SPECTRUM OF THE LAPLACE–BELTRAMI OPERATOR FOR CERTAIN CONGRUENCE SUBGROUPS OF THE MODULAR GROUP

V. V. GOLOVCHANSKII AND M. N. SMOTROV

ABSTRACT. It is shown that, up to multiplicity, the spectra of automorphic Laplacians coincide in the case of the pairs of congruence subgroups $\{\Gamma_0(16N), \Gamma_0(8N)\}$ and $\{\Gamma_0(64N), \Gamma_0(32N)\}$ of the modular group, where N is an odd integer. A formula is obtained for the dimension of the subspaces of automorphic forms for the subgroups $\Gamma_0(16N)$ and $\Gamma_0(64N)$.

§1. INTRODUCTION

In contrast to the classical theory of regular forms, in the theory of real-analytic automorphic forms (i.e., of eigenfunctions of the automorphic Laplacian) we still have no formulas for the dimension of the subspaces of automorphic forms corresponding to a given eigenvalue of the Laplace–Beltrami operator. This is not surprising, because the regularity of forms is used substantially in the deduction of the dimension formula in the regular case.

The first computations of the discrete spectrum of the Laplace–Beltrami operator date back to the early 1970s. Since then, many publications (including those of the present authors) have been devoted to calculation of the eigenvalues of the Laplace–Beltrami operator for various Fuchsian subgroups of the 1st kind, originating from number-theoretic and statistical mechanics problems. It turned out that all calculated eigenvalues have multiplicity one; however, no theoretical results are available so far.

In this paper, we find two series of congruence subgroups in which the spectrum points and their multiplicities are uniquely determined by the spectrum points (and their multiplicities) for two congruence subgroups of a lower level. Namely, the following is true.

Theorem 1. *For any odd N , the discrete spectra (without multiplicities) of the automorphic Laplacians for the following pairs of congruence subgroups coincide:*

$$\Gamma_0(16N); \Gamma_0(8N) \quad \text{and} \quad \Gamma_0(64N); \Gamma_0(32N).$$

Moreover, let $\text{Dim}(2^s N, \lambda)$ be the dimension of the subspace of automorphic forms corresponding to a point λ of the discrete spectrum of the automorphic Laplacian for the congruence subgroup $\Gamma_0(2^s N)$; then

$$(1) \quad \text{Dim}(2^s N, \lambda) = 3 \text{Dim}(2^{s-1} N, \lambda) - 2 \text{Dim}(2^{s-2} N, \lambda) \quad \text{for } s = 4, 6.$$

In physical language, this theorem says that for $s = 4, 6$ the Riemann surfaces $\Gamma_0(2^s N) \setminus H$ and $\Gamma_0(2^{s-1} N) \setminus H$ sound the same, though they have different volumes

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and different signatures. The proof is based on the Selberg trace formula. We write the trace formula in the form

$$\sum_{j=1}^{\infty} h(\lambda_j) = F(h, \Gamma).$$

It turns out that

$$F(h; \Gamma_0(2^s N)) - 3F(h; \Gamma_0(2^{s-1} N)) + 2F(h; \Gamma_0(2^{s-2} N)) \equiv 0$$

if N is odd and $s = 4, 6$.

The Selberg zeta function $Z(s, \Gamma)$ is closely related to the Selberg trace formula and satisfies a relation similar to (1).

Theorem 2. *For any odd N we have*

$$(2) \quad Z(s, \Gamma_0(2^k N)) = \frac{Z^3(s, \Gamma_0(2^{k-1} N))}{Z^2(s, \Gamma_0(2^{k-2} N))} \quad \text{for } k = 4, 6.$$

It is natural to ask about the reason for which precisely these congruence subgroups have equal discrete spectra. In our opinion, the next informal arguments explain this coincidence.

Let

$$f(z) = \sum_{n \neq 0} \rho(n) f_n(\lambda, z)$$

be the Fourier series expansion of the parabolic form of the group $\Gamma_0(N)$. In the case of classical regular forms, in [2, Proposition 3.54] Shimura constructed new parabolic forms for the group $\Gamma_0(M)$ from parabolic forms for the group $\Gamma_0(N)$, where $N|M$. Obviously, that proposition remains valid also for real-analytic forms. Then the next 6 forms (5 old ones and Shimura's form) are automorphic relative to the group $\Gamma_0(16N)$:

$$f(z), f(2z), f(4z), f(8z), f(16z), u(z) = f_{\chi_4}(z)$$

(the form $f_{\chi_2}(z)$ can be expressed linearly in terms of the first 3 forms on the list), where

$$u(z) = f_{\chi_4}(z) = \sum_{n \neq 0} \chi_4(n) \rho(n) f_n(\lambda, z).$$

The first 3 forms on that list are automorphic relative to $\Gamma_0(2^2 N)$, and the first 4 of them are automorphic relative to $\Gamma_0(2^3 N)$. The above list of forms satisfies relation (1): $6 = 3 * 4 - 2 * 3$.

For the group $\Gamma_0(2^6 N)$ this list is supplemented by the following 6 forms (2 old ones and 4 Shimura's): $f(32z), f(64z), u(2z), u(4z), f_{\chi_8}(z), f_{\chi_{-8}}(z)$. The extended list of forms still satisfies (1): $12 = 3 * 8 - 2 * 6$. For the eigenforms (i.e., neither old nor Shimura's) of the groups $\Gamma_0(2^s N)$, $s = 1, 2, 3, 5$, relation (1) can be checked similarly.

Apparently, the congruence subgroups indicated in the theorem are unique, because, for the other groups $\Gamma_0(p^s N)$, if s grows by one, then (5) shows that the number of eigenfunctions is multiplied by p , while a given function can give rise to at most two new ones.

§2. PRELIMINARIES

We fix the model of the hyperbolic plane in the form of the upper half-plane $H = \{x + iy\}$, $y > 0$, with the metric $ds^2 = (dx^2 + dy^2)/y^2$. In this model, the orientation-preserving motions are the linear-fractional transformations with coefficient matrices in $\mathrm{SL}_2(R)$.

Let $\Gamma \subset \mathrm{SL}_2(R)$ be a discrete subgroup the fundamental domain of which with respect to the invariant measure $d\mu(z) = dx dy/y^2$ is finite (a Fuchsian group of the first kind).

Every such discrete subgroup generates a closed Riemann surface with finitely many pricked points, obtained by an appropriate pairwise gluing of the sides of the fundamental domain. The Laplace-Beltrami operator L on this Riemann surface acts in the Hilbert space of Γ -automorphic, square integrable functions $\mathcal{H} = L_2(\Gamma \backslash H, d\mu(z))$. The operator $-L$ is self-adjoint and nonnegative, and the space $\mathcal{H}$ splits into the sum of two orthogonal invariant subspaces $\mathcal{H} = \mathcal{H}^c \oplus \mathcal{H}^d$, where $\mathcal{H}^c$ is the continuous spectrum subspace of the operator $-L$, formed by the Eisenstein series extended analytically to the $\frac{1}{2}$ -line $s = 1/2 + it$, and $\mathcal{H}^d$ is the discrete spectrum subspace (the discrete spectrum itself is written as $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$). The discrete spectrum has finite multiplicity and has no finite accumulation points. Information about the Fuchsian groups and spectral theory of automorphic functions can be found in [3] and [4], respectively.

For a Fuchsian group Γ of the first kind, the Selberg trace formula and the logarithmic derivative of the Selberg zeta function can be written as follows (see [6]):

$$(3) \quad \sum_{\lambda_j} h\left(\sqrt{\lambda_j - 1/4}\right) = \frac{h(0)}{4} \operatorname{tr}(I - \Phi(1/2)) + 2 \sum_{\mathbf{N}} B(\mathbf{N}) g(\log \mathbf{N}) + \int_{-\infty}^{+\infty} R(t) h(t) dt,$$

$$(4) \quad \frac{Z'}{Z}(s) = 2 \sum_{\mathbf{N}} \frac{B(\mathbf{N})}{\mathbf{N}^{s-1/2}} \quad \text{for } \operatorname{Re}(s) > 1,$$

where $h(r)$ is an even function regular in the strip $|\operatorname{Im}(r)| \leq \frac{1}{2} + \varepsilon$, $\varepsilon > 0$, and satisfying $h(r) = O(r^{-2-\delta})$ with $\delta > 0$ as $|r| \rightarrow \infty$ in that strip, g is the Fourier transform of h , $R(t)$ is a smooth function, $\Phi(s)$ is the scattering matrix for the Eisenstein series, and summation on the right in (3) and (4) is over all different norms ($\mathbf{N} = (|\operatorname{tr}(P)| + \sqrt{\operatorname{tr}(P)^2 - 4})/2$) of the hyperbolic elements P of the group Γ . The coefficients $B(\mathbf{N})$ are given by

$$B(\mathbf{N}) = \frac{\log \mathbf{N}^{1/2}}{\mathbf{N}^{1/2} - \mathbf{N}^{-1/2}} \sum_{k|m} \frac{k}{m} \nu(\mathbf{N}^{\frac{k}{m}}).$$

Here $\nu(\mathbf{N})$ is the number of classes of primitive hyperbolic elements of Γ with a given norm $\mathbf{N}$, and m is determined uniquely by the relation $\mathbf{N} = \mathbf{N}_0^m$, where $\mathbf{N}_0$ is a primitive norm (not a power of some norm of the group Γ).

It is well known [4] that, for the congruence subgroups of the modular group $\operatorname{SL}_2(\mathbb{Z})$, the distribution function of the discrete spectrum is of the Weyl type:

$$(5) \quad \sum_{\lambda_j \leq T} 1 = \frac{\operatorname{Vol}(\Gamma \backslash H)}{4\pi} T + O(\sqrt{T} \log T),$$

so that the space of parabolic forms is infinite-dimensional. Recall that the congruence subgroup $\Gamma_0(N)$ is defined as follows:

$$\Gamma = \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

For the subgroups $\Gamma_0(N)$, the norms are positive powers of fundamental units of real quadratic fields. The explicit formulas obtained in [5] express the number of classes of primitive hyperbolic elements of the subgroup $\Gamma_0(N)$ in terms of the number of classes of binary indefinite quadratic forms; basing on these formulas, in [6] we obtained a relationship between $B(L, p^s N)$ and $B(L, N)$, where $(N, p) = 1$.

This relationship will be used in what follows; we present it below for the reader's convenience. We need some notation. Let L denote the trace of an element $P \in \Gamma_0(N)$.

We may assume that $L \geq 3$ because $-I \in \Gamma_0(N)$. Then the norms and the traces are in one-to-one correspondence, and the notations $B(L, N)$ and $\nu(L, N)$ are consistent. The representation $L^2 - 4 = Q^2 D$, where D is the fundamental discriminant of the quadratic form, is unique. We factor out the power of p involved in Q : $Q = p^\alpha Q_1$. In this notation, the following is true.

Theorem ([6]). *For any $L \geq 3$, we have*

$$B(L, p^s N) = \delta(p, s, \alpha) B(L, N),$$

and if (L, Q) is the fundamental solution of the Pellian equation $t^2 - Du^2 = 4$, then

$$\nu(L, p^s N) = \delta(p, s, \alpha) \nu(L, N),$$

where

a) *if $\left(\frac{D}{p}\right) = 1$ for $p \neq 2$ and $D \equiv 1 \pmod{8}$ the $p = 2$, then*

$$\delta(p, s, \alpha) = \begin{cases} 2p^\alpha & \text{if } 2\alpha < s; \\ p^{[s/2]} + p^{[(s-1)/2]} & \text{if } 2\alpha \geq s; \end{cases}$$

b) *if $\left(\frac{D}{p}\right) = -1$ for $p \neq 2$ and $D \equiv 5 \pmod{8}$ for $p = 2$, then*

$$\delta(p, s, \alpha) = \begin{cases} 0 & \text{if } 2\alpha < s; \\ (p^{[s/2]} + p^{[(s-1)/2]}) \frac{p^{\alpha+1} + p^\alpha - (p^{[(s+1)/2]} + p^{[s/2]})}{p^{\alpha+1} + p^\alpha - 2} & \text{if } 2\alpha \geq s; \end{cases}$$

c) *if $p \mid D$, then*

$$\delta(p, s, \alpha) = \begin{cases} 0 & \text{if } 2\alpha < s-1; \\ (p^{[s/2]} + p^{[(s-1)/2]}) \frac{2p^{\alpha+1} - (p^{[(s+1)/2]} + p^{[s/2]})}{2p^{\alpha+1} - 2} & \text{if } 2\alpha \geq s-1. \end{cases}$$

§3. PROOF OF THE THEOREMS

First, we show that the spectrum points with their multiplicities (except for $\lambda = 1/4$) are uniquely determined by the hyperbolic elements contribution to the trace formula (3). We take the function

$$h(t, \lambda, \alpha) = \exp(-\alpha(t^2 + 1/4 - \lambda)^2).$$

Obviously, it satisfies the conditions imposed on the function h in the trace formula. Upon such a choice, in the strip $|t| \leq 1/2$, the expression $h(t, \lambda, \alpha)$ converges as $\alpha \rightarrow \infty$ to the function equal to 1 at the points $\pm\sqrt{\lambda^2 - 1/4}$ and to 0 at the other points of the strip. Then, by (3), for $\lambda = \lambda_j \neq 1/4$, the limit passage as $\alpha \rightarrow \infty$ yields

$$\sum_{\lambda=\lambda_j \neq 1/4} 1 = 2 \lim_{\alpha \rightarrow \infty} \sum_{L=3}^{\infty} B(L, N) g(2 \ln((L + \sqrt{L^2 - 4})/2), \lambda_j, \alpha),$$

showing that the discrete spectrum of the operator, except for $\lambda_j = 1/4$, is fully determined by the contribution of the hyperbolic elements of Γ .

We form a linear combination of the trace formulas for the groups $\Gamma_0(2^s N)$, $\Gamma_0(2^{s-1} N)$, and $\Gamma_0(2^{s-2} N)$:

$$\left(\sum_{\lambda_j} h(\lambda_j, \lambda_j, \alpha) \right)_{2^s N} - 3 \left(\sum_{\lambda_j} h(\lambda_j, \lambda_j, \alpha) \right)_{2^{s-1} N} + 2 \left(\sum_{\lambda_j} h(\lambda_j, \lambda_j, \alpha) \right)_{2^{s-2} N}.$$

Letting $\alpha \rightarrow \infty$, we get

$$\begin{aligned}
 (6) \quad & \left(\sum_{\lambda=\lambda_j} 1 \right)_{2^s N} - 3 \left(\sum_{\lambda=\lambda_j} 1 \right)_{2^{s-1} N} + 2 \left(\sum_{\lambda=\lambda_j} 1 \right)_{2^{s-2} N} \\
 &= 2 \lim_{\alpha \rightarrow \infty} \sum_{L=3}^{\infty} (B(L, 2^s N) - 3B(L, 2^{s-1} N) \\
 &\quad + 2B(L, 2^{s-2} N)) \cdot g(2 \ln((L + \sqrt{L^2 - 4})/2), \lambda_j, \alpha).
 \end{aligned}$$

We shall show that

$$(7) \quad B(L, 2^s N) - 3B(L, 2^{s-1} N) + 2B(L, 2^{s-2} N) = 0$$

for $s = 4, 6$ and any L ; by (6), this implies the claim of Theorem 1 immediately. Applying the theorem proved in [6], we express $B(L, 2^k N)$ in terms of $B(L, N)$:

$$\begin{aligned}
 & B(L, 2^s N) - 3B(L, 2^{s-1} N) + 2B(L, 2^{s-2} N) \\
 &= (\delta(2, s, \alpha) - 3\delta(2, s-1, \alpha) + 2\delta(2, s-2, \alpha))B(L, N) \\
 &\equiv \Delta(s, \alpha)B(L, N).
 \end{aligned}$$

It remains to check that $\Delta(s, \alpha) = 0$ for $s = 4, 6$. We recall the definition of the fundamental discriminant D . We write $L^2 - 4 = Q^2 d$, where d is free of squares. Then if $d \equiv 1 \pmod{4}$, then $D = d$, and $D = 4d$ otherwise.

a) $D \equiv 1 \pmod{8}$. Then there are no L such that $\alpha < 3$. Indeed, if L odd, then $L^2 - 4 \equiv 5 \pmod{8}$. Suppose $L = 4k$; then $L^2 - 4 = 4(4k^2 - 1)$ and $4k^2 - 1 \equiv 3 \pmod{4}$, so that $4 \mid D$. Suppose $L = 4k + 2$; then $L^2 - 4 = 16k(k+1)$, whence $2^5 \mid (L^2 - 4)$, so that for $D \equiv 1 \pmod{4}$ we must have $2^6 \mid (L^2 - 4)$, and, consequently, $\alpha \geq 3$, implying $2\alpha \geq s$. By item a) of the theorem proved in [6], we obtain

$$\begin{aligned}
 \Delta(s, \alpha) &= (2^{\lfloor s/2 \rfloor} + 2^{\lfloor (s-1)/2 \rfloor}) - 3(2^{\lfloor (s-1)/2 \rfloor} + 2^{\lfloor (s-2)/2 \rfloor}) + 2(2^{\lfloor (s-2)/2 \rfloor} + 2^{\lfloor (s-3)/2 \rfloor}) \\
 &= \begin{cases} (2^2 + 2) - 3(2 + 2) + 2(2 + 1) = 0 & \text{if } s = 4; \\ (2^3 + 2^2) - 3(2^2 + 2^2) + 2(2^2 + 1) = 0 & \text{if } s = 6. \end{cases}
 \end{aligned}$$

b) $D \equiv 5 \pmod{8}$. In this case there are no L such that $\alpha = 1, 2$. Indeed, if L is odd, then $\alpha = 0$. If L is even, then $\alpha \geq 3$ by the above. Item b) of the theorem obtained in [6] shows that for $\alpha = 0$ all 3 summands are equal to zero, and if $2\alpha \geq s$, then

$$(8) \quad \delta(2, s, \alpha) = C_1(\alpha)(2^{\lfloor s/2 \rfloor} + 2^{\lfloor (s-1)/2 \rfloor}) + C_2(\alpha)(2^{\lfloor s/2 \rfloor} + 2^{\lfloor (s-1)/2 \rfloor})(2^{\lfloor (s+1)/2 \rfloor} + 2^{\lfloor s/2 \rfloor}).$$

As was checked in a) above, if we sum the first term on the right in (8) over s , $s-1$, and $s-2$, we get 0. Then

$$\Delta(s, \alpha) = C_2(\alpha) \begin{cases} (2^2 + 2)2^3 - 3 * 2^2(2^2 + 2) + 2(2 + 1)2^2 = 0 & \text{if } s = 4; \\ (2^3 + 2^2)2^4 - 3 * 2^3(2^3 + 2^2) + 2^3(2^2 + 2) = 0 & \text{if } s = 6. \end{cases}$$

c) $2 \mid D$. Item c) of the theorem in [6] shows that if $2\alpha < s-2$, then all 3 summands are zero, and if $2\alpha = s-2$, then $\delta(2, s, (s-2)/2) = 0$ and

$$\begin{aligned}
 \Delta(s, \alpha) &= -6 * 2^{s-2} \frac{2 - (2+1)/2}{2^{s/2} - 1} + 2^s(1 + 1/2) \frac{1 - 1/2}{2^{s/2} - 1} \\
 &= \begin{cases} (-3 * 2^2 + 3 * 2^2)/3 = 0 & \text{if } s = 4; \\ (-3 * 2^4 + 3 * 2^4)/7 = 0 & \text{if } s = 6. \end{cases}
 \end{aligned}$$

If $2\alpha > s-2$, then $\delta(2, s, \alpha)$ admits the representation (8) with other constants $C_3(\alpha)$ and $C_4(\alpha)$, whence $\Delta(s, \alpha) = 0$.

It remains to consider the case where $\lambda_j = 1/4$. Then, in view of (3), the extra term $\text{tr}(I - \Phi(1/2))/4$ arises on the right-hand side of (6). The trace $\text{tr}(I)$ is equal to the number ν_∞ of the nonequivalent parabolic vertices. For $\Gamma_0(N)$, the quantity ν_∞ was calculated in [2]. For us it is convenient to write ν_∞ in the form

$$\nu_\infty(N) = \prod_{i=1}^{\omega(N)} (p_i^{[s_i/2]} + p_i^{[(s_i-1)/2]}), \quad \text{where} \quad N = \prod_{i=1}^{\omega(N)} p_i^{s_i}.$$

Then $\nu_\infty(2^s N) = \nu_\infty(2^s) \nu_\infty(N)$, and $\nu_\infty(2^s) = 12, 8, 6, 4, 3$ for $s = 6, 5, 4, 3, 2$, respectively. This implies the formula

$$\nu_\infty(2^s N) = 3\nu_\infty(2^{s-1} N) - 2\nu_\infty(2^{s-2} N)$$

for $s = 4, 6$.

The trace of the scattering matrix $\Phi(1/2 + it)$ at $t = 0$ for the groups $\Gamma_0(N)$ was calculated in [7]. The function $\Phi(1/2, N)$ is multiplicative in N , and

$$\Phi(1/2, 2^a) = d(2^a) + d(2^{a-4}) + 2d(2^{a-6}),$$

where $d(n)$ is the number of divisors function. We have $\Phi(1/2, 2^s) = 12, 8, 6, 4, 3$ for $s = 6, 5, 4, 3, 2$, respectively. It is easy to check that

$$\Phi(1/2, 2^s N) = 3\Phi(1/2, 2^{s-1} N) - 2\Phi(1/2, 2^{s-2} N)$$

for $s = 4, 6$. Thus, Theorem 1 is proved completely.

Using (4) and identity (7) proved above, we see that

$$\frac{Z'}{Z}(s, \Gamma_0(2^k N)) = 3 \frac{Z'}{Z}(s, \Gamma_0(2^{k-1} N)) - 2 \frac{Z'}{Z}(s, \Gamma_0(2^{k-2} N))$$

if $\text{Re}(s) > 1$ and $k = 4, 6$. Now to get (2) it suffices to integrate and exponentiate. Since the Selberg zeta function is analytic in s , (2) is true for every s .

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KHABAROVSK DEPARTMENT, INSTITUTE OF APPLIED MATHEMATICS, DZERZHINSKIĬ STREET 54, Khabarovsk 680000, RUSSIA

E-mail address: `gsm@iam.khv.ru`

KHABAROVSK DEPARTMENT, INSTITUTE OF APPLIED MATHEMATICS, DZERZHINSKIĬ STREET 54, Khabarovsk 680000, RUSSIA

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