PARABOLIC EQUATIONS WITH VARIABLY PARTIALLY VMO COEFFICIENTS

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ABSTRACT. The $W_p^{1,2}$ -solvability of second-order parabolic equations in nondivergence form in the whole space is proved for $p \in (1, \infty)$. The leading coefficients are assumed to be measurable in one spatial direction and have vanishing mean oscillation (VMO) in the orthogonal directions and the time variable in each small parabolic cylinder with direction allowed to depend on the cylinder. This extends a recent result by Krylov for elliptic equations. The novelty in the current paper is that the restriction $p > 2$ is removed.

§1. INTRODUCTION

We consider the $W_p^{1,2}$ -solvability of parabolic equations in nondivergence form:

$$(1.1) \quad Pu - \lambda u = f,$$

where $\lambda \geq 0$ is a constant, $f \in L_p$, and

$$Pu = -u_t + a^{ij}D_{ij}u + b^jD_ju + cu$$

is a uniformly nondegenerate parabolic operator with bounded coefficients.

The L_p -theory of second-order parabolic and elliptic equations has been studied extensively under various regularity assumptions on the coefficients. For equations with uniformly continuous leading coefficients, the solvability has been known for a long time; see, for example, Agmon, Douglis, and Nirenberg [3] and Ladyzhenskaya, Solonnikov and Ural'tseva [20]. With VMO coefficients, the solvability theorems of nondivergence form equations were established in the early 1990s by Chiarenza, Frasca, and Longo [5, 6] and by Bramanti and Cerutti [4]. The main technical tool in these papers was the theory of singular integrals, in particular, certain estimates from the Calderón–Zygmund theorem and the Coifman–Rochberg–Weiss commutator theorem. For divergence form equations with VMO coefficients, we refer the reader to Di Fazio [10], Byun and Wang [2], Acerbi and Mingione [1], and the references therein.

On the contrary, the theory of elliptic and parabolic equations with *partially* VMO coefficients was quite new and originated by Kim and Krylov in [14] and [15]. In [14], the authors established the W_p^2 -solvability of elliptic equations in nondivergence form under the assumption that the coefficients a^{ij} are measurable with respect to x^1 and VMO with respect to the remaining variables. This result was extended to parabolic equations by the same authors in [15]. The arguments in [14] and [15] were based on the method previously developed by Krylov in [17] and [18], where the author gave a unified approach to studying the L_p -solvability of both divergence and nondivergence form parabolic equations with leading coefficients measurable in the time variable and

2010 *Mathematics Subject Classification*. Primary 35K15, 35R05.

Key words and phrases. Second-order equations, vanishing mean oscillation, partially VMO coefficients, Sobolev spaces.

Partially supported by NSF Grant DMS-0635607 from IAS and NSF Grant DMS-0800129.

VMO in the spatial variables. Unlike the arguments in [5, 6, 4], the proofs in [17] and [18] rely mainly on pointwise estimates of sharp functions of spatial derivatives of solutions.¹ We also mention that the results in [15, 17, 18] were improved by Kim in [11, 12, 13], where most leading coefficients are measurable in the time variable and one spatial variable, and VMO in the other variables. In Dong and Krylov [9], the W_p^2 -solvability was obtained for equations with leading coefficients measurable in *two* spatial variables and VMO in the others, when $p > 2$ is sufficiently close to 2.

The W_p^1 and $\mathcal{H}_p^1$ -solvability of elliptic and parabolic equations in divergence form with partially BMO coefficients was obtained in the recent papers [8] and [7].

The result in [14] was generalized very recently by Krylov [19] to nondivergence form elliptic equations with *variably* partially VMO coefficients. More precisely, the leading coefficients were assumed to be measurable in one direction and VMO in the orthogonal directions in each small ball with direction depending on the ball. Roughly speaking, the main idea in [19] was to use in a localized way a pointwise sharp function estimate of a portion of the Hessian D^2u , proved in [14], and apply a generalized version of the Fefferman–Stein theorem on sharp functions. Rather than looking for as weak regularity assumptions on coefficients as possible, we emphasize that one of the main points of these papers was to put forth a new technique, which turned out to be more powerful than singular integrals in several occasions.

As was pointed out in [19], a restriction of the result there (and also in papers mentioned above regarding nondivergence equations with partially VMO coefficients) was that p must be greater than 2. This restriction is due to the following reason. The sharp function estimate in [14] was deduced from the W_2^2 -solvability of equations with a^{ij} depending only on x^1 , which was obtained by using the method of the Fourier transformation. In its turn, the right-hand side of the estimate contains maximal functions of q th power of D^2u for some $q > 2$. Therefore, to apply the Fefferman–Stein theorem and the Hardy–Littlewood maximal function theorem one requires $p > q > 2$.

It is natural to ask if we still have the W_p^2 -solvability under the same assumptions of the coefficients when p goes below 2. In this paper, we give a positive answer to this question. We obtain the W_p^2 -solvability of nondivergence form elliptic equations for any $p \in (1, \infty)$ (stated in Theorem 2.5), under the same assumptions as in [19], i.e., the leading coefficients are variably partially VMO. In fact, we shall establish the corresponding $W_p^{1,2}$ -solvability of parabolic equations, from which the result of elliptic equations follows in a standard way. Here the leading coefficients of parabolic equations are assumed to be measurable in one spatial direction and have vanishing mean oscillation in the orthogonal directions and the time variable in each small parabolic cylinder, with the direction allowed to depend on the cylinder. For the precise statement of this result, see Theorem 2.2. This theorem generalizes the result in [15] as well. Additionally, in the spirit of [18], we also obtain the solvability of parabolic equations in the mixed-norm Sobolev spaces $W_{q,p}^{1,2}$ for $q \geq p$ under the same assumptions (see Theorem 5.1).

The proofs follow the approach in [19] with two new ideas outlined below to get around the obstacles. First, in order to go below 2, we shall establish for any $p \in (1, \infty)$ the $W_p^{1,2}$ -solvability of nondivergence form parabolic equations with a^{ij} depending only on x^1 (stated in Theorem 3.4). For this purpose, we observe that with this structure the equation can be rewritten into a divergence form after a suitable change of variables. This enables us to apply a result recently proved in [7] for divergence form parabolic equations with partially VMO coefficients. Second, to get a sharp function estimate we need to bound the Hölder norms of a portion of D^2u when u satisfies the homogeneous

¹See also [22] for a straightforward treatment of nondivergence form elliptic and parabolic equations in the Morrey space without using any singular integrals.

equation (see Theorem 3.7). The proof of a similar estimate in Lemma 4.8 in [14], which was used crucially in [19], is only for $p = 2$. To this end, we use a novel bootstrap argument with the aid of Theorem 3.4 and an embedding type estimate. We combine Theorems 3.4 and 3.7 to prove Theorem 2.2 by applying the aforementioned generalized Fefferman–Stein theorem obtained in [19].

A possible application of the results is the solvability of elliptic and parabolic equations in domains, with the Dirichlet or Neumann boundary condition; see [14] and [8] for various discussions. We believe that the results in this paper enable us to obtain the solvability of equations with partially VMO coefficients for which the measurable direction is close to the normal direction near the boundary. We do not intend to pursue this in the current paper.

A brief outline of the paper: in the next section, we introduce the notation and state the main results. §3 contains a few preliminary estimates, including Theorems 3.4 and 3.7, which are the main ingredients of the proof. We finish the proof of the $W_p^{1,2}$ -solvability in §4 by combining the results of §3. Finally, we state and prove the $W_{q,p}^{1,2}$ -solvability of parabolic equations in the last section.

§2. NOTATION AND MAIN RESULTS

We begin the section by introducing some notation. Let $d \geq 2$ be an integer. A typical point in $\mathbb{R}^d$ is denoted by $x = (x^1, x^2, \dots, x^d) = (x^1, x')$. By $Du = (D_i u)$ and $D^2 u = (D_{ij} u)$ we mean the gradient and the Hessian matrix of u . On many occasions we need to take these objects relative to only part of variables. We also use the following notation:

$$D_{x'} u = u_{x'}, \quad D_{x^1 x'} u = u_{x^1 x'}, \quad D_{xx'} u = u_{xx'}.$$

Throughout the paper, we always assume that $1 < p, q < \infty$ unless explicitly specified otherwise. Writing $N(d, p, \dots)$, we mean that N is a constant depending only on the prescribed quantities $d, p, \dots$. For a (matrix-valued) function $f(t, x)$ in $\mathbb{R}^{d+1}$, we set

$$(f)_{\mathcal{D}} = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} f(t, x) dx dt = \oint_{\mathcal{D}} f(t, x) dx dt,$$

where $\mathcal{D}$ is an open subset in $\mathbb{R}^{d+1}$ and $|\mathcal{D}|$ is the $(d+1)$ -dimensional Lebesgue measure of $\mathcal{D}$. For $-\infty \leq S < T \leq \infty$ and $\Omega \subset \mathbb{R}^d$, we set

$$L_{q,p}((S, T) \times \Omega) = L_q((S, T), L_p(\Omega)),$$

i.e., $f(t, x) \in L_{q,p}((S, T) \times \Omega)$ if

$$\|f\|_{L_{q,p}((S,T) \times \Omega)} = \left(\int_S^T \left(\int_{\Omega} |f(t, x)|^p dx \right)^{q/p} dt \right)^{1/q} < \infty.$$

Denote

$$\begin{aligned} L_p((S, T) \times \Omega) &= L_{p,p}((S, T) \times \Omega), \\ W_{q,p}^{1,2}((S, T) \times \Omega) &= \{u : u, u_t, Du, D^2 u \in L_{q,p}((S, T) \times \Omega)\}, \\ W_p^{1,2}((S, T) \times \Omega) &= W_{p,p}^{1,2}((S, T) \times \Omega). \end{aligned}$$

We also use the abbreviations $L_p = L_p(\mathbb{R}^{d+1})$, $W_p^{1,2} = W_p^{1,2}(\mathbb{R}^{d+1})$, etc. For $\alpha \in (0, 1]$, we denote the $C^{\alpha/2, \alpha}$ -seminorm by

$$[u]_{C^{\alpha/2, \alpha}((S,T) \times \Omega)} := \sup_{\substack{(t,x), (s,y) \in (S,T) \times \Omega \\ (t,x) \neq (s,y)}} \frac{|u(t, x) - u(s, y)|}{|t - s|^{\alpha/2} + |x - y|^{\alpha}},$$

and the $C^{\alpha/2, \alpha}$ -norm by

$$\|u\|_{C^{\alpha/2, \alpha}((S, T) \times \Omega)} := [u]_{C^{\alpha/2, \alpha}((S, T) \times \Omega)} + \|u\|_{L_\infty((S, T) \times \Omega)}.$$

For any $T \in (-\infty, \infty]$ and $\Omega \subset \mathbb{R}^d$, we denote

$$\mathbb{R}_T = (-\infty, T), \quad \mathbb{R}_T^{d+1} = \mathbb{R}_T \times \mathbb{R}^d.$$

Let

$$B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}, \quad Q_r(t, x) = (t - r^2, t) \times B_r(x).$$

Set $B_r = B_r(0)$, $Q_r = Q_r(0, 0)$, and let $|B_r|$ and $|Q_r|$ be the volume of B_r and Q_r respectively. Let $\mathcal{Q} = \{Q_r(t, x) : (t, x) \in \mathbb{R}^{d+1}, r \in (0, \infty)\}$. For a function g defined on $\mathbb{R}^{d+1}$, we denote its (parabolic) maximal and sharp function, respectively, by

$$\begin{aligned} \mathbb{M}g(t, x) &= \sup_{Q \in \mathcal{Q}: (t, x) \in Q} \int_Q |g(s, y)| dy ds, \\ g^\#(t, x) &= \sup_{Q \in \mathcal{Q}: (t, x) \in Q} \int_Q |g(s, y) - (g)_Q| dy ds. \end{aligned}$$

Next we state our assumptions on the coefficients precisely. We assume that all the coefficients are bounded and measurable, and the a^{jk} are symmetric uniformly elliptic, i.e.,

$$(2.1) \quad \begin{aligned} |b^j| &\leq K, \quad |c| \leq K, \\ \delta|\xi|^2 &\leq a^{jk}\xi^j\xi^k \leq \delta^{-1}|\xi|^2. \end{aligned}$$

Denote by $\mathcal{A}$ the set of symmetric $(d \times d)$ -matrix-valued measurable functions $\bar{a} = (\bar{a}^{ij}(y^1))$ of one spatial variable such that (2.1) holds with $\bar{a}$ in place of a .

Let Ψ be the set of $C^{1,1}$ -diffeomorphisms $\psi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that the mappings ψ and $\phi = \psi^{-1}$ satisfy

$$(2.2) \quad |D\psi| + |D^2\psi| \leq \delta^{-1}, \quad |D\phi| + |D^2\phi| \leq \delta^{-1}.$$

Assumption 2.1 (γ). *There exists a positive constant $R_0 \in (0, 1]$ such that, for any parabolic cylinder Q of radius less than R_0 , one can find $\bar{a} \in \mathcal{A}$ and $\psi = (\psi^1, \dots, \psi^d) \in \Psi$ such that*

$$(2.3) \quad \int_Q |a(t, x) - \bar{a}(\psi^1(x))| dx dt \leq \gamma|Q|.$$

Next we state the main result of the paper.

Theorem 2.2. *For any $p \in (1, \infty)$, there exists $\gamma = \gamma(d, \delta, p) > 0$ such that, under Assumption 2.1 (γ), for any $T \in (-\infty, +\infty]$ the following holds.*

i) *For any $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$,*

$$\lambda\|u\|_{L_p(\mathbb{R}_T^{d+1})} + \sqrt{\lambda}\|Du\|_{L_p(\mathbb{R}_T^{d+1})} + \|D^2u\|_{L_p(\mathbb{R}_T^{d+1})} + \|u_t\|_{L_p(\mathbb{R}_T^{d+1})} \leq N\|Pu - \lambda u\|_{L_p(\mathbb{R}_T^{d+1})},$$

provided that $\lambda \geq \lambda_0$, where $\lambda_0 \geq 0$ and N depend only on d, δ, p, K , and R_0 .

ii) *For any $\lambda > \lambda_0$ and $f \in L_p(\mathbb{R}_T^{d+1})$, there exists a unique solution $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$ of equation (1.1) in $\mathbb{R}_T^{d+1}$.*

Theorem 2.2 yields the following solvability result of the initial value problem for parabolic equations.

Theorem 2.3. *For any $p \in (1, \infty)$, there exists a $\gamma = \gamma(d, \delta, p) > 0$ such that under Assumption 2.1 (γ) for any $T \in (0, \infty)$ the following is true. For any $f \in L_p((0, T) \times \mathbb{R}^d)$, there exists a unique solution $u \in W_p^{1,2}((0, T) \times \mathbb{R}^d)$ of the initial value problem*

$$Pu = f \text{ in } (0, T) \times \mathbb{R}^d, \quad u(0, \cdot) = 0.$$

Moreover, we have

$$\|u\|_{W_p^{1,2}((0,T)\times\mathbb{R}^d)} \leq N\|f\|_{L_p((0,T)\times\mathbb{R}^d)},$$

where N depends only on d, δ, p, K, T , and R_0 .

Indeed, by considering $v := ue^{-(\lambda_0+1)t}$ instead of u , the operator P becomes $P - (\lambda_0 + 1)I$. Now we extend f to be zero for $t < 0$ and solve the equation for v in $\mathbb{R}_T^{d+1}$ using Theorem 2.2. By uniqueness of solutions in $W_p^{1,2}(\mathbb{R}_0^{d+1})$, automatically $v = 0$ when $t \leq 0$. Thus, $u = ve^{(\lambda_0+1)t}$ solves the original initial value problem and the estimate also follows. The uniqueness of u is equivalent to the uniqueness of v . We extend v to be zero for $t < 0$. Then v satisfies $Pv - (\lambda_0 + 1)v = \chi_{t>0}e^{-(\lambda_0+1)t}f$ in $\mathbb{R}_T^{d+1}$, and so the uniqueness follows from Theorem 2.2.

In the case where all the coefficients are time-independent, we also consider the W_p^2 -solvability of elliptic equations in nondivergence form:

$$(2.4) \quad Lu - \lambda u = f,$$

where

$$Lu = a^{jk}D_{jk}u + b^jD_ju + cu.$$

Assumption 2.4 (γ). *There exists a positive constant $R_0 \in (0, 1]$ such that for any ball B of radius less than R_0 one can find $\bar{a} \in \mathcal{A}$ and $\psi = (\psi_1, \dots, \psi_d) \in \Psi$ such that*

$$\int_B |a(x) - \bar{a}(\psi^1(x))| dx \leq \gamma|B|.$$

The following W_p^2 -solvability theorem for elliptic equations is an immediate corollary to Theorem 2.2, which generalizes Theorem 1.4 of [19] by dropping the condition $p > 2$.

Theorem 2.5. *For any $p \in (1, \infty)$, there exists $\gamma = \gamma(d, \delta, p) > 0$ such that under Assumption 2.4 (γ) the following holds true.*

i) *For any $u \in W_p^2(\mathbb{R}^d)$,*

$$(2.5) \quad \lambda\|u\|_{L_p(\mathbb{R}^d)} + \sqrt{\lambda}\|Du\|_{L_p(\mathbb{R}^d)} + \|D^2u\|_{L_p(\mathbb{R}^d)} \leq N\|Lu - \lambda u\|_{L_p(\mathbb{R}^d)},$$

provided that $\lambda \geq \lambda_0$, where $\lambda_0 \geq 0$ and N depend only on d, δ, p, K , and R_0 .

ii) *For any $\lambda > \lambda_0$ and $f \in L_p(\mathbb{R}^d)$, there exists a unique solution $u \in W_p^2(\mathbb{R}^d)$ of equation (2.4) in $\mathbb{R}^d$.*

Proof. We use the idea that solutions to elliptic equations can be viewed as steady state solutions to parabolic equations. By the method of continuity, it suffices to verify the *a priori* estimate. Let λ_0 be the constant in Theorem 2.2. Take $\lambda \geq \lambda_0$. Let η be a smooth function on $\mathbb{R}$ supported on $[-2, 2]$, and let $\eta(t) = 1$ on $[-1, 1]$. For a fixed $T > 0$, denote $v(t, x) = u(x)\eta(t/T)$. Then $v \in W_p^{1,2}$ satisfies

$$(P - \lambda)v(t, x) = -T^{-1}\eta'(t/T)u(x) + \eta(t/T)(L - \lambda)u(x).$$

From Theorem 2.2, we have

$$\lambda\|v\|_{L_p} + \sqrt{\lambda}\|Dv\|_{L_p} + \|D^2v\|_{L_p} \leq \|\eta(t/T)f - T^{-1}\eta'(t/T)u\|_{L_p}.$$

Combined with the triangle inequality, this gives

$$\begin{aligned} & \|\eta(t/T)\|_{L_p(\mathbb{R})}(\lambda\|u\|_{L_p(\mathbb{R}^d)} + \sqrt{\lambda}\|Du\|_{L_p(\mathbb{R}^d)} + \|D^2u\|_{L_p(\mathbb{R}^d)}) \\ & \leq N\|\eta(t/T)\|_{L_p(\mathbb{R})}\|f\|_{L_p(\mathbb{R}^d)} + NT^{-1}\|\eta'(t/T)\|_{L_p(\mathbb{R})}\|u\|_{L_p(\mathbb{R}^d)}. \end{aligned}$$

Dividing both sides by $\|\eta(t/T)\|_{L_p(\mathbb{R})}$ and letting $T \rightarrow \infty$ yield (2.5). The theorem is proved. $\square$

The theorems above generalize several previously known results on nondivergence form equations with discontinuous coefficients to a large extent. In particular, one can get the solvability of equations to which the results in [14] and [15] are not applicable. We refer the reader to an interesting example given in the end of the Introduction in [19]. Here we give another example for $d = 2$.

In the polar coordinates (ρ, θ) , consider the union of two graphs $\rho = e^{-\theta/\varepsilon}$ and $\rho = e^{-(\theta+\pi)/\varepsilon}$ for some $\varepsilon > 0$. This curve divides the plane into two connected components. It becomes flat if ε is small. Let a^{ij} be different constants in these two components. If $\varepsilon = \varepsilon(\gamma)$ is sufficiently small, it is easy to check that the a^{ij} satisfy Assumption 2.4 (γ). But there exists no common diffeomorphism ψ that works for all small balls centered at the origin. Therefore, the result in [14] is not applicable in this case even if one uses a partition of the unity.

Remark 2.6. In [9], the W_p^2 -solvability was obtained for equations with leading coefficients measurable in two spatial variables and VMO in the others, when $p > 2$ is sufficiently close to 2. An interesting problem is whether that result can be extended to equations with variably partially VMO coefficients.

§3. NEW ESTIMATES FOR NONDIVERGENCE FORM EQUATIONS

First, we recall the following embedding type result and a parabolic version of Poincaré's inequality (see, for example, [18] or [20]).

Lemma 3.1. *Let $q \geq 1$, and let*

$$\frac{1}{q} < \frac{1}{d+2} + \frac{1}{p}.$$

Then there is a constant $N = N(d, p, q, r, R)$ such that for any $u \in W_{q, \text{loc}}^{1,2}$ and $0 < r < R < \infty$ we have

$$\|u\|_{L_p(Q_r)} + \|Du\|_{L_p(Q_r)} \leq N \|u\|_{W_q^{1,2}(Q_R)}.$$

Lemma 3.2 (Lemma 5.4 [18]). *Let $p \in [1, \infty)$. Then there is a constant $N = N(d, p)$ such that for any $r \in (0, \infty)$ and $u \in C_{\text{loc}}^\infty(\mathbb{R}^{d+1})$ we have*

$$\begin{aligned} \int_{Q_r} |Du(t, x) - (Du)_{Q_r}|^p dx dt &\leq Nr^p \int_{Q_r} (|D^2u(t, x)|^p + |u_t|^p) dx dt, \\ \int_{Q_r} |u(t, x) - (u)_{Q_r} - x^j (D_j u)_{Q_r}|^p dx dt &\leq Nr^{2p} \int_{Q_r} (|D^2u(t, x)|^p + |u_t|^p) dx dt. \end{aligned}$$

Let

$$P_0 u = -u_t + a^{ij} D_{ij} u, \quad \tilde{P}_0 u = -u_t + D_i (a^{ij} D_j u),$$

where $a^{ij} = a^{ij}(x^1)$. The proposition below is a special case of Corollary 5.5 in [7].

Proposition 3.3 (Corollary 5.5 in [7]). *Suppose $p \in (1, \infty)$, $u \in C_0^\infty$, and $f, g \in L_p$. Then there exists a constant $N > 0$, depending only on d, p and δ , such that we have*

$$\lambda \|u\|_{L_p} + \sqrt{\lambda} \|Du\|_{L_p} \leq N \sqrt{\lambda} \|g\|_{L_p} + N \|f\|_{L_p},$$

provided that $\lambda \geq 0$ and $\tilde{P}_0 u - \lambda u = f + \text{div } g$. In particular, if $\lambda = 0$ and $f = 0$, then

$$\|Du\|_{L_p} \leq N \|g\|_{L_p}.$$

Our proof relies on the following solvability theorem.

Theorem 3.4. *Let $p \in (1, \infty)$ and $T \in (-\infty, \infty]$. Then for any $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$ and $\lambda \geq 0$, we have*

$$(3.1) \quad \lambda \|u\|_{L_p(\mathbb{R}_T^{d+1})} + \sqrt{\lambda} \|Du\|_{L_p(\mathbb{R}_T^{d+1})} + \|D^2u\|_{L_p(\mathbb{R}_T^{d+1})} + \|u_t\|_{L_p(\mathbb{R}_T^{d+1})} \leq N \|P_0u - \lambda u\|_{L_p(\mathbb{R}_T^{d+1})},$$

where $N = N(d, p, \delta) > 0$. Moreover, for any $f \in L_p(\mathbb{R}_T^{d+1})$ and $\lambda > 0$ there is a unique $u \in W_p^{1,2}(\mathbb{R}_T^{d+1})$ solving $P_0u - \lambda u = f$ in $\mathbb{R}_T^{d+1}$.

Proof. First we assume $T = \infty$. By the method of continuity, it suffices to prove the a priori estimate (3.1) for $u \in C_0^\infty$. Let

$$(3.2) \quad f = P_0u - \lambda u.$$

The idea is to use the estimate for the corresponding divergence form operator. We make a change of variables:

$$y^1 = \varphi(x^1) := \int_0^{x^1} \frac{1}{a^{11}(s)} ds, \quad y^j = x^j, \quad j \geq 2.$$

It is easily seen that φ is a bi-Lipschitz function and

$$\delta \leq y^1/x^1 \leq \delta^{-1}, \quad D_{y^1} = a^{11}(x^1)D_{x^1}.$$

Denote

$$v(t, y^1, y') = u(t, \varphi^{-1}(y^1), y'), \quad \tilde{a}^{ij}(y^1) = a^{ij}(\varphi^{-1}(y^1)), \\ \tilde{f}(t, y) = f(t, \varphi^{-1}(y^1), y').$$

We define a divergence form operator $\tilde{P}_0$ by

$$\tilde{P}_0v = -v_t + D_1 \left(\frac{1}{\tilde{a}^{11}} D_1 v \right) + \sum_{j=2}^d D_j \left(\frac{\tilde{a}^{1j} + \tilde{a}^{j1}}{\tilde{a}^{11}} D_1 v \right) + \sum_{i,j=2}^d D_j (\tilde{a}^{ij} D_i v).$$

Clearly, v satisfies

$$\tilde{P}_0v - \lambda v = \tilde{f}$$

in $\mathbb{R}^{d+1}$, and $\tilde{P}_0$ is uniformly nondegenerate. By Proposition 3.3, we have

$$\lambda \|v\|_{L_p} + \sqrt{\lambda} \|Dv\|_{L_p} \leq N \|\tilde{f}\|_{L_p}.$$

Therefore,

$$(3.3) \quad \lambda \|u\|_{L_p} + \sqrt{\lambda} \|Du\|_{L_p} \leq N \|f\|_{L_p}.$$

Next we estimate D^2u . Notice that, for each $k = 2, \dots, d$, D_kv satisfies

$$\tilde{P}_0(D_kv) - \lambda D_kv = D_k \tilde{f}.$$

Again by using Proposition 3.3, we get

$$\|D_{yy^k}v\|_{L_p} \leq N \|\tilde{f}\|_{L_p},$$

which implies

$$(3.4) \quad \|D_{xx'}u\|_{L_p} \leq N \|f\|_{L_p}.$$

Finally, to estimate D_1^2u , we return to the equation in the original coordinates. From (3.2), we see that $w := D_1u$ satisfies

$$-w_t + D_1(a^{11}D_1w) + \Delta_{d-1}w - \lambda w = D_1f + \sum_{ij>1} D_1((\delta_{ij} - a^{ij})D_{ij}u).$$

We use Proposition 3.3 again to get

$$(3.5) \quad \|D_1^2 u\|_{L_p} \leq \|Dw\|_{L_p} \leq N\|f\|_{L_p} + N \sum_{ij>1} \|D_{ij}u\|_{L_p}.$$

Combining (3.3), (3.4), and (3.5) yields (3.1) by bearing in mind that

$$u_t = a^{ij} D_{ij}u - \lambda u - f.$$

For general $T \in (-\infty, \infty]$, we use the fact that $u = w$ for $t < T$, where $w \in W_p^{1,2}$ solves

$$P_0 w - \lambda w = \chi_{t < T}(P_0 u - \lambda u).$$

The theorem is proved. $\square$

As a direct consequence of Theorem 3.4, arguing as after Theorem 2.3, we obtain the following.

Corollary 3.5. *Let $p \in (1, \infty)$ and $T \in (0, \infty)$. Then for any $f \in L_p((0, T) \times \mathbb{R}^d)$ and $\lambda \geq 0$ there is a unique $u \in W_p^{1,2}((0, T) \times \mathbb{R}^d)$ solving $P_0 u - \lambda u = f$ in $(0, T) \times \mathbb{R}^d$ and $u(0, \cdot) = 0$. Moreover, we have*

$$\begin{aligned} \lambda \|u\|_{L_p((0,T) \times \mathbb{R}^d)} + \sqrt{\lambda} \|Du\|_{L_p((0,T) \times \mathbb{R}^d)} + \|D^2 u\|_{L_p((0,T) \times \mathbb{R}^d)} + \|u_t\|_{L_p((0,T) \times \mathbb{R}^d)} \\ \leq N \|P_0 u - \lambda u\|_{L_p((0,T) \times \mathbb{R}^d)}, \end{aligned}$$

where $N = N(d, p, \delta) > 0$.

Corollary 3.6. *Let $p \in (1, \infty)$ and $u \in W_{p,\text{loc}}^{1,2}$. Then for any $(t_0, x_0) \in \mathbb{R}^{d+1}$ and $0 < r < R < \infty$,*

$$\begin{aligned} \|u_t\|_{L_p(Q_r(t_0, x_0))} + \|D^2 u\|_{L_p(Q_r(t_0, x_0))} \\ \leq N \left(\|P_0 u\|_{L_p(Q_R(t_0, x_0))} + \|u\|_{L_p(Q_R(t_0, x_0))} + \|Du\|_{L_p(Q_R(t_0, x_0))} \right), \end{aligned}$$

where $N = N(d, \delta, r, R)$.

Proof. It suffices to apply Theorem 3.4 to $u\eta$, where η is a suitable smooth cutoff function. $\square$

Now we state and prove the following useful theorem, which gives a Hölder estimate of $D_{xx'}u$ for ‘harmonic’ u . This is analogous to Lemma 4.8 in [14], which was used crucially in [19]. However, it turns out that the proof of Lemma 4.8 in [14] (see also the proof of Lemma 4.7 in [15]) only works when $p = 2$ and employs the special structure of scalar equations. Instead, here we use a bootstrap argument which works for any $p \in (1, \infty)$. This approach also makes it possible to extend the results to systems and higher order equations.

Theorem 3.7. *Let $p \in (1, \infty)$ and $u \in C_0^\infty$. Assume $P_0 u = 0$ in $Q_2(t_0, x_0)$ for some $(t_0, x_0) \in \mathbb{R}^{d+1}$. Then for any $\alpha \in (0, 1)$, we have*

$$(3.6) \quad \|D_{xx'}u\|_{C^{\alpha/2, \alpha}(Q_1(t_0, x_0))} \leq N(\|u\|_{L_p(Q_2(t_0, x_0))} + \|Du\|_{L_p(Q_2(t_0, x_0))})$$

for some constant $N = N(d, \delta, \alpha, p)$.

Proof. Without loss of generality, we may assume $(t_0, x_0) = (0, 0)$. We prove the lemma by a bootstrap argument. Take an increasing sequence $p_j \in (1, \infty)$, $j = 0, 1, \dots, m$, where m depends only on d and α , such that

$$p_0 = p, \quad p_m > \frac{d+2}{1-\alpha}, \quad \frac{1}{p_{j+1}} < \frac{1}{p_j} - \frac{1}{d+2}.$$

Also, we take a sequence of shrinking cylinders

$$Q^{(j)}, \quad j = 0, 1, 2, \dots, 3m + 2,$$

such that $Q^{(0)} = Q_2$ and $Q^{(3m+2)} = Q_1$.

By Corollary 3.6, we have

$$(3.7) \quad \|u\|_{W_{p_0}^{1,2}(Q^{(1)})} \leq N(\|u\|_{L_p(Q^{(0)})} + \|Du\|_{L_p(Q^{(0)})}) := NI.$$

Then from Lemma 3.1 it follows that

$$(3.8) \quad \|u\|_{L_{p_1}(Q^{(2)})} + \|Du\|_{L_{p_1}(Q^{(2)})} \leq N\|u\|_{W_{p_0}^{1,2}(Q^{(1)})} \leq NI.$$

Since $D_{x'}u$ satisfies the same equation in Q_2 , by (3.8) and (3.7) we have

$$(3.9) \quad \|D_{x'}u\|_{L_{p_1}(Q^{(3)})} \leq N(\|D_{x'}u\|_{L_{p_0}(Q^{(1)})} + \|D_{xx'}u\|_{L_{p_0}(Q^{(1)})}) \leq NI.$$

Now we write

$$-u_t + a^{11}D_{11}u + \Delta_{d-1}u = \Delta_{d-1}u - \sum_{ij>1} a^{ij}D_{ij}u$$

and use Corollary 3.6 with p_1 in place of p . Together with (3.8) and (3.9), this gives

$$(3.10) \quad \|u\|_{W_{p_1}^{1,2}(Q^{(4)})} \leq N(\|u\|_{L_{p_1}(Q^{(3)})} + \|Du\|_{L_{p_1}(Q^{(3)})} + \|D_{xx'}u\|_{L_{p_1}(Q^{(3)})}) \leq NI.$$

We can iterate (3.8)–(3.10) along with the increasing sequence p_j and shrinking cylinders $Q^{(j)}$. After m steps, we reach

$$\|u\|_{W_{p_m}^{1,2}(Q^{(3m+1)})} \leq NI.$$

Again, since $D_{x'}u$ satisfies the same equation in Q_2 , we have

$$(3.11) \quad \|D_{x'}u\|_{W_{p_m}^{1,2}(Q^{(3m+2)})} \leq N(\|D_{x'}u\|_{L_p(Q^{(1)})} + \|D_{xx'}u\|_{L_p(Q^{(1)})}) \leq NI.$$

Finally, due to the classical Sobolev embedding theorem of parabolic type, (3.11) gives

$$\|D_{xx'}u\|_{C^{\alpha/2,\alpha}(Q_1)} \leq NI.$$

The theorem is proved. $\square$

Corollary 3.8. *Let $p \in (1, \infty)$ and $u \in C_0^\infty$. Suppose $P_0u = 0$ in $Q_2(t_0, x_0)$ for some $(t_0, x_0) \in \mathbb{R}^{d+1}$. Then for any $\alpha \in (0, 1)$, we have*

$$(3.12) \quad [D_{xx'}u]_{C^{\alpha/2,\alpha}(Q_1(t_0, x_0))} \leq N(\|u_t\|_{L_p(Q_2(t_0, x_0))} + \|D^2u\|_{L_p(Q_2(t_0, x_0))})$$

for some constant $N = N(d, \delta, \alpha, p)$.

Proof. Again we assume $(t_0, x_0) = (0, 0)$. We notice that $v := u - (u)_{Q_2} - x^j(D_ju)_{Q_2}$ satisfies the same equation as u in Q_2 . Therefore, by (3.6),

$$(3.13) \quad \begin{aligned} [D_{xx'}u]_{C^{\alpha/2,\alpha}(Q_1)} &= [D_{xx'}v]_{C^{\alpha/2,\alpha}(Q_1)} \\ &\leq N\|u - (u)_{Q_2} - x^j(D_ju)_{Q_2}\|_{L_p(Q_2)} + N\|Du - (Du)_{Q_2}\|_{L_p(Q_2)}. \end{aligned}$$

By Lemma 3.2, we see that the right-hand side of (3.13) is less than the right-hand side of (3.12). The corollary is proved. $\square$

Corollary 3.9. *Suppose $p \in (1, \infty)$, $\kappa \geq 2$, $r \in (0, \infty)$, and $u \in C_0^\infty$. Assume that $P_0u = 0$ in $Q_{\kappa r}(t_0, x_0)$ for some $(t_0, x_0) \in \mathbb{R}^{d+1}$. Then for any $\alpha \in (0, 1)$, we have*

$$(|D_{xx'}u - (D_{xx'}u)_{Q_r(t_0, x_0)}|^p)_{Q_r(t_0, x_0)} \leq N\kappa^{-\alpha p} (|u_t|^p + |D^2u|^p)_{Q_{\kappa r}(t_0, x_0)}$$

for some constant $N = N(d, \delta, \alpha, p)$.

Proof. We may assume $(t_0, x_0) = (0, 0)$. By a scaling argument, it suffices to consider the case where $r = 2/\kappa \leq 1$. By Corollary 3.8, we get

$$\begin{aligned} (|D_{xx'}u - (D_{xx'}u)_{Q_{2/\kappa}}|^p)_{Q_{2/\kappa}} &\leq N\kappa^{-p\alpha} [D_{xx'}u]_{C^{\alpha/2, \alpha}(Q_{2/\kappa})}^p \\ &\leq N\kappa^{-p\alpha} [D_{xx'}u]_{C^{\alpha/2, \alpha}(Q_1)}^p \leq N\kappa^{-p\alpha} (|u_t|^p + |D^2u|^p)_{Q_2}. \end{aligned} \quad \square$$

We have a similar estimate for u_t .

Lemma 3.10. *Suppose $p \in (1, \infty)$, $\kappa \geq 2$, and $r > 0$. Assume that $u \in C_0^\infty$ and $P_0u = 0$ in $Q_{\kappa r}(t_0, x_0)$. Then there exist constants $N = N(d, p, \delta)$ and $\alpha = \alpha(d, p, \delta) \in (0, 1]$ such that*

$$(|u_t - (u_t)_{Q_r(t_0, x_0)}|^p)_{Q_r(t_0, x_0)} \leq N\kappa^{-p\alpha} (|u_t|^p)_{Q_{\kappa r}(t_0, x_0)}.$$

Proof. We may assume $(t_0, x_0) = (0, 0)$. By scaling, we reduce the general situation to that where $r = 1$. Since $Lu_t = 0$ in $Q_{\kappa r}$, by the classical results [16, Lemma 4.2.4] and [21, Theorem 7.21],

$$\operatorname{osc}_{Q_{1/\kappa}} u_t \leq N\kappa^{-\alpha} \|u_t\|_{L_p(Q_1)}$$

with α and N as in the statement. Scaling this estimate shows that

$$\operatorname{osc}_{Q_1} u_t \leq N\kappa^{-\alpha} (|u_t|^p)_{Q_\kappa}^{1/p}.$$

It only remains to observe that

$$(|u_t - (u_t)_{Q_1}|^p)_{Q_1} \leq N(\operatorname{osc}_{Q_1} u_t)^p.$$

The lemma is proved. $\square$

Proposition 3.11. *Suppose $p \in (1, \infty)$, $\kappa \geq 4$, $r > 0$, and $u \in C_0^\infty$. Let α be the constant in Lemma 3.10. Then for any $(t_0, x_0) \in \mathbb{R}^{d+1}$, we have*

$$\begin{aligned} &(|D_{xx'}u - (D_{xx'}u)_{Q_r(t_0, x_0)}|^p)_{Q_r(t_0, x_0)} + (|u_t - (u_t)_{Q_r(t_0, x_0)}|^p)_{Q_r(t_0, x_0)} \\ &\leq N\kappa^{d+2} (|P_0u|^p)_{Q_{\kappa r}(t_0, x_0)} + N\kappa^{-\alpha p} (|u_t|^p + |D^2u|^p)_{Q_{\kappa r}(t_0, x_0)} \end{aligned}$$

for some constant $N = N(d, \delta, p)$.

Proof. The proposition follows from Corollary 3.5, Corollary 3.9, and Lemma 3.10 by decomposing u into a ‘harmonic’ part and a ‘nonharmonic’ part; see, for instance, the proof of Proposition 5.3 below or the proof of Theorem 4.5 in [9]. $\square$

We finish this section by recalling a generalized version of the Fefferman–Stein theorem. To state this theorem, let

$$\mathbb{C}_n = \{C_n(i_0, i_1, \dots, i_d), i_0, \dots, i_d \in \mathbb{Z}\}, \quad n \in \mathbb{Z},$$

be the filtration of partitions given by parabolic dyadic cubes, where

$$C_n(i_0, i_1, \dots, i_d) = [i_0 2^{-2n}, (i_0+1) 2^{-2n}) \times [i_1 2^{-n}, (i_1+1) 2^{-n}) \times \dots \times [i_d 2^{-n}, (i_d+1) 2^{-n}).$$

Theorem 3.12 (Theorem 2.7 in [19]). *Let $p \in (1, \infty)$, $U, V, H \in L_1$. Assume that $V \geq |U|$, $H \geq 0$, and that for any $n \in \mathbb{Z}$ and $C \in \mathbb{C}_n$ there exists a measurable function U^C on C such that $|U| \leq U^C \leq V$ on C and*

$$\int_C |U^C - (U^C)_C| dx dt \leq \int_C H dx dt.$$

Then we have

$$\|U\|_{L_p}^p \leq N \|H\|_{L_p} \|V\|_{L_p}^{p-1},$$

provided that $H, V \in L_p$.

§4. PROOF OF THEOREM 2.2

With the preparations in the previous section, we complete the proof of Theorem 2.2 by adapting the idea in [19].

Lemma 4.1. *Suppose $p \in (1, \infty)$, $\hat{a} \in \mathcal{A}$, and $\psi \in \Psi$. Let α be the constant in Lemma 3.10. Denote*

$$\hat{P}u(t, x) = -u_t(t, x) + \hat{a}^{kl}(y^1)D_{y^k}\phi^i(y)D_{y^i}\phi^j(y)D_{ij}u(t, x),$$

where $y = \psi(x)$ and $\phi = \psi^{-1}$. Then there exist constants $N = N(d, p, \delta)$ and $\nu = \nu(d, \delta) \geq 1$ such that, for any $\kappa \geq 4$, $r > 0$, and $u \in C_0^\infty$, we have

$$(4.1) \quad \sum_{ij>1} (|u_{ij} - (u_{ij})_{Q_r}|^p)_{Q_r} + (|u_t - (u_t)_{Q_r}|^p)_{Q_r} \\ \leq N\kappa^{d+2}(|\hat{P}u|^p + |Du|^p)_{Q_{\nu\kappa r}} + N\kappa^{-\alpha p}(|u_t|^p + |D^2u|^p)_{Q_{\nu\kappa r}},$$

where

$$(4.2) \quad u_{ij}(t, x) = (D_{y^i y^j} v)(t, \psi(x)), \quad v(t, y) = u(t, \phi(y)).$$

Proof. We deduce the lemma from Proposition 3.11 by following the proof of Lemma 3.2 in [19] with some modifications. For the sake of completeness, we give the details. By a translation of the coordinates, we may assume that $\phi(0) = \psi(0) = 0$. We define $f = \hat{P}u$ and $\hat{P}_y v = -v_t + \hat{a}^{kl}(y_1)D_{y^k y^l} v(t, y)$. Note that

$$\hat{P}_y v(t, y) = f(t, \phi(y)) + \hat{a}^{kl}D_{y^k y^l} \phi^i(D_{x^i} u)(t, \phi(y)).$$

By applying Proposition 3.11 to $\hat{P}_y$, for any $\rho > 0$, we get

$$(4.3) \quad (|D_{yy'} v - (D_{yy'} v)_{Q_\rho}|^p)_{Q_\rho} + (|v_t - (v_t)_{Q_\rho}|^p)_{Q_\rho} \\ \leq N\kappa^{d+2}(|\hat{P}_y v|^p)_{Q_{\kappa\rho}} + N\kappa^{-\alpha p}(|v_t|^p + |D_y^2 v|^p)_{Q_{\kappa\rho}}.$$

Bearing in mind that the scales in x and y coordinates are comparable, we can find constants N and ν , which depend only on d and δ , such that for any measurable function g we have

$$\int_{Q_\rho} |g(t, x)| dx dt \leq N \int_{Q_{\sqrt{\nu}\rho}} |g(t, \phi(y))| dy dt, \\ \int_{Q_\rho} |g(t, \phi(y))| dy dt \leq N \int_{Q_{\sqrt{\nu}\rho}} |g(t, x)| dx dt.$$

This and the triangle inequality imply

$$\int_{Q_r} |u_{ij} - (u_{ij})_{Q_r}|^p dx dt \\ \leq \int_{Q_r} \int_{Q_r} |u_{ij}(t_1, x_1) - u_{ij}(t_2, x_2)|^p dx_1 dx_2 dt_1 dt_2 \\ \leq N \int_{Q_{\sqrt{\nu}r}} \int_{Q_{\sqrt{\nu}r}} |D_{y^i y^j} v(t_1, y_1) - D_{y^i y^j} v(t_2, y_2)|^p dy_1 dy_2 dt_1 dt_2 \\ \leq N \int_{Q_{\sqrt{\nu}r}} |D_{y^i y^j} v(t, y) - (D_{y^i y^j} v)_{\sqrt{\nu}r}|^p dy dt.$$

Similarly,

$$\int_{Q_r} |u_t - (u_t)_{B_r}|^p dx dt \leq N \int_{Q_{\sqrt{\nu}r}} |v_t(t, y) - (v_t)_{Q_{\sqrt{\nu}r}}|^p dy dt.$$

Moreover, for $y = \psi(x)$ we obviously have

$$|D_y^2 v(y)| \leq N|D_x^2 u(x)| + N|D_x u(x)|, \quad |\hat{P}_y v(y)| \leq |\hat{P}u(x)| + N|D_x u(x)|.$$

By combining the estimates above, we easily deduce (4.1) from (4.3). $\square$

The next result can be viewed as an extension of Lemma 4.1.

Proposition 4.2. *Let $p \in (1, \infty)$, $\gamma > 0$, and let $\tau, \sigma \in (1, \infty)$ satisfy $1/\tau + 1/\sigma = 1$. Let α be the constant in Lemma 3.10 and $\nu = \nu(d, \delta) > 1$ the constant in Lemma 4.1. Assume $b^i = c = 0$ and $u \in C_0^\infty$. Then, under Assumption 2.1 (γ), for any $r \in (0, \infty)$ and $(t_0, x_0) \in \mathbb{R}^{d+1}$ there exists a diffeomorphism $\psi \in \Psi$ and a positive constant $N = N(d, p, \delta, \tau)$ such that for any $\kappa \geq 4$,*

$$(4.4) \quad \begin{aligned} & \sum_{ij>1} (|u_{ij} - (u_{ij})_{Q_r(t_0, x_0)}|^p)_{Q_r(t_0, x_0)} + (|u_t - (u_t)_{Q_r(t_0, x_0)}|^p)_{Q_r(t_0, x_0)} \\ & \leq N\kappa^{d+2} (|Pu|^p + |Du|^p)_{Q_{\nu\kappa r}(t_0, x_0)} + N\kappa^{d+2}\gamma^{1/\sigma} (|D^2 u|^{p\tau})_{Q_{\nu\kappa r}(t_0, x_0)}^{1/\tau} \\ & \quad + N\kappa^{-p\alpha} (|u_t|^p)_{Q_{\nu\kappa r}(t_0, x_0)} + N(\kappa^{d+2}R^p + \kappa^{-p\alpha}) (|D^2 u|^p)_{Q_{\nu\kappa r}(t_0, x_0)}, \end{aligned}$$

provided that u vanishes outside Q_R for some $R \in (0, R_0]$. Here the u_{ij} are defined in (4.2).

Proof. We fix $\kappa \geq 4$ and $r \in (0, \infty)$. Choose Q to be $Q_{\nu\kappa r}(t_0, x_0)$ if $\nu\kappa r < R$ and Q_R if $\nu\kappa r \geq R$. Let (t^*, x^*) be the center of Q . By Assumption 2.1 (γ) and (2.1), we can find $\psi \in \Psi$ and $\bar{a} = \bar{a}(s) \in \mathcal{A}$ satisfying (2.3). We set

$$\hat{a}^{kl}(s) = \bar{a}^{ij}(s)(D_i \psi^k)(x^*)(D_j \psi^l)(x^*), \quad y^* = \psi(x^*).$$

By Lemma 4.1 with a shift of the coordinates, for $ij > 1$ we have

$$(4.5) \quad \begin{aligned} & (|u_{ij} - (u_{ij})_{Q_r(t_0, x_0)}|^p)_{Q_r(t_0, x_0)} + (|u_t - (u_t)_{Q_r(t_0, x_0)}|^p)_{Q_r(t_0, x_0)} \\ & \leq N\kappa^{d+2} (|\hat{P}u|^p + |Du|^p)_{Q_{\nu\kappa r}(t_0, x_0)} + N\kappa^{-\alpha p} (|u_t|^p + |D^2 u|^p)_{Q_{\nu\kappa r}(t_0, x_0)}, \end{aligned}$$

where N depends only on d and δ . By the definition of $\hat{P}$,

$$(4.6) \quad \int_{Q_{\nu\kappa r}(t_0, x_0)} |\hat{P}u|^p dx dt \leq N \int_{Q_{\nu\kappa r}(t_0, x_0)} |Pu|^p dx dt + NI + NJ,$$

where

$$(4.7) \quad \begin{aligned} I &:= \int_{Q_{\nu\kappa r}(t_0, x_0) \cap Q_R} |((D_{y^k} \phi^i D_{y^l} \phi^j)(\psi) - (D_{y^k} \phi^i D_{y^l} \phi^j)(y^*)) \hat{a}^{kl}(\psi^1) D_{ij} u|^p dx dt \\ &\leq N \| (D_{y^k} \phi^i D_{y^l} \phi^j)(\psi) - (D_{y^k} \phi^i D_{y^l} \phi^j)(y^*) \|_{L^\infty(Q_R)}^p \int_{Q_{\nu\kappa r}(t_0, x_0)} |D_{ij} u|^p dx dt \\ &\leq NR^p \int_{Q_{\nu\kappa r}(t_0, x_0)} |D_{ij} u|^p dx dt, \end{aligned}$$

and

$$J := \int_{Q_{\nu\kappa r}(t_0, x_0) \cap Q_R} |(\bar{a}^{ij}(\psi^1) - a^{ij}(t, x)) D_{ij} u|^p dx dt.$$

We have used (2.2) in (4.7). By Hölder's inequality, we estimate J by

$$(4.8) \quad J \leq N J_1^{1/\sigma} J_2^{1/\tau},$$

where

$$J_1 = \sum_{i,j} \int_{Q_{\nu\kappa r}(t_0, x_0) \cap Q_R} |\bar{a}^{ij}(\psi^1) - a^{ij}|^{p\sigma} dx dt, \quad J_2 = \int_{Q_{\nu\kappa r}(t_0, x_0)} |D^2 u|^{p\tau} dx dt.$$

By Assumption 2.1 (γ) ,

$$J_1 \leq \sum_{i,j} \int_Q |\bar{a}^{ij}(\psi^1) - a^{ij}|^{p\sigma} dx dt \leq N\gamma|Q| \leq N(\nu\kappa r)^{d+2}\gamma.$$

Combined with (4.5)–(4.8), this yields (4.4). The theorem is proved. $\square$

From Proposition 4.2 we obtain the following corollary by simply using the triangle inequality.

Corollary 4.3. *Suppose $q \in (1, \infty)$, $\gamma > 0$, $\kappa \geq 4$, and let $\tau, \sigma \in (1, \infty)$ satisfy $1/\tau + 1/\sigma = 1$. Let α be the constant in Lemma 3.10. Suppose Assumption 2.1 (γ) is satisfied. Assume $b^i = c = 0$. Then for any $n \in \mathbb{Z}$ and $C \in \mathbb{C}_n$, there exists a diffeomorphism $\psi \in \Psi$ and a constant $N = N(d, \delta, q, \tau)$ such that, with any $u \in C_0^\infty$ vanishing outside Q_R for some $R \in (0, R_0]$, we have*

$$(4.9) \quad (|u_t - (u_t)_C|)_C + \sum_{ij>1} (|u_{ij} - (u_{ij})_C|)_C \leq N(\mathbf{g})_C,$$

where the $u_{ij}(t, x)$ are defined by (4.2) and

$$\begin{aligned} \mathbf{g} = & \kappa^{\frac{d+2}{q}} (\mathbb{M}(|Pu|^q))^{\frac{1}{q}} + \kappa^{\frac{d+2}{q}} (\mathbb{M}(|Du|^q))^{\frac{1}{q}} + \kappa^{\frac{d+2}{q}} \gamma^{\frac{1}{\sigma q}} (\mathbb{M}(|D^2u|^{q\tau}))^{\frac{1}{q\tau}} \\ & + \kappa^{-\alpha} (\mathbb{M}(|u_t|^q))^{\frac{1}{q}} + (\kappa^{\frac{d+2}{q}} R + \kappa^{-\alpha}) (\mathbb{M}(|D^2u|^q))^{\frac{1}{q}}. \end{aligned}$$

Moreover, we have

$$(4.10) \quad |u_t| + |D^2u| \leq N \sum_{ij>1} |u_{ij}| + N|u_t| + N|Du| + N|Pu|.$$

Proposition 4.4. *Let $p \in (1, \infty)$. Assume $b^i = c = 0$. Then there exist positive constants γ , N and $R \in (0, 1]$ depending only on d, p , and δ such that under Assumption 2.1 (γ) , for any $u \in C_0^\infty$ vanishing outside Q_{RR_0} we have*

$$(4.11) \quad \|u_t\|_{L_p} + \|D^2u\|_{L_p} \leq N\|Pu\|_{L_p} + N\|Du\|_{L_p}.$$

Proof. Set $f = Pu \in C_0^\infty$. Let $\gamma > 0$, $\kappa \geq 4$, and $R \in (0, 1]$ be constants to be specified later. Let $q = (1 + p)/2 \in (1, p)$ and $\tau = 2(1 + 2p)/(3 + 3p) > 1$ such that $p > q\tau$. We take $n \in \mathbb{Z}$, and $C \in \mathbb{C}_n$, and let $\psi \in \Psi$ be the diffeomorphism occurring in Corollary 4.3. Recall the definition of u_{ij} in (4.2). We set

$$U = |u_t| + |D^2u|, \quad U^C = |u_t| + \sum_{ij>1} |u_{ij}| + |Du| + |f|, \quad V = |u_t| + |D^2u| + |Du| + |f|.$$

From (4.10), we have $U \leq NU^C$. By using the triangle inequality and (4.9),

$$\begin{aligned} (|U^C - (U^C)_C|)_C & \leq 2(|u_t - (u_t)_C|)_C + 2 \sum_{ij>1} (|u_{ij} - (u_{ij})_C|) \\ & \quad + 2(|Du - (Du)_C|)_C + 2(|f - (f)_C|)_C \\ & \leq N(\mathbf{g} + |Du| + |f|)_C. \end{aligned}$$

Now Theorem 3.12 with $H = \mathbf{g} + |Du| + f$ yields

$$\|u_t\|_{L_p}^p + \|D^2u\|_{L_p}^p \leq N\|U\|_{L_p}^p \leq N\|H\|_{L_p}\|V\|_{L_p}^{p-1} \leq N(\varepsilon)\|H\|_{L_p}^p + \varepsilon\|V\|_{L_p}^p.$$

By taking a small $\varepsilon > 0$, it follows that

$$(4.12) \quad \|u_t\|_{L_p}^p + \|D^2u\|_{L_p}^p \leq N\|\mathbf{g}\|_{L_p}^p + N\|Du\|_{L_p}^p + N\|f\|_{L_p}^p.$$

We use the definition of g and the Hardy–Littlewood maximal function theorem (recall that $p > q\tau > q$) to deduce the following inequality from (4.12):

$$(4.13) \quad \begin{aligned} \|u_t\|_{L_p} + \|D^2u\|_{L_p} &\leq N\kappa^{\frac{d+2}{q}} \|Pu\|_{L_p} + N\kappa^{\frac{d+2}{q}} \|Du\|_{L_p} + N\kappa^{-\alpha} \|u_t\|_{L_p} \\ &\quad + N(\kappa^{\frac{d+2}{q}} \gamma^{\frac{1}{\sigma q}} + \kappa^{\frac{d+2}{q}} RR_0 + \kappa^{-\alpha}) \|D^2u\|_{L_p}. \end{aligned}$$

By choosing κ sufficiently large, and then taking γ and R in (4.13) sufficiently small and such that

$$N(\kappa^{\frac{d+2}{q}} \gamma^{\frac{1}{\sigma q}} + \kappa^{\frac{d+2}{q}} R + \kappa^{-\alpha}) \leq 1/2,$$

we come to (4.11). The theorem is proved. $\square$

Proof of Theorem 2.2. For $T = \infty$, the theorem follows in a by now standard way from Proposition 4.4 by using a partition of unity and an idea by S. Agmon. For general $T \in (-\infty, +\infty]$, we again use the argument at the end of the proof of Theorem 3.4. $\square$

§5. SOBOLEV SPACES WITH MIXED NORMS

In this section we consider parabolic equations in Sobolev spaces with mixed norms in the spirit of [18]. As pointed out in [18], the interest in results concerning equations in spaces with mixed Sobolev norms arises, for example, when one wants to get better regularity of traces of solutions for each time slide (see, for instance, [23, 24] and the references therein).

Our objective is to prove the following theorem, which generalizes Theorem 2.2.

Theorem 5.1. *For any $1 < p \leq q < \infty$ there exists $\gamma = \gamma(d, \delta, p, q) > 0$ such that, under Assumption 2.1 (γ), for any $T \in (-\infty, +\infty]$ the following is true.*

i) *For any $u \in W_{q,p}^{1,2}(\mathbb{R}_T^{d+1})$,*

$$\begin{aligned} \lambda \|u\|_{L_{q,p}(\mathbb{R}_T^{d+1})} + \sqrt{\lambda} \|Du\|_{L_{q,p}(\mathbb{R}_T^{d+1})} + \|D^2u\|_{L_{q,p}(\mathbb{R}_T^{d+1})} + \|u_t\|_{L_{q,p}(\mathbb{R}_T^{d+1})} \\ \leq N \|Pu - \lambda u\|_{L_{q,p}(\mathbb{R}_T^{d+1})}, \end{aligned}$$

provided that $\lambda \geq \lambda_0$, where $\lambda_0 \geq 0$ and N depend only on d, δ, p, q, K , and R_0 .

ii) *For any $\lambda > \lambda_0$ and $f \in L_{q,p}(\mathbb{R}_T^{d+1})$, there exists a unique solution $u \in W_{q,p}^{1,2}(\mathbb{R}_T^{d+1})$ of equation (1.1) in $\mathbb{R}_T^{d+1}$.*

Since the case where $p = q$ was covered in Theorem 2.2, in the sequel we assume $p < q$. We make a few preparations before the proof.

Lemma 5.2. *Let $p, q \in (1, \infty)$, $b = c = 0$. Then there exists a constant $\gamma_0 = \gamma_0(d, p, q, \delta) > 0$ such that, under Assumption 2.1 (γ_0), for any $r \in (0, R_0]$ and $u \in W_{q,\text{loc}}^{1,2}$ satisfying $Pu = 0$ in Q_{2r} we have $D^2u \in L_p(Q_r)$ and*

$$(|D^2u|^p)_{Q_r}^{1/p} \leq N(|D^2u|^q)_{Q_{2r}}^{1/q},$$

where N depends only on d, p, q , and δ .

Proof. First we assume $R_0 = 1$. In this case, the lemma is proved in Corollary 6.4 of [18] with the only difference that the coefficients a^{ij} are assumed to be in VMO_x in that paper. The proof of that in [18] employs the L_p -solvability of equations with VMO_x coefficients. Since the solvability is already established with coefficients satisfying Assumption 2.1 (γ_0) with γ_0 depending on d, p, q , and δ , we can just reproduce the proof with almost no change.

For general $R_0 \in (0, 1]$, we make a change of variables $(t, x) \rightarrow (R_0^2 t, R_0 x)$ and notice that the new coefficients satisfy Assumption 2.1 (γ_0) with R_0 replaced by 1. The lemma is proved. $\square$

The next proposition improves Proposition 4.2.

Proposition 5.3. *Let $p \in (1, \infty)$ and $\gamma > 0$. Let α be the constant in Lemma 3.10, $\nu = \nu(d, \delta) > 1$ the constant in Lemma 4.1, and $\gamma_0 = \gamma_0(d, p, 2p, \delta)$ the constants in Lemma 5.2. Assume $b^i = c = 0$ and $u \in C_0^\infty$. Then, under Assumption 2.1 (γ) with $\gamma \in (0, \gamma_0]$, the following is true. For any $R \in (0, R_0]$, $\kappa \geq 16$, $r \in (0, R\kappa^{-1}\nu^{-1}]$, and $(t_0, x_0) \in \mathbb{R}^{d+1}$, there exists a diffeomorphism $\psi \in \Psi$ such that*

$$(5.1) \quad \begin{aligned} & \sum_{ij>1} (|u_{ij} - (u_{ij})_{Q_r(t_0, x_0)}|^p)_{Q_r(t_0, x_0)} + (|u_t - (u_t)_{Q_r(t_0, x_0)}|^p)_{Q_r(t_0, x_0)} \\ & \leq N\kappa^{d+2} (|Pu|^p + |Du|^p)_{Q_{\nu\kappa r}(t_0, x_0)} + N\kappa^{-p\alpha} (|u_t|^p)_{Q_{\nu\kappa r}(t_0, x_0)} \\ & \quad + N(\kappa^{d+2}\gamma^{1/2} + \kappa^{d+2}R^p + \kappa^{-p\alpha}) (|D^2u|^p)_{Q_{\nu\kappa r}(t_0, x_0)}, \end{aligned}$$

where the u_{ij} are defined in (4.2) and $N = N(d, p, \delta) > 0$.

Proof. Without loss of generality, we assume that $(t_0, x_0) = (0, 0)$. We may also assume that the a^{ij} are infinitely differentiable by using standard mollifications if necessary.

Let $f := Pu$. Take a cutoff function $\eta \in C_0^\infty$ such that $\eta = 1$ on $Q_{\nu\kappa r/2}$ and $\eta = 0$ outside the closure of $Q_{\nu\kappa r} \cup (-Q_{\nu\kappa r})$. By Theorem 2.3, there exists a unique solution $w \in W_p^{1,2}((-1, 0) \times \mathbb{R}^d)$ of

$$Pw = f\eta \text{ on } (-1, 0) \times \mathbb{R}^d, \quad w(-1, \cdot) = 0.$$

By the classical theory, we have $w \in C^\infty((-1, 0) \times \mathbb{R}^d)$. Let $h = u - w$; this function is also smooth and satisfies

$$Ph = f(1 - \eta) \text{ on } (-1, 0) \times \mathbb{R}^d, \quad Ph = 0 \text{ on } Q_{\nu\kappa r/2}.$$

First, we estimate h . Choose Q to be $Q_{\nu\kappa r/4}$. By Assumption 2.1 (γ) and (2.1), we can find $\psi \in \Psi$ and $\bar{a} = \bar{a}(s) \in \mathcal{A}$ satisfying (2.3). By repeating the proof of Proposition 4.2 with h in place of u (recall $\kappa/4 \geq 4$), we get

$$(5.2) \quad \begin{aligned} & \sum_{ij>1} (|h_{ij} - (h_{ij})_{Q_r}|^p)_{Q_r} + (|h_t - (h_t)_{Q_r}|^p)_{Q_r} \\ & \leq N\kappa^{d+2} (|Dh|^p)_{Q_{\nu\kappa r/4}} + N\kappa^{d+2}\gamma^{1/2} (|D^2h|^{2p})_{Q_{\nu\kappa r/4}}^{1/2} \\ & \quad + N\kappa^{-p\alpha} (|h_t|^p)_{Q_{\nu\kappa r/4}} + N(\kappa^{d+2}R^p + \kappa^{-p\alpha}) (|D^2h|^p)_{Q_{\nu\kappa r/4}} \\ & \leq N\kappa^{d+2} (|Dh|^p)_{Q_{\nu\kappa r}} + N\kappa^{d+2}\gamma^{1/2} (|D^2h|^p)_{Q_{\nu\kappa r}} \\ & \quad + N\kappa^{-p\alpha} (|h_t|^p)_{Q_{\nu\kappa r}} + N(\kappa^{d+2}R^p + \kappa^{-p\alpha}) (|D^2h|^p)_{Q_{\nu\kappa r}}, \end{aligned}$$

where the h_{ij} are defined in the same way as u_{ij} . In the last inequality we used the relation $Ph = 0$ on $Q_{\nu\kappa r/2}$ and Lemma 5.2.

Next we estimate w . By Theorem 2.3, we have

$$\|w\|_{W_p^{1,2}((-1, 0) \times \mathbb{R}^d)} \leq N\|f\eta\|_{L_p((-1, 0) \times \mathbb{R}^d)} \leq N\|f\|_{L_p(Q_{\nu\kappa r})}.$$

Therefore,

$$(5.3) \quad (|w_t|^p + |Dw|^p + |D^2w|^p)_{Q_r} \leq N\kappa^{d+2} (|f|^p)_{Q_{\nu\kappa r}},$$

$$(5.4) \quad (|w_t|^p + |Dw|^p + |D^2w|^p)_{Q_{\nu\kappa r}} \leq N(|f|^p)_{Q_{\nu\kappa r}}.$$

Since $|w_{ij}| \leq N|Dw| + N|D^2w|$, combining (5.2), (5.3) and (5.4) and using the triangle inequality, we get

$$\begin{aligned}
& \sum_{ij>1} (|u_{ij} - (u_{ij})_{Q_r}|^p)_{Q_r} + (|u_t - (u_t)_{Q_r}|^p)_{Q_r} \\
& \leq N \sum_{ij>1} (|h_{ij} - (h_{ij})_{Q_r}|^p)_{Q_r} + N (|h_t - (h_t)_{Q_r}|^p)_{Q_r} \\
& \quad + N (|w_t|^p + |Dw|^p + |D^2w|^p)_{Q_r} \\
& \leq N \kappa^{d+2} (|Dh|^p)_{Q_{\nu\kappa r}} + N (\kappa^{d+2}\gamma^{1/2} + \kappa^{d+2}R^p + \kappa^{-p\alpha}) (|D^2h|^p)_{Q_{\nu\kappa r}} \\
& \quad + N \kappa^{-p\alpha} (|h_t|^p)_{Q_{\nu\kappa r}} + N \kappa^{d+2} (|f|^p)_{Q_{\nu\kappa r}} \\
& \leq N \kappa^{d+2} (|Du|^p)_{Q_{\nu\kappa r}} + N (\kappa^{d+2}\gamma^{1/2} + \kappa^{d+2}R^p + \kappa^{-p\alpha}) (|D^2u|^p)_{Q_{\nu\kappa r}} \\
& \quad + N \kappa^{-p\alpha} (|u_t|^p)_{Q_{\nu\kappa r}} + N \kappa^{d+2} (|f|^p)_{Q_{\nu\kappa r}},
\end{aligned}$$

which is exactly the right-hand side of (5.1). The theorem is proved. $\square$

The next corollary is deduced from Proposition 5.3.

Corollary 5.4. *Under the assumptions of Proposition 5.3, for any $R \in (0, R_0]$, $\kappa \geq 16$ and any interval $[S, T)$ such that $(T - S)^{1/2} =: r \in (0, R\kappa^{-1}\nu^{-1}]$, we have*

$$\begin{aligned}
(5.5) \quad & \int_{(S,T)} \int_{(S,T)} |\varphi(t) - \varphi(s)|^p dt ds \\
& \leq N \kappa^{d+2} \int_{(T-(\nu\kappa r)^2, T)} \zeta(t)^p dt \\
& \quad + N (\kappa^{d+2}R^p + \kappa^{-p\alpha} + \kappa^{d+2}\gamma^{1/2}) \int_{(T-(\nu\kappa r)^2, T)} \rho(t)^p dt,
\end{aligned}$$

where $N = N(d, p, \delta) > 0$, and

$$\begin{aligned}
\zeta(t)^p &= \int_{\mathbb{R}^d} |Pu(t, x)|^p + |Du(t, x)|^p dx, \\
\rho(t)^p &= \int_{\mathbb{R}^d} |D^2u(t, \cdot)|^p + |u_t(t, \cdot)|^p dx, \\
\varphi(t)^p &= \int_{\mathbb{R}^d} \int_{B_r(y)} |u_t(t, \cdot)|^p + \sum_{ij>1} |u_{ij}^{(y)}(t, \cdot)|^p dx dy.
\end{aligned}$$

Here for each $y \in \mathbb{R}^d$, the $u_{ij}^{(y)}$ are defined in (4.2) with $Q = Q_{\nu\kappa r/4}(T, y)$ and ψ in Assumption 2.1.

Proof. By the triangle inequality, the left-hand side of (5.5) is less than a constant N times

$$\begin{aligned}
& \int_{(S,T)} \int_{(S,T)} \int_{\mathbb{R}^d} \int_{B_r(y)} |u_t(t, \cdot) - u_t(s, \cdot)|^p + \sum_{ij>1} |u_{ij}^{(y)}(t, \cdot) - u_{ij}^{(y)}(s, \cdot)|^p dx dy dt ds \\
& \leq N \int_{\mathbb{R}^d} \sum_{ij>1} (|u_{ij}^{(y)} - (u_{ij}^{(y)})_{Q_r(T,y)}|^p)_{Q_r(T,y)} + (|u_t - (u_t)_{Q_r(T,y)}|^p)_{Q_r(T,y)} dy.
\end{aligned}$$

By Proposition 5.3, the last expression is less than N times

$$\int_{\mathbb{R}^d} \kappa^{d+2} (|Pu|^p + |Du|^p)_{Q_{\nu\kappa r}(T,y)} + (\kappa^{d+2}(R^p + \gamma^{\frac{1}{2}}) + \kappa^{-p\alpha}) (|D^2u|^p + |u_t|^p)_{Q_{\nu\kappa r}(T,y)} dy,$$

which is less than the right-hand side of (5.5). $\square$

Finally, we prove the following estimate, which implies Theorem 5.1 in the same way as Proposition 4.4 implies Theorem 2.2.

Proposition 5.5. *Let $1 < p < q < \infty$. Assume $b^i = c = 0$. Then there exist positive constants γ , N , and $R \in (0, 1]$ depending only on d, p, q , and δ such that under Assumption 2.1 (γ), for any $u \in C_0^\infty$ vanishing outside $(-R^4 R_0^2, 0) \times \mathbb{R}^d$, we have*

$$(5.6) \quad \|u_t\|_{L_{q,p}} + \|D^2 u\|_{L_{q,p}} \leq N \|Pu\|_{L_{q,p}} + N \|Du\|_{L_{q,p}}.$$

Proof. Set $f = Pu$. Recall the definitions of φ , ρ , and ζ in Corollary 5.4. Let $\gamma \in (0, \gamma_0]$, $\kappa \in [16, \infty)$, and $R \in (0, 1]$ be numbers to be chosen later, where $\gamma_0 = \gamma_0(d, p, 2p, \delta)$ is taken from Lemma 5.2. Assume u vanishes outside $(-R^4 R_0^2, 0) \times \mathbb{R}^d$. Denote $f = Pu$ and

$$\begin{aligned} \mathbf{g} = \mathbf{g}(t) &= \left(\kappa^{\frac{d+2}{p}} R_0 R + \kappa^{-\alpha} + \kappa^{\frac{d+2}{p}} \gamma^{\frac{1}{2p}} + (R\kappa)^{2(1-\frac{1}{p})} \right) (\mathbb{M}(\rho^p))^{\frac{1}{p}} \\ &\quad + N \left(\kappa^{\frac{d+2}{p}} + (R\kappa)^{2(1-\frac{1}{p})} \right) (\mathbb{M}(\zeta^p))^{\frac{1}{p}}. \end{aligned}$$

Let $\mathbb{C}_n$ be the filtration of partitions given by the dyadic intervals

$$\{[j2^{-2n}, (j+1)2^{-2n}), j \in \mathbb{Z}\}.$$

For any $C = [S, T) \in \mathbb{C}_n$, we set

$$\begin{aligned} U(t) &= \|u_t(t, \cdot)\|_{L_p(\mathbb{R}^d)} + \|D^2 u(t, \cdot)\|_{L_p(\mathbb{R}^d)}, \\ V(t) &= \|u_t(t, \cdot)\|_{L_p(\mathbb{R}^d)} + \|D^2 u(t, \cdot)\|_{L_p(\mathbb{R}^d)} + \|Du(t, \cdot)\|_{L_p(\mathbb{R}^d)} + \|f(t, \cdot)\|_{L_p(\mathbb{R}^d)}, \end{aligned}$$

and

$$U^C(t) = \begin{cases} \varphi(t) + \|Du(t, \cdot)\|_{L_p(\mathbb{R}^d)} + \|f(t, \cdot)\|_{L_p(\mathbb{R}^d)} & \text{if } 2^{-n} \leq RR_0 \kappa^{-1} \nu^{-1}, \\ V(t) & \text{otherwise.} \end{cases}$$

It is easily seen that $U \leq NU^C \leq NV$ in C . We claim

$$(5.7) \quad (U^C - (U^C)_C)_C \leq N (\mathbf{g} + \|Du(t, \cdot)\|_{L_p(\mathbb{R}^d)} + \|f(t, \cdot)\|_{L_p(\mathbb{R}^d)})_C.$$

Indeed, if $2^{-n} \leq R_0 R \kappa^{-1} \nu^{-1}$, by a shift of the origin we get (5.7) from Corollary 5.4. Otherwise, we have

$$\begin{aligned} (U^C - (U^C)_C)_C &\leq 2 \int_{(S,T)} \chi_{(-R_0^2 R^4, 0)} |V(t)| dt \\ &\leq 2 \left(\int_{(S,T)} \chi_{(-R_0^2 R^4, 0)} dt \right)^{1-1/p} \left(\int_{(S,T)} |V(t)|^p dt \right)^{1/p} \\ &\leq N (R\kappa)^{2(1-1/p)} (\mathbb{M}(\rho^p + \zeta^p)(t_0))^{1/p} \end{aligned}$$

for any $t_0 \in C$. This proves the claim.

Now by Theorem 3.12 with $d = 0$, q in place of p and

$$H(t) := \mathbf{g}(t) + \|Du(t, \cdot)\|_{L_p(\mathbb{R}^d)} + \|f(t, \cdot)\|_{L_p(\mathbb{R}^d)},$$

we get

$$\|u_t\|_{L_{q,p}}^q + \|D^2 u\|_{L_{q,p}}^q \leq N \|U\|_{L_q}^q \leq N \|H\|_{L_q} \|V\|_{L_q}^{q-1} \leq N(\varepsilon) \|H\|_{L_q}^q + \varepsilon \|V\|_{L_q}^q.$$

Taking a small $\varepsilon > 0$, we see that

$$(5.8) \quad \|u_t\|_{L_{q,p}}^q + \|D^2 u\|_{L_{q,p}}^q \leq N \|\mathbf{g}\|_{L_q}^q + N \|Du\|_{L_{q,p}}^q + N \|f\|_{L_{q,p}}^q.$$

We use the definition of g and the Hardy–Littlewood maximal function theorem (recall $p < q$) to deduce from (5.8) the estimate

$$(5.9) \quad \begin{aligned} \|u_t\|_{L_{q,p}} + \|D^2 u\|_{L_{q,p}} &\leq N\left(\kappa^{\frac{d+2}{p}} + (R\kappa)^{2(1-\frac{1}{p})}\right)(\|f\|_{L_{q,p}} + \|Du\|_{L_{q,p}}) \\ &\quad + N\left(\kappa^{\frac{d+2}{p}} R_0 R + \kappa^{-\alpha} + \kappa^{\frac{d+2}{p}} \gamma^{\frac{1}{2p}} + (R\kappa)^{2(1-\frac{1}{p})}\right)(\|u_t\|_{L_{q,p}} + \|D^2 u\|_{L_{q,p}}). \end{aligned}$$

By choosing κ sufficiently large, and then γ and R in (5.9) sufficiently small and such that

$$N\left(\kappa^{\frac{d+2}{p}} R + \kappa^{-\alpha} + \kappa^{\frac{d+2}{p}} \gamma^{\frac{1}{2p}} + (R\kappa)^{2(1-\frac{1}{p})}\right) \leq 1/2,$$

we come to (5.6). The theorem is proved. $\square$

ACKNOWLEDGMENT

The author is sincerely grateful to Nicolai V. Krylov for very helpful comments on the first draft of the article. The author also would like to thank Alexander Nazarov for his careful reading and useful remarks.

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Received 20/JUN/2010

Originally published in English