](x) = \tau(g(x))h(x) \quad \text{for all } h = h(x) \in H; \quad v(g(\cdot))(x) = b(g(x)).$$

Therefore, the representation of $\text{Aff}(H)$ in the Fock space $\text{EXP}(H)$ induces a unitary representation U of the group G^X in this space.

It is well known that the representation U of the group G^X associated in this way with a pair (τ, b) is irreducible provided that the original representation τ of the group G is irreducible and the 1-cocycle b is nontrivial. However, U is irreducible under weaker conditions on the pair (τ, b) (see [32, 26]).

6.2. Definition of the groups $G_u(Y; K)$. Now we proceed to the definition of the “large” groups $G_u(Y; K)$ involved in the construction of representations of current groups in quasi-Poisson spaces. Assume that we are given a punctured Hilbert space (K, h) and a space Y with a measure μ . Here we assume that μ is an *infinite σ -finite measure*.

With (Y, μ) and (K, h) we associate the space K^Y of all measurable sections $y \mapsto v(y) \in K_y$, $K_y = K$, of the fiber bundle over Y with fibers $(K_y, h_y) = (K, h)$ and denote by $H = H(Y; K)$ the Hilbert space $H(Y; K) = L^2(Y, \mu; K)$ of all such sections with the norm

$$\|v\|^2 = \int_Y \|v(y)\|_{K_y}^2 d\mu(y) < \infty,$$

i.e., the space

$$H(Y; K) = \int_Y^\oplus K_y d\mu(y), \quad K_y = K.$$

Denote by $\mathfrak{U}(K) = \mathfrak{U}$ and $\mathfrak{U}(H(Y; K))$ the groups of all unitary operators on K and $H(Y; K)$, respectively. In what follows, for brevity, we often write H and $\mathfrak{U}(H)$ instead of $H(Y; K)$ and $\mathfrak{U}(H(Y; K))$.

The group $\mathfrak{U}(H)$ contains two distinguished subgroups: the subgroup $\mathfrak{U}(Y; K)$ of operators preserving the fibers $K_y = K$ of the fiber bundle over Y and the subgroup $\text{Aut}(Y, \mu)$ of operators permuting these fibers. By definition, the action of operators $T \in \mathfrak{U}(Y; K)$ on H is given by the formula

$$(Tv)(y) = T_y v(y),$$

where T_y is the component of T in the fiber K_y . The group $\text{Aut}(Y, \mu)$ is the group of all μ -preserving transformations $a : y \mapsto ay$ of the space Y . Its elements can be realized as unitary operators on H acting by the rule

$$(av)(y) = v(a^{-1}y).$$

(They are unitary because μ is invariant.)

The group $\text{Aut}(Y, \mu)$ acts on $\mathfrak{U}(Y, K)$ as the group of automorphisms $T \mapsto T^a$, where $(T^a)_y = T_{a^{-1}y}$. Thus, $\mathfrak{U}(Y, K)$ and $\text{Aut}(Y, \mu)$ generate a subgroup in $\mathfrak{U}(H)$ that can be written as the semidirect product

$$\mathcal{G}(Y; K) = \text{Aut}(Y, \mu) \ltimes \mathfrak{U}(Y; K),$$

i.e., the group of pairs (a, T) , $a \in \text{Aut}(Y, \mu)$, $T \in \mathfrak{U}(Y; K)$, with the multiplication law

$$(a_1, T_1)(a_2, T_2) = (a_1 a_2, T_1 T_2^{a_1}).$$

The “large” groups $G_u(Y; K)$ are subgroups of $\mathcal{G}(Y; K)$. To define them, we consider a section $b : \mathfrak{U}(Y; K) \rightarrow K$ associated with fixed vectors $h_y \in K_y$:

$$(6.3) \quad b(T, y) = T_y h_y - h_y.$$

Obviously, b is a 1-cocycle of $\mathfrak{U}(Y; K)$:

$$(6.4) \quad b(T_1 T_2) = b(T_1) + T_1 b(T_2) \quad \text{for all } T_1, T_2 \in \mathfrak{U}(Y; K).$$

Definition 6.1. With a nonnegative function $u(y)$ on Y satisfying conditions (2.5), that is,

$$\int_Y \min(1, u(y)e^{-u(y)}) d\mu(y) < \infty \quad \text{and} \quad \int_Y e^{-u(y)} d\mu(y) = \infty,$$

we associate the subgroup $B_u(Y; K)$ of elements $T \in \mathfrak{U}(Y, K)$ satisfying the condition $b(T) \in L^1(Y, e^{-u(y)} d\mu(y); K)$, i.e.,

$$(6.5) \quad \int_Y \|T_y h_y - h_y\|_{K_y} e^{-u(y)} d\mu(y) < \infty.$$

(The fact that $B_u(Y; K)$ is a subgroup follows from property (6.4) of the 1-cocycle b .)

Note that condition (6.5) is satisfied if

$$\|T_y h_y - h_y\|_{K_y} < c(T)u(y).$$

In the special case where $\dim K = 1$, we write $B_u(Y)$ instead of $B_u(Y; K)$. In this case, $B_u(Y)$ is the group of all measurable complex-valued functions $a(y)$ on Y such that $|a(y)| \equiv 1$ and

$$\int_Y |a(y) - 1| e^{-u(y)} d\mu(y) < \infty.$$

Remark. The group $B_u(Y; K)$ contains the subgroup of operators $T \in \mathfrak{U}(Y; K)$ such that $\mu(\{y \in Y : T_y \neq \text{id}\}) < \infty$.

Proposition 6.1. *The subgroup $B_u(Y; K)$ is invariant under the subgroup $\text{Aut}_u(Y, \mu) \subset \text{Aut}(Y, \mu)$ of the μ -preserving transformations $a : y \mapsto ay$ of Y such that*

$$(6.6) \quad \int_Y |e^{-u(ay)} - e^{-u(y)}| d\mu(y) < \infty.$$

Proof. By (6.6), the integral

$$\int_Y (e^{-u(ay)} - e^{-u(y)}) \|T_y h_y - h_y\|_{K_y} d\mu(y)$$

converges for every $T \in \mathfrak{U}(Y; K)$. Hence, if $T \in B_u(Y; K)$, i.e., condition (6.5) is satisfied, then

$$\int_Y \|T_y h_y - h_y\|_{K_y} e^{-u(ay)} d\mu(y) < \infty,$$

which is equivalent, by the invariance of μ , to the condition

$$\int_Y \|T_{a^{-1}y} h - h\|_{K_y} e^{-u(y)} d\mu(y) < \infty.$$

Thus, the section $y \mapsto T_{a^{-1}y}$ also belongs to $B_u(Y; K)$. □

Definition 6.2. We define a subgroup $G_u(Y; K) \subset \mathcal{G}(Y; K)$ as the semidirect product

$$G_u(Y; K) = \text{Aut}_u(Y, \mu) \ltimes B_u(Y; K),$$

i.e., as the group of pairs (a, T) , $a \in \text{Aut}_u(Y, \mu)$, $T \in B_u(Y; K)$, with the multiplication law $(a_1, T_1)(a_2, T_2) = (a_1 a_2, T_1 T_2^{a_1})$.

If u satisfies the condition

$$(6.7) \quad \int_Y (1 - e^{-u(y)}) d\mu(y) < \infty,$$

in particular, if $u \equiv 0$, then (6.6) holds true for every $a \in \text{Aut}(Y, \mu)$, whence $G_u(Y; K) = \text{Aut}(Y, \mu) \ltimes B_u(Y; K)$.

6.3. A property of the group $G_u(Y; K)$. The following proposition is an immediate consequence of the definition of $B_u(Y; K)$.

Proposition 6.2. *The operators $T \in B_u(Y; K)$ satisfy the condition*

$$(6.8) \quad \int_Y \|T_y h_y - h_y\|_{K_y}^2 e^{-u(y)} d\mu(y) < \infty,$$

i.e., $b(T) \in L^2(Y, e^{-u(y)} d\mu(y); K)$, where $b(T)$ is given by (6.3), and the condition

$$(6.9) \quad \int_Y |\langle b(T, y), h_y \rangle_{K_y}| e^{-u(y)} d\mu(y) \equiv \int_Y |\langle T_y h_y - h_y, h_y \rangle_{K_y}| e^{-u(y)} d\mu(y) < \infty.$$

With a function $u(y)$ on Y we associate the section f_u of the fiber bundle over Y with fibers $K_y = K$ given by the formula

$$f_u(y) = e^{-\frac{1}{2}u(y)}h_y.$$

Theorem 6.1. *The group $G_u(Y; K)$ has a nontrivial 1-cocycle $b_u : G_u(Y; K) \rightarrow H$, which is defined on the elements $T \in B_u(Y; K)$ and $a \in \text{Aut}_u(Y, \mu)$ by the following formulas:*

$$(6.10) \quad \begin{aligned} b_u(T, y) &= T_y f_u(y) - f_u(y) = e^{-\frac{1}{2}u(y)}(T_y h_y - h_y), \\ b_u(a, y) &= (e^{-\frac{1}{2}u(a^{-1}y)} - e^{-\frac{1}{2}u(y)})h_y. \end{aligned}$$

Moreover,

$$(6.11) \quad \int_Y |\langle b_u(T, y), f_u(y) \rangle_{K_y}| d\mu(y) < \infty.$$

Proof. The fact that b_u is a 1-cocycle follows immediately from the definition. We show that $b_u(g) \in H$ for every $g \in G_u(Y; K)$, i.e., $\|b_u(g)\| < \infty$. Indeed, for $g = T \in B_u(Y; K)$, this immediately follows from (6.8). In the case where $g = a \in \text{Aut}_u(Y, \mu)$, this estimate follows from the condition imposed on u and the obvious inequality $(e^{-\frac{1}{2}u(a^{-1}y)} - e^{-\frac{1}{2}u(y)})^2 \leq |e^{-u(a^{-1}y)} - e^{-u(y)}|$. Thus, $b_u(g) \in H$, i.e., b_u is an H -valued 1-cocycle. It is nontrivial because $f_u \notin L^2(Y, \mu; K)$.

Estimate (6.11) is equivalent to (6.9). $\square$

Corollary. *There exists a nontrivial embedding τ of the group $G_u(Y; K) \subset \mathfrak{U}(H)$ into the affine group $\text{Aff}(H) = \mathfrak{U}(H) \ltimes H$, namely,*

$$\tau g = (g, b_u(g)), \quad g \in G_u(Y; K).$$

6.4. Relationship between the “large” groups $G_u(Y; K)$ and $\text{Aff}(H(X; \mathcal{K}))$. Consider the following situation:

$$(Y, \mu) = (S, ds) \times (X, m),$$

where (S, ds) is a space with an infinite measure and (X, m) is a space with a probability measure. Then, there is a simple relationship between the groups $G_u(Y; K)$ and $\text{Aff}(H(X; \mathcal{K}))$, where

$$\mathcal{K} = \int_S^{\oplus} K_s ds, \quad K_s = K,$$

i.e., $\mathcal{K}$ is the Hilbert space of sections $s \mapsto v(s) \in K_s$ of the fiber bundle over S with fibers $K_s = K$.

This relationship is a consequence of the natural isomorphism of the Hilbert spaces $H(Y; K) = L^2(S \times X, \mu; K)$ and $H(X; \mathcal{K}) = L^2(X, m; \mathcal{K})$,

$$H(Y; K) \cong H(X; \mathcal{K}),$$

which takes every vector $v(s, x) \in L^2(S \times X, \mu; K)$ to the vector $v(x) \in L^2(X, m; \mathcal{K})$ defined by the formula $v(x) = \int_S v(s, x) ds$.

This isomorphism induces an isomorphism of the corresponding groups of unitary operators $\mathfrak{U}(H(Y; K)) \rightarrow \mathfrak{U}(H(X; \mathcal{K}))$. Thus, the embeddings $G_u(Y; K) \rightarrow \mathfrak{U}(H(Y; K))$ introduced above, along with the nontrivial embedding $G_u(Y; K) \rightarrow \text{Aff}(H(Y; K))$, give rise to an embedding $G_u(Y; K) \rightarrow \mathfrak{U}(H(X; \mathcal{K}))$ and a nontrivial embedding $G_u(Y; K) \rightarrow \text{Aff}(H(X; \mathcal{K}))$.

6.5. The representation of the group $G_u(Y; K)$ in a quasi-Poisson space. We shall construct a unitary representation of the group $G_u(Y; K) = \text{Aut}_u(Y, \mu) \ltimes B_u(Y; K)$ in the quasi-Poisson space associated with the triple (Y, μ, u) and the punctured space (K, h) ,

$$\text{QPS}(\mathcal{E}(Y), \sigma; K) = \int_{\mathcal{E}(Y)}^{\oplus} K_{\omega}^{\otimes} d\sigma(\omega),$$

i.e., in the space of measurable sections $\omega \mapsto \varphi(\omega)$ of the fiber bundle over $\mathcal{E}(Y)$ with fibers

$$K_{\omega}^{\otimes} = \bigotimes_{y \in \omega} (K_y, h_y), \quad (K_y, h_y) = (K, h),$$

endowed with the norm

$$\|\varphi\|^2 = \int_{\mathcal{E}(Y)} \|\varphi(\omega)\|^2 d\sigma(\omega).$$

Recall that σ is the quasi-Poisson measure with characteristic functional

$$\int_{\mathcal{E}(Y)} e^{-\sum_{y \in \omega} f(y)} d\sigma(\omega) = \exp \left(\int_Y (e^{-f(y)} - e^{-u(y)}) d\mu(y) \right).$$

We begin with a description of a representation of the subgroup $B_u(Y; K)$.

Consider the subspace $L(Y; K) = L^1(Y, e^{-\frac{u}{2}} \mu; K)$, i.e., the space of measurable sections $w : y \mapsto w(y) \in K_y$ of the fiber bundle over Y with fibers $K_y = K$ satisfying the conditions

$$(6.12) \quad \int_Y \|w(y)\| e^{-\frac{1}{2}u(y)} d\mu(y) < \infty.$$

Next, we denote by $L_u(Y; K)$ the set of vectors of the form $v = f_u + w$, where $f_u(y) = e^{-\frac{1}{2}u(y)} h_y$ and $w \in L(Y; K)$.

Obviously, the space $L(Y; K)$ is invariant under the group $G_u(Y; K)$ and contains all sections $b_u(T, y)$, $T \in B_u(Y; K)$. The formula $T(f_u + w) = f_u + (Tw + b(T))$ shows that the set $L_u(Y; K)$ of vectors of the form $v = f_u + w$ is also invariant under the operators $T \in B_u(Y; K)$.

Note that the space $L_u(Y; K)$ contains, in particular, the sections $v(y)$ such that $v(y) = f_y(u)$ outside of a subset $A \subset Y$ of finite measure.

Definition 6.3. With a configuration $\omega \in \mathcal{E}(Y)$ and an element $v \in L_u(Y; K)$ we associate the infinite tensor product

$$(6.13) \quad F_v(\omega) = \bigotimes_{y \in \omega} v(y), \quad v \in L_u(Y; K).$$

Proposition 6.3. For almost every configuration $\omega \in \mathcal{E}(Y)$, the tensor products $F_v(\omega)$, $v \in L_u(Y; K)$, lie in $K_{\omega}^{\otimes}$ and form a total subset in this space.

Proof. From the definition of $L_u(Y; K)$ and Campbell's theorem for the Poisson measure ν associated with the pair $(Y, e^{-u(y)} d\mu(y))$ it follows that for every $v \in L_u(Y; K)$ we have $\sum_{y \in \omega} \|v(y) - h_y\| < \infty$ for almost all configurations $\omega \in \mathcal{E}(Y)$. This implies that the infinite product $\prod_{y \in \omega} \|v(y)\|$ converges. Therefore, $F_v(\omega) \in K_{\omega}^{\otimes}$ and $\|F_v(\omega)\| = \prod_{y \in \omega} \|v(y)\| < \infty$. The totality in $K_{\omega}^{\otimes}$ of the set of elements $F_v(\omega)$ is obvious. $\square$

Definition 6.4. For almost every configuration $\omega \in \mathcal{E}(Y)$, consider the following operators $U_{\omega}(T)$, $T \in B_u(Y; K)$, on the total set of vectors $F_v(\omega) \in K_{\omega}^{\otimes}$:

$$(6.14) \quad U_{\omega}(T)F_v(\omega) = F_{Tv}.$$

Theorem 6.2. *The operators $U_\omega(T)$ are unitary, i.e.,*

$$(6.15) \quad \langle U_\omega(T)F_{v_1}(\omega), U_\omega(T)F_{v_2}(\omega) \rangle = \langle F_{v_1}(\omega), F_{v_2}(\omega) \rangle \text{ for any } v_1, v_2 \in L(Y; K),$$

and they generate a unitary representation of the group $B_u(Y; K)$ on the entire space $K_\omega^\otimes$.

Indeed, from (6.14) it follows immediately that these operators satisfy the group property and relations (6.15). Next, they do generate a unitary representation of the group $B_u(Y; K)$ on the space $K_\omega^\otimes$ because the set of vectors $F_v(\omega)$ is total in $K_\omega^\otimes$.

Definition 6.5. We define a unitary representation $U(T)$ of the group $B_u(Y; K)$ in the quasi-Poisson space

$$\text{QPS}(\mathcal{E}(Y), \sigma; K) = \int_{\mathcal{E}(Y)}^\oplus K_\omega^\otimes d\sigma(\omega)$$

associated with the triple (Y, μ, u) as the direct integral with respect to the measure $\sigma(\omega)$ on $\mathcal{E}(Y)$ of the representations $U_\omega(T)$ of the group $B(Y; \mathfrak{U})$ in the spaces $K_\omega^\otimes$:

$$U(T) = \int_{\mathcal{E}(Y)}^\oplus U_\omega(T) d\sigma(\omega).$$

Then, we define a unitary representation of the group $\text{Aut}_u(Y, \mu)$ in the quasi-Poisson space $\text{QPS}(\mathcal{E}(Y), \sigma; K)$ by the formula

$$(6.16) \quad (U(a)\varphi)(\omega) = e^{-\frac{1}{2} \text{Log } a} \varphi(a^{-1}\omega), \quad a \in \text{Aut}_u(Y, \mu).$$

The operators $U(a)$ obviously satisfy the group property, and they are unitary because of the projective invariance of the measure σ with respect to the group $\text{Aut}_u(Y, \mu)$.

The next theorem follows immediately from the description of the corresponding representations of $\text{Aut}_u(Y, \mu)$ and $\mathcal{B}_u(Y; K)$.

Theorem 6.3. *The representations of the groups $\text{Aut}_u(Y, \mu)$ and $\mathcal{B}_u(Y; K)$ in the quasi-Poisson space $\text{QPS}(\mathcal{E}(Y), \sigma; K)$ generate a unitary representation U of their semidirect product $G_u(Y; K) = \text{Aut}_u(Y, \mu) \ltimes \mathcal{B}_u(Y; K)$.*

Remark. In contrast to the Fock model, the resulting representation of the “large” group $G_u(Y; K)$ is ordinary (not projective).

Theorem 6.4. *The representation U of the group $G_u(Y; K) = \text{Aut}_u(Y, \mu) \ltimes \mathcal{B}_u(Y; K)$ in the space $\text{QPS}(\mathcal{E}(Y), \sigma; K)$ is irreducible.*

Proof. The description of the representations $U_\omega(T)$ of the group $B_u(Y; K)$ in the spaces $K_\omega^\otimes$ shows that, up to a subset of zero measure in the space $\mathcal{E}(Y)$, these representations are irreducible and pairwise nonequivalent. Hence, every operator A on $\text{QPS}(\mathcal{E}(Y), \sigma; K)$ that commutes with the operators $U(T) = \int_{\mathcal{E}(Y)} U_\omega(T) d\sigma(\omega)$ for $T \in B_u(Y; K)$ is scalar on every fiber $K_\omega^\otimes$ of the fiber bundle $\text{QPS}(\mathcal{E}(Y), \sigma; K)$, i.e., is the multiplication by a function $f(\omega)$. If A commutes also with the operators $U(a)$, $a \in \text{Aut}_u(Y, \mu)$, then $f(\omega)$ is constant on the orbits of the group $\text{Aut}_u(Y, \mu)$. Since the measure σ is ergodic with respect to $\text{Aut}_u(Y, \mu)$, it follows that A is a scalar operator. $\square$

6.6. A formula for the spherical function. The *spherical function* of the representation U of the group $G_u(Y; K)$ in the space $\text{QPS}(\mathcal{E}(Y), \sigma; K)$ is the following function Ψ on $G_u(Y; K)$:

$$\Psi(g) = \langle U(g)\Omega, \Omega \rangle, \quad \text{where} \quad \Omega(\omega) = \bigotimes_{y \in \omega} (e^{-\frac{1}{2}u(r)} h_y).$$

Obviously, $\|\Omega\| = 1$. The formulas for the representation U and the characteristic functional of the measure σ easily imply the following result.

Theorem 6.5. *The spherical function of the representation U of the group $G_u(Y; K)$ in the space $\text{QPS}(\mathcal{E}(Y), \sigma; K)$ has the following form:*

$$(6.17) \quad \Psi(T) = \exp \left(\int_Y \langle T_y h_y - h_y, h_y \rangle e^{-u(y)} d\mu(y) \right) \text{ for } T \in \mathcal{B}_u(Y; K);$$

$$(6.18) \quad \Psi(a) = \exp \left(-\frac{1}{2} \int_Y (e^{-\frac{1}{2}u(a^{-1}y)} - e^{-\frac{1}{2}u(y)})^2 d\mu(y) \right) \text{ for } a \in \text{Aut}_u(Y, \mu).$$

6.7. The group $G_0(Y; K)$ and its representation in a Poisson space. The definitions and constructions described above remain valid also for $u(y) \equiv 0$. In this case, $G_u(Y; K)$ is the group

$$G_0(Y; K) = \text{Aut}(Y, \mu) \ltimes B_0(Y; K),$$

where $B_0(Y; K)$ is the subgroup of operators $T \in \mathfrak{U}(Y; K)$ satisfying the condition

$$\int_Y \|T_y h_y - h_y\|_{K_y} d\mu(y) < \infty.$$

This group has a nontrivial 1-cocycle, which vanishes on the elements of $\text{Aut}(Y, \mu)$, and for $T \in B_0(Y; K)$ is given by the formula $b_0(T, y) = T_y h_y - h_y$.

If $u(y) \equiv 0$, the Hilbert space in which the representation of $G_0(Y; K)$ is realized degenerates into the Poisson space $\text{PS}(Y, \nu; K)$, where ν is the Poisson measure with the intensity measure $d\mu(y)$. The operators corresponding to elements of the subgroup $B_0(Y; K)$ act in the same way as for an arbitrary u , and the operators $U(a)$ corresponding to elements of the subgroup $\text{Aut}(Y, \mu)$ act by the formula $(U(a)\varphi)(\omega) = \varphi(a^{-1}\omega)$ (because $\text{Log } a = 0$ for $u = 0$). Accordingly, the spherical function of this representation of $G_0(Y; K)$ has the following form: $\Psi(a) \equiv 1$ for $a \in \text{Aut}(Y, \mu)$, and

$$\Psi(T) = \exp \left(\int_Y \langle T_y h_y - h_y, h_y \rangle d\mu(y) \right) \text{ for } T \in \mathcal{B}_0(Y; K).$$

§7. UNITARY REPRESENTATIONS OF CURRENT GROUPS G^X , WHERE $G = S \ltimes G_0$, IN QUASI-POISSON SPACES

7.1. Unitary representations of groups of the form $G = S \ltimes G_0$ and the summability property. Let G be a locally compact group that can be presented as a semidirect product of groups $G = S \ltimes G_0$, where S is a locally compact (noncompact) group with left-invariant measure ds .

The subgroup S acts on G_0 as the group of automorphisms $g \mapsto g^s = sgs^{-1}$, $s \in S$, so that every unitary representation T_e of G_0 gives rise to a family T_s , $s \in S$, of unitary representations of G_0 in the same space, defined by the formula

$$T_s(g) = T_e(g^s), \quad g^s = sgs^{-1}.$$

We say that these representations T_s are *conjugate* to the original representation T_e .

Let $u(s)$ be an arbitrary continuous positive function on S satisfying the following condition.

Condition A.

- 1) The measure $e^{-u(s)} ds$ on S is infinite, and the measure $\min(1, u(s)) e^{-u(s)} ds$ on S is finite, i.e.,

$$\int_S e^{-u(s)} ds = \infty \quad \text{and} \quad \int_S \min(1, u(s)) e^{-u(s)} ds < \infty.$$

2) The integral

$$(7.1) \quad \text{Log}_u s_0 \equiv \int_S (e^{-u(s_0 s)} - e^{-u(s)}) ds$$

converges absolutely for every $s_0 \in S$.

Note that the function $\text{Log}_u s$ is an additive character on the group S , i.e.,

$$\text{Log}(s_1 s_2) = \text{Log } s_1 + \text{Log } s_2 \quad \text{for any } s_1, s_2 \in S.$$

Definition 7.1. We say that a unitary representation T_e of the subgroup G_0 in a Hilbert space K is *summable with respect to a function u satisfying condition A* if it is irreducible and there exists a nonzero vector $h \in K$, $\|h\| = 1$, such that

$$(7.2) \quad \int_S \|T_s(g)h - h\| e^{-u(s)} ds < \infty \quad \text{for every } g \in G_0.$$

A vector $h \in K$ satisfying (7.2) is said to be *almost invariant* with respect to the representation T_e of G_0 .

By (7.2),

$$(7.3) \quad \int_S \|T_s(g)h - h\|^2 e^{-u(s)} ds < \infty \quad \text{for every } g \in G_0.$$

Every unitary representation of the subgroup G_0 in K summable with respect to a function $u(s)$ and every almost invariant vector $h \in K$ give rise to a unitary representation T of the entire group $G = S \ltimes G_0$ in the direct integral with respect to the measure ds on S of the punctured Hilbert spaces $(K_s, h_s) = (K, h)$,

$$\mathcal{K} = \int_S^\oplus (K_s, h_s) ds,$$

i.e., in the space of sections $s \mapsto f(s)$ of the fiber bundle over S with the fibers $K_s = K$, endowed with the norm $\|f\|^2 = \int_S \|f(s)\|_{K_s}^2 ds$.

By definition, the action of the operators corresponding to the elements of the subgroup G_0 on $\mathcal{K}$ is fiberwise, i.e., $(T(g)f)(s) = T_s(g)f(s)$ for $g \in G_0$; and the operators corresponding to the elements of the subgroup S permute the fibers: $(T(s_0)f)(s) = f(s_0^{-1}s)$. Obviously, the operators T corresponding to the elements of S are unitary; together with the operators corresponding to the elements of G_0 , they generate a unitary representation of the group G in the space $\mathcal{K}$. The resulting representation T of the group G will also be called a *summable representation*.

The operators $T(g)$, $g \in G_0$, are well defined on the set of all sections $s \mapsto f(s) \in K_s$ of the fiber bundle over S with the fibers $K_s = K$. Using this, with every element $g \in G$ we associate the following section $b(g)$ of the fiber bundle over S with the fibers $K_s = K$:

$$(7.4) \quad b_u(g, s) = (T(g)f_u)(s) - f_u(s), \quad \text{where } f_u(s) = e^{-\frac{1}{2}u(s)}h_s.$$

Theorem 7.1. The section $b_u(g)$ defined by (7.4) is a nontrivial 1-cocycle $G \rightarrow \mathcal{K}$ of the representation T of the group G and satisfies the following additional condition:

$$(7.5) \quad |\langle b_u(g), f_u \rangle| \equiv \left| \int_0^\infty \langle b_u(g, s), f_u(s) \rangle ds \right| < \infty.$$

Indeed, it is clear that the section $b_u(g)$ is a 1-cocycle. The fact that $b_u(g) \in \mathcal{K}$ for every $g \in G$ easily follows from (7.3). Thus, $b_u(g)$ is a $\mathcal{K}$ -valued 1-cocycle of T . It is nontrivial, because $f_u \notin \mathcal{K}$. Estimate (7.5) also follows from the summability condition and the conditions imposed on the function u .

7.2. A representation of the group G^X with $G = S \ltimes G_0$ in a quasi-Poisson space. Let $u(s)$ be a function on S satisfying condition A, let T_e be a representation of G_0 in a Hilbert space K summable with respect to u , and let $h \in K$ be an almost invariant vector of this representation. With the triple (u, T_e, h) we associate a representation in a quasi-Poisson space of the current group G^X , where $G = S \ltimes G_0$ and X is a space with a probability measure m .

Consider the space $(Y, \mu) = (S, ds) \times (X, m)$, i.e., the space of pairs (s, x) , $s \in S$, $x \in X$, and extend the function $u(s)$ to the whole of Y by setting $u(s, x) = u(s)$ for all $x \in X$. The function on Y defined in this way satisfies conditions (2.5) imposed on functions $u(y)$ in Subsection 2.3.

The triple (Y, μ, u) and the pair (K, h) give rise to the “large” group $G_u(Y; K) = \text{Aut}_u(Y, \mu) \ltimes B_u(Y; K)$ defined in Subsection 6.1, and in Subsection 6.5 we constructed a unitary representation of this “large” group in the quasi-Poisson space $\text{QPS}_u(\mathcal{E}(Y), \sigma; K)$. Hence, to construct a representation of the group G^X in the same space, it suffices to define an embedding $G^X \rightarrow G_u(Y; K)$. Now we describe this embedding.

Since $G_u(Y, \mu; K, h) = \text{Aut}_u(Y, \mu) \ltimes B_u(Y; K)$ and $G^X = S^X \ltimes G_0^X$, it suffices to define embeddings

$$S^X \rightarrow \text{Aut}_u(Y, \mu) \quad \text{and} \quad G_0^X \rightarrow B_u(Y; K).$$

Theorem 7.2. *The formulas*

$$(7.6) \quad (\tau s_0(\cdot))(s, x) = (s_0(x)s, x) \quad \text{for every } s_0(\cdot) \in S^X,$$

$$(7.7) \quad (\tau g(\cdot))_{s,x} = T_s(g(x)) \quad \text{for every } g(\cdot) \in G_0^X$$

define homomorphisms of groups $\tau : S^X \rightarrow \text{Aut}_u(Y, \mu)$ and $\tau : G_0^X \rightarrow B_u(Y; K)$.

Proof. It suffices to show that the images of the elements of S^X and G_0^X lie in the corresponding groups.

An element $\tau s_0(\cdot)$ lies in $\text{Aut}_u(S \times X, \mu)$ if 1) the transformations $(s, x) \mapsto (s_0(x)s, x)$ on $S \times X$ preserve the measure $ds dm(x)$ and 2) for every $s_0(\cdot) \in S^X$, the integral $\int_X \int_S (e^{-u(s_0(x)s)} - e^{-u(s)}) ds dm(x)$ converges.

The first condition follows from the invariance of the measure ds on S . The second follows from condition A on the function $u(s)$.

The fact that an element $\tau g(\cdot)$ lies in $B_u(S \times X; K)$ follows from the summability of the representation T_e of G_0 . $\square$

Definition 7.2. We define a unitary representation of the group G^X , where $G = S \ltimes G_0$, associated with the triple (u, T_e, h) as the composition of the homomorphism $\tau : G^X \rightarrow G_u(Y; K)$, where $(Y, \mu) = (S, ds) \times (X, m)$, and the representation of the group $G_u(Y; K)$ in the quasi-Poisson space

$$\text{QPS}_u(\mathcal{E}(S \times X), \sigma; K) = \int_{\mathcal{E}(S \times X)}^{\oplus} K_{\omega}^{\otimes} d\sigma(\omega).$$

The description of the representation of $G_u(Y; K)$ given in Subsection 6.5 shows that the operators $U(g(\cdot))$ corresponding to the elements of the subgroup G_0^X act in the fibers

$$K_{\omega}^{\otimes} = \bigotimes_{(s,x) \in \omega} (K_{s,x}, h_{s,x}), \quad (K_{s,x}, h_{s,x}) = (K, h),$$

and that on the total subset of vectors $F_v(\omega) = \bigotimes_{(s,x) \in \omega} v(s, x)$ in $K_{\omega}^{\otimes}$ introduced in Subsection 6.5 they are given by the formula

$$(7.8) \quad U(g(\cdot)) \left(\bigotimes_{(s,x) \in \omega} v(s, x) \right) = \bigotimes_{(s,x) \in \omega} (T_s(g(x))v(s, x)), \quad v \in L_u(Y; K).$$

The operators $U(s_0(\cdot))$ corresponding to the elements of the subgroup S^X on the entire space of functions $\varphi(\omega)$ are given by the formula

$$(7.9) \quad U(s_0(\cdot)\varphi)(\omega) = e^{-\frac{1}{2} \int_X \text{Log}(s_0(x)) dm(x)} \varphi(s_0^{-1}(\cdot)\omega),$$

where $\text{Log } s_0 = \int_S (e^{-u(s_0^{-1}s)} - e^{-u(s)}) ds$ and $\omega \mapsto s_0(\cdot)\omega$ is the transformation of the space of configurations $\mathcal{E}(S \times X)$ induced by the transformation $(s, x) \mapsto (s_0(x)s, x)$ of the space $S \times X$. In particular,

$$(7.10) \quad U(s_0(\cdot)) \left(\bigotimes_{(s,x) \in \omega} v(s, x) \right) = e^{-\frac{1}{2} \int_X \text{Log}(s_0(x)) dm(x)} \bigotimes_{(s,x) \in \omega} v(s_0^{-1}(x)s, x).$$

The action of the operators $U(\tilde{g})$, $\tilde{g} = g(\cdot) \in G^X$, can be expressed by a single formula. For this, we denote by $\tilde{T}(\tilde{g})$ the operators of the representation of the group G^X in the space $\tilde{K} = \int_X \mathcal{K}_x dm(x)$, $\mathcal{K}_x = \mathcal{K}$, induced by the representation T of the group G in the space $\mathcal{K}$, i.e.,

$$\begin{aligned} (\tilde{T}(\tilde{g})v)(s, x) &= T_s(g(x))v(s, x) \quad \text{for } \tilde{g} \in G_0^X; \\ (\tilde{T}(\tilde{s}_0)v)(s, x) &= v(s_0^{-1}(x)s, x) \quad \text{for } \tilde{s}_0 \in S^X. \end{aligned}$$

In this notation, setting $F_v(\omega) = \bigotimes_{(s,x) \in \omega} v(s, x)$, $v(s, x) \in K_{s,x} = K$, we have

$$(7.11) \quad U(\tilde{g})F_v = e^{-\frac{1}{2} \int_X \text{Log}[s(g(x))] dm(x)} F_{\tilde{T}(\tilde{g})v} \quad \text{for every } \tilde{g} \in G^X,$$

where $s(g) \in S$ is the image of an element $g \in G$ under the homomorphism $G \rightarrow S$.

Theorem 7.3. *The representation U of the group $G^X = (S \ltimes G_0)^X$ in the quasi-Poisson space $\text{QPS}_u(\mathcal{E}(Y), \sigma; Kh)$ associated with the triple (u, T_e, h) is irreducible.*

The proof is similar to that of Theorem 6.4.

The formulas from Subsection 6.6 for the spherical function of the representation of the group $G_u(Y; K)$ in the space $\text{QPS}(\mathcal{E}(Y), \sigma; K)$ imply the following theorem.

Theorem 7.4. *The spherical function $\Psi(\tilde{g})$ of the representation U of the group G^X , which is defined by the formulas*

$$\Psi(\tilde{g}) = \langle U(\tilde{g})\Omega, \Omega \rangle, \quad \Omega(\omega) = \bigotimes_{(s,x) \in \omega} f_u(s, x),$$

where $f_u(s, x) = e^{-\frac{1}{2}u(s)} h_{s,x}$, on the elements of the subgroups S^X and G_0^X is given by the formulas

$$(7.12) \quad \Psi(g(\cdot)) = \exp \left(\int_{S \times X} \langle T_s(g(x))h - h, h \rangle e^{-u(s)} ds dm(x) \right) \quad \text{for } g(\cdot) \in G_0^X;$$

$$(7.13) \quad \Psi(s_0(\cdot)) = \exp \left(-\frac{1}{2} \int_{S \times X} (e^{-\frac{1}{2}u(s_0^{-1}(x)s)} - e^{-\frac{1}{2}u(s)})^2 ds dm(x) \right) \quad \text{for } s_0(\cdot) \in S^X.$$

Corollary. *The spherical function can be expressed by the following general formula:*

$$(7.14) \quad \Psi(g(\cdot)) = \exp \left(-\int_X \left(\frac{1}{2} \|b_u(g(x))\|_{\mathcal{K}_x}^2 - i \text{Im} \langle b_u(g(x), f_u(x)) \rangle_{\mathcal{K}_x} \right) dm(x) \right) \\ \text{for every } g(\cdot) \in G^X,$$

where $b_u(g)$ is the 1-cocycle of the representation T of the group G in the space $\mathcal{K}$ given by (7.4).

7.3. The case where $S = \mathbb{R}_+^*$. If $S = \mathbb{R}_+^*$, the invariant measure on the group is $d^*r = r^{-1}dr$, and condition A is satisfied, e.g., for the function $u(r) = r$.

For definiteness, in what follows we assume that $u(r) = r$. Then $(Y, \mu) = (\mathbb{R}_+^*, d^*r) \times (X, m)$, and the characteristic functional of the quasi-Poisson measure σ associated with the pair (Y, μ) and $u(r) = r$ equals

$$\int_{\mathcal{E}(\mathbb{R}_+^* \times X)} e^{-\sum_{(r,x) \in \omega} f(r,x)} d\sigma(\omega) = \exp \left(\int_X \int_0^\infty (e^{-f(r,x)} - e^{-r}) d^*r d\mu(x) \right).$$

Proposition 7.1. *The measure σ is projectively invariant under the group A^X of transformations $a(\cdot) : (r, x) \mapsto (a(x)r, x)$ such that the integral*

$$(7.15) \quad \text{Log } a = \int_X \log a(x) dm(x)$$

converges; namely, $d\sigma(a(\cdot)\omega) = e^{\text{Log } a} d\sigma(\omega)$.

Indeed, this measure is projectively invariant under the group A^X of transformations $a(\cdot) : (r, x) \mapsto (a(x)r, x)$ such that the integral $\int_X \int_0^\infty (e^{-u(a^{-1}(x)r)} - e^{-r}) d^*r d\mu(x)$ converges. Integration with respect to r (see (2.10)) reduces this integral to (7.15).

In accordance with the general definition, an irreducible representation T_1 of a subgroup $G_0 \subset G$ in a space K , where $G = \mathbb{R}_+^* \ltimes G_0$, is said to be summable (with respect to a function u) if there exists a vector $h \in K$, $\|h\| = 1$, such that

$$\int_0^\infty \|T_r(g)h - h\| e^{-r} d^*r < \infty \quad \text{for every } g \in G_0,$$

where the T_r are the representations of G_0 conjugate to the original representation T_1 with respect to the action on G_0 of the group of automorphisms $\mathbb{R}_+^*$.

In particular, this condition is satisfied if for some $\alpha > 0$ we have

$$\|T_r(g)h - h\| < c(g) r^\alpha \quad \text{for every } g \in G_0.$$

Every pair (T_1, h) , where T_1 is a summable representation of G_0 in a space K and h is an almost invariant vector, gives rise to a unitary representation U of the current group $G^X = (\mathbb{R}_+^* \ltimes G_0)^X$ in the quasi-Poisson space

$$\text{QPS}(\mathcal{E}(\mathbb{R}_+^* \times X), \sigma; K) = \int_{\mathcal{E}(\mathbb{R}_+^*)}^\oplus K_\omega^\otimes d\sigma(\omega).$$

The operators $U(g(\cdot))$ corresponding to the elements of the subgroup G_0^X act in the fibers

$$K_\omega^\otimes = \bigotimes_{(r,x) \in \omega} (K_{r,x}, h_{r,x}), \quad (K_{r,x}, h_{r,x}) = (K, h),$$

and on the total subset of vectors

$$F_v(\omega) = \bigotimes_{(s,x) \in \omega} v(s, x)$$

in $K_\omega^\otimes$ they are given by formula (7.8) with $s = r$.

The operators $U(r_0(\cdot))$ corresponding to the elements of the subgroup $(\mathbb{R}_+^*)^X$ on the entire space of functions $\varphi(\omega)$ are given by formula (7.9) with $s = r$ and $s_0 = r_0$.

The ergodic property of the quasi-Poisson measure σ with respect to the group of automorphisms $\text{Aut}_u(Y, \mu)$ implies the following theorem.

Theorem 7.5. *The representation U of the current group $(\mathbb{R}_+^* \ltimes G_0)^X$ in the quasi-Poisson space $\text{QPS}(\mathcal{E}(\mathbb{R}_+^* \times X), \sigma; K)$ associated with a summable representation of the subgroup G_0 in a space K is irreducible.*

§8. MODELS OF REPRESENTATIONS OF THE GROUP P^X , WHERE P IS THE MAXIMAL PARABOLIC SUBGROUP OF $O(n, 1)$

As examples of groups G that can be presented as semidirect products $S \ltimes G_0$ and have summable representations, we mention the maximal parabolic subgroups P of simple Lie groups of rank 1. In this and the next sections, we describe summable representations of the maximal parabolic subgroups $P \subset O(n, 1)$, where⁹ $n > 2$, and $P \subset U(n, 1)$, and present a construction of the corresponding representations of the current groups P^X in quasi-Poisson spaces.

The maximal parabolic subgroup $P \subset O(n, 1)$, where $n > 2$, is isomorphic to the group of triples

$$(r, w, c), \quad \text{where } r \in \mathbb{R}^*, \quad w \in O(n-1), \quad c \in \mathbb{R}^{n-1},$$

with the multiplication law

$$(r_1, w_1, c_1)(r_2, w_2, c_2) = (r_1 r_2, w_1 w_2, c_1 + r_1 c_2 w_1).$$

Thus, P can be presented as a semidirect product with noncompact factors in two ways:

$$P = \mathbb{R}_+^* \ltimes P_0, \quad \text{where } P_0 = O(n-1) \ltimes \mathbb{R}^{n-1}$$

and an element $r \in \mathbb{R}_+^*$ acts on P_0 as the automorphism $(w, c) \mapsto (w, c)^r = (w, rc)$; and

$$P = D \ltimes \mathbb{R}^{n-1}, \quad \text{where } D = \mathbb{R}_+^* \times O(n-1)$$

and an element $d = (r, w) \in D$ acts on $\mathbb{R}^{n-1}$ as the automorphism $c \mapsto c^d = rcw$.

In the first case, a summable representation exists for the subgroup $P_0 = O(n-1) \ltimes \mathbb{R}^{n-1}$; in the second case, it exists for the commutative subgroup $\mathbb{R}^{n-1}$. Each of these decompositions of P into a semidirect product gives rise to a unitary representation of the current group P^X in a quasi-Poisson space. To describe these models, it suffices to describe a summable representation of the subgroup P_0 or $\mathbb{R}^{n-1}$, respectively. We shall describe them separately.

8.1. The model associated with the decomposition $P = \mathbb{R}\mathbb{R}_+^* \ltimes P_0$, where $P_0 = O(n-1) \ltimes \mathbb{R}\mathbb{R}^{n-1}$. The subgroup P_0 has a unique, up to conjugation with respect to the group of automorphisms $\mathbb{R}_+^*$, summable irreducible unitary representation T_1 . It is realized in the Hilbert space K of functions $f(s)$ on the sphere $S^{n-2} \subset \mathbb{R}^{n-1}$ with the norm

$$\|f\|^2 = \int_{S^{n-2}} |f(s)|^2 ds,$$

where ds is the invariant measure on S^{n-2} normalized so that $\int_{S^{n-2}} ds = 1$. The operators of this representation act by the formulas

$$\begin{aligned} (T_1(c)f)(s) &= e^{i\langle c, s \rangle} f(s) \quad \text{for } c \in \mathbb{R}^{n-1}, \\ (T_1(w)f)(s) &= f(sw) \quad \text{for } w \in O(n-1). \end{aligned}$$

Accordingly, the representations conjugate to T_1 are given by the formulas

$$(T_r(c)f)(s) = e^{ir\langle c, s \rangle} f(s) \quad \text{and} \quad T_r(w) = T_1(w).$$

Obviously,

$$\|T_r(g)h - h\| < c(g)r, \quad \text{where } h = h(s) \equiv 1.$$

This estimate implies that the representation T_1 is summable with respect to the function $u(r) = r$ on $\mathbb{R}_+^*$. Hence, it gives rise to a unitary representation U of the current group P^X in the quasi-Poisson space $\text{QPS}(\mathcal{E}(Y), \sigma; K)$ associated with the triple (Y, μ, u) , where $(Y, \mu) = (\mathbb{R}_+^*, d^*r) \times (X, m)$, and the punctured space (K, h) .

⁹The group $O(2, 1)$ is locally isomorphic to $SU(1, 1)$, and this case is considered in the next section.

By Theorem 7.5, this representation of P^X is irreducible. The general formulas of Subsection 7.2 imply the following theorem.

Theorem 8.1. *The spherical function of the representation U of the group P^X , which is defined by the formula*

$$\Psi(\tilde{g}) = \langle U(\tilde{g})\Omega, \Omega \rangle, \quad \text{where} \quad \Omega(\omega) = \bigotimes_{(r,x) \in \omega} (e^{-\frac{r}{2}} h_{r,x}),$$

can be written as

$$(8.1) \quad \Psi(\tilde{g}) = \exp \left(- \int_X \psi(c(x)) dm(x) \right) \quad \text{for} \quad \tilde{g} = (\tilde{w}, \tilde{c}) \in P_0^X,$$

where

$$(8.2) \quad \psi(c) = \int_{S^{n-2}} \log(1 + i\langle c, s \rangle) ds,$$

$$(8.3) \quad \Psi(\tilde{r}) = \exp \left(- \int_X \log \left(\frac{r^{1/2}(x) + r^{-1/2}(x)}{2} \right) dm(x) \right) \quad \text{for} \quad \tilde{r} \in (\mathbb{R}_+^*)^X.$$

Another (integral) model of this representation was given in [12].

8.2. The model associated with the decomposition $P = D \ltimes P_0$, where $D = \mathbb{R}\mathbb{R}_+^* \times O(n-1)$. The construction of this model is based on an arbitrary one-dimensional representation $T_{1,e}$ of the commutative group $\mathbb{R}^{n-1}$; it is given by the formula $c \mapsto e^{i\langle c, c_0 \rangle}$, where $c_0 \neq 0$ is an arbitrary fixed vector in $\mathbb{R}^{n-1}$. The conjugate one-dimensional representations $T_{r,w}$ have the form $c \mapsto e^{i\langle rcw, c_0 \rangle}$, where $r \in \mathbb{R}_+^*$ and $w \in O(n-1)$. Note that $T_{r,w} = T_{r,ww_0}$ if w_0 is an element of the stationary subgroup of the vector $c_0 \in \mathbb{R}^{n-1}$. Hence, the family of one-dimensional representations of $\mathbb{R}^{n-1}$ conjugate to $T_{1,e}$ is equivalent to the space of pairs (r, s) , $s \in S^{n-2}$.

Clearly, the triple $(u, T_{1,e}, h)$, where $h = 1$ and $u(r, w) = r$, satisfies the summability condition and hence gives rise to a representation of the group P^X in a quasi-Poisson space; this role is played by the space

$$\text{QPS}(\mathcal{E}(Y), \sigma) = L^2(\mathcal{E}(Y), \sigma), \quad \text{where} \quad (Y, \mu) = (\mathbb{R}_+^* \times O(n-1), d^*r dw) \times (X, m),$$

i.e., the space of scalar functions $\varphi(\omega)$ on $\mathcal{E}(Y)$ with the norm $\|\varphi\|^2 = \int_{\mathcal{E}(Y)} |\varphi(\omega)|^2 d\sigma(\omega)$.

The operators of this representation are given by the formulas

$$(U(\tilde{c})\varphi)(\omega) = \exp \left(i \sum_{(r,w;x) \in \omega} \langle c(x), rc_0w \rangle \right) \varphi(\omega) \quad \text{for} \quad \tilde{c} \in (\mathbb{R}^{n-1})^X,$$

$$U(\tilde{r}_0, \tilde{w}_0) \left(\bigotimes_{(r,w;x) \in \omega} v(r, w; x) \right) = e^{-\frac{1}{2} \int_X \text{Log } r_0(x) dm(x)} \bigotimes_{(r,w;x) \in \omega} v(r_0^{-1}(x)r, w_0^{-1}(x)w; x),$$

where $\text{Log } r_0 = \int_0^\infty (e^{-r_0^{-1}r} - e^{-r}) d^*r = \log r_0$.

Another model of this representation was given in [8].

We shall show that the spherical functions of the resulting representations of P^X coincide and, consequently, the representations are equivalent. For this, we consider the representation of the group P associated with the one-dimensional representation $T_{1,e}$ of the subgroup $\mathbb{R}^{n-1}$. Since the family of representations of $\mathbb{R}^{n-1}$ conjugate to $T_{1,e}$ is the space $\mathbb{R}_+^* \times S^{n-2}$, this representation of P acts in the same space as the representation from the first model. It is easy to check that the operators of these representations of P and, consequently, their 1-cocycles also coincide. Since the spherical functions of the representations of P^X can be expressed in terms of the 1-cocycles of the representations of P , these spherical functions also coincide.

§9. MODELS OF REPRESENTATIONS OF THE GROUP P^X , WHERE P IS THE MAXIMAL PARABOLIC SUBGROUP OF $U(n, 1)$

9.1. Preliminary definitions. The maximal parabolic subgroup $P \subset U(n, 1)$ has a structure of a semidirect product:

$$P = D \ltimes N, \quad \text{where} \quad D = \mathbb{C}^* \times U(n-1)$$

and N is the Heisenberg group of order $2n-1$.

We shall write elements of N as pairs (it, z) , $t \in \mathbb{R}$, $z \in \mathbb{C}^{n-1}$, or, which is more convenient, as pairs

$$(\zeta, z), \quad \text{where} \quad \zeta = it - \frac{1}{2}|z|^2, \quad t \in \mathbb{R}, \quad z \in \mathbb{C}^{n-1}.$$

With this notation, the group operation in N takes the form

$$(\zeta_1, z_1)(\zeta_2, z_2) = (\zeta_1 + \zeta_2 - z_1 z_2^*, z_1 + z_2),$$

where $z_1 z_2^* = \langle z_1, z_2 \rangle$.

By definition, elements $s \in \mathbb{C}^*$ and $u \in U(n-1)$ act on N as the automorphisms

$$(\zeta, z) \mapsto (|s|^2 \zeta, \bar{s}z) \quad \text{and} \quad (\zeta, z) \mapsto (\zeta, zu).$$

Thus, the group P can be presented as the semidirect product

$$P = \mathbb{R}_+^* \ltimes P_0, \quad P_0 = D_0 \ltimes N,$$

where $D_0 = U(1) \times U(n-1)$.

9.2. Summable irreducible representations of the group P_0 . To construct unitary irreducible representations of the group P^X in quasi-Poisson spaces, it suffices to construct summable representations of P_0 . We shall briefly describe these representations (see [12]).

For $n > 1$, up to conjugation with respect to the group of automorphisms $\mathbb{R}_+^*$, there are two countable families $T_m^\pm$, $m = 0, 1, \dots$, of summable irreducible unitary representations of the group P_0 . First, we describe the representations T_m^+ .

Consider a unitary representation of the group P_0 in the Hilbert space $H = H^+$ of functions $f(z)$ on $\mathbb{C}^{n-1}$ with the norm

$$(9.1) \quad \|f\|^2 = \int_{\mathbb{C}^{n-1}} |f(z)|^2 e^{-zz^*} d\mu(z), \quad zz^* = \sum z_i \bar{z}_i.$$

The operators of this representation are given by the formulas

$$(9.2) \quad (T^+(g)f)(z) = e^{\zeta_0 - zz_0^*} f(z + z_0) \quad \text{for} \quad g = (\zeta_0, z_0) \in N,$$

$$(9.3) \quad (T^+(g)f)(z) = f(\bar{s}zu) \quad \text{for} \quad g = (s, u) \in D_0.$$

Accordingly, the operators T_r^+ conjugate to T^+ with respect to the group of automorphisms $\mathbb{R}_+^*$ have the form

$$(9.4) \quad (T_r^+(g)f)(z) = e^{r^2 \zeta_0 - rzz_0^*} f(z + rz_0) \quad \text{for} \quad g = (\zeta_0, z_0) \in N,$$

$$(9.5) \quad (T_r^+(g)f)(z) = f(\bar{s}zu) \quad \text{for} \quad g = (s, u) \in D_0.$$

The space H^+ can be presented as the direct sum of irreducible pairwise nonequivalent invariant subspaces:

$$H^+ = \bigoplus_{m=1}^{\infty} H_m^+,$$

where H_m^+ is the invariant subspace cyclically generated by the homogeneous polynomials in $\bar{z}_1, \dots, \bar{z}_{n-1}$ of homogeneous degree m . In particular, the elements of H_0^+ are entire analytic functions on $\mathbb{C}^{n-1}$.

Denote by T_m^+ the restrictions of the operators T^+ to these subspaces.

Proposition 9.1. *The representation T_m^+ of the group P_0 in the space H_m^+ is summable, with almost invariant vector of the form $h_m(z) = l_m^{n-2}(zz^*)$, where l_m^{n-2} is a Laguerre polynomial.*

Indeed, it is easy to show that every polynomial $f(z)$ in zz^* satisfies the bound $\|(T_r^+(g)f)(z) - f(z)\| < c(g)r$ for small r . It follows that each of these polynomials is almost invariant. Laguerre polynomials arise when orthogonalizing the sequence $\{(zz^*)^m\}$ with respect to the norm in H .

Every representation T_m^+ of the subgroup P_0 gives rise to an irreducible unitary representation of the group P in the Hilbert space $\mathcal{H}_m^+ = \int_0^\infty H_m^+ d^*r$; this representation is given by the formulas

$$\begin{aligned} (T_m^+(g)f)(r, z) &= e^{r^2\zeta_0 - rzz_0^*} f(r, z + rz_0) \quad \text{for } g = (\zeta_0, z_0) \in N, \\ (T_m^+(g)f)(r, z) &= f(|s|^2r, \bar{s}zu) \quad \text{for } (s, u) \in D. \end{aligned}$$

The second family T_m^- of summable irreducible unitary representations of P_0 can be obtained from T_m^+ by complex conjugation; namely, the representation T_m^- acts in the space of functions $\overline{f(z)}$, where $f(z) \in H_m^+$, by operators of the form

$$T^-(g)\bar{f} = \overline{T^+(g)f}.$$

In particular, the representation T_0^- acts in the space of entire antianalytic functions on $\mathbb{C}^{n-1}$.

In accordance with the general construction, every representation $T_m^\pm$ of the subgroup P_0 gives rise to an irreducible unitary representation of the current group P^X in the quasi-Poisson space $\text{QPS}_u(\mathcal{E}(Y), \sigma H_m^\pm)$, where $u(r) = r$, $(Y, \mu) = (\mathbb{R}_+^*, d^*r) \times (X, m)$.

These constructions of summable representations of $U(n, 1)$ apply also for $n = 1$. In this case, the subgroup P_0 is one-dimensional and has two (up to conjugation) summable one-dimensional representations. They give rise to two irreducible unitary representations of the current group P^X .

§10. CONSTRUCTING A REPRESENTATION OF THE GROUP G^X IN THE QUASI-POISSON SPACE $\text{QPS}(X; K)$; THE GENERAL CASE

In this section, the notion of a summable representation introduced in Subsection 7.1 for groups G of the form $G = S \ltimes G_0$ is extended to a wider class of groups, which includes $U(n, 1)$ and $O(n, 1)$; for every summable representation T of a group G in this class, we construct the associated representation of the current group G^X in a quasi-Poisson space. In the case of $U(n, 1)$ and $O(n, 1)$, these representations are extensions of the representations of the group P^X , where P is the maximal parabolic subgroup in $U(n, 1)$ and $O(n, 1)$, respectively, described in the previous sections.

10.1. Summable representations of a group G . Fix a punctured finite-dimensional or separable Hilbert space (K, h) and a locally compact (noncompact) group S with a left-invariant measure ds .

With the pair (K, S) we associate the Hilbert space

$$\mathcal{K} = L^2(S, ds; K) = \int_S^\oplus (K_s, h_s) ds, \quad (K_s, h_s) = (K, h),$$

i.e., the space of measurable sections $f \mapsto f(s)$ of the fiber bundle over S with the fibers $(K_s, h_s) = (K, h)$, endowed with the norm $\|f\|^2 = \int_S \|f(s)\|_{K_s}^2 ds$.

Fix an arbitrary positive function $u(s)$ on S satisfying Condition A (see Subsection 7.1); with u we associate the section $s \mapsto f_u(s) \in K_s$ of this fiber bundle given by the formula $f_u(s) = e^{-\frac{1}{2}u(s)}h_s$, $h_s = h$.

Let G be an arbitrary locally compact group and T a unitary representation of G in a Hilbert space $\mathcal{K}$. We assume that the action of G is defined on the space $\tilde{\mathcal{K}}$ of all measurable sections of the fiber bundle over S with the fibers $K_s = K$.

With each element $g \in G$ we associate the following section of this fiber bundle:

$$(10.1) \quad b_u(g) = T(g)f_u - f_u, \quad \text{where} \quad f_u(s) = e^{-\frac{1}{2}u(s)}h_s.$$

Obviously, $b_u(g)$ has the 1-cocycle property:

$$b_u(g_1g_2) = b_u(g_1) + T(g_1)b_u(g_2) \quad \text{for any} \quad g_1, g_2 \in G.$$

Definition 10.1. We say that the pair (T, b_u) is *summable* if

$$b_u(g) \in L^2(S, ds; K) \cap L^1(S, e^{-\frac{1}{2}u(s)}ds; K) \quad \text{for every} \quad g \in G,$$

i.e.,

$$(10.2) \quad \int_S \|(T(g)f_u)(s) - f_u(s)\|^2 ds < \infty, \quad \int_S \|(T(g)f_u)(s) - f_u(s)\| e^{-\frac{1}{2}u(s)} ds < \infty,$$

and the set of vectors $b_u(g)$, $g \in G$, is total in $\mathcal{K}$.

In the special case where $G = S \ltimes G_0$, the summability of a representation T_e of G_0 implies that the pair (T, b_u) , where T is the representation of G associated with T_e and b_u is a 1-cocycle of T , is summable in the sense of Definition 10.1.

Theorem 10.1. *The map $b_u : G \rightarrow \tilde{\mathcal{K}}$ is a nontrivial 1-cocycle of the representation T , and*

$$(10.3) \quad |\langle b_u(g), f_u \rangle| \equiv \left| \int_S \langle b_u(g, s), f_u(s) \rangle_{K_s} ds \right| < \infty.$$

Indeed, the fact that $b_u(g)$ is a 1-cocycle and the required bound follow from the summability condition. This 1-cocycle is nontrivial, because $f_u \notin L^2(S, ds; K)$ by Condition A on the function u .

Remark. The condition

$$b_u(g) \in L^1(S, e^{-\frac{1}{2}u(s)}ds; K) \quad \text{for every} \quad g \in G$$

in the definition of summability can be replaced by the weaker condition (10.3).

10.2. The representation of the group G^X in the space $L^2(X, m; \mathcal{K})$. We associate a unitary representation of the current group G^X in a quasi-Poisson space with every summable pair (T, b_u) , where T is a representation of the group G in a space $\mathcal{K} = L^2(S, ds; K) = \int_S^\oplus (K_s, h_s) ds$ and $b_u(g)$ is the 1-cocycle of this representation given by (10.1).

For this, first we introduce a (reducible) unitary representation $\tilde{T}$ of the group G^X in the Hilbert space

$$H = L^2(X, m; \mathcal{K}) = \int_X^\oplus \mathcal{K}_x dm(x), \quad \mathcal{K}_x = \mathcal{K},$$

i.e., in the space of sections of the fiber bundle over X with the fibers $\mathcal{K}_x = \mathcal{K}$, endowed with the norm $\|f\|^2 = \int_X \|f(x)\|_{\mathcal{K}_x}^2 dm(x)$. By definition, the operators of this representation act in the fibers $f(x)$ of this fiber bundle by the formula

$$(\tilde{T}(g(\cdot))f)(x) = T(g(x))f(x).$$

The 1-cocycle $b_u(g)$ of the representation T of the group G gives rise to a 1-cocycle $\tilde{b}_u(g(\cdot)) : G^X \rightarrow H$ of the representation $\tilde{T}$, which is defined by the formula $\tilde{b}_u(g(\cdot), x) = b_u(g(x))$. For simplicity, in what follows we shall write $\tilde{b}$ instead of $\tilde{b}_u$.

The summability of (T, b_u) and the condition on the group G^X show that the pair $(\tilde{T}, \tilde{b})$, $\tilde{b} = \tilde{b}_u$, has similar summability properties; namely,

- 1) $\tilde{b}(g) \in L^2(X, m; \mathcal{K}) \cap L^1(X, m; \mathcal{K})$ for every $g \in G^X$;
- 2) the set of vectors $\tilde{b}(g)$, $g \in G^X$, is total in H ;
- 3) $|\langle \tilde{b}(g), f_u \rangle| \equiv |\int_X \langle \tilde{b}(g, x), f_u(x) \rangle_{\mathcal{K}_x} dm(x)| < \infty$ for every $g \in G^X$.

There exists a natural isomorphism of Hilbert spaces

$$L^2(X, m; \mathcal{K}) \approx L^2(S \times X, ds dm(x); K),$$

where

$$L^2(S \times X, ds dm(x); K) = \int_{S \times X}^{\oplus} K_{s,x} ds dm(x), \quad K_{s,x} = K,$$

i.e., this is the space of sections $v(s, x)$ of the fiber bundle over $S \times X$ with the fibers $K_{s,x} = K$, endowed with the norm $\|v\|^2 = \int_{S \times X} \|v(s, x)\|^2 d^*r dm(x)$. Namely, every vector $v(s, x) \in L^2(S \times X, ds dm(x); K)$ is sent to the vector $v(s) = \int_X^{\oplus} v(s, x) dm(x) \in L^2(X, m; \mathcal{K})$.

In view of this isomorphism, the representation $\tilde{T}$ of the group G^X can be realized in the Hilbert space of functions $v(s, x)$ on $S \times X$ with values in $K_{s,x} = K$ with the norm

$$\|v\|^2 = \int_{S \times X} \|v(s, x)\|^2 ds dm(x),$$

i.e., in the space

$$\tilde{K} = \int_{S \times X} K_{s,x} ds dm(x), \quad K_{s,x} = K.$$

In this new realization, the cocycle $\tilde{b}$ of the representation $\tilde{T}$ is given by the formula

$$(10.4) \quad \tilde{b}(g(\cdot), s, x) = (\tilde{T}(g(\cdot))f_u)(s, x) - f_u(s, x), \quad \text{where} \quad f_u(s, x) = e^{-\frac{1}{2}u(s)}h_{s,x}.$$

10.3. Constructing a unitary representation of the group G^X in the quasi-Poisson space $\text{QPS}(X; K)$. Starting with the pair $(\tilde{T}, \tilde{b}_u)$ defined in Subsection 10.2, we construct a representation of the current group G^X in the quasi-Poisson space

$$\text{QPS}(X; K) = \text{QPS}_u(\mathcal{E}(S \times X), \sigma; K)$$

associated with the triple $(S \times X, \mu, u)$, where $d\mu(s, x) = ds dm(x)$ and u is an extension to $S \times X$ of the original function $u(s)$ on S : $u(s, x) = u(s)$. Condition A imposed on $u(s)$ implies that the function $u(s, x)$ satisfies condition (2.5).

Definition 10.2. Let

$$L(X; K) = L^2(S \times X, ds dm(x); K) \cap L^1(S \times X, e^{-\frac{1}{2}u(s)} ds dm(x); K);$$

$$L_u(X; K) = f_u + L(X; K), \quad \text{where} \quad f_u(s, x) = e^{-\frac{1}{2}u(s)}h_{s,x}.$$

With each element $v \in L_u(X; K)$ and each configuration $\omega \in \mathcal{E}(S \times X)$ we associate the infinite tensor product

$$(10.5) \quad F_v(\omega) = \bigotimes_{(s,x) \in \omega} v(s, x), \quad v \in L_u(X; K).$$

From the definition of the representation T of the group G and the summability condition it follows that

- 1) $L(X; K)$ is a linear dense subspace in $L^2(S \times X, ds dm(x); K)$ invariant under the operators $\tilde{T}$ of the representation of G^X ;
  - 2)
- $$(10.6) \quad \tilde{b}(g) \in L(X; K) \quad \text{for all } g \in G^X,$$

and, consequently, the set $L_u(X; K)$ is also invariant under the operators $\tilde{T}$ of the representation of G^X : if $w \in L(X; K)$, then

$$\tilde{T}(g)(f_u + w) = f_u + (\tilde{w} + \tilde{b}(g)), \quad \tilde{w} + \tilde{b}(g) \in L(X; K);$$

- 3) the vectors F_v are related to the vectors $\text{qps } v$ belonging to the quasi-Poisson space $\text{QPS}(X; K) \equiv \text{QPS}(\mathcal{E}(S \times X), \sigma; K)$ (see Subsection 4.4) by the following formula:

$$(10.7) \quad F_v = e^{\langle w, f_u \rangle} \text{qps } w, \quad \text{where } w = v - f_u \in L(X; K).$$

In its turn, this implies that the set $\mathcal{L}(X; K)$ of vectors F_v , $v \in L(X; K)$, of the form (10.5) is contained and total in the space $\text{QPS}(X; K)$. Hence, in order to describe a representation of the group G^X in the quasi-Poisson space $\text{QPS}(X; K)$, it suffices to define the operators of this representation only on the total subset $\mathcal{L}(X; K)$ of vectors F_v , $v \in L_u(X; K)$.

Definition 10.3. We define an action of the operators $U(\tilde{g})$, $\tilde{g} \in G^X$, on the total subset $\mathcal{L}(X; K)$ in $\text{QPS}(X; K)$ by the following formula:

$$(10.8) \quad U(\tilde{g})F_v = e^{\lambda(\tilde{g}, v)} F_{\tilde{T}(\tilde{g})v}, \quad v \in L_u(X; K),$$

where

$$(10.9) \quad \lambda(\tilde{g}, v) = -\frac{1}{2} \|\tilde{b}(\tilde{g})\|^2 - \text{Re} \langle \tilde{b}(\tilde{g}), f_u \rangle - \langle \tilde{T}(\tilde{g})w, \tilde{T}(\tilde{g})f_u \rangle + \langle w, f_u \rangle, \quad w = v - f_u.$$

Theorem 10.2. The operators (10.8) are unitary on the set $\mathcal{L}(X; K)$, i.e.,

$$(10.10) \quad \langle U(\tilde{g})F_{v_1}, U(\tilde{g})F_{v_2} \rangle = \langle F_{v_1}, F_{v_2} \rangle \quad \text{for any } v_1, v_2 \in L_u(X; K) \quad \text{and } \tilde{g} \in G^X;$$

they satisfy the relation

$$(10.11) \quad U(\tilde{g}_1 \tilde{g}_2)F_v = e^{i\rho(\tilde{g}_1, \tilde{g}_2)} U(\tilde{g}_1)U(\tilde{g}_2)F_v \quad \text{for any } \tilde{g}_1, \tilde{g}_2 \in G^X \quad \text{and } v \in L_u(X; K),$$

where

$$\rho(\tilde{g}_1, \tilde{g}_2) = -\text{Im}(\langle \tilde{b}(\tilde{g}_2), \tilde{b}(\tilde{g}_1^{-1}) \rangle) + \text{Im} \langle \tilde{b}(\tilde{g}_1 \tilde{g}_2) - \tilde{b}(\tilde{g}_1) - \tilde{b}(\tilde{g}_2), f_u \rangle.$$

Therefore, they can be extended to unitary operators on the entire quasi-Poisson space $\text{QPS}(X; K)$ forming a projective representation of the group G^X .

Proof. We use the isomorphism between the quasi-Poisson space $\text{QPS}(X; K)$ and the Fock space $\text{EXP}(H)$, $H = L^2(S \times X, ds dm(x); k)$ that sends a vector $\text{qps } w \in \text{QPS}(X; K)$ to the vector $\exp w \in \text{EXP}(H)$. In view of this isomorphism, the unitary projective representation of the group G^X in the space $\text{EXP}(H)$ is equivalent to the unitary projective representation of this group in the quasi-Poisson space $\text{QPS}(X; K)$ whose operators are defined on the total subset of vectors $\text{qps } w$, $w \in L(X, K)$, by the formula

$$V(\tilde{g}) \text{qps } w = e^{\lambda'(\tilde{g}, w)} \text{qps}(\tilde{T}(\tilde{g})w + \tilde{b}(\tilde{g})),$$

where

$$\lambda'(\tilde{g}, w) = -\frac{1}{2}\|\tilde{b}(\tilde{g})\|^2 - \langle \tilde{T}(\tilde{g})w, \tilde{b}(\tilde{g}) \rangle,$$

and satisfy the relation

$$V(\tilde{g}_1\tilde{g}_2)e^{-i\operatorname{Im}\langle \tilde{b}(\tilde{g}_2), \tilde{b}(\tilde{g}_1^{-1}) \rangle}V(\tilde{g}_1)V(\tilde{g}_2).$$

Passing from the vectors qps w to the original vectors F_v (see (10.7)), we see that the operators $V(\tilde{g})$ act on F_v by the formula

$$V(\tilde{g})F_v = e^{\lambda''(\tilde{g}, w)}F_{\tilde{T}(\tilde{g})v},$$

where $\lambda''(\tilde{g}, w) = \lambda'(\tilde{g}, w) + \langle w, f_u \rangle - \langle \tilde{T}(\tilde{g})w + \tilde{b}(\tilde{g}), f_u \rangle$. This expression for $\lambda''(\tilde{g}, w)$ can be written as follows:

$$\lambda''(\tilde{g}, w) = -\frac{1}{2}\|\tilde{b}(\tilde{g})\|^2 - \langle \tilde{b}(\tilde{g}), f_u \rangle - \langle \tilde{T}(\tilde{g})w, \tilde{T}(\tilde{g})f_u \rangle + \langle w, f_u \rangle.$$

Comparing the formulas for the operators $V(\tilde{g})$ and $U(\tilde{g})$ on the set of vectors F_v , we see that they satisfy

$$(10.12) \quad U(\tilde{g}) = e^{i\operatorname{Im}\langle \tilde{b}(\tilde{g}), f_u \rangle}V(\tilde{g}).$$

The theorem follows. $\square$

In the proof of Theorem 10.2 we have also established a projective equivalence between the representation U of the group G^X in the space $\operatorname{QPS}(X; K)$ constructed in this section and the Fock representation V of this group. Namely, the operators of these representations satisfy relation (10.12).

Proposition 10.1. *In the case where $G = S \ltimes G_0$ and T is the representation of the group G associated with a summable representation T_e of the subgroup G_0 , the constructed representation U of the group G^X coincides with the representation described in Subsection 7.2. Thus, in contrast to the Fock model, it is not projective.*

Proof. Observe that formulas (10.8) and (7.11) for the operators of these representations coincide up to coefficients. Hence, it suffices to check that the coefficients in these formulas also coincide, i.e., that $\lambda(\tilde{g}, v) = 0$ for $\tilde{g} \in G_0^X$ and $\lambda(\tilde{s}, v) = -\frac{1}{2}\int_X \operatorname{Log} \tilde{s}(x) dm(x)$ for $\tilde{s} \in S^X$.

First, we verify that

$$(10.13) \quad -\frac{1}{2}\|\tilde{b}(\tilde{g})\|^2 - \operatorname{Re}\langle \tilde{b}(\tilde{g}), f_u \rangle = -\frac{1}{2}\operatorname{Log} s(\tilde{g}) \quad \text{for every } \tilde{g} \in G^X.$$

Indeed, for $\tilde{g} \in G_0^X$ this follows from the formula $\frac{1}{2}\|b(g)\|_K^2 + \operatorname{Re}\langle b(g), f_u \rangle_K = 0$ for $g \in G_0$. For $\tilde{g} \in S^X$, (10.13) follows from the equation

$$\frac{1}{2}\|b(s_0)\|_K^2 + \operatorname{Re}\langle b(s_0), f_u \rangle_K = \frac{1}{2}\int_S (e^{-u(s_0s)} - e^{-u(s)}) ds = \frac{1}{2}\operatorname{Log} s_0 \quad \text{for } g = s_0 \in S.$$

It remains to prove that

$$(10.14) \quad \langle \tilde{T}(\tilde{g})w, \tilde{T}(\tilde{g})f_u \rangle = \langle w, f_u \rangle \quad \text{for every } \tilde{g} \in G^X.$$

For $\tilde{g} \in G_0^X$ this is obvious. If $\tilde{g} = \tilde{s}_0 \in S^X$, we have

$$\langle \tilde{T}(\tilde{s}_0)w, \tilde{T}(\tilde{s}_0)f_u \rangle = \int_{S \times X} \langle w(s_0(x)s, x), f_u(s_0(x)s, x) \rangle_{K_{s,x}} ds dm(x),$$

and (10.14) follows from the left invariance of the measure ds . $\square$

Remark. For an arbitrary group G , relation (10.14) is not true in general.

10.4. A formula for the spherical function. The vector $\Omega \in \text{QPS}(X, K)$ given by the formula

$$\Omega(\omega) = \bigotimes_{(s,x) \in \omega} f_u(s, x), \quad \text{where} \quad f_u(s, x) = e^{-\frac{1}{2}u(s)} h_{s,x},$$

is called the *vacuum* vector. Obviously, $\|\Omega\| = 1$. The *spherical function* of the representation U of the group G^X is the function Ψ on G^X given by the formula $\Psi(g(\cdot)) = \langle U(g(\cdot))\Omega, \Omega \rangle$. If Ω is a cyclic vector of U , then this representation is uniquely determined up to equivalence by the spherical function.

Theorem 10.3. *The spherical function of the representation U of the group G^X associated with a summable representation T of the group G and a 1-cocycle b of this representation equals*

$$(10.15) \quad \Psi(\tilde{g}) = \exp\left(-\frac{1}{2}\|\tilde{b}(\tilde{g})\|^2 + i \operatorname{Im} \langle \tilde{b}(g), f_u \rangle\right),$$

where $\tilde{b}$ is the 1-cocycle of the representation $\tilde{T}$ of G^X in $L^2(X; K)$ generated by the pair (T, b) .

Proof. By (10.8), we have

$$\langle \tilde{T}(g(\cdot))\Omega(\omega), \Omega(\omega) \rangle_{K_\omega^\otimes} = e^{\lambda(g)} \prod_{(s,x) \in \omega} \langle (\tilde{T}(g(\cdot))f_u)(r, x), f_u(r, x) \rangle,$$

where $\lambda(g) = -\frac{1}{2}\|\tilde{b}(g)\|^2 - \operatorname{Re} \langle \tilde{b}(g), f_u \rangle$. Integrating with respect to the quasi-Poisson measure σ yields, by an analog of Campbell's formula,

$$\Psi(g) = e^{\lambda(g)} \exp\left(\int_{\mathbb{R}_+^* \times X} \langle (\tilde{T}(g(\cdot))f_u)(r, x) - f_u(r, x), f_u(r, x) \rangle_{K_{r,x}} d^*r dm(x)\right).$$

Since

$$\int_{\mathbb{R}_+^* \times X} \langle (\tilde{T}(g(\cdot))f_u)(r, x) - f_u(r, x), f_u(r, x) \rangle_{K_{r,x}} d^*r dm(x) = \langle \tilde{b}(g), f_u \rangle,$$

this implies (10.15). □

§11. UNITARY REPRESENTATIONS OF THE CURRENT GROUP $U(n, 1)^X$ IN A QUASI-POISSON SPACE

In accordance with the considerations of §10, in order to describe representations of the current group G^X in quasi-Poisson spaces, it suffices to describe summable representations of the group G . In this and the next sections, we give such a description for the groups $U(n, 1)$ and $O(n, 1)$.

11.1. Basic definitions. By definition, $U(n, 1)$ is the group of linear transformations in $\mathbb{C}^{n+1}$ preserving a Hermitian form of signature $(n, 1)$. For instance, we may take the form

$$|x|^2 = x_1 \bar{x}_{n+1} + x_{n+1} \bar{x}_1 + |x_2|^2 + \cdots + |x_n|^2$$

and write elements of the group $U(n, 1)$ as block matrices $g = \|g_{ij}\|_{i,j=1,2,3}$, where the diagonal contains square matrices of orders 1, $n-1$, and 1, respectively.

To construct summable representations of the group $U(n, 1)$, consider the homogeneous space $L = U(n, 1)/C$ of this group, where C is the maximal compact subgroup (the multidimensional complex Lobachevsky space). In the adopted matrix realization of $U(n, 1)$, we have $C = U(n, 1) \cap U(n+1)$,

$$L = \{v = (a, b) \in \mathbb{C} \oplus \mathbb{C}^{n-1} \mid a + \bar{a} + b^* b < 0\}, \quad \text{where} \quad b^* b = \sum \bar{b}_i b_i$$

(b is a column vector), and the action $v \mapsto gv$ on the space L of elements $g = \|g_{ij}\|_{i,j=1,2,3}$ of $U(n, 1)$ is given by the formula $g(a, b) = (a', b')$, where

$$\begin{aligned} a' &= (g_{11} + g_{12}b + g_{13}a)^{-1} (g_{31} + g_{32}b + g_{33}a), \\ b' &= (g_{11} + g_{12}b + g_{13}a)^{-1} (g_{21} + g_{22}b + g_{23}a). \end{aligned}$$

The maximal compact subgroup C is the stationary subgroup of the point $v_0 = (-1, 0)$. Denote

$$(11.1) \quad p(v_1, v_2) = -(a_1 + \bar{a}_2 + b_2^* b_1) \quad \text{for } v_i = (a_i, b_i) \in L.$$

It is easily seen that

$$(11.2) \quad \operatorname{Re} p(v_1, v_2) > 0 \quad \text{for any } v_1, v_2 \in L.$$

Proposition 11.1. *The function $p(v_1, v_2)$ satisfies the relation*

$$(11.3) \quad p(gv_1, gv_2) = p(v_1, v_2) (g_{11} + g_{12}b_1 + g_{13}a_1)^{-1} \overline{(g_{11} + g_{12}b_2 + g_{13}a_2)}^{-1}$$

for any $v_1 = (a_1, b_1), v_2 = (a_2, b_2) \in L$ and $g \in U(n, 1)$.

Corollary. *The function Q of a quadruple of points $v_i, v'_i \in L$, $i = 1, 2$, given by the formula*

$$(11.4) \quad Q(v_1, v_2; v'_1, v'_2) = \frac{p(v_1, v'_2) p(v_2, v'_1)}{p(v_1, v'_1) p(v_2, v'_2)}$$

is invariant under the action of the group $U(n, 1)$ on L , i.e.,

$$(11.5) \quad Q(gv_1, gv_2; gv'_1, gv'_2) = Q(v_1, v_2; v'_1, v'_2) \quad \text{for every } g \in U(n, 1).$$

11.2. Summable irreducible representations of $U(n, 1)$. There are two summable unitary irreducible representations $T^\pm$ of the group $U(n, 1)$. The representation T^+ can be realized in the space $\mathcal{K}^+$ of functions $f(r, z)$ on $\mathbb{R}_+^* \times \mathbb{C}^{n-1}$ entire analytic in z , endowed with the norm

$$(11.6) \quad \|f\|^2 = \int_0^\infty \int_{\mathbb{C}^{n-1}} |f(r, z)|^2 e^{-zz^*} d\mu(z) d^*r,$$

where $zz^* = \sum z_i \bar{z}_i$ and $d\mu(z)$ is the Lebesgue measure on $\mathbb{C}^{n-1}$ normalized so that $\int_0^\infty \int_{\mathbb{C}^{n-1}} e^{-zz^*} d\mu(z) = 1$.

Thus, $\mathcal{K}^+$ is the direct integral of the Bargmann spaces $K_r^+ = K^+$ of entire analytic functions $f(z)$ on $\mathbb{C}^{n-1}$ with the norm $\|f\|^2 = \int_0^\infty |f(z)|^2 e^{-zz^*} d\mu(z)$, i.e.,

$$\mathcal{K}^+ = \int_0^\infty K_r^+ d^*r.$$

Recall that the same space $\mathcal{K}^+$ is the space of the irreducible representation T_0^+ of the maximal parabolic subgroup $P \subset U(n, 1)$ (see Subsection 9.2). We shall show that the representation T^+ of $U(n, 1)$ defined in this section is an extension of T_0^+ .

The second summable representation T^- can be obtained from T^+ by complex conjugation and, consequently, is realized in the space $\mathcal{K}^-$ of functions $f(r, z)$ on $\mathbb{R}_+^* \times \mathbb{C}^{n-1}$ entire antianalytic in z , endowed with the same norm (11.6). For definiteness, in what follows we confine ourselves to the representation T^+ and omit the plus sign.

To describe the representation T of the group $U(n, 1)$ in the space $\mathcal{K}$, we introduce a total subset $M \subset \mathcal{K}$ and define the operators of T only on this subset.

With every element $v = (a, b) \in L$ we associate the following function on $\mathbb{R}_+^* \oplus \mathbb{C}^{n-1}$:

$$(11.7) \quad f_v(r, z) = e^{r^2 a + r(z, b)}, \quad \text{where } (z, b) = \sum z_i b_i.$$

Proposition 11.2. *For any fixed $v \in L$ and r , the function $f_v(r, z)$ lies in the space K_r ; the inner product in K_r of two such functions equals*

$$(11.8) \quad \langle f_{v_1}, f_{v_2} \rangle_{K_r} = e^{-r^2 p(v_1, v_2)} \quad \text{for } v_i = (a_i, b_i),$$

where $p(v_1, v_2)$ is given by (11.1). In particular, $\|f_v\|^2_K = e^{-r^2 p(v, v)} < \infty$.

Proposition 11.2 and the relation

$$\int_0^\infty (e^{-ar^2} - e^{-br^2}) d^*r = \frac{1}{2} \log(b/a) \quad \text{for } \operatorname{Re} a, \operatorname{Re} b > 0$$

imply the following result.

Proposition 11.3. *Functions of the form $f_{v_1} - f_{v_2}$, where $v_1, v_2 \in L$, lie in the space $\mathcal{K}$; the inner product of two such functions equals*

$$(11.9) \quad \langle f_{v_1} - f_{v_2}, f_{v'_1} - f_{v'_2} \rangle = \frac{1}{2} \log Q(v_1, v_2; v'_1, v'_2),$$

where $Q(v_1, v_2; v'_1, v'_2)$ is given by (11.4). In particular,

$$(11.10) \quad \|f_{v_1} - f_{v_2}\|^2 = \frac{1}{2} \log Q(v_1, v_2; v_1, v_2) = \frac{1}{2} \log \frac{|p(v_1, v_2)|^2}{p(v_1, v_1)p(v_2, v_2)}.$$

(By $\log$ we denote the branch of the logarithm on the plane $\mathbb{C}$ cut along the negative real axis such that $\log 1 = 0$.)

It is easy to check that the set M of functions of the form $f_{v_1} - f_{v_2}$ is total in $\mathcal{K}$.

Definition 11.1. We define an action of the operators $\tilde{T}(g)$, $g \in U(n, 1)$, on the set M of functions of the form $f_{v_1} - f_{v_2}$ by the formula

$$(11.11) \quad \tilde{T}(g)(f_{v_1} - f_{v_2}) = f_{gv_1} - f_{gv_2} \quad \text{for any } v_1, v_2 \in L.$$

This definition shows that the operators $\tilde{T}(g)$ preserve the set M and satisfy the group relation. Propositions 11.3 and 11.1 (corollary) imply the following result.

Theorem 11.1. *The operators $\tilde{T}(g)$ preserve the inner product in M and, consequently, generate a unitary representation of the group $U(n, 1)$ in the space $\mathcal{K}$.*

The representation $\tilde{T}$ of $U(n, 1)$ defined in this way has a nontrivial 1-cocycle $\beta : U(n, 1) \rightarrow \mathcal{K}$ of the form $\beta(g) = f_{gv_0} - f_{v_0}$, where $v_0 = (-1, 0)$. In terms of the elements of a block matrix $g \in U(n, 1)$, this 1-cocycle is given by the formula

$$(11.12) \quad \beta(g; r, z) = e^{r^2 a(g) + (z, b(g))} - e^{-r^2},$$

where

$$(11.13) \quad a(g) = (g_{11} - g_{13})^{-1} (g_{31} - g_{33}), \quad b(g) = (g_{11} - g_{13})^{-1} (g_{21} - g_{23}).$$

Theorem 11.2. *The pair $(\tilde{T}, \beta)$ satisfies the summability condition and, consequently, gives rise to a unitary representation U of the current group $U(n, 1)^X$ in the quasi-Poisson space $\operatorname{QPS}(\mathcal{E}(Y), \sigma; K)$ associated with $u(r) = r^2$ and $(Y, \mu) = (\mathbb{R}_+^*, d^*r) \times (X, m)$.*

Proof. By Definition 10.1, it suffices to check that $\beta(g) \in L^1(\mathbb{R}_+^*, e^{-r^2} d^*r; K)$ for every $g \in U(n, 1)$, i.e.,

$$(11.14) \quad \int_0^\infty \|\beta(g; r)\|_{K_r} e^{-r^2} d^*r < \infty.$$

By (11.8), we have

$$\|(g, r)\|^2 = e^{r^2 (2 \operatorname{Re} a(g) + |b(g)|^2)} - e^{r^2 (a(g) - 1)} - e^{r^2 (\overline{a(g)} - 1)} + e^{-2r^2}.$$

Therefore, $\|b(g, r)\| < c(g)r$ for small r . Since $2\operatorname{Re} a(g) + |b(g)|^2 < 0$ and $\operatorname{Re}(a(g) - 1) < 0$ for every $g \in U(n, 1)$, the integral (11.14) converges. $\square$

11.3. The restriction of the representation $\tilde{T}^+$ of $U(n, 1)$ to the maximal parabolic subgroup $P \subset U(n, 1)$. In the adopted matrix realization, P is the subgroup of lower block triangular matrices and can be presented as the semidirect product

$$P = D \ltimes N,$$

where N is the subgroup of matrices of the form

$$h = \begin{pmatrix} 1 & 0 & 0 \\ -z^* & e_{n-1} & 0 \\ it - \frac{1}{2}zz^* & z & 1 \end{pmatrix}, \quad t \in \mathbb{R}, \quad z \in \mathbb{C}^{n-1}$$

(the Heisenberg group of order $2n - 1$) and $D \cong \mathbb{C}^* \times U(n - 1)$ is the subgroup of block diagonal matrices of the form $d = \operatorname{diag}(\bar{s}^{-1}, u, s)$, $s \in \mathbb{C}^*$, $u \in U(n - 1)$.

We shall write elements of N as pairs (ζ, z) , where $z \in \mathbb{C}^{n-1}$ and $\zeta = it - \frac{|z|^2}{2}$, and elements of D as pairs (s, u) , where $s \in \mathbb{C}^*$, $u \in U(n - 1)$. In this notation, P coincides with the group described in §9.

Proposition 11.4. *The restriction of the representation $\tilde{T}$ of the group $U(n, 1)$ to the subgroup P coincides with the representation T_0^+ of P described in Subsection 9.2.*

Proof. Since $\tilde{T}(g)f_v = f_{gv}$ for any $v \in L$ and $g \in U(n, 1)$, we have

$$\tilde{T}(g)e^{r^2a+r(z,b)} = e^{r^2a'+r(z,b')},$$

where

$$\begin{aligned} a' &= a + \zeta_0 + (z_0, b), & b' &= b - z_0^* \quad \text{for } g = (\zeta_0, z_0) \in N; \\ a' &= |s|^2a, & b' &= ub\bar{s} \quad \text{for } (s, u) \in D. \end{aligned}$$

It follows that the extensions of these operators to the entire space of functions $f(r, z)$ have the form

$$\begin{aligned} (\tilde{T}(g)f)(r, z) &= e^{r^2\zeta_0 - rz z_0^*} f(r, z + rz_0) \quad \text{for } g = (\zeta_0, z_0) \in N; \\ (\tilde{T}(g)f)(r, z) &= f(|s|^2r, \bar{s}zu) \quad \text{for } (s, u) \in D. \end{aligned}$$

These formulas coincide with the formulas for the representation T_0^+ of P obtained in Subsection 9.2. $\square$

Corollary. *The representation U of the group $U(n, 1)^X$ in the quasi-Poisson space associated with the pair $(\tilde{T}, \beta)$ is an extension of the representation of the subgroup P^X constructed in §9. Since the representation of P^X is irreducible, it follows that U is irreducible.*

11.4. Formulas for the raising and lowering operators. The raising and lowering operators of the Heisenberg subgroup $N \subset U(n, 1)$ in the space $\mathcal{K}^+$ of the representation T^+ of the group $U(n, 1)$ are defined by the formulas

$$a_k^+ v = \left. \frac{\partial(T^+(g)v)}{\partial \bar{z}_k} \right|_{z=0}, \quad a_k^- v = \left. \frac{\partial(T^+(g)v)}{\partial z_k} \right|_{z=0}, \quad k = 1, \dots, n,$$

for $g = (t, z) \in N$. The explicit formulas for the operators corresponding to elements of N show that

$$(a_k^+ v)(r, z) = -rz_k v(r, z), \quad (a_k^- v)(r, z) = \frac{\partial v(r, z)}{\partial z_k}.$$

In the representation of the current group $U(n, 1)^X$ in the quasi-Poisson space

$$\text{QPS}(\mathcal{E}(\mathbb{R}_+^* \times X), \sigma; K)$$

associated with T^+ , the corresponding operators are defined on the total subset of functionals of the form

$$\varphi(\omega; v) = \bigotimes_{(r, x) \in \omega} v(r, x; z)$$

by the following formulas:

$$\begin{aligned} (a_k^+ \varphi)(\omega; v) &= - \sum_{(r_s, x_s) \in \omega} (v(r_1, x_1; z) \otimes \cdots \otimes v(r_{s-1}, x_{s-1}; z) \otimes r_s z_k v(r_s, x_s; z) \otimes v(r_{s+1}, x_{s+1}; z) \otimes \cdots); \\ (a_k^- \varphi)(\omega; v) &= \sum_{(r_s, x_s) \in \omega} \left(v(r_1, x_1; z) \otimes \cdots \otimes v(r_{s-1}, x_{s-1}; z) \otimes r_s \frac{\partial v(r_s, x_s; z)}{\partial z_k} \otimes v(r_{s+1}, x_{s+1}; z) \otimes \cdots \right). \end{aligned}$$

§12. UNITARY REPRESENTATIONS OF THE CURRENT GROUP $O(n, 1)^X$, $n > 2$, IN QUASI-POISSON SPACES

12.1. Basic definitions. By definition, $O(n, 1)$ is the group of linear transformations in $\mathbb{R}^{n+1}$ preserving a quadratic form of signature $(n, 1)$.

For $n > 2$, this group has one summable representation.¹⁰ To describe this representation, it is convenient to realize $G = O(n, 1)$ as the group of linear transformations in $\mathbb{R}^{n+1}$ preserving the quadratic form $2x_1x_{n+1} + x_2^2 + \cdots + x_n^2$ and write its elements as block matrices $g = \|g_{ij}\|_{i,j=1,2,3}$, where the diagonal contains square matrices of orders 1, $n-1$, and 1, respectively. In this matrix realization, the maximal nilpotent subgroup in G is the group $N \cong \mathbb{R}^{n-1}$ of lower triangular matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ -\gamma^* & e_{n-1} & 0 \\ -\frac{1}{2}|\gamma|^2 & \gamma & 1 \end{pmatrix},$$

and $C = O(n, 1) \cap O(n+1)$ is the maximal compact subgroup in G .

We shall identify elements of the subgroup $D = \text{diag}(r^{-1}, e_{n-1}, r)$, $r \in \mathbb{R}_+^*$, with elements of $\mathbb{R}_+^*$, and elements of the subgroup N with vectors $\gamma \in \mathbb{R}^{n-1}$.

Proposition 12.1. *Every element $g \in G$ can be uniquely written as a product*

$$g = r\gamma u, \quad \text{where } r \in D, \quad \gamma \in N, \quad u \in C.$$

Here

$$(12.1) \quad r(g) = (g_{11}^2 + g_{12}g_{12}^* + g_{13}^2)^{-1/2}, \quad \gamma(g) = (g_{11} - g_{13})^{-1}(g_{23}^* - g_{21}^*),$$

where the g_{ij} are the entries of a block matrix g .

¹⁰In the special case of $n = 2$, the group $O(2, 1)$ is locally isomorphic to $SU(1, 1)$ and, like $U(1, 1)$, has two summable representations.

12.2. The summable representation of $O(n, 1)$. There is a unique summable irreducible unitary representation T of the group $G = O(n, 1)$. We briefly describe it (see [9] for details).

There exists a representation T of G in the space $\tilde{\mathcal{H}}$ of measurable functions $f(\gamma)$ on $N \cong \mathbb{R}^{n-1}$ by operators of the form

$$(T(g)f)(\gamma) = f(\gamma\bar{g}) \left| -\frac{|\gamma|^2}{2}g_{13} + \gamma g_{23} + g_{33} \right|^{1-n},$$

where

$$\gamma\bar{g} = \left(-\frac{|\gamma|^2}{2}g_{13} + \gamma g_{23} + g_{33} \right)^{-1} \left(-\frac{|\gamma|^2}{2}g_{12} + \gamma g_{22} + g_{32} \right).$$

Denote by $\mathcal{H}$ the Hilbert space of functions $f(\gamma) \in \tilde{\mathcal{H}}$ satisfying the condition

$$\int_{\mathbb{R}^{n-1}} f(\gamma) d\gamma = 0,$$

where $d\gamma$ is the Lebesgue measure on $\mathbb{R}^{n-1}$. The norm in $\mathcal{H}$ is given by the formula

$$(12.2) \quad \|f\|^2 = \int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} \log |\gamma' - \gamma''| f(\gamma') \overline{f(\gamma'')} d\gamma' d\gamma''.$$

This subspace is invariant, and the operators T on $\mathcal{H}$ determine an irreducible unitary representation of G , which has the following nontrivial 1-cocycle:

$$b(g, \gamma) = (T(g)f)(\gamma) - f(\gamma), \quad \text{where} \quad f(\gamma) = \left(1 + \frac{|\gamma|^2}{2} \right)^{1-n}.$$

This cocycle is nontrivial because $f \notin \mathcal{H}$.

It is crucial that the vector f is invariant under the operators corresponding to the elements of the maximal compact subgroup C , so that $b(g) = 0$ for $g \in C$.

A summable representation of the group G can be obtained by passing from functions $f(\gamma)$ in $\mathcal{H}$ to their Fourier transforms in the variable γ . In this new realization, the space $\mathcal{H}$ can be written as the direct integral with respect to the measure d^*r on $\mathbb{R}_+^*$ of Hilbert spaces H_r ,

$$\mathcal{H} = \int_0^\infty H_r d^*r, \quad H_r = H,$$

where $H_r = H$ is the Hilbert space of functions on the unit sphere $S^{n-2} \subset \mathbb{R}^{n-1}$ with the norm

$$\|f\|^2 = \int_{S^{n-2}} |f(\omega)|^2 d\omega.$$

Thus, the elements of $\mathcal{H}$ are functions $\varphi(r, \omega)$ on $\mathbb{R}_+^* \times S^{n-2}$ with the norm

$$\|\varphi\|^2 = \int_0^\infty \int_{S^{n-2}} |\varphi(r, \omega)| d\omega d^*r.$$

The operators of this representation are defined on the elements of the subgroups N and $\mathbb{R}_+^*$ by the following formulas:

$$\begin{aligned} (T(\gamma)\varphi)(r, \omega) &= e^{-ir\langle\gamma, \omega\rangle} \varphi(r, \omega) \quad \text{for } \gamma \in N, \\ (T(r_0)\varphi)(r, \omega) &= \varphi(r_0r, \omega) \quad \text{for } r_0 = \text{diag}(r_0^{-1}, e_{n-1}, r). \end{aligned}$$

In the new realization, the nontrivial 1-cocycle of the representation T takes the form

$$b(g, r, \omega) = (T(g)\varphi)(r, \omega) - \varphi(r, \omega),$$

where $\varphi(r, \omega)$ is the Fourier transform of the function $f(\gamma) = \left(1 + \frac{|\gamma|^2}{2} \right)^{1-n}$.

There exists an explicit formula for $\varphi(r, \omega)$ in terms of the Bessel function $K_{\frac{n-1}{2}}(r)$ (see [1, Vol. 2]):

$$\varphi(r, \omega) = r^{\frac{n-1}{2}} K_{\frac{n-1}{2}}(\sqrt{2}r).$$

Proposition 12.1 implies the following result.

Proposition 12.2. *The nontrivial 1-cocycle $b(g)$ is given by the following explicit formula: for an arbitrary $g \in G$,*

$$(12.3) \quad b(g, r, \omega) = e^{-ir\langle \omega, \gamma(g) \rangle} \varphi(r_0(g)r) - \varphi(r), \quad \text{where} \quad \varphi(r) = r^{\frac{n-1}{2}} K_{\frac{n-1}{2}}(\sqrt{2}r)$$

and $r_0(g)$ and $\gamma(g)$ are given by (12.1).

Theorem 12.1. *The pair (T, b) is summable.*

Proof. It suffices to check that the integrals

$$\int_0^\infty \int_{S^{n-2}} |b(g, r, \omega)| d\omega d^*r$$

converge for every $g \in G$. This follows from the explicit formula (12.3) for the 1-cocycle $b(g)$ and the properties of the function $\varphi(r) = r^{\frac{n-1}{2}} K_{\frac{n-1}{2}}(r)$: on the semiaxis $r > 0$, this function is real-valued and positive, it decays exponentially as $r \rightarrow \infty$, and $\varphi(r) \sim c_0 + c_1 r$ as $r \rightarrow 0$. $\square$

12.3. The restriction to the maximal parabolic subgroup. In the adopted matrix realization, the maximal parabolic subgroup P of $O(n, 1)$ is the subgroup of lower block triangular matrices and can be written as the semidirect product $P = \mathbb{R}_+^* \ltimes (D_0 \ltimes N)$, where $\mathbb{R}_+^*$ and D_0 are the subgroups of block diagonal matrices of the form $\text{diag}(r^{-1}, e_{n-1}, r)$ and $\text{diag}(\epsilon, u, \epsilon)$, $\epsilon = \pm 1$, $u \in O(n)$, respectively. Thus, the description of P coincides with that given in §8. On the other hand, the formulas for the representation described in this section show that its restriction to the subgroup P coincides with the representation described in Subsection 8.1 and, consequently, is irreducible. This implies the following result.

Theorem 12.2. *The unitary representation of the current group $O(n, 1)^X$ associated with the pair (T, b) is irreducible.*

REFERENCES

- [1] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions*. Vols. I, II, Robert E. Krieger Publ. Co., Inc., Melbourne, FL, 1981. MR0698779 (84h:33001a); MR0698780 (84h:33001b)
- [2] F. A. Berezin, *Representations of a continuous direct product of universal coverings of the group of motions of the complex ball*, Trudy Moskov. Mat. Obshch. **36** (1978), 275–293; English transl. in Trans. Moscow Math. Soc. **1979**, no. 2. MR0507574 (80e:22009)
- [3] A. M. Vershik, *Does a Lebesgue measure in an infinite-dimensional space exist?* Trudy Mat. Inst. Steklova **259** (2007), 256–281; English transl. in Proc. Steklov Inst. Math. **259** (2007), no. 1. MR2433687 (2011i:28014)
- [4] A. M. Vershik, I. M. Gel'fand, and M. I. Graev, *Representations of the group $SL(2, R)$, where R is a ring of functions*, Uspekhi Mat. Nauk **28** (1973), no. 5, 83–128; English transl., Russian Math. Surveys **28** (1973), no. 5, 87–132. MR0393337 (52:14147)
- [5] ———, *Irreducible representations of the group G^X and cohomology*, Funktsional. Anal. i Prilozhen. **8** (1974), no. 2, 67–69; English transl., Funct. Anal. Appl. **8** (1974), 151–153. MR0348032 (50:530)
- [6] ———, *Representations of the group of diffeomorphisms*, Uspekhi Mat. Nauk **30** (1975), no. 6, 3–50; English transl. in Russian Math. Surveys **30** (1975), no. 6. MR0399343 (53:3188)
- [7] ———, *Commutative model of the representation of the group of flows $SL(2, \mathbb{R})^X$ connected with a unipotent subgroup*, Funktsional. Anal. i Prilozhen. **17** (1983), no. 2, 70–72; English transl., Funct. Anal. Appl. **17** (1983), no. 2, 137–139. MR0705049 (85j:22037b)
- [8] A. M. Vershik and M. I. Graev, *A commutative model of a representation of the group $O(n, 1)^X$ and a generalized Lebesgue measure in a distribution space*, Funktsional. Anal. i Prilozhen. **39** (2005), no. 2, 1–12; English transl., Funct. Anal. Appl. **39** (2005), no. 2, 81–90. MR2161512 (2006h:22013)
- [9] ———, *The structure of complementary series and special representations of the groups $O(n, 1)$ and $U(n, 1)$* , Uspekhi Mat. Nauk **61** (2006), no. 5, 3–88; English transl., Russian Math. Surveys **61** (2006), no. 5, 799–884. MR2328257 (2009d:22020)

- [10] ———, *Integral models of representations of current groups*, Funktsional. Anal. i Prilozhen. **42** (2008), no. 1, 22–32; English transl., Funct. Anal. Appl. **42** (2008), no. 1, 19–27. MR2423975 (2009e:22029)
- [11] ———, *Integral models of unitary representations of current groups with values in semidirect products*, Funktsional. Anal. i Prilozhen. **42** (2008), no. 4, 37–49; English transl., Funct. Anal. Appl. **42** (2008), no. 4, 279–289. MR2492425 (2010m:22009)
- [12] ———, *Integral models of representations of current subalgebras of simple Lie groups*, Uspekhi Mat. Nauk **64** (2009), no. 2, 5–72; English transl., Russian Math. Surveys **64** (2009), no. 2, 205–271. MR2530917 (2011c:22035)
- [13] A. M. Vershik and S. I. Karpushev, *Cohomology of groups in unitary representations, neighborhood of the identity and conditionally positive definite functions*, Mat. Sb. (N. S.) **119** (1982), no. 4, 521–533; English transl. in Math. USSR-Sb. **47** (1984), no. 2. MR0682497 (84d:22010)
- [14] A. M. Vershik and N. V. Tsilevich, *Fock factorizations and decompositions of the L^2 spaces over general Lévy processes*, Uspekhi Mat. Nauk **58** (2003), no. 3, 3–50; English transl., Russian Math. Surveys **58** (2003), no. 3, 427–472. MR1998773 (2004e:46087)
- [15] I. M. Gel'fand and M. I. Graev, *Special representations of the group $SU(n, 1)$ and projective unitary representations of the current group $SU(n, 1)^X$* , Dokl. Akad. Nauk **332** (1993), no. 3, 280–282; English transl., Russian Acad. Sci. Dokl. Math. **48** (1994), no. 2, 291–295. MR1257017 (95b:22013)
- [16] A. Guichardet, *Cohomologie des groupes topologiques et des algèbres de Lie*, Textes Math., vol. 2, CEDIC, Paris, 1980. MR0644979 (83f:22004)
- [17] D. A. Kazhdan, *On the connection of the dual space of a group with the structure of its closed subgroups*, Funktsional. Anal. i Prilozhen. **1** (1967), no. 1, 71–74. (Russian) MR0209390 (35:288)
- [18] J. F. C. Kingman, *Poisson processes*, Oxford Stud. in Probab., vol. 3, The Clarendon Press, Oxford Univ. Press, New York, 1993. MR1207584 (94a:60052)
- [19] A. A. Kirillov, *Lectures on the orbit method*, Novosibirsk, 2002; English transl., Grad. Stud. in Math., vol. 64, Amer. Math. Soc., Providence, RI, 2004. MR2069175 (2005c:22001)
- [20] V. F. Molchanov, *Canonical representations and overgroups for hyperboloids*, Funktsional. Anal. i Prilozhen. **39** (2005), no. 4, 48–61; English transl., Funct. Anal. Appl. **39** (2005), no. 4, 284–295. MR2197513 (2007d:22018)
- [21] Yu. A. Neretin, *Categories of symmetries and infinite-dimensional groups*, Editorial URSS, Moscow, 1998; English transl., London Math. Soc. Monogr. (N. S.), vol. 16, Clarendon Press, Oxford Univ. Press, New York, 1996. MR1418863 (98b:22003)
- [22] A. M. Perelomov, *Generalized coherent states and their applications*, Nauka, Moscow, 1987; English transl., Springer-Verlag, Berlin, 1986. MR0899736 (88i:22029); MR0858831 (87m:22035)
- [23] N. E. Hart, *Geometric quantization in action*, Math. Appl. (East European Ser.), vol. 8, D. Reidel Publ. Co., Dordrecht–Boston, MA, 1983. MR0689710 (84f:58053)
- [24] H. Araki, *Factorizable representation of current algebra. Non commutative extension of the Lévy–Kinchin formula and cohomology of a solvable group with values in a Hilbert space*, Publ. Res. Inst. Math. Sci. **5** (1969/1970), 361–422. MR0263326 (41:7931)
- [25] P. Delorme, *1-cohomologie des représentations unitaires des groupes de Lie semi-simples et résolubles. Produits tensoriels continus de représentations*, Bull. Soc. Math. France **105** (1977), 281–336. MR0578893 (58:28272)
- [26] ———, *Irreductibilité de certaines représentations de G^X* , J. Funct. Anal. **30** (1978), 36–47. MR0513476 (80a:22022)
- [27] I. M. Gel'fand, M. I. Graev, and A. M. Vershik, *Representations of the group of smooth mappings of a manifold X into a compact Lie group*, Compositio Math. **35** (1977), 299–334. MR0578652 (58:28257)
- [28] ———, *Representations of the group of functions taking values in a compact Lie group*, Compositio Math. **42** (1981), 217–243. MR0596877 (83g:22002)
- [29] ———, *Models of representations of current groups*, Representations of Lie Groups and Lie Algebras (Budapest, 1971), Akad. Kiadó, Budapest, 1985, pp. 121–179. MR0829048 (88a:22032)
- [30] M. I. Graev and A. M. Vershik, *The basic representation of the current group $O(n, 1)^X$ in the L^2 space over the generalized Lebesgue measure*, Indag. Math. **16** (2005), 499–529. MR2313636 (2008g:22028)
- [31] A. Guichardet, *Symmetric Hilbert spaces and related topics*, Lecture Notes in Math., vol. 261, Springer-Verlag, Berlin–New York, 1972. MR0493402 (58:12416)
- [32] R. S. Ismagilov, *Representations of infinite-dimensional groups*, Transl. Math. Monogr., vol. 152, Amer. Math. Soc., Providence, RI, 1996. MR1393939 (97g:22014)

- [33] K. R. Parthasarathy and K. Schmidt, *Positive definite kernels, continuous tensor products, and central limit theorems of probability theory*, Lecture Notes in Math., vol. 272, Springer-Verlag, Berlin–New York, 1972. MR0622034 (58:29849)
- [34] N. Tsilevich, A. Vershik, and M. Yor, *An infinite-dimensional analogue of the Lebesgue measure and distinguished properties of the gamma process*, J. Funct. Anal. **185** (2001), 274–296. MR1853759 (2002g:46071)
- [35] A. Vershik, *Invariant measures for the continual Cartan subgroup*, J. Funct. Anal. **255** (2008), 2661–2682. MR2473272 (2010f:37002)
- [36] ———, *The behavior of the Laplace transform of the invariant measure on the hypersphere of high dimension*, J. Fixed Point Theory Appl. **3** (2008), 317–329. MR2434451 (2009g:58016)
- [37] A. Vershik and B. Tsirelson, *Examples of nonlinear continuous tensor products of measure spaces and non-Fock factorizations*, Rev. Math. Phys. **10** (1998), 81–145. MR1606855 (99c:60085)
- [38] J. A. Wolf, *Unitary representations of maximal parabolic subgroups of the classical groups*, Mem. Amer. Math. Soc. **8** (1976), no. 180, 193 pp. MR0444847 (56:3194)

ST. PETERSBURG BRANCH, STEKLOV MATHEMATICAL INSTITUTE, RUSSIAN ACADEMY OF SCIENCES,
 FONTANKA 27, ST. PETERSBURG 191023, RUSSIA
E-mail address: `vershik@pdmi.ras.ru`

SCIENTIFIC-RESEARCH INSTITUTE FOR SYSTEM STUDIES, RUSSIAN ACADEMY OF SCIENCES, NAKHI-
 MOVSKII PROSPEKT 36–1, MOSCOW 117218, RUSSIA
E-mail address: `graev_36@mtu-net.ru`

Received 29/JUN/2010

Translated by N. V. TSILEVICH