OVERGROUPS OF $E(m, R) \otimes E(n, R)$ I. LEVELS AND NORMALIZERS

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ABSTRACT. The study of subgroups H such that

$$E(m, R) \otimes E(n, R) \leq H \leq G = \mathrm{GL}(mn, R)$$

is started, provided that the ring R is commutative and $m, n \geq 3$. The principal results of this part can be summarized as follows. The group $\mathrm{GL}_m \otimes \mathrm{GL}_n$ is described by equations, and it is proved that the elementary subgroup $E(m, R) \otimes E(n, R)$ is normal in $(\mathrm{GL}_m \otimes \mathrm{GL}_n)(R)$. Moreover, when $m \neq n$, the normalizers of all three subgroups $E(m, R) \otimes e$, $e \otimes E(n, R)$, and $E(m, R) \otimes E(n, R)$ in $\mathrm{GL}(mn, R)$ coincide with $(\mathrm{GL}_m \otimes \mathrm{GL}_n)(R)$. With each such intermediate subgroup H , a uniquely defined level (A, B, C) is associated where A, B, C are ideals in R such that $mA, A^2 \leq B \leq A$ and $nA, A^2 \leq C \leq A$. Conversely, a level determines a perfect intermediate subgroup $\mathrm{EE}(m, n, R, A, B, C)$. It is shown that each intermediate subgroup contains a unique largest subgroup of this type. Next, the normalizer $N_G(\mathrm{EE}(m, n, R, A))$ of these perfect intermediate subgroups is calculated completely in the crucial case, where $A = B = C$. The standard answer to the above problem can now be stated as follows. Every such intermediate subgroup H is contained in the normalizer $N_G(\mathrm{EE}(m, n, R, A, B, C))$. In the special case where $n \geq m + 2$, such a standard description will be established in the second part of the present work.

FOREWORD

In the present paper we initiate the study of the subgroups H of the general linear group $\mathrm{GL}(n, R)$ over a commutative ring R that contain the tensor product of elementary linear groups,

$$E(n_1, R) \otimes \cdots \otimes E(n_t, R) \leq H \leq \mathrm{GL}(n, R),$$

where $n = n_1 \dots n_t$.

Recall that $E(n, R)$ denotes the elementary subgroup of $\mathrm{GL}(n, R)$ generated by all elementary transvections (the basic notation is recalled in §§ 2–4), whereas tensor products should be interpreted as tensor products of *linear* groups. In other words, the tensor product $H \otimes G$ consists of all *Kronecker* products $x \otimes y$, where $x \in H$, $y \in G$. Initially, we expected the ultimate objective of the upcoming series of papers would be a complete description of such overgroups when all n_i are greater than or equal to 3.

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The research of the second author was started in the framework of the RFBR projects 09-01-00878 “Overgroups of reductive groups in algebraic groups over rings” and 09-01-90304 “Structure theory of classical and algebraic groups”. Apart from that, at the initial stage of the work he was supported by EPSRC EP/D03695X/1 (first grant scheme of Roozbeh Hazrat) and SFB-701 at the Uni. Bielefeld, and at the final stage he was supported by the RFBR projects 08-01-00756, 09-01-00762, 09-01-00784, 09-01-91333, and 10-01-90016. The third author acknowledges support of the RFBR project 10-01-92651 “Higher composition laws, algebraic K -theory, and exceptional groups”.

However, in the meantime we discovered that to bring such a program to a completion is a formidable technical challenge. It soon became clear that as a first step one has to completely settle the special case where $t = 2$. Next, it turned out that even in this special case it is highly nontrivial even to simply *state* a right answer. Over rings, fresh difficulties crop up, *not in the least* apparent in the field case, considered by Shang Zhi Li. This is where one has to start: locate these new difficulties and state the correct answer. As of today, our immediate goal is somewhat less assuming, classification of subgroups

$$E(m, R) \otimes E(n, R) \leq H \leq \mathrm{GL}(mn, R),$$

provided that $m, n \geq 3$.

In this first part of our paper we prove several general results, which allow us to construct levels of intermediate subgroups, and thus, to *state* a complete answer to this last problem. Furthermore, we calculate the normalizers of arising intermediate subgroups. For the time being, even for the case of two factors, we could describe intermediate subgroups only under some simplifying assumptions, such as $n \geq m + 2$.

The present first part is approximately one half of the total proof for this case. The second part contains two other major ingredients of the proof, extraction of transvections and inclusion in the normalizer, provided $m \geq 3$, $n \geq m + 2$. Partially, the content of the present paper was announced in [1].

§1. INTRODUCTION

In the real world, as in dreams, nothing is quite what it seems.

The Book of Sorrows

Before passing to a detailed description of the results of the present paper and its sequels, we start with several general observations, to put our results in context, and record the techniques we have to bring out for their proofs.

From a most general viewpoint, somewhat naive but overall accurate, our forthcoming series of papers constitutes a long-awaited pendant to the papers [2, 3] by Zenon Borevich and the second author. In those papers, we described subgroups of the general linear group $\mathrm{GL}(n, R)$ over a commutative ring R that contain the direct sum of elementary linear groups

$$E(n_1, R) \oplus \cdots \oplus E(n_t, R) \leq H \leq \mathrm{GL}(n, R),$$

where $n = n_1 + \cdots + n_t$. Obviously, here too, direct sums should be understood as the direct sums of *linear* groups. Thus, the left-hand side of the above relation consists of all block diagonal matrices of the form

$$x_1 \oplus \cdots \oplus x_t = \mathrm{diag}(x_1, \dots, x_t), \quad x_i \in E(n_i, R).$$

Recall that in [3] such subgroups H were completely described under the assumption that all n_i are at least 3. This description was given in terms of some systems of ideals compatible with the partition $\nu = (n_1, \dots, n_t)$ of degree n , the so-called ν -*nets* of ideals. Morally, they are simply the subalgebras of the full matrix algebra $M(n, R)$ that contain the block diagonal subalgebra

$$M(\nu) = M(n_1, R) \oplus \cdots \oplus M(n_t, R).$$

The description can be stated as follows. With every net σ we can associate two subgroups in $G = \mathrm{GL}(n, R)$ containing

$$E(\nu) = E(n_1, R) \oplus \cdots \oplus E(n_t, R),$$

namely, the subgroup $E(\sigma)$ spanned by elementary generators, and the subgroup $G(\sigma)$, defined by *linear* congruences. Their normalizers coincide,

$$N(\sigma) = N_G(E(\sigma)) = N_G(G(\sigma)) = \text{Tran}_G(E(\sigma), G(\sigma)),$$

and conditions for a matrix to belong to $N(\sigma)$ are also expressed by congruences, this time *quadratic*. The bottom line is that for each subgroup H , $E(\nu) \leq H \leq \text{GL}(n, R)$, there exists a unique ν -net σ , or, in other words, a unique subalgebra containing $M(\nu)$, such that $E(\sigma) \leq H \leq N(\sigma)$.

Later, this result served as a model for numerous variations and generalizations, including the work by Vladimir Koibaev [23], Igor Golubchik [19], and Alexei Stepanov [34].

In Chapter V of the Habilitation Thesis [4] of the second author, the results of [2, 3] were generalized to other classical Chevalley groups. These results were announced in [5] and [7], and the detailed proofs appeared in [6], [8]–[10]. To produce proofs in this more general setting, we had to rethink and simplify the original proofs for the model case of GL_n . The main points of such a simplified proof were set forth in [80]. It follows the same plan of [3], but group-theoretic arguments were used to avoid some of the most onerous matrix calculations. Subsequently, two independent proofs of the same result were proposed, a geometric one [73], and a localization one [74].

In the present paper we start to systematically transcribe these results from direct sums to tensor products. The prospects of such a generalization were outlined in [79, 80, 73]. For instance, in [80] this was articulated as follows: “there is no doubt, that ... one can obtain analogous results for an arbitrary commutative ring R and arbitrary number of factors t provided that $n_1, \dots, n_t \geq 3$ ”. This comment was accompanied by the following imprudent forecast: “The answer involves ideals of R and is similar to, but easier than, that for the case of subsystem subgroups.”

However, reality went contrary to our expectations. As a matter of fact, this problem, as also description of overgroups of classical groups, proved themselves *significantly* harder than we fancied when writing [79], when writing [80], or even when writing [73]. As a total surprise for us, the actual situation was exactly opposite to that predicted in the preceding paragraph. Both in conceptual and technical aspects, description of overgroups of tensor products turned out to be *terribly* much trickier than the works by Zenon Borevich, the second author, and Alexei Stepanov on overgroups of subsystem subgroups [2]–[10], [33, 34] and, for that matter, even trickier than the works by the second author, Victor Petrov, and Hong You on overgroups of classical groups [14]–[16], [30, 66], [88]–[90].

Actually, additional difficulties, as compared to [3], crop up already at the very first steps. What is worse, even many subsidiary statements require an entirely different level of technical strain, comparable to the effort one would rather expect in the analysis of exceptional groups [11, 12], [25]–[27]. Below we list some of the most fundamental differences.

- Direct sums of elementary groups contain unipotents of residue 1. Thus, in the field case, description of their overgroups easily follows from the description of irreducible subgroups generated by root subgroups. At the same time, the smallest residue of a quadratic unipotent contained in tensor products of elementary subgroups can be *arbitrarily* large. This means that, even in the field case, description of their overgroups should rely either on the full force of the tremendous infinite analog of quadratic pairs, as developed by Franz Timmesfeld [76], or on the *extremely* intricate and abstruse matrix computations by Shang Zhi Li [58], [61]–[63].

- *Linear* congruences defining connected intermediate subgroups should be replaced by *quadratic* ones. Accordingly, *quadratic* congruences defining their normalizers should be replaced by *quartic* ones!

- Whereas the group of R -points of the affine group scheme $\mathrm{GL}_m \oplus \mathrm{GL}_n$ coincides with $\mathrm{GL}(m, R) \oplus \mathrm{GL}(n, R)$, nothing like that can be expected for tensor products in general. The subgroup of $\mathrm{GL}(mn, R)$ consisting of all Kronecker products of matrices of degrees m and n over R , can be strictly smaller than the group of R -points of the affine group scheme $\mathrm{GL}_m \otimes \mathrm{GL}_n$. In other words, in general,

$$\mathrm{GL}(m, R) \otimes \mathrm{GL}(n, R) \not\subseteq (\mathrm{GL}_m \otimes \mathrm{GL}_n)(R).$$

This means that, unlike [3], where we could content ourselves with the elementary matrix techniques, here we must take into account the effects of the theory of algebraic groups, right from the outset!

- It is next to impossible to predict the occurring answer from the first principles, without a detailed calculation of levels we perform in the present paper. In addition, the emerging difficulties do not manifest themselves at the field level. Indeed, as opposed to a commutative ring, a field *very* seldom *simultaneously* has characteristic 2 and characteristic m or n . As a result, all pathologies occurring for $m = 2$ or $n = 2$ over fields of characteristic 2 are attributed *exclusively* to the exceptional behavior of the group $\mathrm{GL}(2, R)$, instead of being viewed as part of the *standard* answer. At the same time, over commutative rings similar phenomena do occur for all m and n and must be regarded as PART OF THE STANDARD ANSWER, rather than exceptions.¹

- Apart from the usual structure results, such as the Suslin normality theorem, or the Wilson–Golubchik theorem on normal subgroups, we have to invoke much harder results of structure theory of linear groups. Among other things, the proofs in the present paper rely on the description of epimorphisms of linear groups, obtained by Waterhouse, Petechuk, Golubchik–Mikhalev, and Zelmanov, or on the characterization of the elementary subgroup $E(n, R)$ as the largest perfect subgroup in $\mathrm{GL}(n, R)$, obtained by Bak, Hazrat, and the second author. In their turn, these results rely on the full force of localization-completion, nontrivial ring-theoretic computations, etc. Moreover, some of these results are only available in the literature under additional assumptions, such as $2 \in R^*$, which sometimes presents additional impediments.

- In the proofs of some subsidiary results, we substantially invoke certain very recent results on description of overgroups for other Aschbacher classes, most notably, the work by Alexei Stepanov on overgroups of subring subgroups [70]–[72].

Now we turn to the content of the present paper, which is an exposition of term papers by the first author and the third author, under the supervision of the second author. As stated above, here we study subgroups

$$E(m, R) \otimes E(n, R) \leq H \leq \mathrm{GL}(mn, R).$$

Guided by the analogy with results by Victor Petrov and the second author [14]–[16], [30, 66], one might expect that under the assumption $m, n \geq 3$, for every such subgroup H there exists a unique ideal A in R such that H lies between the group

$$\mathrm{EE}(m, n, R, A) = (E(m, R) \otimes E(n, R))E(mn, R, A)$$

and its normalizer in $\mathrm{GL}(mn, R)$:

$$\mathrm{EE}(m, n, R, A) \leq H \leq N_G(\mathrm{EE}(m, n, R, A)).$$

¹It was a major mental obstacle for us to apprehend this phenomenon and its implications. Probably, this is a main reason why the results of the present paper have not been obtained much earlier.

This is the case, sometimes, under appropriate simplifying assumptions on the ground ring such as, say, $m, n \in R^*$. However, in general, the answer is stated in terms of *three* ideals of the ground ring, rather than one ideal. Two of these ideals play exactly the same role as *relative form parameters* in the description of normal subgroups in classical groups over rings where 2 is not invertible.

Namely, with every intermediate subgroup H we associate a triple of ideals

$$(A, B, C), \quad mA, A^2 \leq B \leq A, \quad nA, A^2 \leq C \leq A,$$

which suggests itself as the level of H . Conversely, any such triple gives rise to the corresponding elementary subgroup $EE(m, n, R, A, B, C) \leq H$, which is the largest subgroup of H spanned by root unipotents. Now the standard answer to the above problem can be stated as follows: H is contained in the normalizer of $N_G(EE(m, n, R, A, B, C))$. Thus, without the assumption $m, n \in R^*$, it is nontrivial even to correctly *state* a complete answer.

We list the main results of the present paper. Their precise statements are postponed till we recall/introduce the necessary notation.

- Calculation of tensor products $GL_m \otimes GL_n$, in the category of algebraic groups, Theorems 1 and 2. Observe that the group of points $(GL_m \otimes GL_n)(R)$ is, in general, strictly larger than the tensor product $GL(m, R) \otimes GL(n, R)$ in the category of linear groups.

- Calculation of the normalizers of the elementary subgroups $E(m, R) \otimes e$ and $e \otimes E(n, R)$, Theorem 3, and a more difficult calculation of the normalizer of the elementary subgroup $E(m, R) \otimes E(n, R)$. Under assumption that $m \neq n$ this normalizer coincides with the normalizer of $GL(m, R) \otimes GL(n, R)$ which, in turn, coincides with $(GL_m \otimes GL_n)(R)$, Theorem 4. However, for $m = n$ it is somewhat larger.

- Description of transvections in the subgroups of $GL(mn, R)$ that contain $E(m, R) \otimes E(n, R)$. Namely, we construct *perfect* subgroups $EE(m, n, R, A, B, C)$ and associate with every intermediate subgroup H its level. The level of H can be defined as the largest admissible triple (A, B, C) such that $EE(m, n, R, A, B, C) \leq H$, Theorem 5.

- Finally, in Theorem 6 we completely calculate the normalizer of the elementary subgroup $EE(m, n, R, A) = EE(m, n, R, A, A, A)$ in the special case where $A = B = C$. It turns out that this normalizer coincides with the full preimage $C(m, n, R, A)$ of the group of points $(GL_m \otimes GL_n)(R)$ with respect to the reduction homomorphism modulo A . This normalizer is described by simple quadratic *congruences* on matrix entries.

Apparently, these results may be of independent interest. On the other hand, for us all of them are staple tools in the description of overgroups of $E(m, R) \otimes E(n, R)$. Namely, in the second part of this paper we design to carry through the extraction of transvections and herewith to complete the description of overgroups of $E(m, R) \otimes E(n, R)$, provided that $m \geq 3, n \geq m + 2$. This case is vastly more accessible than the case where $n = m + 1$, let alone the case where $n = m$. The reason is that under the assumption $n \geq m + 2$ it is easy to stabilize a column using minors [1]. This trick was proposed in 1987 by Alexei Stepanov and the second author [33, 4]; see also [73, 13].

Observe that in the field case, $R = K$, our main results boil down to a special case of results by Shang Zhi Li [58, 61], see also [56, 62, 63]. However, the proof by Shang Zhi Li consists of highly nontrivial matrix computations which, in their turn, rely *in particular* on the results of [57] and [60]. Moreover, some details of calculations are only published in Chinese [63]. As a matter of fact, Shang Zhi Li describes overgroups of $E(m, K) \otimes E(n, K)$ under the assumption $m, n \geq 2$. However, in the case where one of the degrees m or n equals 2, nonstandard intermediate subgroups occur. Anyway, as is

well known, there is no hope to improve the bound $m, n \geq 3$ for arbitrary commutative rings.

The present paper is organized as follows. In §2 we discuss the general context and the place of the problem in question among other similar problems on intermediate subgroups. In §§3–5 we recall the basic notation and some general facts concerning the groups $\mathrm{GL}(n, R)$, their elementary subgroups, and tensor products of linear groups. In §§6 and 7 we prove Theorems 1 and 2, respectively. In §8 we begin to study the elementary subgroup $E(m, R) \otimes E(n, R)$. In §§9 and 10 we recall some fundamental structure results of the theory of linear groups over rings, such as description of normal subgroups, and description of epimorphisms, in the form that will be used in our proofs. In §§11 and 12 we calculate the normalizers of the elementary subgroups and prove Theorems 3 and 4. In §§13–15 we study elementary transvections in subgroups containing $E(m, R) \otimes E(n, R)$, introduce perfect intermediate subgroups $\mathrm{EE}(m, n, R, A, B, C)$, and prove Theorem 5. Finally, in §16 we prove Theorem 6, which calculates the normalizer of $\mathrm{EE}(m, n, R, A)$.

§2. ASCHBACHER CLASSES

Let us mention a broader context for our results. Namely, over the last decades, a central problem of the theory of finite groups was description of their maximal subgroups. In 1984, Michael Aschbacher [36] established a subgroup structure theorem, which asserts that every maximal subgroup of a finite classical group either falls into one of 8 explicitly described *Aschbacher classes* $\mathcal{C}_1 - \mathcal{C}_8$, or else is an (almost) simple group in an irreducible representation. The philosophy behind this theorem was described in a very suggestive survey by Aschbacher [37]. Later, Martin Liebeck and Gary Seitz proposed a simpler proof, based on the theory of algebraic groups [65]. From the viewpoint of Aschbacher's subgroup structure theorem, in the present paper we consider overgroups of subgroups from the Aschbacher classes $\mathcal{C}_4 + \mathcal{C}_7$.

For algebraically closed and for finite fields, overgroups of groups from Aschbacher classes have been studied intensely. Over a finite field, maximality of subgroups from Aschbacher classes was completely settled in the book [53] by Peter Kleidman and Martin Liebeck. Many publications by Roger Dye, Oliver King, Shang Zhi Li and others address the problem of maximality of subgroups from Aschbacher classes over arbitrary fields. Sometimes, these subgroups are not maximal, and then the issue was to describe their overgroups, see references in [17, 80].

Ideologically, the present paper is related to the work of the Saint Petersburg school on subgroups of classical groups over commutative rings; see [2]–[10], [14]–[18], [30, 66], [70]–[72]. A detailed account of such results on description of overgroups, the methods used therein, and unsolved problems can be found in the surveys by the second author and Alexei Stepanov [17, 80], as also in the forthcoming papers [64, 74].

- From the viewpoint of the Maximal Subgroup Classification Project, the above result by Zenon Borevich and the second author, as also subsequent results by Alexei Stepanov and others on overgroups of subsystem subgroups, pertain to the Aschbacher classes $\mathcal{C}_1 + \mathcal{C}_2$.

Until recently, over rings little was known about the description of overgroups for other classes. To a certain extent, the situation has changed over the last decade. To put our results in the context, we give a very brief overview of related developments for other classes.

- For commutative rings it does not make any sense to consider the class $\mathcal{C}_6$.

- For the class $\mathcal{C}_8$, overgroups were completely described in recent papers by the second author and Victor Petrov [14]–[16], [30, 66] and independently by Hong You [88]–[90] and his students [81].² From a technical viewpoint, these papers turned out to be much harder than the works on overgroups of subsystem subgroups, and advanced the whole area to a completely different technological level. These works were brilliantly resumed by Alexander Luzgarev [25]–[27], who obtained similar results for exceptional groups. Technically, the present paper is *very* largely inspired by [15, 16, 26].

- Quite recently, Alexei Stepanov [70]–[72] obtained absolutely amazing results concerning subring subgroups, the class $\mathcal{C}_5$. It turned out that for the general linear group and for the even orthogonal group, description of their overgroups is *very* seldom standard [70, 71].³ Quite to the contrary, for the symplectic group $\mathrm{Sp}(2l, R)$ and for the odd orthogonal group $\mathrm{SO}(2l + 1, R)$, $l \geq 2$, as also for the Chevalley group of type F_4 , such a description is *always* standard [72], at least under the assumption $2 \in R^*$. Further still, the work by Stepanov leaves little doubt that for $l \geq 3$ the description will remain standard even without the assumption $2 \in R^*$. Evidently, in general one has to replace usual ring ideals by form ideals, in the statement of the standard description.

- Finally, description of overgroups for groups in the class $\mathcal{C}_3$, ring extension subgroups, is *tremendously* difficult. Recall that for the general linear group it amounts to describing overgroups of $E(n, S)$ in $\mathrm{GL}(mn, R)$, where S/R is a commutative ring extension such that S is a free R -module of rank m . For arbitrary finite field extensions L/K this was done by Shang Zhi Li [57, 60] under the assumption $n \geq 2$. We believe that the techniques developed in the work of the second author, Petrov, Stepanov, and Hong You, together with the methods of the present paper, will eventually make it possible to solve this problem for arbitrary commutative rings under the assumption $n \geq 3$, but it will be a long way.

- As we have already mentioned, for the field case, the overgroups for the class $\mathcal{C}_4$ — but not the class $\mathcal{C}_7$ — were described by Shang Zhi Li [58], [61]–[63]. In this connection, one could mention also the paper [24] by Alexei Korotkevich, which stands somewhat apart. That paper addresses description of overgroups of $E(m, R) \otimes e$ in $\mathrm{GL}(mn, R)$, which, as we can now assess, is *extremely* difficult!

- Yet another closely related series of works are the papers by Vladimir Platonov, Dragomir Djoković, Robert Guralnick, William Waterhouse, and others [42]–[48], [67, 68, 85] on overgroups of semisimple subgroups, emerging in the context of linear preserver problems. In those papers, they consider still *vastly* harder problems than the one we study in the present paper. A paradigmatic problem there is description of overgroups of the elementary group $E(n, R)$ in $\mathrm{GL}(n^2, R)$, in the adjoint representation. Clearly, in the above papers, these problems were completely solved only over fields — mostly even only over *classical* fields such as $\mathbb{C}$ or $\mathbb{R}$.

§3. BASIC NOTATION

Mostly, the notation we use is utterly standard and coincides with that used in [3, 7], [14]–[17], [73]. A common reference, where one can find the general setup, all definitions, and most of the general facts we need, is the remarkable monograph by Alexander Hahn

²Observe, though, that the result of [81] is *much* weaker than the results of the earlier paper by Victor Petrov [66]. Apparently, the authors of [81] have not noticed that the *local* stable rank of commutative rings equals 1, so that their result is *completely* superceded by the result of [66]. Also, [81] contains several further successful impairments, as compared to [66].

³Roughly speaking, it is standard only for algebraic extensions, and for rings of dimension 1.

and Timothy O'Meara [49]. Nevertheless, to make the present paper completely independent of [15] and [16], below we reproduce the basic notation pertaining to subgroups of $\mathrm{GL}(n, R)$.

First, let G be an arbitrary group, and let $C(G)$ be its center. By the commutator of two elements $x, y \in G$ we always understand their left-normed commutator $[x, y] = xyx^{-1}y^{-1}$. By ${}^xy = xyx^{-1}$ and $y^x = x^{-1}yx$ we denote the left and the right conjugates of y by x , respectively. We write $H \leq G$ to denote that H is a subgroup of G , and $H \trianglelefteq G$ to denote that H is a normal subgroup. For a subset $X \subseteq G$, we denote by $\langle X \rangle$ the subgroup it generates. For a subgroup $H \leq G$, we denote by $\langle X \rangle^H$ the smallest subgroup of G containing X and normalized by H . For two subgroups $F, H \leq G$, we denote by $[F, H]$ their mutual commutator subgroup spanned by $[f, h]$, $f \in F$, $h \in H$. Multiple commutators are also normed on the left, so that, in particular, $[E, F, H] = [[E, F], H]$.

Let E, F be two subgroups of a group G . Recall that the *transporter* of the subgroup E to the subgroup F is the subset

$$\mathrm{Tran}_G(E, F) = \{g \in G \mid E^g \leq F\}.$$

As a rule, we use this notation in the case where $E \leq F$, and then

$$\mathrm{Tran}_G(E, F) = \{g \in G \mid [g, E] \leq F\}.$$

Recall that a group G is called the central product of its subgroups $F, H \leq G$ if the following three conditions are satisfied:

$$\langle F, H \rangle = G, \quad [F, H] = 1, \quad F, H \trianglelefteq G.$$

In this case, we usually write $G = F \circ H$. It is easily seen that the condition $[F, H] = 1$ implies

$$F \cap H = C(F) \cap C(H) \leq C(G).$$

Now, let R be an arbitrary associative ring with 1; by default, it will be assumed commutative. As usual, $M(m, n, R)$ denotes the R -bimodule of $(m \times n)$ -matrices with entries in R , $M(n, R) = M(n, n, R)$ is the full matrix ring of degree n over R . Next, $R^\bullet = R \setminus \{0\}$ is the set of nonzero elements of the ring R , and R^* is its multiplicative group. As always, $G = \mathrm{GL}(n, R) = M(n, R)^*$ is the general linear group of degree n over R . For a commutative ring R , we consider also the special linear group $\mathrm{SL}(n, R)$ of degree n over R .

As usual, for a matrix $g \in G$ we denote by g_{ij} its entry at the position (i, j) . In other words, $g = (g_{ij})$, $1 \leq i, j \leq n$. As always, $g^{-1} = (g'_{ij})$ denotes the inverse of g and g^t denotes its transpose. When upper indices are employed in the notation of matrix entries, we usually write $\tilde{g}_{ij}$ instead of g'_{ij} . Next, $g_{*j} = (g_{1j}, \dots, g_{nj})^t$ denotes the j th column of a matrix g , and $g_{i*} = (g_{i1}, \dots, g_{in})$ denotes its i th row.

As always, e denotes the identity matrix and e_{ij} is a standard matrix unit, i.e., the matrix that has 1 at the position (i, j) and zeros elsewhere. By $t_{ij}(\xi) = e + \xi e_{ij}$, for $\xi \in R$ and $1 \leq i \neq j \leq n$, we denote an *elementary transvection*. By $X_{ij} = \{t_{ij}(\xi), \xi \in R\}$, $i \neq j$, we denote an elementary *root subgroup*.

Below, without any further reference we use standard relations among elementary transvections, such as additivity $t_{ij}(\xi)t_{ij}(\zeta) = t_{ij}(\xi + \zeta)$ or Chevalley commutator formula $[t_{ij}(\xi), t_{jh}(\zeta)] = t_{ih}(\xi\zeta)$, see, for instance, [32].

Now, let $C(n, R)$ be the center of the group $\mathrm{GL}(n, R)$, consisting of scalar matrices εe , $\varepsilon \in R^*$. Usually, the class of a matrix $g \in \mathrm{GL}(n, R)$ in the quotient group $\mathrm{GL}(n, R)/C(n, R)$ of the general linear group $\mathrm{GL}(n, R)$ modulo the center is denoted by $[g]$. When parentheses are already used in the notation of g , in the notation of its class $[g]$ they are omitted: instead of $[(g_{ij})]$ one simply writes $[g_{ij}]$. By definition,

$$[hg] = [h][g], \quad [g^{-1}] = [g]^{-1}, \quad [\lambda g] = [g],$$

for each $\lambda \in R^*$. However, when the whole calculation takes place in the quotient group $\mathrm{GL}(n, R)/C(n, R)$, we often simply write $\bar{g}$ instead of $[g]$, not to overburden notation with brackets.

Further, $\mathrm{PGL}(n, R)$ denotes the *projective linear group* which, by definition, is the group of R -points of the quotient group of the *algebraic* group GL_n modulo its center $C(\mathrm{GL}_n) \cong \mathbb{G}_m$. Clearly, $\mathrm{GL}(n, R)/C(n, R) \leq \mathrm{PGL}(n, R)$ and over fields this inclusion actually becomes an identity, $\mathrm{PGL}(n, K) = \mathrm{GL}(n, K)/C(n, K)$. However, over general commutative rings the group $\mathrm{PGL}(n, R)$ is almost invariably *strictly larger* than $\mathrm{GL}(n, R)/C(n, R)$! Morally, this is accounted by the fact that epimorphisms of algebraic groups may fail to be surjective on points. Over a commutative ring, every element of the group $\mathrm{PGL}(n, R)$ can also be expressed as $[g]$, for some matrix g of degree n . Note only that here, in general, $g \in \mathrm{GL}(n, S)$ for some extension S of the ground ring R .

In the sequel we consider the left action of $\mathrm{GL}(n, R)$ on the *right* free R -module $V = R^n$ of rank n , consisting of all columns of height n over R . Following Paul Cohn, we denote by nR the *left* free R -module V^* of rank n , consisting of all rows of length n over R . The standard bases of R^n and nR are denoted by $e_1, \dots, e_n$ and $e^1, \dots, e^n$, respectively. In other words, $e_i = e_{*i}$ is the i th column of the identity matrix e of degree n , whereas $e^i = e_{i*}$ is the i th row of e . In this notation, $g_{*j} = ge_j$, $g_{i*} = e^i g$, and $g_{ij} = e^i ge_j$. It is quite useful to bear these obvious formulas in mind for cross-cultural communication. While in linear algebra it is customary to talk of rows and columns of matrices from a group G , experts in algebraic K -theory and algebraic groups usually talk of the orbit of e^1 or e_1 , under the action of G , known as the “orbit of the highest weight vector”.

A column $v = (v_1, \dots, v_n)^t \in R^n$ is said to be *unimodular* if $Rv_1 + \dots + Rv_n = R$ or, in other words, if the *left* ideal generated by $v_1, \dots, v_n$ coincides with R . Similarly, a row $u = (u_1, \dots, u_n) \in {}^nR$ is *unimodular* if $u_1R + \dots + u_nR = R$ or, in other words, if the *right* ideal generated by $u_1, \dots, u_n$ coincides with R . Unimodularity of a row/column is a *necessary* condition for this row/column to be a row/column of an invertible matrix. However, apart from some very special classes of rings, this condition is not sufficient. As was observed above, to actually be a column of an invertible matrix, this column must lie in the orbit of e_1 .

§4. ELEMENTARY SUBGROUPS

The subgroup of $\mathrm{GL}(n, R)$ generated by all elementary transvections $t_{ij}(\xi)$ for $\xi \in R$ and $1 \leq i \neq j \leq n$, is called the [absolute] *elementary subgroup* and is denoted by $E(n, R)$.

The next claim immediately follows from the fact that for $n \geq 2$ the group $E(n, R)$ is absolutely irreducible, or, what is the same in this case, additively generates $M(n, R)$. Recall that $\mu_n(R)$ denotes the group of roots of 1 of degree n in the ring R .

Lemma 1. *We have*

$$C_{M(n, R)}(E(n, R)) = Re, \quad C_{\mathrm{GL}(n, R)}(E(n, R)) = C(n, R).$$

In particular, $C(E(n, R)) = \mu_n(R)$.

The quotient group $E(n, R)/C(E(n, R))$ is denoted by $\mathrm{PE}(n, R)$ and is called the *projective elementary group*. Clearly,

$$\mathrm{PE}(n, R) \leq \mathrm{GL}(n, R)/C(n, R) \leq \mathrm{PGL}(n, R).$$

As a matter of fact, we need a slightly more precise result, asserting that already matrices commuting with the elementary transvections of the form $t_{ij}(1)$, $1 \leq i \neq j \leq n$, are scalar.

Lemma 2. *We have*

$$C_{M(n,R)}(E(n, \mathbb{Z}1)) = Re, \quad C_{\mathrm{GL}(n,R)}(E(n, \mathbb{Z}1)) = C(n, R).$$

Proof. Clearly, if $g = (g_{ij}) \in M(n, R)$ commutes with all $t_{ij}(1) \in E(n, \mathbb{Z}1)$, then it commutes with all e_{ij} , $1 \leq i \neq j \leq n$. Now, $e_{ij}g = ge_{ij}$ implies that $g_{ji} = 0$ and that $g_{ii} = g_{jj}$. $\square$

Next, let $I \trianglelefteq R$ be an ideal of R . Denote by $E(n, I)$ the subgroup of G generated by all elementary transvections of level I ,

$$E(n, I) = \langle t_{ij}(\xi), \xi \in I, 1 \leq i \neq j \leq n \rangle.$$

A paramount role throughout this paper is played by the *relative elementary groups* $E(n, R, I)$. Recall that, by definition, $E(n, R, I)$ is the normal closure of $E(n, I)$ in $E(n, R)$,

$$E(n, R, I) = \langle t_{ij}(\xi), \xi \in I, 1 \leq i \neq j \leq n \rangle^{E(n, R)}.$$

In the sequel, sometimes without any further reference we use some classical facts on elementary subgroups. The following fact, first established in [35], will be quoted as *Suslin's theorem*.

Lemma 3. *If the ring R is commutative and $n \geq 3$, then the group $E(n, R, I)$ is normal in $\mathrm{GL}(n, R)$.*

The following result was proved for the first time by Leonid Vaserstein and Andrei Suslin, and in the context of Chevalley groups, by Michael Stein [69].

Lemma 4. *For $n \geq 3$, the group $E(n, R, I)$ is generated by all transvections of the form*

$$z_{ij}(\xi, \zeta) = t_{ji}(\zeta)t_{ij}(\xi)t_{ji}(-\zeta), \quad \xi \in I, \zeta \in R, 1 \leq i \neq j \leq n.$$

The following fact is a common knowledge. It is valid for GL_n simply because this is an affine group scheme. On the other hand, it is valid for $E(n, R)$ because it is generated by the groups of points of affine group schemes X_{ij} .

Lemma 5. *Let R_i , $i \in I$, be an inductive system of rings. Then*

$$\mathrm{GL}(n, \varinjlim R_i) = \varinjlim \mathrm{GL}(n, R_i), \quad E(n, \varinjlim R_i) = \varinjlim E(n, R_i).$$

Mostly, we shall apply this argument in the following situation. Consider the inductive system R_i of all finitely generated subrings of a ring R , with respect to inclusion. Then $R = \varinjlim R_i$, which allows us to reduce many relevant problems to the case of Noetherian rings.

The next claim immediately follows from the Chevalley commutator formula.

Lemma 6. *Suppose R is an arbitrary commutative ring and $n \geq 3$. Then the elementary subgroup $E(n, R)$ is perfect.*

The following characterization of the elementary subgroup plays a key role in many subsequent calculations. It was first obtained by Anthony Bak [39]; see also [51, 40] for easier proofs of more general results. We are not aware of *any* elementary proof of this fact that would not invoke the full force of localization methods. Formally, this result was not stated this way in [39, 51]. Yet, it immediately follows from the existence of nilpotent filtrations of $\mathrm{GL}(n, R)$, constructed in those papers.

Lemma 7. *Suppose R is an arbitrary Noetherian commutative ring and $n \geq 3$. Then the group $E(n, R)$ is the largest perfect subgroup of $\mathrm{GL}(n, R)$.*

In other words, in this case the group $E(n, R)$ is the *perfect radical* of $\mathrm{GL}(n, R)$. Of course, this immediately implies the following important consequence.

Lemma 8. *Suppose R is an arbitrary Noetherian commutative ring and $n \geq 3$. Then the group $E(n, R)$ is a fully characteristic subgroup of $\mathrm{GL}(n, R)$.*

Observe that our proof of Lemma 8 does *not* refer to the description of automorphisms of $\mathrm{GL}(n, R)$ (but relies on the full force of localization methods!). In its turn, the next result immediately follows from the description of automorphisms, which we recall in § 10.

Lemma 9. *Suppose R is an arbitrary commutative ring and $n \geq 3$. In the case where $n = 3$, assume, moreover, that $2 \in R^*$. Then the group $E(n, R)$ is a characteristic subgroup of $\mathrm{GL}(n, R)$.*

Thus, in the sequel we always have an alternative, to refer to Lemma 9, or to refer to Lemmas 5 and 8 instead. However, both Lemma 8 and Lemma 9 depend on *difficult* external results.

§5. THE GROUP $\mathrm{GL}(m, R) \otimes \mathrm{GL}(n, R)$

In the present section we describe the tensor product of $\mathrm{GL}(m, R)$ and $\mathrm{GL}(n, R)$ as *linear* groups, to avoid any possible confusion, *abstract* linear groups. One of the first major differences between the present paper and the paper [3] by Borevich and the second author is precisely that the tensor product of $\mathrm{GL}(m, R)$ and $\mathrm{GL}(n, R)$ as linear *algebraic* groups is, generally speaking, strictly larger, than $\mathrm{GL}(m, R) \otimes \mathrm{GL}(n, R)$.

With any four natural numbers l, m, n, p we associate a binary operation

$$\otimes : M(l, m, R) \times M(n, p, R) \longrightarrow M(lm, np, R)$$

that assigns the *Kronecker product* $x \otimes y$ to a pair of matrices $x \in M(l, m, R)$, $y \in M(n, p, R)$. The easiest way to introduce this operation is to think of $x \otimes y$ as the following block matrix

$$x \otimes y = \begin{pmatrix} x_{11}y & \cdots & x_{1m}y \\ & \cdots & \\ x_{l1}y & \cdots & x_{lm}y \end{pmatrix} \in M(lm, np, K).$$

This operation is particularly useful for square matrices. It should be mentioned that for $x \in M(m, R)$, $y \in M(n, R)$ we have

$$x \otimes y \in M(m, M(n, R)), \quad \text{whereas} \quad y \otimes x \in M(n, M(m, R)).$$

Both rings may be identified with $M(mn, R)$. Nevertheless, $x \otimes y \neq y \otimes x$ in general. Actually, $y \otimes x$ is obtained from $x \otimes y$ by an appropriate permutation of columns and rows.

Below, the following well known (and obvious) identities relating the Kronecker product with other matrix operations will be used systematically without any further reference.

Lemma 10. *The Kronecker product is subject to the following identities.*

- *Associativity.* For all $x \in M(l, R)$, $y \in M(m, R)$, $z \in M(n, R)$ we have

$$(x \otimes y) \otimes z = x \otimes (y \otimes z)$$

(upon identification of both sides with matrices in $M(lmn, R)$).

- *Distributivity with respect to addition:*

$$(x + y) \otimes z = x \otimes z + y \otimes z, \quad x \otimes (y + z) = x \otimes y + x \otimes z.$$

- *Homogeneity:* $\alpha x \otimes y = \alpha(x \otimes y) = x \otimes \alpha y$.

• *Mutual distributivity of the Kronecker product and the usual multiplication of matrices:*

$$xy \otimes uv = (x \otimes u)(y \otimes v).$$

• *If x and y are both invertible, then*

$$(x \otimes y)^{-1} = x^{-1} \otimes y^{-1}.$$

In particular, the two last identities imply that $\mathrm{GL}(m, R) \otimes \mathrm{GL}(n, R)$ is a subgroup of $\mathrm{GL}(mn, R)$.

• Sometimes, in our calculations we shall apply the explicit formula for the determinant of a Kronecker product:

$$\det(x \otimes y) = \det(x)^n \det(y)^m.$$

Having studied several dozens of papers on Kronecker products, we decided to stick to the *blockwise* numbering of the entries of matrices in $\mathrm{GL}(mn, R)$, compatible with the above partition into $(m \times m)$ -blocks of sizes $n \times n$, see, in particular, [75, 77, 87]. Namely, the upper pair of indices indicates the position of a block in the block matrix of size $m \times m$, whereas the lower pair of indices indicates the position of an entry inside such a block of size $n \times n$.

Explicitly, the correspondence between the above notation and the usual notation of the entries of $(mn \times mn)$ -matrices is described by the following formula:

$$g_{pq}^{ij} = g_{n(i-1)+p, n(j-1)+q},$$

where $1 \leq i, j \leq m$, $1 \leq p, q \leq n$. In particular, if $g = x \otimes y$ for some $x \in \mathrm{GL}(m, R)$ and $y \in \mathrm{GL}(n, R)$, then

$$g_{pq}^{ij} = x_{ij} y_{pq}.$$

In this notation, the formula for the usual matrix multiplication takes the following form

$$(fg)_{pq}^{ij} = \sum_{h=1}^m \sum_{r=1}^n f_{pr}^{ih} g_{rq}^{hj}.$$

As we agreed in §3, the entries of the inverse matrix will now be written as

$$(g^{-1})_{pq}^{ij} = \tilde{g}_{pq}^{ij}.$$

In §2 we have already recalled the notation for rows and columns of a matrix. This notation naturally extends to block matrices. Below, we frequently use the following shorthand notation.

- g_{**}^{ij} — the (i, j) th block.
- g_{pq}^{**} — the entries at the positions (p, q) , in all blocks.
- g_{p*}^{i*} — the p th row in the i th horizontal strip.
- g_{*q}^{*j} — the q th column in the j th vertical strip.
- g_{*q}^{i*} — the pieces of the q th columns in the i th horizontal strip.
- g_{p*}^{*j} — the pieces of the p th rows in the j th vertical strip.
- g_{**}^{i*} — the i th horizontal strip.
- g_{**}^{*j} — the j th vertical strip.
- g_{p*}^{**} — the p th rows in all horizontal strips.
- g_{*q}^{**} — the q th columns in all vertical strips.

For instance, in this notation the formula for matrix multiplication takes the form

$$(fg)_{pq}^{ij} = f_{p*}^{i*} \cdot g_{*q}^{*j}.$$

The (i, j) th block of the product equals

$$(fg)_{**}^{ij} = f_{**}^{i*} \cdot g_{**}^{*j}.$$

Similarly, the matrix consisting of entries of all blocks at the position (p, q) equals

$$(fg)_{pq}^{**} = f_{p*}^{**} \cdot g_{*q}^{**}.$$

Practicing a few minutes, the reader will be able to produce dozens of further similar formulas.

§6. THE GROUP $\mathrm{GL}_m \otimes \mathrm{GL}_n$

In the present section we describe the tensor product of $\mathrm{GL}(m, R)$ and $\mathrm{GL}(n, R)$ as linear *algebraic* groups. A general reference, where one can find the necessary background on algebraic groups, is [82]. On the other hand, our proofs of Theorems 1 and 2 bluntly imitate [86].

Namely, we are about to show that, as an algebraic subgroup of GL_{mn} , the group $\mathrm{GL}_m \otimes \mathrm{GL}_n$ is defined by the equations

$$g_{pq}^{ij} g_{rs}^{hk} = g_{pq}^{hk} g_{rs}^{ij}.$$

Here and below $1 \leq i, j, k, l \leq m$, $1 \leq p, q, r, s \leq n$.

In other words, we are going to show that for every commutative ring we have

$$(\mathrm{GL}_m \otimes \mathrm{GL}_n)(R) = \left\{ (g_{pq}^{ij}) \in \mathrm{GL}(mn, R) \mid g_{pq}^{ij} g_{rs}^{kl} = g_{rs}^{kl} g_{pq}^{ij} \right\}.$$

Denote the set on the right-hand side by $G(R)$.

Lemma 11. *For every commutative ring R , the set $G(R)$ is multiplicatively closed and*

$$\mathrm{GL}(m, R) \otimes \mathrm{GL}(n, R) \leq G(R).$$

Proof. Indeed, let $f, g \in G(R)$. Then

$$(fg)_{pq}^{ij} \cdot (fg)_{rs}^{hk} = \sum_{l=1}^m \sum_{t=1}^n f_{pt}^{il} g_{tq}^{lj} \cdot \sum_{u=1}^m \sum_{v=1}^n f_{rv}^{hu} g_{vs}^{uk}.$$

Changing the order of summation and applying equations for f and g , we see that this sum equals

$$\sum_{u=1}^m \sum_{t=1}^n f_{pt}^{hu} g_{tq}^{uk} \cdot \sum_{l=1}^m \sum_{v=1}^n f_{rv}^{il} g_{vs}^{lj} = (fg)_{pq}^{hk} \cdot (fg)_{rs}^{ij}.$$

Thus, $fg \in G(R)$.

To establish the inclusion, observe that if $g = x \otimes y$ for some $x \in \mathrm{GL}(m, R)$, $y \in \mathrm{GL}(n, R)$, then $g_{pq}^{ij} = x_{ij} y_{pq}$, and thus

$$g_{pq}^{ij} g_{rs}^{kl} = x_{ij} y_{pq} x_{kl} y_{rs} = x_{ij} y_{rs} x_{kl} y_{pq} = g_{rs}^{kl} g_{pq}^{ij}. \quad \square$$

It is somewhat harder to verify that $G(R)$ is a group (this is a sly identity for minors, similar to the identities of Lewis Carroll or Sylvester). However, there is no need to prove this directly. Instead, we shall prove a much more precise statement, namely, that the affine schemes $\mathrm{GL}_m \otimes \mathrm{GL}_n$ and G coincide. Since the first of them is a group scheme, obviously, the second will be a group scheme as well.

Theorem 1. *We have equality $\mathrm{GL}_m \otimes \mathrm{GL}_n = G$ of affine group schemes over $\mathbb{Z}$.*

It is well known that to prove equality of two group schemes, it suffices to prove equality of their groups of points over all local rings. Thus, the theorem is an immediate consequence of the following sharper result.

Proposition 1. *For a local ring R we have*

$$(\mathrm{GL}_m \otimes \mathrm{GL}_n)(R) = \mathrm{GL}(m, R) \otimes \mathrm{GL}(n, R) = G(R).$$

Proof. The first identity is obvious. Indeed, as an algebraic group, $\mathrm{GL}_m \otimes \mathrm{GL}_n$ is a quotient group of the direct product $\mathrm{GL}_m \times \mathrm{GL}_n$ modulo the central diagonal subgroup $(\lambda e_m, \lambda e_n)$, isomorphic to the multiplicative group GL_1 . The exact sequence of affine group schemes

$$1 \longrightarrow \mathrm{GL}_1 \longrightarrow \mathrm{GL}_m \times \mathrm{GL}_n \longrightarrow \mathrm{GL}_m \otimes \mathrm{GL}_n \longrightarrow 1$$

begets the exact sequence in Galois cohomology

$$\begin{aligned} 1 \longrightarrow R^* \longrightarrow \mathrm{GL}(m, R) \times \mathrm{GL}(n, R) &\longrightarrow (\mathrm{GL}_m \otimes \mathrm{GL}_n)(R) \\ &\longrightarrow H^1(R, \mathrm{GL}_1) \longrightarrow H^1(R, \mathrm{GL}_m \times \mathrm{GL}_n) \longrightarrow H^1(R, \mathrm{GL}_m \otimes \mathrm{GL}_n). \end{aligned}$$

It is well known that $H^1(R, \mathrm{GL}_1) = \mathrm{Pic}(R)$ is the Picard group of the ring R . For a local ring R the Picard group is trivial. Thus, in this case the homomorphism

$$\mathrm{GL}(m, R) \times \mathrm{GL}(n, R) \longrightarrow (\mathrm{GL}_m \otimes \mathrm{GL}_n)(R)$$

is in fact surjective, as claimed.

Now, let the ring R be local and $g = (g_{pq}^{ij}) \in G(R)$. The determinant of the matrix g is invertible and is a homogeneous polynomial in $\{g_{pq}^{ij}\}$. Since all noninvertible elements of the ring R form an ideal, there exist i, j, p, q such that $g_{pq}^{ij} \in R^*$. We fix such indices. Then the equations defining $G(R)$ can be rewritten as follows:

$$g_{rs}^{kl} = (g_{pq}^{ij})^{-1} g_{rs}^{ij} g_{pq}^{kl},$$

whence

$$g = (g_{pq}^{kl})_{1 \leq k, l \leq m} \otimes \left((g_{pq}^{ij})^{-1} g_{rs}^{ij} \right)_{1 \leq r, s \leq n}.$$

It remains to observe that, since the Kronecker product is invertible, both factors are also invertible. $\square$

§7. THE GROUP OF POINTS $(\mathrm{GL}_m \otimes \mathrm{GL}_n)(R)$

In view of nonsurjectivity of epimorphisms of algebraic groups on points, the group $G(R)$ is in general *strictly larger* than $\mathrm{GL}(m, R) \otimes \mathrm{GL}(n, R)$. As can be seen from the proof of Proposition 1, the matrices in $G(R)$ still can be expressed as tensor products of two matrices of degrees m and n , respectively. However, the entries of those matrices may fail to belong to the ring R itself. They can be taken from its extension S that splits the corresponding element of the Picard group. This means that Theorem 1 can be rephrased in the following down to earth fashion.

Theorem 2. *For any commutative ring R , the set $G(R)$ is a group and*

$$\mathrm{GL}(m, R) \otimes \mathrm{GL}(n, R) \trianglelefteq G(R).$$

Every element $g \in G(R)$ can be expressed in the form

$$g = x \otimes y, \quad x \in \mathrm{GL}(m, S), \quad y \in \mathrm{GL}(n, S),$$

where S is an extension of the ground ring R .

All claims of this theorem, apart from the normality of the group $\mathrm{GL}(m, R) \otimes \mathrm{GL}(n, R)$ in $G(R)$, immediately follow from Theorem 1 (or from its proof). On the other hand, normality is an immediate consequence of the following stronger result, asserting that not only the tensor product itself, but both factors are already normal in $G(R)$.

Proposition 2. *For any commutative ring R we have*

$$\mathrm{GL}(m, R) \otimes e, e \otimes \mathrm{GL}(n, R) \trianglelefteq G(R).$$

Proof. We prove that for $y \in \mathrm{GL}(n, R)$ and $g \in (\mathrm{GL}_m \otimes \mathrm{GL}_n)(R)$ the matrix $g(e \otimes y)g^{-1}$ belongs to $e \otimes \mathrm{GL}(n, R)$. Obviously, the case of $g(x \otimes e)g^{-1}$ can be considered in exactly the same way. We wish to verify that

$$(g(e \otimes y)g^{-1})_{kl}^{ij} = 0, \quad (g(e \otimes y)g^{-1})_{kl}^{ii} = (g(e \otimes y)g^{-1})_{kl}^{jj}$$

for all $1 \leq k, l \leq n, 1 \leq i \neq j \leq m$.

Clearly,

$$(g(e \otimes y)g^{-1})_{kl}^{ij} = \sum_{u,v=1}^n y_{uv} \sum_{s=1}^m g_{ku}^{is} \tilde{g}_{vl}^{sj}.$$

Thus, to prove our statement it suffices to verify the following identities:

$$\sum_{s=1}^m g_{ku}^{is} \tilde{g}_{vl}^{sj} = 0, \quad \sum_{s=1}^m g_{ku}^{is} \tilde{g}_{vl}^{si} = \sum_{s=1}^m g_{ku}^{js} \tilde{g}_{vl}^{sj},$$

where $1 \leq k, l, u, v \leq n, 1 \leq i \neq j \leq m$.

By the last statement of Theorem 2 there exists an extension S of the ring R and matrices $x \in \mathrm{GL}(m, S), z \in \mathrm{GL}(n, S)$ such that $g = x \otimes z$. But then, obviously,

$$\sum_{s=1}^m g_{ku}^{is} \tilde{g}_{vl}^{sj} = \sum_{s=1}^m x_{is} z_{ku} x'_{sj} z'_{vl} = z_{ku} z'_{vl} \sum_{s=1}^m x_{is} x'_{sj} = \delta_{ij} z_{ku} z'_{vl},$$

as claimed. $\square$

§8. THE GROUP $E(m, R) \otimes E(n, R)$

The decomposition

$$E(m, R) \otimes E(n, R) = (E(m, R) \otimes e)(e \otimes E(n, R))$$

implies that the group $E(m, R) \otimes E(n, R)$ is generated by the following two types of large transvections:

$$T^{ij}(\xi) = e + \xi \sum_{p=1}^m e_{pp}^{ij} = t_{ij}(\xi) \otimes e, \quad 1 \leq i \neq j \leq m, \quad \xi \in R,$$

and

$$T_{pq}(\xi) = e + \xi \sum_{i=1}^n e_{pq}^{ii} = e \otimes t_{pq}(\xi), \quad 1 \leq p \neq q \leq n, \quad \xi \in R,$$

respectively.

More precisely,

$$E(m, R) \otimes e = \langle T^{ij}(\xi), 1 \leq i \neq j \leq m, \xi \in R \rangle,$$

$$e \otimes E(n, R) = \langle T_{pq}(\xi), 1 \leq p \neq q \leq n, \xi \in R \rangle.$$

The following relations follow immediately from the mutual distributivity of the Kronecker product and the usual matrix multiplication.

- For any pairwise distinct indices $1 \leq i, j, h \leq m$ and any $\xi, \zeta \in R$ we have

$$[T^{ij}(\xi), T^{jh}(\zeta)] = T^{ih}(\xi\zeta).$$

- For any pairwise distinct indices $1 \leq p, q, r \leq n$ and any $\xi, \zeta \in R$ we have

$$[T_{pq}(\xi), T_{qr}(\zeta)] = T_{pr}(\xi\zeta).$$

- For any indices $1 \leq i \neq j \leq m$, $1 \leq p \neq q \leq n$ and any $\xi, \zeta \in R$ we have

$$[T^{ij}(\xi), T_{pq}(\zeta)] = e.$$

In the following sections we calculate the normalizer of the group $E(m, R) \otimes E(n, R)$ in $\text{GL}(mn, R)$. We start with some subsidiary results which will be used in this calculation.

Lemma 12. *Let R be a Noetherian commutative ring, and let $m, n \geq 3$. Then the group $E(m, R) \otimes E(n, R)$ is the largest perfect subgroup of $\text{GL}(m, R) \otimes \text{GL}(n, R)$.*

Proof. If H is a perfect subgroup of $\text{GL}(m, R) \otimes \text{GL}(n, R)$, then its projections to both factors are perfect, and thus are contained in $E(m, R)$ and $E(n, R)$, respectively. $\square$

In particular, this implies the following important fact.

Lemma 13. *Let R be a Noetherian commutative ring, and let $m, n \geq 3$. Then the group $E(m, R) \otimes E(n, R)$ is a fully characteristic subgroup of $\text{GL}(m, R) \otimes \text{GL}(n, R)$.*

Suslin's theorem shows that, for $m, n \geq 3$,

$$E(m, R) \otimes E(n, R) \trianglelefteq \text{GL}(m, R) \otimes \text{GL}(n, R).$$

Now, Lemma 4 immediately implies a more precise result, asserting that this elementary subgroup is in fact normal in the corresponding algebraic group. Surely, this is a very special case of Theorem 1 in the paper by Petrov and Stavrova [31]. The proof in that paper, as ultimately our proof as well, is based on localization methods. With some perseverance, one could give a direct elementary proof, using equations in the style of [73].

Proposition 3. *Let R be an arbitrary commutative ring, and let $m, n \geq 3$. Then*

$$E(m, R) \otimes e, e \otimes E(n, R) \trianglelefteq (\text{GL}_m \otimes \text{GL}_n)(R).$$

Proof. Indeed, from Proposition 2 we know that $\text{GL}(m, R) \otimes e$ and $e \otimes \text{GL}(n, R)$ are both normal in $(\text{GL}_m \otimes \text{GL}_n)(R)$. However, a characteristic subgroup of a normal subgroup is itself a normal subgroup. Thus, to finish the proof it only remains to refer to Lemma 9, or else to Lemmas 8 and 5. $\square$

Proposition 4. *Let R be an arbitrary commutative ring, and let $m, n \geq 3$. Then*

$$E(m, R) \otimes E(n, R) \trianglelefteq (\text{GL}_m \otimes \text{GL}_n)(R).$$

Proof. The product of two normal subgroups is usually a normal subgroup itself. $\square$

§9. DECOMPOSITION OF $E(n, R)$ INTO A CENTRAL PRODUCT

Here we start preparation to the calculation of the normalizer of the group $E(m, R) \otimes E(n, R)$ in $\text{GL}(n, R)$. For this, we recall two classical structure results on the groups $\text{GL}(n, R)$, description of their normal subgroups, and description of their epimorphisms.

As usual, let R be a commutative ring, and let $I \trianglelefteq R$ be an ideal of R . Consider the factor-ring R/I and denote by $\rho_I : R \rightarrow R/I$ the canonical projection, which maps $\lambda \in R$ to $\bar{\lambda} = \lambda + I \in R/I$. Application of this projection to all matrix entries gives rise to the reduction homomorphism $\rho_I : \text{GL}(n, R) \rightarrow \text{GL}(n, R/I)$ modulo I , assigning to a matrix $a = (a_{ij})$ its class $\bar{a} = (\bar{a}_{ij})$ modulo I . The kernel of the homomorphism ρ_I is denoted by $\text{GL}(n, R, I)$ and is called the *principal congruence subgroup* of level I in G .

The full preimage of the center of the group $\text{GL}(n, R/I)$ is denoted by $C(n, R, I)$ and is called the *full congruence subgroup* of level I . The group $C(n, R, I)$ consists of all matrices congruent to a scalar matrix modulo I . The results of the present paper heavily depend on the following structure results, known as the *Wilson–Golubchik theorem* and the *standard commutator formula*.

Lemma 14. *If R is commutative and $n \geq 3$, then for every subgroup $H \leq \mathrm{GL}(n, R)$ normalized by $E(n, R)$, there exists a unique ideal $I \trianglelefteq R$ such that*

$$E(n, R, I) \leq H \leq C(n, R, I).$$

Lemma 15. *If R is commutative and $n \geq 3$, then*

$$[E(n, R), C(n, R, I)] = E(n, R, I).$$

Easy proofs of these results can be found, for instance, in the papers by Vaserstein [78], Borevich–Vavilov [3] or Stepanov–Vavilov [73].

Now we exemplify a typical use of Lemma 14.

Lemma 16. *Suppose R is commutative and $n \geq 3$. Also, suppose that $\mathrm{GL}(n, R)$ is decomposed into a central product $\mathrm{GL}(n, R) = F \circ H$ of two subgroups F and H . Then the ring R can be decomposed into a direct sum $R = A \oplus B$ of two ideals A and B in such a way that*

$$E(n, A) \leq F \leq \mathrm{GL}(n, A)B^*, \quad E(n, B) \leq H \leq \mathrm{GL}(n, B)A^*.$$

Proof. Indeed, being factors of a central product, the two subgroups F and H are normal in $\mathrm{GL}(n, R)$, so that by Lemma 13 there exist ideals A and B such that

$$E(n, R, A) \leq F \leq C(n, R, A), \quad E(n, R, B) \leq H \leq C(n, R, B).$$

It remains to verify that the ring R is their direct sum. For this, observe that

$$E(n, R, A \cap B) \leq E(n, R, A) \cap E(n, R, B) \leq F \cap H \leq C(n, R),$$

whence $A \cap B = 0$. Similarly,

$$\mathrm{GL}(n, R) = FH \leq C(n, R, A) \cdot C(n, R, B) \leq C(n, R, A + B),$$

whence $A + B = R$.

Now, the decomposition $R = A \oplus B$ implies the identities

$$\begin{aligned} E(n, R, A) &= E(n, A), & E(n, R, B) &= E(n, B), \\ \mathrm{GL}(n, R, A) &= \mathrm{GL}(n, A), & \mathrm{GL}(n, R, B) &= \mathrm{GL}(n, B), \\ C(n, R, A) &= \mathrm{GL}(n, A)B^*, & C(n, R, B) &= \mathrm{GL}(n, B)A^*. \end{aligned} \quad \square$$

Observe that the decomposition $R = A \oplus B$ out of hand implies also that

$$\mathrm{GL}(n, A) \cap E(n, R) = E(n, A), \quad \mathrm{GL}(n, B) \cap E(n, R) = E(n, B),$$

and thus,

$$\mathrm{GL}(n, A)B^* \cap E(n, R) = E(n, A)\mu_n(B), \quad \mathrm{GL}(n, B)A^* \cap E(n, R) = E(n, B)\mu_n(A).$$

As a matter of fact, this lemma is a key step in the description of isomorphisms of general linear groups. Thus, in [21, Proposition 1], Golubchik and Mikhalev established a slightly weaker statement for arbitrary associative rings (at least under the additional assumption $2 \in R^*$) regardless of the standard description of normal subgroups, including some cases where this standard description even fails, in general!

§10. EPIMORPHISMS BETWEEN GROUPS $\mathrm{GL}(n, R)$

Now, we recall the *standard* answer to the problem of description of *epimorphisms* between groups $\mathrm{GL}(n, R)$. For that, we list four obvious types of homomorphisms from $\mathrm{GL}(n, R)$ to $\mathrm{GL}(n, S)$.

- *Ring homomorphisms.* The assignment $R \rightsquigarrow \mathrm{GL}(n, R)$ is a *functor* from the category of commutative rings to the category of groups. In particular, any ring homomorphism $\rho \in \mathrm{Hom}(R, S)$ begets the corresponding group homomorphism $\rho : \mathrm{GL}(n, R) \rightarrow \mathrm{GL}(n, S)$. Two most important special cases should be mentioned.

- A ring automorphism $\rho \in \mathrm{Aut}(R)$ begets the corresponding group automorphism $\rho \in \mathrm{Aut}(\mathrm{GL}(n, R))$.

- The reduction homomorphism $\rho_I : R \rightarrow R/I$ begets the reduction homomorphism $\rho_I : \mathrm{GL}(n, R) \rightarrow \mathrm{GL}(n, R/I)$, which we saw in the previous section.

- *Inner automorphisms.* Let S be a ring extension of R . In accordance with the previous item, $\mathrm{GL}(n, R) \leq \mathrm{GL}(n, S)$. Let $g \in \mathrm{GL}(n, S)$ be a matrix normalizing $\mathrm{GL}(n, R)$. Then the map $\iota_g : x \mapsto g x g^{-1}$ is called the *inner automorphism* of the group $\mathrm{GL}(n, R)$ represented by (or else, induced by) the matrix g .

Understandably, in general, such an automorphism ι_g may fail to be an inner automorphism of the *abstract* group $\mathrm{GL}(n, R)$, in the usual sense. However, working with groups of points of algebraic groups over rings (and often even over non algebraically closed fields!), it is an absolute must to stand on an *algebraic* viewpoint, rather than an abstract (= set-theoretical) viewpoint.

- *Graph automorphisms.* Recall that $x^{-t} = (x^t)^{-1} \in \mathrm{GL}(n, R^\circ)$ is called the matrix contragradient to $x \in \mathrm{GL}(n, R)$. For a commutative ring R , the contragradient map $\mathrm{GL}(n, R) \rightarrow \mathrm{GL}(n, R)$, $x \mapsto x^{-t}$, is an automorphism of $\mathrm{GL}(n, R)$. In the same vein, if $R = A \oplus B$, then the map $(x_1, x_2) \mapsto (x_1, x_2^{-t})$ determines an automorphism of $\mathrm{GL}(n, R)$, identical on $\mathrm{GL}(n, A)$ and contragradient on $\mathrm{GL}(n, B)$. Automorphisms of this type are called *graph automorphisms*.

- *Central automorphisms.* Finally, let $C(n, R) \cong R^*$ be the center of the group $\mathrm{GL}(n, R)$, and $\chi : \mathrm{GL}(n, R) \rightarrow R^*$ a linear character of $\mathrm{GL}(n, R)$. Then the map $\hat{\chi} : G \rightarrow G$, $x \mapsto \chi(x)x$, is an automorphism of $\mathrm{GL}(n, R)$, called a central automorphism.

Since the group $E(n, R)$ is perfect, whereas the group R^* is Abelian, it follows that the group $E(n, R)$ does not have any nontrivial central automorphisms.

Compositions of the above four types of automorphisms are called the *standard automorphisms* of the group $\mathrm{GL}(n, R)$. The standard description of automorphisms is the claim that all abstract automorphisms of the group $\mathrm{GL}(n, R)$ are actually standard. Essentially, this is indeed the case for commutative rings. Under the assumption $2 \in R^*$, the following result was first established in full generality in a paper by William Waterhouse [83], and for $n \geq 4$ it was first proved by Vasily Petechuk [28]. Here, we do not discuss noncommutative generalizations of this result, in particular, the remarkable groundbreaking papers by Igor Golubchik, A. V. Mikhalev and Efim Zelmanov [20]–[22], etc.

Lemma 17. *Suppose R is a commutative ring and $n \geq 3$. If $n = 3$, assume additionally that $2 \in R^*$. Then every automorphism of the group $\mathrm{GL}(n, R)$ is standard.*

Lemma 18. *Let $E(m, A) \cong E(n, B)$ for some commutative rings A, B and some $m, n \geq 3$. Then $m = n$. If, moreover, $m \geq 4$ or $2 \in A^*$, then $A \cong B$.*

Observe, though, that in the case where $m = n = 3$ some proviso clauses are necessary here. As a matter of fact, Petechuk, Fu An Li and Zun Xian Li [54, 55] constructed an

isomorphism $\mathrm{SL}(3, \mathbb{Z}/4\mathbb{Z}) \cong \mathrm{SL}(3, \mathbb{F}_0 2[d])$ between the special linear groups of degree 3 over the residue ring modulo 4 and over the ring $\mathbb{F}_0 2[d]$, $d^2 = 0$, of dual numbers.

It will be essential for us that, for commutative rings the first claim of the above result is true for arbitrary *epimorphisms*; see [29]. We state this result in the form that will be used in the calculation of the normalizer of $E(m, R) \otimes E(n, R)$.

Lemma 19. *Assume that the rings A, B are commutative and $m, n \geq 3$. If there exists an epimorphism of elementary groups $\phi : E(m, A) \longrightarrow \mathrm{PE}(n, B)$, then $m = n$.*

§11. THE NORMALIZERS OF $E(m, R) \otimes e$ AND $e \otimes E(n, R)$

We proceed to prepare the calculation of the normalizer of $E(m, R) \otimes E(n, R)$. With this end in mind, first we calculate the normalizers of $E(m, R) \otimes e$ and $e \otimes E(n, R)$. It is very easy to calculate the centralizers of these subgroups.

Lemma 20. *Suppose R is a commutative ring and $m, n \geq 2$.*

• *If $g \in \mathrm{GL}(mn, R)$ commutes with all large transvections $T^{ij}(1)$, $1 \leq i \neq j \leq m$, then $g \in (\mathrm{GL}_1 \otimes \mathrm{GL}_n)(R)$.*

• *If $g \in \mathrm{GL}(mn, R)$ commutes with all large transvections $T_{pq}(1)$, $1 \leq p \neq q \leq n$, then $g \in (\mathrm{GL}_m \otimes \mathrm{GL}_1)(R)$.*

Proof. We prove the second claim; the proof of the first is quite similar.

The assumption that g commutes with the large transvections $T_{pq}(1)$, $1 \leq p \neq q \leq n$, amounts to saying that all blocks $g_{**}^{ij} \in M(n, R)$ of the matrix g commute with all usual transvections $t_{pq}(1)$, $1 \leq p \neq q \leq n$. But then, by Lemma 2, all blocks are central, in other words, are of the form $g_{**}^{ij} = \lambda x_{ij} e$ for some $\lambda, x_{ij} \in S$, $1 \leq i, j \leq n$, from an appropriate extension S/R . But this means precisely that $g = x \otimes \lambda$, where $x = (x_{ij}) \in M(n, S)$. Since g is invertible, we have $x \in \mathrm{GL}(n, S)$, $\lambda \in S^*$, as claimed. $\square$

Now, we are all set to prove the main result of the present section. We could have proved this result directly, without any reference to deep external results. This could be done by using equations in the style of §§ 6 and 7. There is little doubt, though, that it is *much* easier to summon up the standard description of automorphisms of the group $E(n, R)$, as we do below.

Theorem 3. *Suppose R is a commutative ring and $m, n \geq 3$.*

• *If $m = 3$, assume additionally that $2 \in R^*$. Then*

$$N_G(E(m, R) \otimes e) = (\mathrm{GL}_m \otimes \mathrm{GL}_n)(R).$$

• *If $n = 3$, assume additionally that $2 \in R^*$. Then*

$$N_G(e \otimes E(n, R)) = (\mathrm{GL}_m \otimes \mathrm{GL}_n)(R).$$

Proof. We limit ourselves to the proof of the second claim; the proof of the first one is quite similar.

Proposition 2 asserts that $(\mathrm{GL}_m \otimes \mathrm{GL}_n)(R)$ normalizes $e \otimes E(n, R)$. In other words, it is contained in $N_G(e \otimes E(n, R))$. Conversely, suppose that $g \in \mathrm{GL}(mn, R)$ normalizes $e \otimes E(n, R)$. In this case, for every $y \in E(n, R)$ we have

$$g(e \otimes y)g^{-1} = e \otimes \beta(y) \in E(n, R).$$

Clearly, $\beta : E(n, R) \longrightarrow E(n, R)$ is an automorphism of $E(n, R)$.

Now, we can invoke the standard description of the automorphisms of $E(n, R)$. By Lemma 20, the ring automorphism part of the automorphism β is trivial.

A similar argument assures that the graph automorphism part of the automorphism β is also trivial. Indeed, if this is not the case, then β must induce a nontrivial graph

automorphism of the group $E(n, R/M)$ for some maximal ideal $M \in \text{Max}(R)$. Denote by π the canonical projection $R \rightarrow R/M$. By assumption, conjugation by $\pi(g) \in \text{GL}(mn, R/M)$ induces a nontrivial graph automorphism of the group $E(n, R/M)$ over a field.

It is well known that this is impossible. One can easily check this by exactly the same argument as in the proof of the previous lemma. For this, we transfer the matrix g to the right-hand side of the relation

$$gT_{pq}(1)g^{-1} = T_{n+1-q, n+1-p}(\pm 1)$$

and then compare the respective entries of matrices on the left and on the right. Now, recalling that $n \geq 3$, we see that all blocks of g are zero, $g_{**}^{ij} = 0$, which is impossible.

Thus, the automorphism β must be inner. In other words, there exists an extension S/R and a matrix $h \in \text{GL}(n, S)$ such that $\beta(y) = hyh^{-1}$ for all $y \in E(n, R)$. Then $g(e \otimes h)^{-1}$ belongs to centralizer of $e \otimes E(n, R)$, and to finish the proof it only remains to invoke the previous lemma. $\square$

§12. THE NORMALIZER OF $E(m, R) \otimes E(n, R)$

Now, we are all set to prove one of the main results of this part of our paper. It is both interesting in itself and instrumental in the proof of our main results on description of overgroups. For simplicity, we limit ourselves with the statement for the easier case where $m \neq n$, because only this case will be used in the next part of the present work.

Theorem 4. *Suppose R is an arbitrary commutative ring, $m, n \geq 3$, and $m \neq n$. Denote*

$$E = E(m, R) \otimes E(n, R), \quad \Gamma = \text{GL}(m, R) \otimes \text{GL}(n, R),$$

$G = \text{GL}(mn, R)$. *Then*

$$N_G(E) = N_G(\Gamma) = \text{Tran}_G(E, \Gamma) = (\text{GL}_m \otimes \text{GL}_n)(R).$$

Proof. Obviously, $N_G(E), N_G(\Gamma) \leq \text{Tran}_G(E, \Gamma)$. On the other hand, let $g \in \text{GL}(mn, R)$ be such that $gEg^{-1} \leq \Gamma$. Since the subgroup gEg^{-1} is perfect, it is contained in E by Lemma 7. Since the same is true with g replaced by g^{-1} , we have $gEg^{-1} = E$. Thus, $\text{Tran}_G(E, \Gamma) \leq N_G(E)$, whence $N_G(E) = \text{Tran}_G(E, \Gamma)$.

To avoid cumbersome notation, below we shall work in the projective linear group $\text{PGL}(mn, R)$. For brevity, we denote the image of $g \in \text{GL}(mn, R)$ in $\text{PGL}(mn, R)$ not by $[g]$, as usual, but rather by $\bar{g}$. Similar notation will be employed for projectivizations of subgroups $H \leq \text{GL}(mn, R)$. In particular, $\bar{G} = \text{GL}(mn, R)/C(mn, R)$; we recall that, generally speaking, $\bar{G} \neq \text{PGL}(mn, R)$. Similarly, the projectivization of the group $E = E(m, R) \otimes E(n, R)$ will be denoted by $\bar{E}$. Upon projectivization, a central product becomes a direct product, so that

$$\bar{E} = \text{PE}(m, R) \times \text{PE}(n, R).$$

Now, let $g \in N_G(E)$, and, accordingly, $\bar{g} \in N_{\bar{G}}(\bar{E})$. Then, for every $\bar{x} \in \bar{E}(m, R)$,

$$\bar{g}(\overline{x \otimes e})\bar{g}^{-1} = \alpha(\bar{x}) \times \beta(\bar{x})$$

for some maps

$$\alpha : \text{PE}(m, R) \rightarrow \text{PE}(m, R), \quad \beta : \text{PE}(m, R) \rightarrow \text{PE}(n, R),$$

where the binary operation $\times$ is induced by $\otimes$. It is easily seen that α, β are in fact homomorphisms:

$$\begin{aligned} \alpha(\bar{u}\bar{v}) \times \beta(\bar{u}\bar{v}) &= \bar{g}(\overline{uv \otimes e})\bar{g}^{-1} = \bar{g}(\overline{u \otimes e})\bar{g}^{-1} \cdot \bar{g}(\overline{v \otimes e})\bar{g}^{-1} \\ &= (\alpha(\bar{u}) \times \beta(\bar{u}))(\alpha(\bar{v}) \times \beta(\bar{v})) = \alpha(\bar{u})\alpha(\bar{v}) \times \beta(\bar{u})\beta(\bar{v}). \end{aligned}$$

Obviously, $\text{Ker}(\alpha) \cap \text{Ker}(\beta) = 1$.

By the same token, for every $\bar{y} \in \text{PE}(n, R)$ we have

$$\bar{g}(\overline{e \otimes y})\bar{g}^{-1} = \gamma(\bar{y}) \times \delta(\bar{y})$$

for some homomorphisms

$$\gamma : \text{PE}(n, R) \longrightarrow \text{PE}(m, R), \quad \delta : \text{PE}(n, R) \longrightarrow \text{PE}(n, R)$$

such that $\text{Ker}(\gamma) \cap \text{Ker}(\delta) = 1$.

Evidently,

$$\text{PE}(m, R) \times \text{PE}(n, R) = \bar{g}(\text{PE}(m, R) \times e)\bar{g}^{-1} \times \bar{g}(e \times \text{PE}(n, R))\bar{g}^{-1},$$

which yields the direct product decompositions

$$\text{PE}(m, R) = \alpha(\text{PE}(m, R)) \times \gamma(\text{PE}(n, R)),$$

$$\text{PE}(n, R) = \beta(\text{PE}(m, R)) \times \delta(\text{PE}(n, R)).$$

Next, denote by $\alpha(E(m, R))$ the full preimage of $\alpha(\text{PE}(m, R))$ in $E(m, R)$, and similarly for β, γ, δ . In this notation, the elementary groups of degrees m and n admit the following central product decompositions:

$$E(m, R) = \alpha(E(m, R)) \circ \gamma(E(n, R)), \quad E(n, R) = \beta(E(m, R)) \circ \delta(E(n, R)).$$

Now, we can apply Lemma 16 to these decompositions. Consider, say, the first of them; the second can be analyzed similarly. Lemma 16 implies that the ring R decomposes into the direct sum $R = A \oplus B$ of its ideals $A, B \trianglelefteq R$ such that

$$E(m, A) \leq \alpha(E(m, R)) \leq C(m, R, A) \cap E(m, R) = E(m, A)\mu_m(B),$$

$$E(m, B) \leq \gamma(E(n, R)) \leq C(m, R, B) \cap E(m, R) = E(m, B)\mu_m(A).$$

Since all occurring elementary subgroups $E(m, A)$, $E(m, B)$, $E(m, R)$, and $E(n, R)$ are perfect, it immediately follows that

$$\alpha(E(m, R)) = E(m, A), \quad \gamma(E(n, R)) = E(m, B),$$

whence

$$\alpha(\text{PE}(m, R)) = \text{PE}(m, A), \quad \gamma(\text{PE}(n, R)) = \text{PE}(m, B).$$

The case where $m \neq n$ is a straightforward consequence of the description of epimorphisms between groups $\text{PE}(n, R)$, recalled in §10. Namely, Lemma 19 implies that for $B \neq 0$, there are no nontrivial epimorphisms $\gamma : \text{PE}(n, R) \longrightarrow \text{PE}(m, B)$. Thus, for $m \neq n$ one can conclude straight away that $\gamma = e$. Obviously, in exactly the same way we see that $\beta = e$ in this case.

This means that if $m \neq n$, then, automatically,

$$g(E(m, R) \otimes e)g^{-1} = E(m, R) \otimes e, \quad g(e \otimes E(n, R))g^{-1} = e \otimes E(n, R)$$

for any $g \in N_G(E)$. Now, recalling that, by assumption, either $m \geq 4$ or $n \geq 4$, we can apply Theorem 3 and conclude that the normalizer of $E = E(m, R) \otimes E(n, R)$ in $G = \text{GL}(mn, R)$ coincides with $(\text{GL}_m \otimes \text{GL}_n)(R)$. $\square$

Notice, that — apart from the last two paragraphs — the above proof is entirely functional also in the case where $m = n$. Now, replacing the reference to the standard description of epimorphisms of elementary groups by the reference to the standard description of automorphisms of $E(m, R \oplus R)/R^*$ (see, in particular, [84]), we get the following result.

Theorem 4bis. *Suppose R is an arbitrary commutative ring and $m, n \geq 3$. If $m = n = 3$, assume additionally that $2 \in R^*$. Then*

$$N_G(E) = N_G(\Gamma) = \text{Tran}_G(E, \Gamma) = \text{Tran}_G(E, N_G(E)).$$

Clearly, for $m = n$ the normalizer of $E(n, R) \otimes E(n, R)$ is not connected, and thus, is strictly larger than $(\text{GL}_n \otimes \text{GL}_n)(R)$. Specifically, this is due to the existence of conjugations that switch the factors.

§13. ELEMENTARY TRANSVECTIONS IN SUBGROUPS NORMALIZED BY $E(m, R) \otimes E(n, R)$

The remaining part of the present paper is devoted to the study of the usual elementary [linear] transvections in $\text{GL}(mn, R)$ contained in subgroups normalized by $E(m, R) \otimes E(n, R)$. In doing this, it will be expedient to revert to the notation for elementary transvections consentaneous with the blockwise notation of matrix entries of Kronecker products. Namely, for $(i, p) \neq (j, q)$ we write

$$t_{pq}^{ij}(\zeta) = e + \zeta e_{ij} \otimes e_{pq}$$

instead of $t_{(i-1)n+p, (j-1)n+q}(\zeta)$. Clearly, every transvection $t_{rs}(\zeta)$, $1 \leq r \neq s \leq mn$, can be expressed in this notation.

In this new notation, the Chevalley commutator formula looks like this.

- For all pairwise distinct pairs of indices (i, p) , (j, q) , (k, r) and all $\xi, \zeta \in R$, we have

$$[t_{pq}^{ij}(\xi), t_{qr}^{jk}(\zeta)] = t_{pr}^{ik}(\xi\zeta), \quad [t_{pq}^{ij}(\xi), t_{rp}^{ki}(\zeta)] = t_{rp}^{ki}(-\zeta\xi).$$

- For all pairs of indices $(i, p) \neq (j, q)$ and $(k, r) \neq (l, s)$ such that $(i, p) \neq (l, s)$ and $(j, q) \neq (k, r)$, and all $\xi, \zeta \in R$, we have

$$[t_{pq}^{ij}(\xi), t_{rs}^{kl}(\zeta)] = e.$$

Next, we produce formulas relating large transvections $T^{ij}(\xi)$ with the usual elementary [linear] transvections $t_{rs}^{kl}(\zeta)$. These identities immediately follow from the usual commutator formulas for elementary transvections together with expressions of large transvections as products of pairwise commuting elementary transvections,

$$T^{ij}(\xi) = \prod_{1 \leq p \leq n} t_{pp}^{ij}(\xi), \quad T_{pq}(\xi) = \prod_{1 \leq i \leq m} t_{pq}^{ii}(\xi).$$

- For all $1 \leq k \neq i \neq j \leq m$ such that $(j, r) \neq (k, s)$, and all $\xi, \zeta \in R$, we have

$$[T^{ij}(\xi), t_{rs}^{jk}(\zeta)] = t_{rs}^{ik}(\xi\zeta), \quad [t_{sr}^{kj}(\xi), T^{ji}(\zeta)] = t_{sr}^{ki}(\xi\zeta).$$

- For all $1 \leq i \neq j \leq m$ such that $(k, r) \neq (l, s)$, and $i \neq l, j \neq k$, and all $\xi, \zeta \in R$, we have

$$[T^{ij}(\xi), t_{rs}^{kl}(\zeta)] = e.$$

Also, we reproduce yet another, a slightly fancier formula. It looks implausible, because it dissimulates commutation of opposite transvections.

- For all $1 \leq i \neq j \leq m$ and $1 \leq r \neq s \leq n$, and all $\xi, \zeta \in R$, we have

$$\begin{aligned} [T^{ij}(\xi), t_{rs}^{ji}(\zeta)] &= [t_{rr}^{ij}(\xi)t_{ss}^{ji}(\xi), t_{rs}^{ji}(\zeta)] = t_{rr}^{ij}(\xi)[t_{ss}^{ji}(\xi), t_{rs}^{ji}(\zeta)] \cdot [t_{rr}^{ij}(\xi), t_{rs}^{ji}(\zeta)] \\ &= t_{rr}^{ij}(\xi)t_{rs}^{ji}(-\zeta\xi)t_{rr}^{ij}(-\xi)t_{rs}^{ji}(\xi\zeta) = t_{rs}^{ij}(-\xi\zeta\xi)t_{rs}^{ji}(-\zeta\xi)t_{rs}^{ii}(\xi\zeta). \end{aligned}$$

One could easily write similar commutator formulas also for the second kind of large transvections.

Lemma 21. *For any ideal $A \trianglelefteq R$, we have*

$$E(mn, R, A) = E(mn, A)^{E(m, R) \otimes E(n, R)}.$$

Proof. Obviously, the left-hand side is contained in the right-hand side. To prove the reverse inclusion, it suffices to verify that for all $1 \leq r \neq s \leq mn$, $\xi \in A$, $\zeta \in R$, the element

$$z_{rs}(\xi, \zeta) = t_{sr}(\zeta)t_{rs}(\xi)$$

belongs to $H = E(mn, A)^{E(m, R) \otimes E(n, R)}$.

As above, instead of $z_{(i-1)n+p, (j-1)n+q}(\zeta)$ we now write $z_{pq}^{ij}(\zeta)$.

Fix an arbitrary elementary transvection $t_{pq}^{ij}(\xi)$. First, we observe that when $i = j$ or $p = q$, the fact that $z_{pq}^{ij}(\xi, \zeta) \in H$ is obvious.

- Indeed, if $i = j$, then $p \neq q$ and, thus,

$$z_{pq}^{ii}(\xi, \zeta) = T_{qp}(\zeta)t_{pq}^{ii}(\xi).$$

- Similarly, if $p = q$, then $i \neq j$ and, thus,

$$z_{pp}^{ij}(\xi, \zeta) = T^{ji}(\zeta)t_{pp}^{ij}(\xi).$$

In general, it is much harder to prove such an inclusion. To do that, first we observe that any transvection $t_{pq}^{ij}(\xi)$ can be expressed as the commutator of two elementary transvections of the above types, namely,

$$t_{pq}^{ij}(\xi) = [t_{pq}^{ii}(1), t_{qq}^{ij}(\xi)].$$

Thus,

$$z_{pq}^{ij}(\xi, \zeta) = [t_{qp}^{ji}(\zeta)t_{pq}^{ii}(1), t_{qp}^{ji}(\zeta)t_{qq}^{ij}(\xi)] = [t_{qq}^{ji}(\zeta)t_{pq}^{ii}(1), t_{qq}^{ji}(\zeta)t_{qp}^{ii}(-\xi\zeta)].$$

To complete the proof, we decompose the right-hand side with the help of the following formula:

$$[ab, cd] = {}^a[b, c] \cdot {}^{ac}[b, d] \cdot [a, c] \cdot {}^c[a, d].$$

Now, straightforward calculations using the Chevalley commutator formula show that all these factors belong to H . Indeed:

- ${}^a[b, c]$ equals

$$t_{qq}^{ji}(\zeta)[t_{pq}^{ii}(1), t_{qq}^{ij}(\xi)] = t_{qq}^{ji}(\zeta)t_{pq}^{ij}(\xi) = t_{pq}^{ij}(\xi)t_{qp}^{ii}(-\xi\zeta) \in H,$$

- ${}^{ac}[b, d]$ equals

$$\begin{aligned} t_{qq}^{ji}(\zeta)t_{qq}^{ij}(\xi)[t_{pq}^{ii}(1), t_{qp}^{ii}(-\xi\zeta)] &= T^{ji}(\zeta)t_{qq}^{ij}(\xi)[t_{pq}^{ii}(1), t_{qp}^{ii}(-\xi\zeta)] \\ &= t_{pq}^{jj}(\xi^2\zeta^2)t_{pp}^{ji}(\xi\zeta^2 - \xi^2\zeta^3)t_{pq}^{ji}(-\xi\zeta^2 - \xi^2\zeta^3) \in H, \end{aligned}$$

- $[a, c]$ equals

$$[t_{qq}^{ji}(\zeta), t_{qq}^{ij}(\xi)] = [T^{ji}(\zeta), t_{qq}^{ij}(\xi)] \in H,$$

- ${}^c[a, d]$ equals

$$t_{qq}^{ij}(\xi)[t_{qq}^{ji}(\zeta), t_{qp}^{ii}(-\xi\zeta)] = t_{qq}^{ij}(\xi)t_{qp}^{ii}(-\xi\zeta^2) \in H. \quad \square$$

§14. ADMISSIBLE TRIPLES

Let H be a subgroup of $\mathrm{GL}(mn, R)$ containing $E(m, R) \otimes E(n, R)$. In the present section we introduce the notion of an admissible triple which, in our situation, serves as an analog of an admissible pair, or a form ideal.

To justify this notion, we consider elementary transvections contained in H . For $(i, p) \neq (j, q)$ set

$$A_{pq}^{ij} = \{\xi \in R \mid t_{pq}^{ij}(\xi) \in H\}.$$

Clearly, all A_{pq}^{ij} are additive subgroups. First, we show that they are actually ideals. Not to stipulate everywhere that $(i, p) \neq (j, q)$, we additionally set $A_{pp}^{ii} = R$.

Lemma 22. *The additive subgroups $A_{pq}^{ij} \trianglelefteq R$ are ideals for all i, j, p, q .*

Proof. Indeed, let $\xi \in A_{pq}^{ij}$, $\zeta \in R$. Then the commutator formulas of the previous section immediately imply the following four claims.

- Suppose that $r \neq q$, and if $i = j$, then, moreover, $r \neq p$. Then

$$t_{pr}^{ij}(\xi\zeta) = [t_{pq}^{ij}(\xi), t_{qr}^{jj}(\zeta)] = [t_{pq}^{ij}(\xi), T_{qr}(\zeta)] \in H.$$

Thus, $A_{pq}^{ij}R \leq A_{pr}^{ij}$, and, in particular, $A_{pq}^{ij} \leq A_{pr}^{ij}$.

- Suppose that $r \neq p$, and if $i = j$, then, moreover, $r \neq q$. Then

$$t_{rq}^{ij}(\zeta\xi) = [t_{rp}^{ii}(\zeta), t_{pq}^{ij}(\xi)] = [T_{rp}(\zeta), t_{pq}^{ij}(\xi)] \in H.$$

Thus, $RA_{pq}^{ij} \leq A_{rq}^{ij}$, and, in particular, $A_{pq}^{ij} \leq A_{rq}^{ij}$.

- Suppose that $k \neq j$, and if $p = q$, then, moreover, $k \neq i$. Then

$$t_{pq}^{ik}(\xi\zeta) = [t_{pq}^{ij}(\xi), t_{qq}^{jk}(\zeta)] = [t_{pq}^{ij}(\xi), T^{jk}(\zeta)] \in H.$$

Thus, $A_{pq}^{ij}R \leq A_{pq}^{ik}$, and, in particular, $A_{pq}^{ij} \leq A_{pq}^{ik}$.

- Suppose that $k \neq i$, and if $p = q$, then, moreover, $k \neq j$. Then

$$t_{pq}^{kj}(\zeta\xi) = [t_{pp}^{ki}(\zeta), t_{pq}^{ij}(\xi)] = [T^{ki}(\zeta), t_{pq}^{ij}(\xi)] \in H.$$

Thus, $RA_{pq}^{ij} \leq A_{pq}^{kj}$, and, in particular, $A_{pq}^{ij} \leq A_{pq}^{kj}$.

Now, if $(i, p) \neq (j, q)$, then there exists either $r \neq p, q$, or $k \neq i, j$. In the first case $A_{pq}^{ij}R \leq A_{pr}^{ij} \leq A_{pq}^{ij}$, whereas in the second case $A_{pq}^{ij}R \leq A_{pq}^{ik} \leq A_{pq}^{ij}$. $\square$

Observe that we have shown not only that all A_{pq}^{ij} are in fact ideals, but also that there are some inclusions among these ideals. Next, we systematically consider all such inclusions.

Lemma 23. *The ideal A_{pq}^{ij} depends not on i, j, p, q themselves, but only on whether $i = j$ and/or $p = q$.*

Proof. The calculations we performed in the proof of the previous lemma show that $A_{pp}^{kl} = A_{pp}^{ij}$ for $i \neq j$, $k \neq l$. Now we verify that even $A_{pp}^{kl} = A_{qq}^{ij}$ and, similarly, $A_{pq}^{ii} = A_{rs}^{jj}$. Indeed,

$$\begin{aligned} t_{pp}^{ij}(-\xi)t_{qq}^{ij}(\xi)t_{qp}^{ij}(-\xi) &= [T_{qp}(1), t_{pq}^{ij}(\xi)] \\ &= [T_{qp}(1), [t_{pp}^{ij}(\xi), t_{pq}^{jj}(1)]] = [T_{qp}(1), [t_{pp}^{ij}(\xi), T_{pq}(1)]] \in H. \end{aligned}$$

Since $A_{pp}^{ij} \leq A_{qp}^{ij}$, we have $t_{pp}^{ij}(-\xi)t_{qq}^{ij}(\xi) \in H$ and, thus, $t_{qq}^{ij}(\xi) \in H$. This means that $A_{pp}^{ij} \leq A_{qq}^{ij}$. The reverse inclusion is established similarly. Combining this with the above identity, we get $A_{pp}^{kl} = A_{pp}^{ij} = A_{qq}^{ij}$. $\square$

Since we are only interested in the ideals outside of the principal diagonal, for which the relations $i = j$ and $p = q$ cannot occur simultaneously, we have established the following.

- All ideals A_{pq}^{ij} for $i \neq j$ and $p \neq q$ are equal. Denote their common value by A .
- All ideals A_{pq}^{ii} for $p \neq q$ are equal. Denote their common value by B .
- All ideals A_{pp}^{ij} for $i \neq j$ are equal. Denote their common value by C .

Lemma 24. *The ideals A, B, C are interrelated as follows:*

$$mA, A^2 \leq B \leq A, \quad nA, A^2 \leq C \leq A.$$

Proof. The embeddings $B, C \leq A$ have been established in the proof of Lemma 22. Now, let $\xi \in A_{pq}^{ij}$, $i \neq j$, $p \neq q$. Then

$$t_{pp}^{ij}(-\xi)t_{qq}^{ij}(\xi) = [T_{qp}(1), t_{pq}^{ij}(\xi)]t_{qp}^{ij}(\xi) \in H.$$

Since $A_{pq}^{ij} = A_{rq}^{ij}$, $r \neq q$, we have

$$t_{qq}^{ij}(n\xi) = T^{ij}(\xi) \prod_{p=1}^n t_{pp}^{ij}(-\xi)t_{qq}^{ij}(\xi) \in H,$$

so that $nA \leq B$. The inclusion $mA \leq C$ is established similarly. The inclusions $A^2 \leq B, C$ immediately follow from the instance of the Chevalley commutator formula used in the proof of Lemma 21. $\square$

A triple of ideals (A, B, C) satisfying the inclusions listed in the above lemma will be called an *admissible triple*. The standard answer to the problem of description of subgroups in $\mathrm{GL}(mn, R)$ containing $E(m, R) \otimes E(n, R)$ will be stated in terms of such admissible triples.

There is a noteworthy analogy between admissible triples and form ideals (I, Γ) introduced by Anthony Bak [38]; see further references in [41, 49, 50, 52]. In the present situation the ideal A plays the role of a ring ideal I , whereas the two ideals B and C play the role of a relative form parameter Γ .

§15. THE GROUP $E(m, n, R, A, B, C)$

Our next move is to associate a perfect intermediate subgroup with each admissible triple of ideals (A, B, C) . First of all, consider the subgroup generated by the elementary transvections of the corresponding levels, in accordance with the definition of ideals A, B, C :

$$\begin{aligned} E(m, n, A, B, C) \\ = \left\langle t_{pq}^{ij}(\xi), \xi \in A, i \neq j, p \neq q; t_{pq}^{ii}(\zeta), \zeta \in B, p \neq q; t_{pp}^{ij}(\eta), \eta \in C, i \neq j \right\rangle. \end{aligned}$$

Actually, in the description of the subgroups in $\mathrm{GL}(mn, R)$ normalized by $E(m, R) \otimes E(n, R)$, not these subgroups themselves but the slightly larger subgroups

$$E(m, n, R, A, B, C) = E(m, n, A, B, C)^{E(m, R) \otimes E(n, R)}$$

serve as basal subgroups.

Since in the present paper we are only interested in subgroups containing $E(m, R) \otimes E(n, R)$, we introduce yet another elementary subgroup of level (A, B, C) , slightly larger still:

$$\mathrm{EE}(m, n, R, A, B, C) = (E(m, R) \otimes E(n, R))E(m, n, R, A, B, C).$$

With this notation, we can summarize the results of the preceding sections as follows, to get another main result of the present paper. Here and thereafter, talking of the order on admissible triples, we always have in mind the componentwise order.

Theorem 5. *Let R be a commutative ring, let $m, n \geq 3$, and let H be a subgroup of $\mathrm{GL}(mn, R)$ containing $E(m, R) \otimes E(n, R)$. Then there exists a unique largest triple (A, B, C) such that*

$$\mathrm{EE}(m, n, R, A, B, C) \leq H.$$

The largest triple (A, B, C) of this type whose existence is secured by the theorem, is called the *level* of H . By definition, if $t_{pq}^{ij}(\xi) \in H$ for some $(i, p) \neq (j, q)$, then

- $\xi \in A$ if $i \neq j$, $p \neq q$;

- $\xi \in B$ if $i = j$, $p \neq q$;
- $\xi \in C$ if $i \neq j$, $p = q$.

Proposition 5. *For each admissible triple (A, B, C) , the corresponding elementary subgroup $\text{EE}(m, n, R, A, B, C)$ is perfect.*

Proof. It suffices to verify that all generators of the subgroup $\text{EE}(m, n, R, A, B, C)$ sit in its derived subgroup. For large transvections, this immediately follows from the fact that the elementary groups $E(m, R)$ and $E(n, R)$ are perfect. On the other hand, for the usual linear transvections $t_{rs}^{ij}(\xi)$, $\xi \in A, B, C$, this follows from the commutator formulas.

- Indeed, since $m \geq 3$, for any $i \neq j$ we can find $l \neq i, j$. Then

$$t_{rs}^{ij}(\xi) = [t_{rs}^{il}(\xi), T^{lj}(1)].$$

- Similarly, since $n \geq 3$, for any $r \neq s$, we can find $p \neq r, s$. Then

$$t_{rs}^{ij}(\xi) = [t_{rp}^{ij}(\xi), T_{ps}(1)]. \quad \square$$

§16. THE NORMALIZER OF $E(m, n, R, A)$

Observe that when all three ideals of an admissible triple coincide, $A = B = C$, Lemma 21 immediately implies that

$$E(m, n, R, A, A, A) = E(mn, R, A).$$

In this case, instead of $\text{EE}(m, n, R, A, A, A)$ we shall simply write

$$\text{EE}(m, n, R, A) = (E(m, R) \otimes E(n, R))E(mn, R, A).$$

Under the assumption $m, n \in R^*$, the description of overgroups of $E(m, R) \otimes E(n, R)$ simplifies *drastically*, because then automatically $A = B = C$. Thus, in this case the level of an intermediate subgroup is determined by *one* ideal, and the usual level reduction routine can be employed successfully.

As the last major result of this first part of our work, we calculate the normalizer of $\text{EE}(m, n, R, A)$ in $G = \text{GL}(mn, R)$, which will appear in our main results on description of overgroups of $E(m, R) \otimes E(n, R)$. Toward this end, we return to examination of the reduction homomorphism $\rho_A : \text{GL}(mn, R) \rightarrow \text{GL}(mn, R/A)$ and denote by

$$C(m, n, R, A) = \rho_A^{-1}((\text{GL}_m \otimes \text{GL}_n)(R))$$

the full preimage of the group $(\text{GL}_m \otimes \text{GL}_n)(R)$ with respect to ρ_A .

In other words, for every $A \trianglelefteq R$ we have

$$C(m, n, R, A) = \left\{ (g_{pq}^{ij}) \in \text{GL}(mn, R) \mid g_{pq}^{ij} g_{rs}^{kl} \equiv g_{rs}^{ij} g_{pq}^{kl} \pmod{A} \right\}.$$

Now we are all set to prove the main results of the first part of the present paper. The proof of the next theorem closely follows the proof of Theorem 2 in [15] and that of Theorem 2 in [16].

Theorem 6. *Suppose $m, n \geq 3$, $m \neq n$. Then for any ideal $A \trianglelefteq R$ we have*

$$N_G(\text{EE}(m, n, R, A)) = C(m, n, R, A).$$

Proof. It only remains to prove the inclusion

$$N_G(\text{EE}(m, n, R, A)) \geq C(m, n, R, A),$$

because the reverse inclusion already follows from Theorem 5.

First, we observe that

$$[(\text{GL}_m \otimes \text{GL}_n)(R) \cdot \text{GL}(mn, R, A), \text{EE}(m, n, R, A)] \leq \text{EE}(m, n, R, A).$$

Indeed, consider the commutator $[xy, hg]$, where

$$x \in (\mathrm{GL}_m \otimes \mathrm{GL}_n)(R), \quad y \in \mathrm{GL}(mn, R, A), h \in E(m, R) \otimes E(n, R), \quad g \in E(mn, R, A).$$

We rewrite this commutator as a product

$$[xy, hg] = {}^x[y, h] \cdot [x, h] \cdot {}^h[xy, g].$$

It is clear that

- $[y, h] \in E(mn, R, A)$, by the standard commutator formula,
- $[x, h] \in E(m, R) \otimes E(n, R)$, by Proposition 3,
- ${}^x[y, h], {}^h[xy, g] \in E(mn, R, A)$, by Lemma 3.

Applying Proposition 3 and the homomorphism theorem, we get

$$[C(m, n, R, A), \mathrm{EE}(m, n, R, A)] \leq \mathrm{EE}(m, n, R, A) \mathrm{GL}(mn, R, A).$$

Next, set

$$H = [[C(m, n, R, A), \mathrm{EE}(m, n, R, A)], [C(m, n, R, A), \mathrm{EE}(m, n, R, A)]].$$

The previous inclusion implies that H is generated by the commutators of the form

$$[uv, [z, y]], \quad u, y \in \mathrm{EE}(m, n, R, A), \quad v \in \mathrm{GL}(mn, R, A), \quad z \in C(m, n, R, A).$$

We show that $H \leq \mathrm{EE}(m, n, R, A)$. For this, observe that

$$[uv, [z, y]] = {}^u[v, [z, y]] \cdot [u, [z, y]].$$

The second commutator is contained in $\mathrm{EE}(m, n, R, A)$ by the above, whereas the first commutator lies in $E(mn, R, A)$ by the standard commutator formula.

Now we are all set to finish the proof. Since the group $\mathrm{EE}(m, n, R, A)$ is perfect by Proposition 5, it suffices to verify the relation

$$[z, [x, y]] \in \mathrm{EE}(m, n, R, A)$$

for all $x, y \in \mathrm{EE}(m, n, R, A)$, $z \in C(m, n, R, A)$. Indeed, the Hall–Witt identity shows that

$$[z, [x, y]] = {}^{xz}[[z^{-1}, x^{-1}], y] \cdot {}^{xy}[[y^{-1}, z], x^{-1}],$$

where the second commutator already belongs to $\mathrm{EE}(m, n, R, A)$ by the above. We rewrite the first commutator as

$$\begin{aligned} {}^{xz}[[z^{-1}, x^{-1}], y] &= x[z[z^{-1}, x^{-1}], {}^zy]x^{-1} = x[[x^{-1}, z], [z, y]z]x^{-1} \\ &= x([x^{-1}, z], [z, y]) \cdot {}^{[z, y]}[[x^{-1}, z], y]x^{-1}. \end{aligned}$$

The two commutators on the right already belong to $\mathrm{EE}(m, n, R, A)$ by the above, while the conjugating element at the second commutator normalizes $\mathrm{EE}(m, n, R, A)$ because it lies in the group $\mathrm{EE}(m, n, R, A) \mathrm{GL}(mn, R, A)$. $\square$

The main step in the description of intermediate subgroups

$$E(m, R) \otimes E(n, R) \leq H \leq \mathrm{GL}(mn, R)$$

in the next part of the present paper is the proof of the *weak* structure theorem for such subgroups, which asserts that

$$\mathrm{EE}(m, n, R, A, B, C) \leq H \leq C(m, n, R, A),$$

where (A, B, C) is the level of H , as constructed in Theorem 5.

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