THE HOCHSCHILD COHOMOLOGY ALGEBRA FOR A FAMILY OF SELF-INJECTIVE ALGEBRAS OF THE TREE CLASS D_n

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ABSTRACT. The Hochschild cohomology algebra for a series of self-injective algebras of the tree class D_n is described in terms of generators and relations. The proof involves the existing description of the minimal bimodule resolvent and of the additive structure for the Hochschild cohomology algebras in question.

§1. INTRODUCTION

Let R be a representation-finite self-injective basic algebra over an algebraically closed field. The stable AR -quiver of such an algebra can be described with the help of an associated tree, which coincides with one of the Dynkin diagrams A_n, D_n, E_6, E_7 , or E_8 (see [1]). It is known [2] that the Hochschild cohomology of the algebra R is periodic. If the associated tree of R has type A_n , then, by [3], R is stably equivalent to a serial self-injective algebra, or to the so-called “Möbius algebra”. In [4], the Hochschild cohomology algebra $\mathrm{HH}^*(R)$ for serial self-injective algebras was calculated. Furthermore, for the Möbius algebra R , the subalgebra $\mathrm{HH}^{*r}(R)$ of $\mathrm{HH}^*(R)$ was calculated in [5], where $\mathrm{HH}^{*r}(R)$ is the subalgebra generated by homogeneous elements of degree divisible by r ; here r is some parameter linked with the defining relations for R . In [4, 5], the following fact was used substantially: a syzygy of an appropriate order for the R -bimodule R can be described as a twisted bimodule.

A more direct approach to the calculation of the Hochschild cohomology for the Möbius algebra R was suggested in [6]. Namely, in that paper the minimal projective resolution was constructed for the algebra R viewed as a Λ -module, where Λ is the enveloping algebra of R , and then, in [7], this resolution was used for the description of an additive structure of the algebra $\mathrm{HH}^*(R)$, i.e., the dimensions of the groups $\mathrm{HH}^t(R)$ were calculated for the Möbius algebra R .

If the algebra R is of the tree class D_n , then, by [8], R is stably equivalent to an algebra of one of five types. Their bound quivers were presented also in [8]. In [9], the approach of [6] was employed to construct the periodic minimal projective resolution for the algebras of one of these types. Then it was used for the calculation of the additive structure of $\mathrm{HH}^*(R)$. The present paper completes the description of the Hochschild cohomology algebra for the algebras considered in [9]. Using the description of the bimodule resolution and the additive structure of the Hochschild cohomology algebra, we find a finite set of generators of $\mathrm{HH}^*(R)$ and the relations they satisfy.

§2. BIMODULE RESOLUTION AND THE ADDITIVE STRUCTURE OF THE HOCHSCHILD COHOMOLOGY ALGEBRA

In this section, we recall the notation and results from [9].

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Assume that k is an algebraically closed field. We consider the following bound quiver (Q, I) . The set of vertices is $Q_0 = \{i \in \mathbb{N} \mid 1 \leq i \leq rn\}$; in the sequel, the elements of Q_0 are specified modulo rn . The set of arrows Q_1 in the quiver Q consists of the elements

$$\alpha_{s,i} : (s-1)n + i \rightarrow (s-1)n + i + 1,$$

where $s \in \{1, \dots, r\}$, $i \in \{1, \dots, n-3\}$, and

$$\gamma_{s,n-1} : sn - 2 \rightarrow sn - 1, \quad \gamma_{s,n} : sn - 2 \rightarrow sn,$$

$$\beta_{s,n-1} : sn - 1 \rightarrow sn + 1, \quad \beta_{s,n} : sn \rightarrow sn + 1,$$

where $s \in \{1, \dots, r\}$. In this notation for arrows, the first index is determined modulo r .

The ideal I is generated by the following elements of the path algebra kQ of Q :

$$\gamma_{s,n-1}\alpha_{s,n-3} \dots \alpha_{s,1}\beta_{s-1,n} \quad \text{and} \quad \gamma_{s,n}\alpha_{s,n-3} \dots \alpha_{s,1}\beta_{s-1,n-1},$$

$$\beta_{s,n-1}\gamma_{s,n-1} - \beta_{s,n}\gamma_{s,n},$$

$$\alpha_{s+1,t} \dots \alpha_{s+1,1}\beta_{s,n-1}\gamma_{s,n-1}\alpha_{s,n-3} \dots \alpha_{s,t},$$

where $s = 1, \dots, r$, $t = 1, \dots, n-3$. We consider the k -algebra $R = kQ/I$ and denote by $\Lambda = R \otimes R^{\text{op}}$ the enveloping algebra of R . Let $e_{i,j}$ ($1 \leq i \leq r$, $1 \leq j \leq n$) be the idempotent of R corresponding to the vertex $(i-1)n + j$ of the quiver Q . In the sequel, the first index in $e_{i,j}$ will be taken modulo r . Then $\{e_{i_1,j_1} \otimes e_{i_2,j_2}\}_{i_1,j_1,i_2,j_2}$ is a full set of orthogonal primitive idempotents of the algebra Λ . Let $P_{i,j} = Re_{i,j}$ be the projective R -module corresponding to the vertex $(i-1)n + j$ of the quiver Q and $S_{i,j}$ the corresponding simple R -module. We denote by $P_{[i_1,j_1][i_2,j_2]} = \Lambda e_{i_1,j_1} \otimes e_{i_2,j_2}$ the projective Λ -module corresponding to the idempotent $e_{i_1,j_1} \otimes e_{i_2,j_2}$. If w is a path in Q starting at the vertex $(i_1-1)n + j_1$ and ending at the vertex $(i_2-1)n + j_2$, then right multiplication by w induces a homomorphism $w^* : P_{i_2,j_2} \rightarrow P_{i_1,j_1}$. For simplicity, we denote this homomorphism also by w . If w_1 is a path starting at the vertex $(i_1-1)n + j_1$ and ending at the vertex $(i_2-1)n + j_2$, and w_2 is a path starting at the vertex $(i_3-1)n + j_3$ and ending at the vertex $(i_4-1)n + j_4$, then $w_1 \otimes w_2 \in \text{Hom}_{\Lambda}(P_{[i_2,j_2][i_3,j_3]}, P_{[i_1,j_1][i_4,j_4]})$.

We introduce the following auxiliary notation:

$$\tau_i = \beta_{i,n}\gamma_{i,n}, \quad \mu_{i,j} = \alpha_{i,j} \dots \alpha_{i,1}, \quad \nu_{i,j} = \alpha_{i,n-3} \dots \alpha_{i,j}.$$

Here, for uniformity reasons, we assume that an empty product of arrows is identified with an appropriate idempotent of the algebra R ; for example, $\mu_{i,0} = e_{i,1}$, $\alpha_{i,j-1} \dots \alpha_{i,j} = e_{i,j}$, $\nu_{i,n-2} = e_{i,n-2}$.

We introduce a k -linear map $\sigma : R \rightarrow R$ such that $\sigma(xy) = \sigma(x)\sigma(y)$ and

$$\sigma(e_{i,j}) = e_{i+n-1,\phi^n(j)}, \quad \sigma(\alpha_{i,j}) = \alpha_{i+n-1,j}, \quad \sigma(\gamma_{i,n-1}) = -\gamma_{i+n-1,\phi^n(n-1)},$$

$$\sigma(\gamma_{i,n}) = -\gamma_{i+n-1,\phi^n(n)}, \quad \sigma(\beta_{i,n-1}) = \beta_{i+n-1,\phi^n(n-1)}, \quad \sigma(\beta_{i,n}) = \beta_{i+n-1,\phi^n(n)},$$

where $\phi(j) = j$ for $1 \leq j \leq n-2$, $\phi(n-1) = n$, and $\phi(n) = n-1$. Clearly, σ is an automorphism of the algebra R .

In [9], it was shown that σ is a finite order automorphism of R . This order was calculated in the same paper. In the sequel, we use the notation

$$(2.1) \quad \lambda = (2n-3) \operatorname{ord}(\sigma).$$

Consider the following modules:

$$\begin{aligned} Q_{2m} &= \bigoplus_{i=1}^r \left(\left(\bigoplus_{j=1}^{n-2-m} P_{[i+m, j+m][i, j]} \right) \oplus \left(\bigoplus_{j=n-1-m}^{n-2} P_{[i+m, j+m-(n-2)][i, j]} \right) \right. \\ &\quad \left. \oplus P_{[i+m, \phi^m(n-1)][i, n-1]} \oplus P_{[i+m, \phi^m(n)][i, n]} \right) \\ &\quad (m = 0, \dots, n-2), \\ Q_{2m+1} &= \bigoplus_{i=1}^r \left(\left(\bigoplus_{j=1}^{n-3-m} P_{[i+m, j+m+1][i, j]} \right) \oplus P_{[i+m, n-1][i, n-2-m]} \oplus P_{[i+m, n][i, n-2-m]} \right. \\ &\quad \left. \oplus \left(\bigoplus_{j=n-1-m}^{n-2} P_{[i+m+1, j+m-(n-2)][i, j]} \right) \oplus P_{[i+m+1, m+1][i, n-1]} \oplus P_{[i+m+1, m+1][i, n]} \right) \\ &\quad (m = 0, \dots, n-3). \end{aligned}$$

Let $\rho\Lambda$ be the category of finitely generated projective (left) Λ -modules. We introduce the k -linear functor $\sigma : \rho\Lambda \rightarrow \rho\Lambda$. Let $V = \bigoplus_{l \in L} P_{[i_2, l, j_2, l][i_1, l, j_1, l]}$. Then $\sigma(V) = \bigoplus_{l \in L} P_{[\sigma(i_2, l, j_2, l)][i_1, l, j_1, l]}$ (here, $\sigma(i, j) = (x, y) \Leftrightarrow \sigma(e_{i, j}) = e_{x, y}$). Assume that V_1 and V_2 are modules in $\rho\Lambda$, $V_1 = \bigoplus_{l \in L} P_{[i_2, l, j_2, l][i_1, l, j_1, l]}$. Let $d : V_1 \rightarrow V_2$ be a homomorphism of Λ -modules such that $d|_{P_{[i_2, l, j_2, l][i_1, l, j_1, l]}} = \sum_{t \in T} u_{l, t} w_{l, t, 1} \otimes w_{l, t, 2}$, where $u_{l, t} \in k$, $w_{l, t, 1}, w_{l, t, 2} \in R$. Then we define $\sigma(d)|_{P_{[\sigma(i_2, l, j_2, l)][i_1, l, j_1, l]}} = \sum_{t \in T} u_{l, t} \sigma(w_{l, t, 1}) \otimes w_{l, t, 2}$.

Let $\mu : Q_0 \rightarrow R$ be a homomorphism defined as follows: for $w_1 \otimes w_2 \in P_{[i, j][i, j]}$, we put $\mu(w_1 \otimes w_2) = w_1 w_2$. The homomorphisms $d_i : Q_{i+1} \rightarrow Q_i$ for $0 \leq i \leq 2n-4$ (with $Q_{2n-3} = \sigma(Q_0)$) were described in [9]. The following theorem was proved in the same paper.

Theorem 1. *The minimal Λ -projective resolution of the module R is presented by the following sequence:*

$$0 \leftarrow R \xleftarrow{\mu} Q_0 \xleftarrow{d_0} Q_1 \xleftarrow{d_1} \dots \xleftarrow{d_{2n-5}} Q_{2n-4} \xleftarrow{d_{2n-4}} Q_{2n-3} \xleftarrow{d_{2n-3}} Q_{2n-2} \xleftarrow{d_{2n-2}} \dots,$$

where the modules Q_i for $i = 0, \dots, 2n-3$ are as described above, and the homomorphisms d_i for $i = 0, \dots, 2n-4$ are as mentioned before this theorem. Moreover, for $0 \leq t \leq 2n-4$ we put $Q_{l(2n-3)+t} = \sigma^l(Q_t)$, $d_{l(2n-3)+t} = \sigma^l(d_t)$.

Furthermore, in the same paper, the following two theorems were proved, in which the additive structure of the Hochschild cohomology algebra of R was described for $r > 1$ and $r = 1$, respectively.

Theorem 2. *Let $\operatorname{HH}^s(R)$ be the s th group of Hochschild cohomology of the algebra R with coefficients in R . Assume that $r > 1$. Then*

$\dim_k \mathrm{HH}^s(R) = 1$ if one of the following conditions is fulfilled:

$$(2.2) \quad \begin{cases} s \in \{2m + l(2n - 3), 2m + 1 + l(2n - 3)\}, & 0 \leq m \leq n - 3, \\ r \mid m + l(n - 1), & 2 \mid m + ln \text{ and either } \operatorname{char} k = 2 \text{ or } 2 \mid l; \end{cases}$$

$$(2.3) \quad \begin{cases} s \in \{2m + 1 + l(2n - 3), 2m + 2 + l(2n - 3)\}, & 0 \leq m \leq n - 3, \\ r \mid m + l(n - 1), & 2 \nmid m + ln \text{ and either } \operatorname{char} k = 2 \text{ or } 2 \nmid l; \end{cases}$$

$$(2.4) \quad s \in \{(l + 1)(2n - 3) - 1, (l + 1)(2n - 3)\}, \quad r \mid (l + 1)(n - 1) - 1, \\ 2 \nmid ln, \quad \operatorname{char} k = 2;$$

$$(2.5) \quad s \in \{(l + 1)(2n - 3) - 1, (l + 1)(2n - 3)\}, \quad r \mid (l + 1)(n - 1) - 1, \\ 2 \nmid (l + 1)n;$$

$\dim_k \mathrm{HH}^s(R) = 2$ if

$$(2.6) \quad s \in \{(l + 1)(2n - 3) - 1, (l + 1)(2n - 3)\}, \quad r \mid (l + 1)(n - 1) - 1, \\ 2 \mid n, \text{ and either } \operatorname{char} k = 2 \text{ or } 2 \mid l;$$

and $\dim_k \mathrm{HH}^s(R) = 0$ if none of the above conditions holds true.

Theorem 3. Assume that $r = 1$. Then

$\dim_k \mathrm{HH}^s(R) = 1$ if one of the following conditions is fulfilled:

$$(2.7) \quad s \in \{2(n - 2) + l(2n - 3), (l + 1)(2n - 3)\}, \quad 2 \nmid l, \text{ and } \operatorname{char} k \neq 2;$$

$$(2.8) \quad s \in \{2(n - 2) + l(2n - 3), (l + 1)(2n - 3)\}, \quad 2 \nmid (l + 1)n;$$

$$(2.9) \quad s = 2m + 1 + l(2n - 3), \quad 0 \leq m \leq n - 3, \quad 2 \nmid m + ln, \text{ and} \\ \text{either } 2 \nmid l \text{ or } \operatorname{char} k = 2;$$

$$(2.10) \quad s = 2m + 1 + l(2n - 3), \quad 0 \leq m \leq n - 3, \quad 2 \mid m + ln, \text{ and} \\ \text{either } 2 \mid l \text{ or } \operatorname{char} k = 2;$$

$$(2.11) \quad s = 2m + l(2n - 3), \quad 1 \leq m \leq n - 3, \quad 2 \mid m + ln, \text{ and } \operatorname{char} k \neq 2;$$

$\dim_k \mathrm{HH}^s(R) = 2$ if one of the following conditions is fulfilled:

$$(2.12) \quad s \in \{2(n - 2) + l(2n - 3), (l + 1)(2n - 3)\}, \quad 2 \mid n, \quad 2 \mid l, \text{ and } \operatorname{char} k \neq 2;$$

$$(2.13) \quad s \in \{2(n - 2) + l(2n - 3), (l + 1)(2n - 3)\}, \quad 2 \nmid ln, \text{ and } \operatorname{char} k = 2;$$

$$(2.14) \quad s = 2m + l(2n - 3), \quad 1 \leq m \leq n - 3, \quad 2 \mid m + ln, \text{ and } \operatorname{char} k = 2;$$

$\dim_k \mathrm{HH}^s(R) = 3$ if

$$(2.15) \quad s \in \{2(n - 2) + l(2n - 3), (l + 1)(2n - 3)\}, \quad 2 \mid n, \text{ and } \operatorname{char} k = 2;$$

$\dim_k \mathrm{HH}^s(R) = n + 1$ if $s = 0$;

and $\dim_k \mathrm{HH}^s(R) = 0$ if none of the above conditions holds true.

§3. CALCULATION OF TRANSLATES

We put $\delta^s = \operatorname{Hom}_\Lambda(d_s, R) : \operatorname{Hom}_\Lambda(Q_s, R) \rightarrow \operatorname{Hom}_\Lambda(Q_{s+1}, R)$, where d_s is the differential in the resolution described in Theorem 1 (see also [9, 127–130]).

Any s -cycle $f \in \operatorname{Ker} \delta^s$ can be lifted (uniquely up to homotopy) to a chain map of complexes $\{\phi_t : Q_{s+t} \rightarrow Q_t\}_{t \geq 0}$. The homomorphism ϕ_t is called the t th translate of f and will be denoted by $T_t(f)$. For any cocycles $f \in \operatorname{Ker} \delta^s$ and $g \in \operatorname{Ker} \delta^t$, we have

$$\operatorname{cl} g \cdot \operatorname{cl} f = \operatorname{cl}(\mu T_0(g) T_t(f)).$$

Hence, we need formulas for translates of some elements of the Hochschild cohomology algebra to deduce relations in this algebra. This section is devoted to the calculation of translates needed for the description of relations in $\mathrm{HH}^*(R)$.

Remark 1. In the sequel, we usually denote by f the cohomology class $\text{cl } f \in \text{HH}^s(R)$ of the s -cocycle $f \in \text{Ker } \delta^s$.

Let w be a path from the vertex $(i_1 - 1)n + j_1$ to the vertex $(i_2 - 1)n + j_2$. We denote by w^* the Λ -homomorphism belonging to $\text{Hom}_\Lambda(P_{[i_2, j_2][i_1, j_1]}, R)$ such that $w^*(e_{i_2, j_2} \otimes e_{i_1, j_1}) = w$. For any $1 \leq i \leq r$, we introduce the following auxiliary homomorphisms:

$$\begin{aligned} w_{i,m,j} &= (\alpha_{i,j+m-1} \cdots \alpha_{i,j})^* & (0 \leq m \leq n-3, 1 \leq j \leq n-2-m); \\ t_{i,m,j} &= (\mu_{i+1,j+m-(n-1)} \tau_i \nu_{i,j})^* & (1 \leq m \leq n-2, n-1-m \leq j \leq n-2); \\ u_{i,m,q} &= (\gamma_{i,q} \nu_{i,n-2-m})^* & (0 \leq m \leq n-3, q \in \{n-1, n\}); \\ v_{i,m,q} &= (\mu_{i+1,m} \beta_{i,q})^* & (0 \leq m \leq n-3, q \in \{n-1, n\}); \\ u_{i,q} &= e_{i,q}^* & (q \in \{n-1, n\}); \\ v_{i,q} &= (\gamma_{i+1,q} \nu_{i+1,1} \beta_{i,q})^* & (q \in \{n-1, n\}). \end{aligned}$$

Now, we pick some elements of the algebra $\text{HH}^*(R)$.

a) Suppose $s = 2m + l(2n - 3)$, $0 \leq m \leq n-2$, $r \mid m + l(n-1)$, $2 \mid m + ln$, and either $\text{char } k = 2$ or $2 \nmid l$. We define an s -cocycle $f_s \in \text{Ker } \delta^s$ by

$$(3.1) \quad f_s = \sum_{i=1}^r \left(\sum_{j=1}^{n-2-m} w_{i,m,j} + u_{i,n-1} + u_{i,n} \right).$$

b) Suppose $s = 2m + 1 + l(2n - 3)$, $0 \leq m \leq n-3$, $r \mid m + l(n-1)$, $2 \nmid m + ln$, and either $\text{char } k = 2$ or $2 \nmid l$. We define an s -cocycle $g_s \in \text{Ker } \delta^s$ by

$$(3.2) \quad g_s = \sum_{i=1}^r \left(\sum_{j=n-1-m}^{n-2} t_{i,m,j} + u_{i,m,n-1} + v_{i,m,n-1} \right).$$

c) Suppose $s = (l+1)(2n-3) - 1$, $r \mid (l+1)(n-1) - 1$, and $2 \nmid (l+1)n$. We define an s -cocycle $h_s \in \text{Ker } \delta^s$ by

$$(3.3) \quad h_s = \sum_{i=1}^r \sum_{j=1}^{n-2} (-1)^j w_{i,0,j}.$$

d) Suppose $s = (l+1)(2n-3) - 1$, $r \mid (l+1)(n-1) - 1$, $2 \mid n$, and either $\text{char } k = 2$ or $2 \nmid l$. We define an s -cocycle $p_s \in \text{Ker } \delta^s$ by

$$(3.4) \quad p_s = \sum_{i=1}^r \left(\sum_{j=1}^{n-2} (-1)^j w_{i,0,j} + u_{i,n-1} \right).$$

Using the description of the homomorphisms d_s in [9], it is easy to verify that f_s , g_s , h_s , and p_s defined above are indeed cocycles.

Remark 2. In the rest of the paper, we suppose that $s = 2m + l(2n - 3)$ or $s = 2m + 1 + l(2n - 3)$, where $0 \leq s - l(2n - 3) \leq 2n - 4$. The additional conditions imposed on s will show each time what version indeed occurs.

Proposition 1. 1) Assume that the element f_s is defined for some s . Then $T_t(f_s) : Q_{t+s} \rightarrow Q_t$ can be defined on the direct summands of Q_{t+s} in the following way:

1.1) if $t = 2q$, $0 \leq q \leq n-2-m$, then

$$\begin{aligned} T_t(f_s)|_{P_{[i+q,j+m+q][i,j]}} &= e_{i+q,j+m+q} \otimes \alpha_{i,j+m-1} \cdots \alpha_{i,j} & (1 \leq j \leq n-2-m-q), \\ T_t(f_s)|_{P_{[i+q,j+m+q-(n-2)][i,j]}} &= 0 & (n-1-m-q \leq j \leq n-2-q), \end{aligned}$$

$$T_t(f_s)|_{P_{[i+q, j+m+q-(n-2)][i, j]}} = \alpha_{i+q, j+m+q-(n-1)} \cdots \alpha_{i+q, j+q-(n-2)} \otimes e_{i, j} \quad (n-1-q \leq j \leq n-2),$$

$$T_t(f_s)|_{P_{[i+q, \phi^q(j)][i, j]}} = e_{i+q, \phi^q(j)} \otimes e_{i, j} \quad (j \in \{n-1, n\});$$

1.2) if $t = 2q + 1$, $0 \leq q \leq n-3-m$, then

$$T_t(f_s)|_{P_{[i+q, j+m+q+1][i, j]}} = e_{i+q, j+m+q+1} \otimes \alpha_{i, j+m-1} \cdots \alpha_{i, j} \quad (1 \leq j \leq n-3-m-q),$$

$$T_t(f_s)|_{P_{[i+q, \phi^q(j)][i, n-2-m-q]}} = e_{i+q, \phi^q(j)} \otimes \alpha_{i, n-3-q} \cdots \alpha_{i, n-2-m-q} \quad (j \in \{n-1, n\}),$$

$$T_t(f_s)|_{P_{[i+q+1, j+m+q-(n-2)][i, j]}} = \mu_{i+q+1, j+m+q-(n-1)} \beta_{i+q, \phi^q(n)} \otimes \alpha_{i, n-3-q} \cdots \alpha_{i, j} \\ - \mu_{i+q+1, j+m+q-(n-1)} \beta_{i+q, \phi^q(n-1)} \otimes \alpha_{i, n-3-q} \cdots \alpha_{i, j} \quad (n-1-m-q \leq j \leq n-2-q),$$

$$T_t(f_s)|_{P_{[i+q+1, j+m+q-(n-2)][i, j]}} = \alpha_{i+q+1, j+m+q-(n-1)} \cdots \alpha_{i+q+1, j+q-(n-2)} \otimes e_{i, j} \quad (n-1-q \leq j \leq n-2),$$

$$T_t(f_s)|_{P_{[i+q+1, m+q+1][i, j]}} = \sum_{z=1}^m \alpha_{i+q+1, m+q} \cdots \alpha_{i+q+1, z+q+1} \otimes \mu_{i+1, z-1} \beta_{i, j} \\ + \alpha_{i+q+1, m+q} \cdots \alpha_{i+q+1, q+1} \otimes e_{i, j} \quad (j \in \{n-1, n\});$$

1.3) if $t = 2n-3-2m+2q$, $0 \leq q \leq m-1$, then

$$T_t(f_s)|_{P_{[i+q+n-1-m, j+q][i, j]}} = (-1)^q (\mu_{i+q+n-1-m, j+q-1} \beta_{i+q+n-2-m, \phi^{q+n-m}(n)} \otimes \alpha_{i, m-q-1} \cdots \alpha_{i, j} \\ - \mu_{i+q+n-1-m, j+q-1} \beta_{i+q+n-2-m, \phi^{q+n-m}(n-1)} \otimes \alpha_{i, m-q-1} \cdots \alpha_{i, j}) \quad (1 \leq j \leq m-q),$$

$$T_t(f_s)|_{P_{[i+q+n-1-m, j+q][i, j]}} = (-1)^q \alpha_{i+q+n-1-m, j+q-1} \cdots \alpha_{i+q+n-1-m, j+q-m} \otimes e_{i, j} \quad (m-q+1 \leq j \leq n-2-q),$$

$$T_t(f_s)|_{P_{[i+q+n-1-m, j+q-(n-2)][i, j]}} = 0 \quad (n-1-q \leq j \leq n-2),$$

$$T_t(f_s)|_{P_{[i+q+n-1-m, \phi^{q+n-m}(j)][i, j]}} = (-1)^{j+q-n} \left(e_{i+q+n-1-m, \phi^{q+n-m}(j)} \otimes \mu_{i+1, m-q-1} \beta_{i, j} \right. \\ + \sum_{z=1}^{m-q-1} \gamma_{i+q+n-1-m, \phi^{q+n-m}(j)} \nu_{i+q+n-1-m, z+q+n-1-m} \otimes \mu_{i+1, z-1} \beta_{i, j} \\ \left. + \gamma_{i+q+n-1-m, \phi^{q+n-m}(j)} \nu_{i+q+n-1-m, q+n-1-m} \otimes e_{i, j} \right) \quad (j \in \{n-1, n\});$$

1.4) if $t = 2n-2-2m+2q$, $0 \leq q \leq m-1$, then

$$T_t(f_s)|_{P_{[i+q+n-1-m, j+q+1][i, j]}} = 0 \quad (1 \leq j \leq m-q-1),$$

$$T_t(f_s)|_{P_{[i+q+n-1-m, j+q+1][i, j]}} = (-1)^q \alpha_{i+q+n-1-m, j+q} \cdots \alpha_{i+q+n-1-m, j+q+1-m} \otimes e_{i, j} \quad (m-q \leq j \leq n-3-q),$$

$$T_t(f_s)|_{P_{[i+q+n-1-m, \phi^{q+n-m}(j)][i, n-2-q]}} = (-1)^q \left(- \sum_{z=n-2-q}^{n-2} \gamma_{i+q+n-1-m, \phi^{q+n-m}(j)} \nu_{i+q+n-1-m, z+q-m+1} \otimes \alpha_{i, z-1} \cdots \alpha_{i, n-2-q} \right. \\ \left. + (-1)^{j-n} e_{i+q+n-1-m, \phi^{q+n-m}(j)} \otimes \gamma_{i, \phi(j)} \nu_{i, n-2-q} \right) \quad (j \in \{n-1, n\}),$$

$$\begin{aligned}
T_t(f_s)|_{P_{[i+q+n-m, j+q-(n-2)][i, j]}} &= (-1)^q (\mu_{i+q+n-m, j+q-n} \beta_{i+q+n-1-m, \phi^q+n-m}(n) \otimes \gamma_{i, n-1} \nu_{i, j} \\
&\quad + \mu_{i+q+n-m, j+q-n} \beta_{i+q+n-1-m, \phi^q+n-m}(n-1) \otimes \gamma_{i, n} \nu_{i, j}) \\
&\quad (n-1-q \leq j \leq n-2),
\end{aligned}$$

$$\begin{aligned}
T_t(f_s)|_{P_{[i+q+n-m, q+1][i, j]}} &= (-1)^{j+q-(n-1)} \mu_{i+q+n-m, q} \beta_{i+q+n-1-m, \phi^q+n-m-1}(j) \otimes e_{i, j} \\
&\quad (j \in \{n-1, n\});
\end{aligned}$$

1.5) if $t = u + v(2n-3)$ with $0 \leq u \leq 2n-4$ and $v \geq 1$, then $T_t(f_s) = \sigma^v(T_u(f_s))$, where $T_u(f_s)$ is described by the formulas in 1.1)-1.4).

2) Assume that the element g_s is defined. Then $T_t(g_s) : Q_{t+s} \rightarrow Q_t$ can be defined on the direct summands of Q_{t+s} as follows:

2.1) if $t = 2q$, $0 \leq q \leq n-3-m$, then

$$T_t(g_s)|_{P_{[i+q, j+m+q+1][i, j]}} = 0 \quad (1 \leq j \leq n-3-m-q),$$

$$T_t(g_s)|_{P_{[i+q, \phi^q(n)][i, n-2-m-q]}} = 0,$$

$$T_t(g_s)|_{P_{[i+q, \phi^q(n-1)][i, n-2-m-q]}} = (-1)^q e_{i+q, \phi^q(n-1)} \otimes \gamma_{i, n-1} \nu_{i, n-2-m-q},$$

$$T_t(g_s)|_{P_{[i+q+1, m+q+1][i, n]}} = 0,$$

$$\begin{aligned}
T_t(g_s)|_{P_{[i+q+1, m+q+1][i, n-1]}} &= (-1)^q \left(\sum_{z=n-1-q}^{n-2} \alpha_{i+q+1, m+q} \cdots \alpha_{i+q+1, z+q-(n-2)} \otimes \mu_{i+1, z-1} \beta_{i, n-1} \right. \\
&\quad \left. + \mu_{i+q+1, m+q} \beta_{i+q, \phi^q(n-1)} \otimes e_{i, n-1} \right),
\end{aligned}$$

$$\begin{aligned}
T_t(g_s)|_{P_{[i+q+1, j+m+q-(n-2)][i, j]}} &= (-1)^q \left(\sum_{z=n-1-q}^j \alpha_{i+q+1, j+m+q-(n-1)} \cdots \alpha_{i+q+1, z+q-(n-2)} \otimes \mu_{i+1, z-1} \tau_i \nu_{i, j} \right. \\
&\quad \left. + \mu_{i+q+1, j+m+q-(n-1)} \beta_{i+q, \phi^q(n-1)} \otimes \gamma_{i, n-1} \nu_{i, j} \right) \\
&\quad (n-1-m-q \leq j \leq n-2);
\end{aligned}$$

2.2) if $t = 2q+1$, $0 \leq q \leq n-3-m$, then

$$\begin{aligned}
T_t(g_s)|_{P_{[i+q+1, j+m+q+1][i, j]}} &= (-1)^{q+1} \left(\sum_{z=1}^j \alpha_{i+q+1, j+m+q} \cdots \alpha_{i+q+1, z+q+1} \otimes \mu_{i+1, z-1} \tau_i \nu_{i, j} \right. \\
&\quad \left. + \alpha_{i+q+1, j+m+q} \cdots \alpha_{i+q+1, q+1} \otimes \gamma_{i, n-1} \nu_{i, j} \right) \\
&\quad (1 \leq j \leq n-3-m-q),
\end{aligned}$$

$$T_t(g_s)|_{P_{[i+q+1, j+m+q-(n-3)][i, j]}} = 0 \quad (n-2-m-q \leq j \leq n-2),$$

$$T_t(g_s)|_{P_{[i+q+1, \phi^q(n)][i, n]}} = 0,$$

$$\begin{aligned}
T_t(g_s)|_{P_{[i+q+1, \phi^q(n-1)][i, n-1]}} &= (-1)^q \left(e_{i+q+1, \phi^q(n-1)} \otimes \mu_{i+1, n-3-q} \beta_{i, n-1} \right. \\
&\quad + \sum_{z=1}^{n-3-q} \gamma_{i+q+1, \phi^q(n-1)} \nu_{i+q+1, z+q+1} \otimes \mu_{i+1, z-1} \beta_{i, n-1} \\
&\quad \left. + \gamma_{i+q+1, \phi^q(n-1)} \nu_{i+q+1, q+1} \otimes e_{i, n-1} \right);
\end{aligned}$$

2.3) if $t = 2n - 4 - 2m$, then

$$\begin{aligned} T_t(g_s)|_{P_{[i+n-1-m, j][i, j]}} &= - \sum_{z=m+1}^j \alpha_{i+n-1-m, j-1} \cdots \alpha_{i+n-1-m, z-m} \otimes \mu_{i+1, z-1} \tau_i \nu_{i, j} \\ &\quad - \mu_{i+n-1-m, j-1} \beta_{i+n-2-m, \phi^{n-1-m}(n)} \otimes \gamma_{i, n-1} \nu_{i, j} \\ &\quad (1 \leq j \leq n-2), \end{aligned}$$

$$T_t(g_s)|_{P_{[i+n-1-m, \phi^{n-1-m}(n)][i, n]}} = 0,$$

$$\begin{aligned} T_t(g_s)|_{P_{[i+n-1-m, \phi^{n-1-m}(n-1)][i, n-1]}} \\ = - \sum_{z=m+1}^{n-2} \gamma_{i+n-1-m, \phi^{n-1-m}(n-1)} \nu_{i+n-1-m, z-m} \otimes \mu_{i+1, z-1} \beta_{i, n-1}; \end{aligned}$$

2.4) if $t = 2n - 4 - 2m + 2q$, $1 \leq q \leq m$, then

$$\begin{aligned} T_t(g_s)|_{P_{[i+q+n-1-m, j+q][i, j]}} \\ = - \sum_{z=m+1}^j \alpha_{i+q+n-1-m, j+q-1} \cdots \alpha_{i+q+n-1-m, z+q-m} \otimes \mu_{i+1, z-1} \tau_i \nu_{i, j} \\ (1 \leq j \leq n-2-q), \end{aligned}$$

$$T_t(g_s)|_{P_{[i+q+n-1-m, \phi^{q+n-1-m}(n)][i, n]}} = 0,$$

$$\begin{aligned} T_t(g_s)|_{P_{[i+q+n-1-m, \phi^{q+n-1-m}(n-1)][i, n-1]}} \\ = \sum_{z=n-1-q}^m \gamma_{i+q+n-1-m, \phi^{q+n-1-m}(n-1)} \nu_{i+q+n-1-m, z+q-m} \otimes \mu_{i+1, z-1} \beta_{i, n-1} \\ - \sum_{z=m+1}^{n-2-q} \gamma_{i+q+n-1-m, \phi^{q+n-1-m}(n-1)} \nu_{i+q+n-1-m, z+q-m} \otimes \mu_{i+1, z-1} \beta_{i, n-1}, \end{aligned}$$

$$\begin{aligned} T_t(g_s)|_{P_{[i+q+n-1-m, 1][i, n-1-q]}} &= e_{i+q+n-1-m, 1} \otimes \mu_{i+1, m-q} \tau_i \nu_{i, n-1-q} \\ &\quad - \beta_{i+q+n-1-m, \phi^{q+n-1-m}(n-1)} \otimes \gamma_{i, n} \nu_{i, n-1-q} \\ &\quad + \beta_{i+q+n-1-m, \phi^{q+n-1-m}(n)} \otimes \gamma_{i, n-1} \nu_{i, n-1-q} \\ &\quad - \sum_{z=n-1-q}^{n-2} \tau_{i+q+n-2-m} \nu_{i+q+n-2-m, z+q-m} \otimes \alpha_{i, z-1} \cdots \alpha_{i, n-1-q}, \end{aligned}$$

$$\begin{aligned} T_t(g_s)|_{P_{[i+q+n-1-m, j+q-(n-2)][i, j]}} &= e_{i+q+n-1-m, j+q-(n-2)} \otimes \mu_{i+1, j+m-(n-1)} \tau_i \nu_{i, j} \\ &\quad (n-q \leq j \leq n-2); \end{aligned}$$

2.5) if $t = 2n - 3 - 2m + 2q$, $0 \leq q \leq m-1$, then

$$T_t(g_s)|_{P_{[i+q+n-1-m, j+q+1][i, j]}} = 0 \quad (1 \leq j \leq n-3-q),$$

$$T_t(g_s)|_{P_{[i+q+n-1-m, \phi^{q+n-1-m}(n-1)][i, n-2-q]}} = 0,$$

$$\begin{aligned} T_t(g_s)|_{P_{[i+q+n-1-m, \phi^{q+n-1-m}(n)][i, n-2-q]}} \\ = -\gamma_{i+q+n-1-m, \phi^{q+n-1-m}(n)} \nu_{i+q+n-1-m, n-2-m} \otimes e_{i, n-2-q}, \end{aligned}$$

$$T_t(g_s)|_{P_{[i+q+n-m, q+1][i, n]}} = 0,$$

$$T_t(g_s)|_{P_{[i+q+n-m, q+1][i, n-1]}} = e_{i+q+n-m, q+1} \otimes \mu_{i+1, m} \beta_{i, n-1},$$

$$T_t(g_s)|_{P_{[i+q+n-m, j+q-(n-2)][i, j]}} = e_{i+q+n-m, j+q-(n-2)} \otimes \mu_{i+1, j+m-(n-1)} \tau_i \nu_{i, j} \\ (n-1-q \leq j \leq n-2);$$

2.6) if $t = u + v(2n-3)$ with $0 \leq u \leq 2n-4$ and $v \geq 1$, then $T_t(g_s) = \sigma^v(T_u(g_s))$, where $T_u(g_s)$ is described by the formulas in 2.1)-2.5).

3) Assume that the element h_s is defined. Then $T_t(h_s) : Q_{t+s} \rightarrow Q_t$ can be defined on the direct summands of Q_{t+s} as follows:

3.1) if $t = 0$, then

$$T_t(h_s)|_{P_{[i, j][i, j]}} = (-1)^j e_{i, j} \otimes e_{i, j} \quad (1 \leq j \leq n-2),$$

$$T_t(h_s)|_{P_{[i, \phi(j)][i, j]}} = 0 \quad (j \in \{n-1, n\});$$

3.2) if $t = 4q$, $1 \leq q \leq (n-3)/2$, then

$$T_t(h_s)|_{P_{[i+2q, j+2q][i, j]}} = (-1)^{j+1} e_{i+2q, j+2q} \otimes e_{i, j} \quad (1 \leq j \leq n-2-2q),$$

$$T_t(h_s)|_{P_{[i+2q, n-1][i, n-1-2q]}} = 0,$$

$$T_t(h_s)|_{P_{[i+2q, n][i, n-1-2q]}} = \sum_{z=\frac{n-1}{2}-q}^{\frac{n-3}{2}} \gamma_{i+2q, n} \nu_{i+2q, 2z+2q-(n-2)} \otimes \alpha_{i, 2z-1} \cdots \alpha_{i, n-1-2q},$$

$$T_t(h_s)|_{P_{[i+2q+1, 2j+2q-(n-1)][i, 2j]}} \\ = \sum_{z=j}^{\frac{n-3}{2}} \mu_{i+2q+1, 2j+2q-n} \tau_{i+2q} \nu_{i+2q, 2z+2q-(n-2)} \otimes \alpha_{i, 2z-1} \cdots \alpha_{i, 2j} \\ - \sum_{z=\frac{n-1}{2}-q}^{j-1} \alpha_{i+2q+1, 2j+2q-n} \cdots \alpha_{i+2q+1, 2z+2q-(n-3)} \otimes \mu_{i+1, 2z} \tau_i \nu_{i, 2j} \\ \left(\frac{n+1}{2} - q \leq j \leq \frac{n-3}{2} \right),$$

$$T_t(h_s)|_{P_{[i+2q+1, 2j+2q-(n-2)][i, 2j+1]}} \\ = - \sum_{z=j}^{\frac{n-3}{2}} \mu_{i+2q+1, 2j+2q-(n-1)} \tau_{i+2q} \nu_{i+2q, 2z+2q-(n-3)} \otimes \alpha_{i, 2z} \cdots \alpha_{i, 2j+1} \\ + \mu_{i+2q+1, 2j+2q-(n-1)} \beta_{i+2q, n-1} \otimes \gamma_{i, n-1} \nu_{i, 2j+1} \\ - \mu_{i+2q+1, 2j+2q-(n-1)} \beta_{i+2q, n} \otimes \gamma_{i, n} \nu_{i, 2j+1} \\ + \sum_{z=\frac{n-1}{2}-q}^j \alpha_{i+2q+1, 2j+2q-(n-1)} \cdots \alpha_{i+2q+1, 2z+2q-(n-2)} \otimes \mu_{i+1, 2z-1} \tau_i \nu_{i, 2j+1} \\ \left(\frac{n-1}{2} - q \leq j \leq \frac{n-3}{2} \right),$$

$$T_t(h_s)|_{P_{[i+2q+1, 2q][i, n]}} = 0,$$

$$T_t(h_s)|_{P_{[i+2q+1, 2q][i, n-1]}} = - \sum_{z=\frac{n-1}{2}-q}^{\frac{n-3}{2}} \alpha_{i+2q+1, 2q-1} \cdots \alpha_{i+2q+1, 2z+2q-(n-3)} \otimes \mu_{i+1, 2z} \beta_{i, n-1};$$

3.3) if $t = 4q+2$, $0 \leq q \leq (n-3)/2$, then

$$T_t(h_s)|_{P_{[i+2q+1, j+2q+1][i, j]}} = (-1)^j e_{i+2q+1, j+2q+1} \otimes e_{i, j} \quad (1 \leq j \leq n-3-2q),$$

$$T_t(h_s)|_{P_{[i+2q+1, n][i, n-2-2q]}} = e_{i+2q+1, n} \otimes \gamma_{i, n-1} \nu_{i, n-2-2q},$$

$$T_t(h_s)|_{P_{[i+2q+1, n-1][i, n-2-2q]}} = \sum_{z=\frac{n-3}{2}-q}^{\frac{n-3}{2}} \gamma_{i+2q+1, n-1} \nu_{i+2q+1, 2z+2q-(n-4)} \otimes \alpha_{i, 2z} \cdots \alpha_{i, n-2-2q} \\ + e_{i+2q+1, n-1} \otimes \gamma_{i, n} \nu_{i, n-2-2q},$$

$$\begin{aligned}
& T_t(h_s)|_{P_{[i+2q+2, 2j+2q-(n-2)][i, 2j]}} \\
&= - \sum_{z=j}^{\frac{n-3}{2}} \mu_{i+2q+2, 2j+2q-(n-1)} \tau_{i+2q+1} \nu_{i+2q+1, 2z+2q-(n-3)} \otimes \alpha_{i, 2z-1} \cdots \alpha_{i, 2j} \\
&\quad + \sum_{z=\frac{n-3}{2}-q}^{j-1} \alpha_{i+2q+2, 2j+2q-(n-1)} \cdots \alpha_{i+2q+2, 2z+2q-(n-4)} \otimes \mu_{i+1, 2z} \tau_i \nu_{i, 2j} \\
&\quad \left(\frac{n-1}{2} - q \leq j \leq \frac{n-3}{2} \right),
\end{aligned}$$

$$\begin{aligned}
& T_t(h_s)|_{P_{[i+2q+2, 2j+2q-(n-3)][i, 2j+1]}} \\
&= \sum_{z=j}^{\frac{n-3}{2}} \mu_{i+2q+2, 2j+2q-(n-2)} \tau_{i+2q+1} \nu_{i+2q+1, 2z+2q-(n-4)} \otimes \alpha_{i, 2z} \cdots \alpha_{i, 2j+1} \\
&\quad - \mu_{i+2q+2, 2j+2q-(n-2)} \beta_{i+2q+1, n} \otimes \gamma_{i, n-1} \nu_{i, 2j+1} \\
&\quad + \mu_{i+2q+2, 2j+2q-(n-2)} \beta_{i+2q+1, n-1} \otimes \gamma_{i, n} \nu_{i, 2j+1} \\
&\quad - \sum_{z=\frac{n-1}{2}-q}^j \alpha_{i+2q+2, 2j+2q-(n-2)} \cdots \alpha_{i+2q+2, 2z+2q-(n-3)} \otimes \mu_{i+1, 2z-1} \tau_i \nu_{i, 2j+1} \\
&\quad \left(\frac{n-1}{2} - q \leq j \leq \frac{n-3}{2} \right),
\end{aligned}$$

$$T_t(h_s)|_{P_{[i+2q+2, 2q+1][i, n]}} = 0,$$

$$T_t(h_s)|_{P_{[i+2q+2, 2q+1][i, n-1]}} = \sum_{z=\frac{n-3}{2}-q}^{\frac{n-3}{2}} \alpha_{i+2q+2, 2q} \cdots \alpha_{i+2q+2, 2z+2q-(n-4)} \otimes \mu_{i+1, 2z} \beta_{i, n-1};$$

3.4) if $t = 4q + 1$, $0 \leq q \leq (n-3)/2$, then

$$\begin{aligned}
& T_t(h_s)|_{P_{[i+2q+1, 2j+2q][i, 2j]}} \\
&= - \sum_{z=j}^{\frac{n-3}{2}-q} \mu_{i+2q+1, 2j+2q-1} \tau_{i+2q} \nu_{i+2q, 2z+2q+1} \otimes \alpha_{i, 2z-1} \cdots \alpha_{i, 2j} \\
&\quad - \sum_{z=0}^{j-1} \alpha_{i+2q+1, 2j+2q-1} \cdots \alpha_{i+2q+1, 2z+2q+2} \otimes \mu_{i+1, 2z} \tau_i \nu_{i, 2j} \\
&\quad (1 \leq j \leq \frac{n-3}{2} - q),
\end{aligned}$$

$$\begin{aligned}
& T_t(h_s)|_{P_{[i+2q+1, 2j+2q+1][i, 2j+1]}} \\
&= \sum_{z=j}^{\frac{n-5}{2}-q} \mu_{i+2q+1, 2j+2q} \tau_{i+2q} \nu_{i+2q, 2z+2q+2} \otimes \alpha_{i, 2z} \cdots \alpha_{i, 2j+1} \\
&\quad + \mu_{i+2q+1, 2j+2q} \beta_{i+2q, n-1} \otimes \alpha_{i, n-3-2q} \cdots \alpha_{i, 2j+1} \\
&\quad + \alpha_{i+2q+1, 2j+2q} \cdots \alpha_{i+2q+1, 2q+1} \otimes \gamma_{i, n-1} \nu_{i, 2j+1} \\
&\quad + \sum_{z=1}^j \alpha_{i+2q+1, 2j+2q} \cdots \alpha_{i+2q+1, 2z+2q+1} \otimes \mu_{i+1, 2z-1} \tau_i \nu_{i, 2j+1} \\
&\quad (0 \leq j \leq \frac{n-3}{2} - q),
\end{aligned}$$

$$\begin{aligned}
& T_t(h_s)|_{P_{[i+2q+1, \phi(j)][i, j]}} = \sum_{z=0}^{\frac{n-5}{2}-q} \gamma_{i+2q+1, \phi(j)} \nu_{i+2q+1, 2z+2q+2} \otimes \mu_{i+1, 2z} \beta_{i, j} \\
&\quad + e_{i+2q+1, \phi(j)} \otimes \mu_{i+1, n-3-2q} \beta_{i, j} \quad (j \in \{n-1, n\}),
\end{aligned}$$

$$T_t(h_s)|_{P_{[i+2q+1, j+2q-(n-2)][i, j]}} = (-1)^{j+1} e_{i+2q+1, j+2q-(n-2)} \otimes e_{i, j} \quad (n-1-2q \leq j \leq n-2);$$

3.5) if $t = 4q + 3$, $0 \leq q \leq (n-5)/2$, then

$$\begin{aligned} T_t(h_s)|_{P_{[i+2q+2, 2j+2q+1][i, 2j]}} &= \sum_{z=j}^{\frac{n-5}{2}-q} \mu_{i+2q+2, 2j+2q} \tau_{i+2q+1} \nu_{i+2q+1, 2z+2q+2} \otimes \alpha_{i, 2z-1} \cdots \alpha_{i, 2j} \\ &\quad + \mu_{i+2q+2, 2j+2q} \beta_{i+2q+1, n} \otimes \alpha_{i, n-4-2q} \cdots \alpha_{i, 2j} \\ &\quad + \sum_{z=0}^{j-1} \alpha_{i+2q+2, 2j+2q} \cdots \alpha_{i+2q+2, 2z+2q+3} \otimes \mu_{i+1, 2z} \tau_i \nu_{i, 2j} \end{aligned} \quad (1 \leq j \leq \frac{n-3}{2} - q),$$

$$\begin{aligned} T_t(h_s)|_{P_{[i+2q+2, 2j+2q+2][i, 2j+1]}} &= - \sum_{z=j}^{\frac{n-5}{2}-q} \mu_{i+2q+2, 2j+2q+1} \tau_{i+2q+1} \nu_{i+2q+1, 2z+2q+3} \otimes \alpha_{i, 2z} \cdots \alpha_{i, 2j+1} \\ &\quad - \alpha_{i+2q+2, 2j+2q+1} \cdots \alpha_{i+2q+2, 2q+2} \otimes \gamma_{i, n-1} \nu_{i, 2j+1} \\ &\quad - \sum_{z=1}^j \alpha_{i+2q+2, 2j+2q+1} \cdots \alpha_{i+2q+2, 2z+2q+2} \otimes \mu_{i+1, 2z-1} \tau_i \nu_{i, 2j+1} \end{aligned} \quad (0 \leq j \leq \frac{n-5}{2} - q),$$

$$\begin{aligned} T_t(h_s)|_{P_{[i+2q+2, j][i, j]}} &= - \sum_{z=0}^{\frac{n-5}{2}-q} \gamma_{i+2q+2, j} \nu_{i+2q+2, 2z+2q+3} \otimes \mu_{i+1, 2z} \beta_{i, j} \\ &\quad (j \in \{n-1, n\}), \end{aligned}$$

$$\begin{aligned} T_t(h_s)|_{P_{[i+2q+2, j+2q-(n-3)][i, j]}} &= (-1)^j e_{i+2q+2, j+2q-(n-3)} \otimes e_{i, j} \\ &\quad (n-2-2q \leq j \leq n-2); \end{aligned}$$

3.6) if $t = u + v(2n-3)$ with $0 \leq u \leq 2n-4$ and $v \geq 1$, then $T_t(h_s) = \sigma^v(T_u(h_s))$, where $T_u(h_s)$ is described by the formulas in 3.1)–3.5).

4) Assume that the element p_s is defined. Then $T_t(p_s) : Q_{t+s} \rightarrow Q_t$ can be defined on the direct summands of Q_{t+s} as follows:

4.1) if $t = 0$, then

$$\begin{aligned} T_t(p_s)|_{P_{[i, j][i, j]}} &= (-1)^j e_{i, j} \otimes e_{i, j} \quad (1 \leq j \leq n-2), \\ T_t(p_s)|_{P_{[i, n][i, n]}} &= 0, \\ T_t(p_s)|_{P_{[i, n-1][i, n-1]}} &= e_{i, n-1} \otimes e_{i, n-1}; \end{aligned}$$

4.2) if $t = 4q$, $1 \leq q \leq (n-2)/2$, then

$$\begin{aligned} T_t(p_s)|_{P_{[i+2q, j+2q][i, j]}} &= (-1)^{j+1} e_{i+2q, j+2q} \otimes e_{i, j} \quad (1 \leq j \leq n-2-2q), \\ T_t(p_s)|_{P_{[i+2q, n][i, n-2q]}} &= 0, \\ T_t(p_s)|_{P_{[i+2q, n-1][i, n-1-2q]}} &= \sum_{z=\frac{n-2}{2}-q}^{\frac{n-4}{2}} \gamma_{i+2q, n-1} \nu_{i+2q, 2z+2q-(n-3)} \otimes \alpha_{i, 2z} \cdots \alpha_{i, n-1-2q} \\ &\quad - e_{i+2q, n-1} \otimes \gamma_{i, n-1} \nu_{i, n-1-2q}, \end{aligned}$$

$$\begin{aligned}
& T_t(p_s)|_{P_{[i+2q+1, 2j+2q-(n-1)][i, 2j]}} \\
&= - \sum_{z=j}^{\frac{n-2}{2}} \mu_{i+2q+1, 2j+2q-n} \tau_{i+2q} \nu_{i+2q, 2z+2q-(n-2)} \otimes \alpha_{i, 2z-1} \cdots \alpha_{i, 2j} \\
&\quad - \mu_{i+2q+1, 2j+2q-n} \beta_{i+2q, n} \otimes \gamma_{i, n} \nu_{i, 2j} \\
&\quad + \sum_{z=\frac{n-2}{2}-q}^{j-1} \alpha_{i+2q+1, 2j+2q-n} \cdots \alpha_{i+2q+1, 2z+2q-(n-3)} \otimes \mu_{i+1, 2z} \tau_i \nu_{i, 2j} \\
&\quad \left(\frac{n}{2} - q \leq j \leq \frac{n-2}{2} \right),
\end{aligned}$$

$$\begin{aligned}
& T_t(p_s)|_{P_{[i+2q+1, 2j+2q-(n-2)][i, 2j+1]}} \\
&= \sum_{z=j}^{\frac{n-4}{2}} \mu_{i+2q+1, 2j+2q-(n-1)} \tau_{i+2q} \nu_{i+2q, 2z+2q-(n-3)} \otimes \alpha_{i, 2z} \cdots \alpha_{i, 2j+1} \\
&\quad - \mu_{i+2q+1, 2j+2q-(n-1)} \beta_{i+2q, n-1} \otimes \gamma_{i, n-1} \nu_{i, 2j+1} \\
&\quad - \sum_{z=\frac{n}{2}-q}^j \alpha_{i+2q+1, 2j+2q-(n-1)} \cdots \alpha_{i+2q+1, 2z+2q-(n-2)} \otimes \mu_{i+1, 2z-1} \tau_i \nu_{i, 2j+1} \\
&\quad \left(\frac{n}{2} - q \leq j \leq \frac{n-4}{2} \right),
\end{aligned}$$

$$T_t(p_s)|_{P_{[i+2q+1, 2q][i, n]}} = 0,$$

$$\begin{aligned}
& T_t(p_s)|_{P_{[i+2q+1, 2q][i, n-1]}} = -\mu_{i+2q+1, 2q-1} \beta_{i+2q, n-1} \otimes e_{i, n-1} \\
&\quad - \sum_{z=\frac{n}{2}-q}^{\frac{n-2}{2}} \alpha_{i+2q+1, 2q-1} \cdots \alpha_{i+2q+1, 2z+2q-(n-2)} \otimes \mu_{i+1, 2z-1} \beta_{i, n-1};
\end{aligned}$$

4.3) if $t = 4q + 2$, $0 \leq q \leq (n-4)/2$, then

$$T_t(p_s)|_{P_{[i+2q+1, j+2q+1][i, j]}} = (-1)^j e_{i+2q+1, j+2q+1} \otimes e_{i, j} \quad (1 \leq j \leq n-3-2q),$$

$$\begin{aligned}
& T_t(p_s)|_{P_{[i+2q+1, n-1][i, n-2-2q]}} = -e_{i+2q+1, n-1} \otimes \gamma_{i, n} \nu_{i, n-2-2q} \\
&\quad - \sum_{z=\frac{n-2}{2}-q}^{\frac{n-2}{2}} \gamma_{i+2q+1, n-1} \nu_{i+2q+1, 2z+2q-(n-3)} \otimes \alpha_{i, 2z-1} \cdots \alpha_{i, n-2-2q},
\end{aligned}$$

$$T_t(p_s)|_{P_{[i+2q+1, n][i, n-2-2q]}} = - \sum_{z=\frac{n-2}{2}-q}^{\frac{n-4}{2}} \gamma_{i+2q+1, n} \nu_{i+2q+1, 2z+2q-(n-4)} \otimes \alpha_{i, 2z} \cdots \alpha_{i, n-2-2q},$$

$$\begin{aligned}
& T_t(p_s)|_{P_{[i+2q+2, 2j+2q-(n-2)][i, 2j]}} \\
&= \sum_{z=j}^{\frac{n-2}{2}} \mu_{i+2q+2, 2j+2q-(n-1)} \tau_{i+2q+1} \nu_{i+2q+1, 2z+2q-(n-3)} \otimes \alpha_{i, 2z-1} \cdots \alpha_{i, 2j} \\
&\quad + \mu_{i+2q+2, 2j+2q-(n-1)} \beta_{i+2q+1, n-1} \otimes \gamma_{i, n} \nu_{i, 2j} \\
&\quad - \sum_{z=\frac{n-2}{2}-q}^{j-1} \alpha_{i+2q+2, 2j+2q-(n-1)} \cdots \alpha_{i+2q+2, 2z+2q-(n-4)} \otimes \mu_{i+1, 2z} \tau_i \nu_{i, 2j} \\
&\quad \left(\frac{n}{2} - q \leq j \leq \frac{n-2}{2} \right),
\end{aligned}$$

$$\begin{aligned}
& T_t(p_s)|_{P_{[i+2q+2, 2j+2q-(n-3)][i, 2j+1]}} \\
&= - \sum_{z=j}^{\frac{n-4}{2}} \mu_{i+2q+2, 2j+2q-(n-2)} \tau_{i+2q+1} \nu_{i+2q+1, 2z+2q-(n-4)} \otimes \alpha_{i, 2z} \cdots \alpha_{i, 2j+1} \\
&\quad + \mu_{i+2q+2, 2j+2q-(n-2)} \beta_{i+2q+1, n} \otimes \gamma_{i, n-1} \nu_{i, 2j+1} \\
&\quad + \sum_{z=\frac{n-2}{2}-q}^j \alpha_{i+2q+2, 2j+2q-(n-2)} \cdots \alpha_{i+2q+2, 2z+2q-(n-3)} \otimes \mu_{i+1, 2z-1} \tau_i \nu_{i, 2j+1} \\
&\quad \left(\frac{n-2}{2} - q \leq j \leq \frac{n-4}{2} \right),
\end{aligned}$$

$$\begin{aligned}
T_t(p_s)|_{P_{[i+2q+2, 2q+1][i, n]}} &= 0, \\
T_t(p_s)|_{P_{[i+2q+2, 2q+1][i, n-1]}} &= \mu_{i+2q+2, 2q} \beta_{i+2q+1, n} \otimes e_{i, n-1} \\
&\quad + \sum_{z=\frac{n-2}{2}-q}^{\frac{n-2}{2}} \alpha_{i+2q+2, 2q} \cdots \alpha_{i+2q+2, 2z+2q-(n-3)} \otimes \mu_{i+1, 2z-1} \beta_{i, n-1};
\end{aligned}$$

4.4) if $t = 4q + 1$, $0 \leq q \leq (n-4)/2$, then

$$\begin{aligned}
&T_t(p_s)|_{P_{[i+2q+1, 2j+2q][i, 2j]}} \\
&= - \sum_{z=j}^{\frac{n-4}{2}-q} \mu_{i+2q+1, 2j+2q-1} \tau_{i+2q} \nu_{i+2q, 2z+2q+1} \otimes \alpha_{i, 2z-1} \cdots \alpha_{i, 2j} \\
&\quad - \mu_{i+2q+1, 2j+2q-1} \beta_{i+2q, n-1} \otimes \alpha_{i, n-3-2q} \cdots \alpha_{i, 2j} \\
&\quad - \sum_{z=0}^{j-1} \alpha_{i+2q+1, 2j+2q-1} \cdots \alpha_{i+2q+1, 2z+2q+2} \otimes \mu_{i+1, 2z} \tau_i \nu_{i, 2j} \\
&\quad \quad \quad (1 \leq j \leq \frac{n-2}{2} - q),
\end{aligned}$$

$$\begin{aligned}
&T_t(p_s)|_{P_{[i+2q+1, 2j+2q+1][i, 2j+1]}} \\
&= \sum_{z=j}^{\frac{n-4}{2}-q} \mu_{i+2q+1, 2j+2q} \tau_{i+2q} \nu_{i+2q, 2z+2q+2} \otimes \alpha_{i, 2z} \cdots \alpha_{i, 2j+1} \\
&\quad + \alpha_{i+2q+1, 2j+2q} \cdots \alpha_{i+2q+1, 2q+1} \otimes \gamma_{i, n-1} \nu_{i, 2j+1} \\
&\quad + \sum_{z=1}^j \alpha_{i+2q+1, 2j+2q} \cdots \alpha_{i+2q+1, 2z+2q+1} \otimes \mu_{i+1, 2z-1} \tau_i \nu_{i, 2j+1} \\
&\quad \quad \quad (0 \leq j \leq \frac{n-4}{2} - q),
\end{aligned}$$

$$\begin{aligned}
&T_t(p_s)|_{P_{[i+2q+1, n-1][i, n-1]}} = -\gamma_{i+2q+1, n-1} \nu_{i+2q+1, 2q+1} \otimes e_{i, n-1} \\
&\quad - \sum_{z=1}^{\frac{n-4}{2}-q} \gamma_{i+2q+1, n-1} \nu_{i+2q+1, 2z+2q+1} \otimes \mu_{i+1, 2z-1} \beta_{i, n-1} \\
&\quad - e_{i+2q+1, n-1} \otimes \mu_{i+1, n-3-2q} \beta_{i, n-1},
\end{aligned}$$

$$T_t(p_s)|_{P_{[i+2q+1, n][i, n]}} = \sum_{z=0}^{\frac{n-4}{2}-q} \gamma_{i+2q+1, n} \nu_{i+2q+1, 2z+2q+2} \otimes \mu_{i+1, 2z} \beta_{i, n},$$

$$\begin{aligned}
T_t(p_s)|_{P_{[i+2q+1, j+2q-(n-2)][i, j]}} &= (-1)^j e_{i+2q+1, j+2q-(n-2)} \otimes e_{i, j} \\
&\quad (n-1-2q \leq j \leq n-2);
\end{aligned}$$

4.5) if $t = 4q + 3$, $0 \leq q \leq (n-4)/2$, then

$$\begin{aligned}
&T_t(p_s)|_{P_{[i+2q+2, 2j+2q+1][i, 2j]}} \\
&= \sum_{z=j}^{\frac{n-4}{2}-q} \mu_{i+2q+2, 2j+2q} \tau_{i+2q+1} \nu_{i+2q+1, 2z+2q+2} \otimes \alpha_{i, 2z-1} \cdots \alpha_{i, 2j} \\
&\quad + \mu_{i+2q+2, 2j+2q} \beta_{i+2q+1, n} \otimes \alpha_{i, n-4-2q} \cdots \alpha_{i, 2j} \\
&\quad - \mu_{i+2q+2, 2j+2q} \beta_{i+2q+1, n-1} \otimes \alpha_{i, n-4-2q} \cdots \alpha_{i, 2j} \\
&\quad + \sum_{z=0}^{j-1} \alpha_{i+2q+2, 2j+2q} \cdots \alpha_{i+2q+2, 2z+2q+3} \otimes \mu_{i+1, 2z} \tau_i \nu_{i, 2j} \\
&\quad \quad \quad (1 \leq j \leq \frac{n-4}{2} - q),
\end{aligned}$$

$$\begin{aligned}
T_t(p_s)|_{P_{[i+2q+2, 2j+2q+2][i, 2j+1]}} &= - \sum_{z=j}^{\frac{n-6}{2}-q} \mu_{i+2q+2, 2j+2q+1} \tau_{i+2q+1} \nu_{i+2q+1, 2z+2q+3} \otimes \alpha_{i, 2z} \cdots \alpha_{i, 2j+1} \\
&\quad - \mu_{i+2q+2, 2j+2q+1} \beta_{i+2q+1, n-1} \otimes \alpha_{i, n-4-2q} \cdots \alpha_{i, 2j+1} \\
&\quad - \alpha_{i+2q+2, 2j+2q+1} \cdots \alpha_{i+2q+2, 2q+2} \otimes \gamma_{i, n-1} \nu_{i, 2j+1} \\
&\quad - \sum_{z=1}^j \alpha_{i+2q+2, 2j+2q+1} \cdots \alpha_{i+2q+2, 2z+2q+2} \otimes \mu_{i+1, 2z-1} \tau_i \nu_{i, 2j+1} \\
&\quad \quad \quad (0 \leq j \leq \frac{n-4}{2} - q),
\end{aligned}$$

$$\begin{aligned}
T_t(p_s)|_{P_{[i+2q+2, n][i, n-1]}} &= \gamma_{i+2q+2, n} \nu_{i+2q+2, 2q+2} \otimes e_{i, n-1} \\
&\quad + \sum_{z=1}^{\frac{n-4}{2}-q} \gamma_{i+2q+2, n} \nu_{i+2q+2, 2z+2q+2} \otimes \mu_{i+1, 2z-1} \beta_{i, n-1},
\end{aligned}$$

$$\begin{aligned}
T_t(p_s)|_{P_{[i+2q+2, n-1][i, n]}} &= - \sum_{z=0}^{\frac{n-6}{2}-q} \gamma_{i+2q+2, n-1} \nu_{i+2q+2, 2z+2q+3} \otimes \mu_{i+1, 2z} \beta_{i, n} \\
&\quad - e_{i+2q+2, n-1} \otimes \mu_{i+1, n-4-2q} \beta_{i, n},
\end{aligned}$$

$$\begin{aligned}
T_t(p_s)|_{P_{[i+2q+2, j+2q-(n-3)][i, j]}} &= (-1)^{j+1} e_{i+2q+2, j+2q-(n-3)} \otimes e_{i, j} \\
&\quad \quad \quad (n-2-2q \leq j \leq n-2);
\end{aligned}$$

4.6) if $t = u + v(2n-3)$ with $0 \leq u \leq 2n-4$ and $v \geq 1$, then $T_t(p_s) = \sigma^v(T_u(p_s))$, where $T_u(p_s)$ is described by the formulas in 4.1)–4.5).

Proof. We need to verify that, for $b \in \{f_s, g_s, h_s, p_s\}$, the identities $\mu T_0(b) = b$ and $d_t T_{t+1}(b) = T_t(b) d_{t+s}$ are valid for all $t \geq 0$. If $t = u + v(2n-3)$, where $0 \leq u \leq 2n-4$ and $v \geq 1$, then $d_t T_{t+1}(b) = \sigma^v(d_u T_{u+1}(b))$, $T_t(b) d_{t+s} = \sigma^v(T_u(b) d_{u+s})$. Hence, we need to prove that $d_t T_{t+1}(b) = T_t(b) d_{t+s}$ only for all $0 \leq t \leq 2n-4$. For this, we consider the cases 1.1)–1.4), 2.1)–2.5), 3.1)–3.5), 4.1)–4.5) separately. Then, by using the formulas for the differentials d_t given in [9], the required identities are proved by direct but cumbersome calculations. We leave this to the reader. $\square$

Remark 3. It is clear that f_0 is the unity of the algebra $\mathrm{HH}^*(R)$. Moreover, the resolution described in Theorem 1 is λ -periodic with λ as in (2.1), and hence, $T_t(f_\lambda)$ is an isomorphism for all $t \geq 0$. Consequently, multiplication by f_λ gives rise to an isomorphism between $\mathrm{HH}^s(R)$ and $\mathrm{HH}^{s+\lambda}(R)$ for all $s \geq 1$.

§4. GENERATORS

In this section, we describe a set of generators for the algebra $\mathrm{HH}^*(R)$. This set is infinite, but later we show how to pick a finite subset in it generating $\mathrm{HH}^*(R)$.

First, we define some additional elements of $\mathrm{HH}^*(R)$.

We define a 1-cocycle $\varepsilon_1 \in \mathrm{Ker} \delta^1$ as follows:

$$\varepsilon_1 = u_{1,0,n-1} + u_{1,0,n}.$$

Assume that $s = l(2n-3)$, $l \geq 1$, $r \mid l(n-1)-1$, $2 \mid ln$, and either $\mathrm{char} k = 2$ or $2 \nmid l$. Let χ_s (respectively χ'_s) be the s -cocycle equal to $v_{r,n}$ on $P_{[1,n][r,n]}$ (respectively, to $v_{r,n-1}$ on $P_{[1,n-1][r,n-1]}$) and to 0 on the remaining direct summands of Q_s .

Assume that $s = l(2n-3)$, $r \mid l(n-1)-1$ and $2 \nmid ln$. Let ξ_s be the s -cocycle equal to $t_{r,n-2,1}$ on $P_{[1,1][r,1]}$ and to 0 on the remaining summands of Q_s .

If $r = 1$, then, for simplicity, we shall omit the first index in notation such as $e_{i,j}$ and $P_{i,j}$. If $1 \leq j \leq n-2$, then we denote by $\varepsilon_0^{(j)}$ the 0-cocycle equal to $t_{n-2,j}$ on $P_{[j][j]}$ and

to 0 on the remaining direct summands of Q_0 . If $q \in \{n-1, n\}$, then we denote by $\varepsilon_0^{(q)}$ the 0-cocycle equal to v_q on $P_{[q][q]}$ and to 0 on the remaining direct summands of Q_0 .

The proofs of Theorems 2 and 3 in [9] show that the maps ε_1 , χ_s , χ'_s , ξ_s , and $\varepsilon_0^{(j)}$ defined above are indeed cocycles.

Remark 4. Let $s = l(2n-3)$ with $l \geq 1$. It is clear that the elements χ_s and χ'_s are defined if and only if the element f_{s-1} is defined, and the element ξ_s is defined if and only if the element h_{s-1} is defined. Moreover, if χ_s and χ'_s are defined, then p_{s-1} is defined if and only if $2 \mid n$.

Lemma 1. 1) Assume that s is a number such that the element f_s is defined and $2n-3 \nmid s+1$. Then $\varepsilon_1 f_s \neq 0$ in $\mathrm{HH}^*(R)$.

2) Assume that s is such that g_s is defined. Then $\varepsilon_1 g_s \neq 0$ in $\mathrm{HH}^*(R)$.

3) Assume that s is such that h_s is defined. Then in $\mathrm{HH}^*(R)$ we have

$$(4.1) \quad \varepsilon_1 h_s = \left(\frac{n-1}{2}\right) \xi_{s+1}.$$

Moreover, $\xi_{s+1} \neq 0$ in $\mathrm{HH}^*(R)$.

4) Assume that s is such that f_s and p_s are defined. Then in $\mathrm{HH}^*(R)$ we have

$$(4.2) \quad \begin{aligned} \varepsilon_1 f_s &= \chi_{s+1} - \chi'_{s+1}, \\ \varepsilon_1 p_s &= \left(\frac{2-n}{2}\right) \chi_{s+1} - \left(\frac{n}{2}\right) \chi'_{s+1}. \end{aligned}$$

Moreover, χ_{s+1} and χ'_{s+1} are linearly independent over k in $\mathrm{HH}^*(R)$.

5) Assume that s is such that the element χ_s is defined and $2 \nmid n$. Then $\chi_s \neq 0$ in $\mathrm{HH}^*(R)$.

Proof. 1) As was noticed at the beginning of the previous section, the element $\varepsilon_1 f_s \in \mathrm{HH}^*(R)$ is determined by the cocycle $\mu T_0(\varepsilon_1) T_1(f_s) = \varepsilon_1 T_1(f_s)$. Using the formulas of item 1.2) of Proposition 1, we see that $\varepsilon_1 T_1(f_s) = u_{1,m,n-1} + u_{1,m,n}$. We must prove that $\varepsilon_1 T_1(f_s) \notin \mathrm{Im} \delta^s$. Assume that this is not true. Then there exists a homomorphism $w \in \mathrm{Hom}_\Lambda(Q_s, R)$ such that $\delta^s(w) = u_{1,m,n-1} + u_{1,m,n}$. The proofs of Theorems 2 and 3 in [9] show that $w = \sum_{i=1}^r \left(\sum_{j=1}^{n-2-m} \varkappa_{i,j} w_{i,m,j} + \varphi_{i,1} u_{i,n-1} + \varphi_{i,2} u_{i,n} \right) + w'$ for some $\varkappa_{i,j}, \varphi_{i,1}, \varphi_{i,2} \in k$, where $w' = 0$ for $r > 1$ and $w' \in \mathrm{Ker} \delta^s$ for $r = 1$. From the same proofs, it follows that if $\delta^s(w) = u_{1,m,n-1} + u_{1,m,n}$, then

$$\begin{aligned} \varkappa_{i,j+1} - \varkappa_{i,j} &= 0, \quad \varkappa_{i+1,1} - \varphi_{i,1} = 0, \quad \varkappa_{i+1,1} - \varphi_{i,2} = 0 \quad (1 \leq j \leq n-3-m, 1 \leq i \leq r); \\ \varphi_{i,1} - \varkappa_{i,n-2-m} &= 0, \quad \varphi_{i,2} - \varkappa_{i,n-2-m} = 0 \quad (2 \leq i \leq r); \\ \varphi_{1,1} - \varkappa_{1,n-2-m} &= 1, \quad \varphi_{1,2} - \varkappa_{1,n-2-m} = 1. \end{aligned}$$

The first two lines imply that $\varphi_{i+1,1} = \varphi_{i,1}$ for $1 \leq i \leq r-1$, i.e., we have $\varphi_{r,1} = \varphi_{1,1}$. But the first and third lines show that, similarly, $\varphi_{1,1} = \varphi_{r,1} + 1$. Hence, we arrive at a contradiction, and this completes the proof of part 1.

2) Using formulas of item 2.2) of Proposition 1, we obtain $\varepsilon_1 T_1(g_s) = v_{r,n-1}$. Assume that $\varepsilon_1 g_s = 0$ in $\mathrm{HH}^*(R)$, i.e., there exists a homomorphism $w \in \mathrm{Hom}_\Lambda(Q_s, R)$ such that $\delta^s(w) = v_{r,n-1}$.

The proofs of Theorems 2 and 3 in [9] show that

$$\begin{aligned} w = \sum_{i=1}^r \left(\sum_{j=1}^{n-3-m} \varkappa_{i,j} w_{i,m+1,j} + \varphi_{i,1} u_{i,m,n} + \varphi_{i,2} u_{i,m,n-1} \right. \\ \left. + \sum_{j=n-1-m}^{n-2} \eta_{i,j} t_{i,m,j} + \zeta_{i,1} v_{i,m,n-1} + \zeta_{i,2} v_{i,m,n} \right) \end{aligned}$$

for some $\varkappa_{i,j}, \varphi_{i,q}, \eta_{i,j}, \zeta_{i,q} \in k$. The same proofs imply that the identity $\delta^s(w) = v_{r,n-1}$ is equivalent to the fact that

$$\begin{aligned} \zeta_{i,1} - \zeta_{i,2} &= \eta_{i,n-2} = \cdots = \eta_{i,n-1-m} = \varphi_{i,2} - \varphi_{i,1}, \\ -\zeta_{i,2} - \sum_{j=1}^{n-3-m} \varkappa_{i+1,j} + \varphi_{i+1,1} &= 0 \quad (1 \leq i \leq r); \\ -\zeta_{i,1} - \sum_{j=1}^{n-3-m} \varkappa_{i+1,j} + \varphi_{i+1,2} &= 0 \quad (1 \leq i \leq r-1); \\ -\zeta_{r,1} - \sum_{j=1}^{n-3-m} \varkappa_{1,j} + \varphi_{1,2} &= 1. \end{aligned}$$

Put $\epsilon_i = \zeta_{i,1} - \zeta_{i,2} = \varphi_{i,2} - \varphi_{i,1}$. Then, by the second and third line, we have $\epsilon_{i+1} = \epsilon_i$ for $1 \leq i \leq r-1$, i.e., $\epsilon_r = \epsilon_1$. But the second and fourth lines imply that $\epsilon_1 = \epsilon_r + 1$. This contradiction completes the proof of part 2).

3) We obtain $\varepsilon_1 T_1(h_s) = \sum_{z=0}^{\frac{n-3}{2}} t_{1,n-2,2z+1}$ with the help of formulas of item 3.4) of Proposition 1. By definition, we have $\xi_{s+1} = t_{r,n-2,1}$. By the proofs of Theorems 2 and 3 in [9], the set $\{t_{i,n-2,j}\}_{1 \leq i \leq r, 1 \leq j \leq n-2}$ is a k -basis of $\text{Ker } \delta^{s+1}$.

Now we find a condition on the elements $\eta_{i,j} \in k$ under which the element $v = \sum_{i=1}^r \sum_{j=1}^{n-2} \eta_{i,j} t_{i,n-2,j} \in \text{Ker } \delta^{s+1}$ equals 0 in $\text{HH}^*(R)$, i.e., there exists $w \in \text{Hom}_\Lambda(Q_s, R)$ such that $\delta^s(w) = v$. Assume that such w exists. From the proofs of Theorems 2 and 3 in [9], it is clear that $w = \sum_{i=1}^r \sum_{j=1}^{n-2} \varkappa_{i,j} w_{i,0,j} + w'$ for some $\varkappa_{i,j} \in k$, where $w' = 0$ for $r > 1$ and $w' \in \text{Ker } \delta^s$ for $r = 1$. The same proofs imply that $\sum_{z=j}^{n-2} \varkappa_{i,z} - \sum_{z=1}^j \varkappa_{i+1,z} = \eta_{i,j}$ with $1 \leq i \leq r$ and $1 \leq j \leq n-2$. Adding all these identities, we get $\sum_{i=1}^r \sum_{j=1}^{n-2} \eta_{i,j} = 0$.

This condition defines a subspace $V \subset \text{Ker } \delta^{s+1}$ of dimension $r(n-2) - 1$. Since $\dim_k \text{Im } \delta^s = r(n-2) - 1$ and $\text{Im } \delta^s \subset V$, we obtain

$$(4.3) \quad \text{Im } \delta^s = V = \left\{ \sum_{i=1}^r \sum_{j=1}^{n-2} \eta_{i,j} t_{i,n-2,j} \mid \sum_{i=1}^r \sum_{j=1}^{n-2} \eta_{i,j} = 0 \right\}.$$

Clearly, $\sum_{z=0}^{\frac{n-3}{2}} t_{1,n-2,2z+1} - \left(\frac{n-1}{2}\right) t_{r,n-2,1} \in \text{Im } \delta^s$ and $t_{r,n-2,1} \notin \text{Im } \delta^s$, i.e., we have $\varepsilon_1 h_s = \left(\frac{n-1}{2}\right) \xi_{s+1}$ and $\xi_{s+1} \neq 0$ in $\text{HH}^*(R)$.

4) Using formulas of items 1.3) and 4.4) of Proposition 1, we see that $\varepsilon_1 T_1(f_s) = v_{r,n} - v_{r,n-1}$ and $\varepsilon_1 T_1(p_s) = -\sum_{z=1}^{\frac{n-2}{2}} t_{1,n-2,2z} - v_{r,n-1}$. By definition, we have $\chi_{s+1} = v_{r,n}$ and $\chi'_{s+1} = v_{r,n-1}$. Hence, $\varepsilon_1 T_1(f_s) = \chi_{s+1} - \chi'_{s+1}$, i.e., $\varepsilon_1 f_s = \chi_{s+1} - \chi'_{s+1}$ in $\text{HH}^*(R)$. The proofs of Theorems 2 and 3 in [9] show that the set

$$\{t_{i,n-2,j}\}_{1 \leq i \leq r, 1 \leq j \leq n-2} \cup \{v_{i,q}\}_{1 \leq i \leq r, q \in \{n-1, n\}}$$

is a k -basis of $\text{Ker } \delta^{s+1}$ for $r > 1$. If $r = 1$, the set

$$\{t_{1,n-2,j}\}_{1 \leq j \leq n-2} \cup \{v_{1,q}\}_{q \in \{n-1, n\}} \cup \{w_0\}$$

with a suitable w_0 in $\text{Ker } \delta^{s+1}$ forms a k -basis of $\text{Ker } \delta^{s+1}$.

Now we find a condition on the elements $\eta_{i,j}, \zeta_{i,n-1}, \zeta_{i,n}, \epsilon \in k$ under which the element

$$v = \sum_{i=1}^r \left(\sum_{j=1}^{n-2} \eta_{i,j} t_{i,n-2,j} + \zeta_{i,n-1} v_{i,n-1} + \zeta_{i,n} v_{i,n} \right) + \epsilon w_0 \in \text{Ker } \delta^{s+1}$$

is equal to 0 in $\text{HH}^*(R)$, i.e., there exists $w \in \text{Hom}_\Lambda(Q_s, R)$ such that $\delta^s(w) = v$ (for $r > 1$ we put $w_0 = 0$ and $\epsilon = 0$). Assume that such w exists. The proofs of Theorems 2

and 3 in [9] show that

$$w = \sum_{i=1}^r \left(\sum_{j=1}^{n-2} \varkappa_{i,j} w_{i,0,j} + \varphi_{i,n-1} u_{i,n-1} + \varphi_{i,n} u_{i,n} \right) + w'$$

with $\varkappa_{i,j}, \varphi_{i,n-1}, \varphi_{i,n} \in k$, where $w' = 0$ for $r > 1$ and $w' \in \text{Ker } \delta^s$ for $r = 1$. The same proofs imply the following identities:

$$\begin{aligned} \sum_{z=j}^{n-2} \varkappa_{i,z} - \varphi_{i,n-1} + \varphi_{i,n} - \sum_{z=1}^j \varkappa_{i+1,z} &= \eta_{i,j} & (1 \leq i \leq r, 1 \leq j \leq n-2); \\ \varphi_{i,n-1} + \sum_{z=1}^{n-2} \varkappa_{i+1,z} - \varphi_{i+1,n-1} &= \zeta_{i,n-1} & (1 \leq i \leq r); \\ -\varphi_{i,n} + \sum_{z=1}^{n-2} \varkappa_{i+1,z} + \varphi_{i+1,n} &= \zeta_{i,n} & (1 \leq i \leq r). \end{aligned}$$

If $r = 1$, then we also have $\epsilon = 0$. We subtract the third identity from the second. Then, summing the identities obtained for $1 \leq i \leq r$, we get $\sum_{i=1}^r (\zeta_{i,n-1} - \zeta_{i,n}) = 0$. The first and second identities imply $\sum_{i=1}^r \left(\sum_{j=1}^{n-2} (-1)^j \eta_{i,j} + \zeta_{i,n-1} \right) = 0$. Clearly, the system

$$\begin{cases} \epsilon = 0, \\ \sum_{i=1}^r (\zeta_{i,n-1} - \zeta_{i,n}) = 0, \\ \sum_{i=1}^r \left(\sum_{j=1}^{n-2} (-1)^j \eta_{i,j} + \zeta_{i,n-1} \right) = 0 \end{cases}$$

determines a subspace $V \subset \text{Ker } \delta^{s+1}$ of dimension $rn - 2$. Since $\dim_k \text{Im } \delta^s = rn - 2$ and $\text{Im } \delta^s \subset V$, we have

$$\text{Im } \delta^s = V = \left\{ \sum_{i=1}^r \left(\sum_{j=1}^{n-2} \eta_{i,j} t_{i,n-2,j} + \zeta_{i,n-1} v_{i,n-1} + \zeta_{i,n} v_{i,n} \right) \mid \sum_{i=1}^r (\zeta_{i,n-1} - \zeta_{i,n}) = 0, \sum_{i=1}^r \left(\sum_{j=1}^{n-2} (-1)^j \eta_{i,j} + \zeta_{i,n-1} \right) = 0 \right\}.$$

It is easily seen that if $y_1 v_{r,n} + y_2 v_{r,n-1} \in \text{Im } \delta^s$ with $y_1, y_2 \in k$, then $y_1 = y_2 = 0$, i.e., the elements χ_{s+1} and χ'_{s+1} are linearly independent over k in $\text{HH}^*(R)$.

Moreover, it is easy to verify that $-\sum_{z=1}^{\frac{n-2}{2}} t_{1,n-2,2z} - v_{r,n-1} + \left(\frac{n-2}{2}\right) v_{r,n} + \left(\frac{n}{2}\right) v_{r,n-1} \in \text{Im } \delta^s$; hence, in $\text{HH}^*(R)$ we have $\varepsilon_1 p_s = \left(\frac{2-n}{2}\right) \chi_{s+1} - \left(\frac{n}{2}\right) \chi'_{s+1}$.

5) Observe that $\chi_s = v_{r,n}$. The proofs of Theorems 2 and 3 in [9] show that the set $\{t_{i,n-2,j}\}_{1 \leq i \leq r, 1 \leq j \leq n-2} \cup \{v_{i,q}\}_{1 \leq i \leq r, q \in \{n-1,n\}}$ is a k -basis of $\text{Ker } \delta^s$ for $r > 1$, and for $r = 1$, the set $\{t_{1,n-2,j}\}_{1 \leq j \leq n-2} \cup \{v_{1,q}\}_{q \in \{n-1,n\}} \cup \{w_0\}$, where w_0 is a suitable element of $\text{Ker } \delta^s$, is a k -basis of $\text{Ker } \delta^s$.

Now we derive a condition on the elements $\eta_{i,j}, \zeta_{i,n-1}, \zeta_{i,n}, \epsilon \in k$ equivalent to the fact that the element

$$v = \sum_{i=1}^r \left(\sum_{j=1}^{n-2} \eta_{i,j} t_{i,n-2,j} + \zeta_{i,n-1} v_{i,n-1} + \zeta_{i,n} v_{i,n} \right) + \epsilon w_0 \in \text{Ker } \delta^s$$

is equal to 0 in $\text{HH}^*(R)$, i.e., there exists $w \in \text{Hom}_\Lambda(Q_{s-1}, R)$ such that $\delta^{s-1}(w) = v$ (if $r > 1$, we put $w_0 = 0$ and $\epsilon = 0$). Assume that such w exists. By the proofs of Theorems 2 and 3 in [9], we can write

$$w = \sum_{i=1}^r \left(\sum_{j=1}^{n-2} \varkappa_{i,j} w_{i,0,j} + \varphi_{i,n-1} u_{i,n-1} + \varphi_{i,n} u_{i,n} \right) + w'$$

with $\varkappa_{i,j}, \varphi_{i,n-1}, \varphi_{i,n} \in k$, where $w' = 0$ for $r > 1$ and $w' \in \text{Ker } \delta^{s-1}$ for $r = 1$. Also, the same proofs imply the identities

$$\begin{aligned} \sum_{z=j}^{n-2} \varkappa_{i,z} - \varphi_{i,n-1} + \varphi_{i,n} - \sum_{z=1}^j \varkappa_{i+1,z} &= \eta_{i,j} & (1 \leq i \leq r, 1 \leq j \leq n-2); \\ \varphi_{i,n-1} + \sum_{z=1}^{n-2} \varkappa_{i+1,z} - \varphi_{i+1,n-1} &= \zeta_{i,n-1} & (1 \leq i \leq r); \\ -\varphi_{i,n} + \sum_{z=1}^{n-2} \varkappa_{i+1,z} + \varphi_{i+1,n} &= \zeta_{i,n} & (1 \leq i \leq r). \end{aligned}$$

If $r = 1$, then $\epsilon = 0$. We add the third identity to the second. Then, summing the identities obtained for $1 \leq i \leq r$, we see that $\sum_{i=1}^r (\zeta_{i,n-1} + \zeta_{i,n}) = 0$. Clearly, the system

$$\begin{cases} \epsilon = 0, \\ \sum_{i=1}^r (\zeta_{i,n-1} + \zeta_{i,n}) = 0 \end{cases}$$

determines a subspace $V \subset \text{Ker } \delta^s$ of dimension $rn - 1$. Since $\dim_k \text{Im } \delta^{s-1} = rn - 1$ and $\text{Im } \delta^{s-1} \subset V$, we have

$$\begin{aligned} \text{Im } \delta^{s-1} &= V \\ (4.4) \quad &= \left\{ \sum_{i=1}^r \left(\sum_{j=1}^{n-2} \eta_{i,j} t_{i,n-2,j} + \zeta_{i,n-1} v_{i,n-1} + \zeta_{i,n} v_{i,n} \right) \mid \sum_{i=1}^r (\zeta_{i,n-1} + \zeta_{i,n}) = 0 \right\}. \end{aligned}$$

It follows that $\chi_s \neq 0$ in $\text{HH}^*(R)$. The lemma is proved. $\square$

Remark 5. In the sequel, we use the following easily proved assertion: if $v \in \text{Ker } \delta^s$ and $v(Q_s) \not\subset \text{Rad } R$, then $v \neq 0$ in $\text{HH}^*(R)$.

Corollary 1. *If the element f_s (respectively, g_s or h_s) is defined for some $s \geq 0$, then f_s (respectively, g_s or h_s) is nonzero in $\text{HH}^*(R)$. Moreover, if p_s and f_s are defined for a fixed s , then they are linearly independent over k in $\text{HH}^*(R)$.*

Proof. The fact that $g_s \neq 0$ follows easily from Lemma 1. All remaining assertions follow from Remark 5. $\square$

We consider the following sets (in each case, we include a variable with index s in the corresponding set if the variable is defined for this s):

$$\mathcal{F} = \{\varepsilon_1, f_s, g_s\}; \quad \mathcal{G}_1 = \{p_s\}; \quad \mathcal{G}_2 = \{h_s\}; \quad \mathcal{H}_1 = \{\chi_s\}; \mathcal{H}_2 = \{\xi_s\}; \quad \mathcal{J} = \{\varepsilon_0^{(j)}\}_{1 \leq j \leq n}.$$

Next, we put

$$\begin{aligned} \mathcal{Z}_1 &= \mathcal{F} \cup \mathcal{G}_1; & \mathcal{Z}_2 &= \mathcal{Z}_1 \cup \mathcal{H}_1; & \mathcal{Z}_3 &= \mathcal{F} \cup \mathcal{G}_2; \\ \mathcal{Z}_4 &= \mathcal{Z}_3 \cup \mathcal{H}_2; & \mathcal{Z}_5 &= \mathcal{Z}_3 \cup \mathcal{H}_1; & \mathcal{Z}_6 &= \mathcal{Z}_3 \cup \mathcal{H}_1 \cup \mathcal{H}_2. \end{aligned}$$

Proposition 2. 1) *Assume that $2 \mid n$, $\text{char } k \nmid n - 1$, and $r > 1$. Then the set $\mathcal{Z}_1$ generates $\text{HH}^*(R)$ as a k -algebra.*

2) *Assume that $2 \mid n$, $\text{char } k \mid n - 1$, and $r > 1$. Then the set $\mathcal{Z}_2$ generates $\text{HH}^*(R)$ as a k -algebra.*

3) *Assume that $2 \nmid n$, $\text{char } k \neq 2$, $\text{char } K \nmid \left(\frac{n-1}{2}\right)$, and $r > 1$. Then the set $\mathcal{Z}_3$ generates $\text{HH}^*(R)$ as a k -algebra.*

4) *Assume that $2 \nmid n$, $\text{char } k \neq 2$, $\text{char } k \mid \left(\frac{n-1}{2}\right)$, and $r > 1$. Then the set $\mathcal{Z}_4$ generates $\text{HH}^*(R)$ as a k -algebra.*

5) Assume that $2 \nmid n$, $\text{char } k = 2$, $\text{char } k \nmid \left(\frac{n-1}{2}\right)$, and $r > 1$. Then the set $\mathcal{Z}_5$ generates $\text{HH}^*(R)$ as a k -algebra.

6) Assume that $2 \nmid n$, $\text{char } k = 2$, $\text{char } k \mid \left(\frac{n-1}{2}\right)$, and $r > 1$. Then the set $\mathcal{Z}_6$ generates $\text{HH}^*(R)$ as a k -algebra.

7) If we replace the condition $r > 1$ by the condition $r = 1$ in any of items 1)–6), then the set $\mathcal{Z}_i \cup \mathcal{J}$, where i is the number of the corresponding item, generates $\text{HH}^*(R)$ as a k -algebra.

Proof. Let $\mathcal{H}$ denote a subalgebra of $\text{HH}^*(R)$ generated by the corresponding set $\mathcal{Z}_i$ for a fixed item among 1)–6) and, respectively, generated by the set $\mathcal{Z}_i \cup \mathcal{J}$ for part 7). Fix any $s \geq 0$. We prove that $\text{HH}^s(R) \subset \mathcal{H}$.

a) First we consider the case where $r > 1$. We may assume that s satisfies one of conditions (2.2)–(2.6). We analyze all cases successively.

1. Assume that $s = 2m + l(2n - 3)$ satisfies condition (2.2). Then the element f_s is defined. We have $f_s \neq 0$ by Corollary 1, and $\dim_k \text{HH}^s(R) = 1$; hence, f_s generates $\text{HH}^s(R)$.

2. Assume that $s = 2m + 1 + l(2n - 3)$ satisfies condition (2.2). Then the element f_{s-1} is defined. We have $\varepsilon_1 f_{s-1} \neq 0$ by Lemma 1, and $\dim_k \text{HH}^s(R) = 1$; hence, $\varepsilon_1 f_{s-1}$ generates $\text{HH}^s(R)$.

3. Assume that $s = 2m + 1 + l(2n - 3)$ satisfies condition (2.3). Then the element g_s is defined. We have $g_s \neq 0$ by Corollary 1, and $\dim_k \text{HH}^s(R) = 1$; hence, g_s generates $\text{HH}^s(R)$.

4. Assume that $s = 2m + 2 + l(2n - 3)$ satisfies condition (2.3). Then the element g_{s-1} is defined. We have $\varepsilon_1 g_{s-1} \neq 0$ by Lemma 1, and $\dim_k \text{HH}^s(R) = 1$; hence, $\varepsilon_1 g_{s-1}$ generates $\text{HH}^s(R)$.

5. Assume that $s = (l + 1)(2n - 3) - 1$ satisfies condition (2.4). Then the element f_s is defined. Since $f_s \neq 0$ by Corollary 1 and $\dim_k \text{HH}^s(R) = 1$, we see that f_s generates $\text{HH}^s(R)$.

6. Assume that $s = l(2n - 3)$ satisfies condition (2.4). Then the element χ_s is defined and the assumption of one of items 5) and 6) is satisfied. Since $\chi_s \neq 0$ by Lemma 1 and $\dim_k \text{HH}^s(R) = 1$, we see that χ_s generates $\text{HH}^s(R)$.

7. Assume that $s = (l + 1)(2n - 3) - 1$ satisfies condition (2.5). Then the element h_s is defined and the assumption of one of items 3)–6) is satisfied. Since $h_s \neq 0$ by Corollary 1 and $\dim_k \text{HH}^s(R) = 1$, we see that h_s generates $\text{HH}^s(R)$.

8. Assume that $s = l(2n - 3)$ satisfies condition (2.5). Then the elements h_{s-1} and ξ_s are defined and the assumption of one of items 3)–6) is satisfied. We have $\dim_k \text{HH}^s(R) = 1$ in this case. If the assumption of one of items 3) and 5) is satisfied, i.e., $\text{char } k \nmid \left(\frac{n-1}{2}\right)$, then $\varepsilon_1 h_{s-1}$ generates $\text{HH}^s(R)$ because $\varepsilon_1 h_{s-1} \neq 0$ by Lemma 1. If the assumption of one of items 4) and 6) is satisfied, then ξ_s generates $\text{HH}^s(R)$, because $\xi_s \neq 0$ by Lemma 1.

9. Assume that $s = (l + 1)(2n - 3) - 1$ satisfies condition (2.6). Then the elements f_s and p_s are defined, and the assumption of one of items 1) and 2) is satisfied. Since f_s and p_s are linearly independent over k by Corollary 1 and $\dim_k \text{HH}^s(R) = 2$, it follows that f_s and p_s generate $\text{HH}^s(R)$.

10. Assume that $s = l(2n - 3)$ satisfies assumption (2.6). Then the elements f_{s-1} , p_{s-1} , χ_s , and χ'_s are defined, and the assumption of one of items 1) and 2) is satisfied. We have $\dim_k \text{HH}^s(R) = 2$ in this case. Since χ_s and χ'_s are linearly independent over k by Lemma 1, they generate $\text{HH}^s(R)$. If the assumption of item 1) is satisfied, i.e., $\text{char } k \nmid n - 1$, then χ_s and χ'_s are expressed in terms of $\varepsilon_1 f_{s-1}$ and $\varepsilon_1 p_{s-1}$ (see (4.2)). Thus, $\varepsilon_1 f_{s-1}$ and $\varepsilon_1 p_{s-1}$ generate $\text{HH}^s(R)$. If the assumption of item 2) is satisfied, then $\varepsilon_1 f_{s-1}$ and χ_s generate $\text{HH}^s(R)$, because $\varepsilon_1 f_{s-1} = \chi_s - \chi'_s$ by formula (4.2).

The claims in items 1)–6) of the proposition are proved.

b) Assume that $r = 1$. We consider the case where $s = 0$. Then $\mathrm{HH}^0(R) = \mathrm{Ker} \delta^0$ and $\dim_k \mathrm{HH}^0(R) = n + 1$. Since the elements $f_0, \varepsilon_0^{(j)}$ ($1 \leq j \leq n$) are linearly independent over k and lie in $\mathrm{Ker} \delta^0$, they generate $\mathrm{HH}^0(R)$.

The cases where $s \geq 1$ satisfies one of conditions (2.7)–(2.15) in Theorem 3 are analyzed much as in part a) of this proof, with the use of Remark 5.

We have shown that $\mathrm{HH}^s(R) \subset \mathcal{H}$ for each of cases 1)–7). Consequently, $\mathrm{HH}^*(R) = \mathcal{H}$. The proposition is proved. $\square$

Corollary 2. *In all cases 1)–7) of Proposition 2, the algebra $\mathrm{HH}^*(R)$ is generated by the elements of the corresponding generating set whose degree does not exceed λ (see (2.1)).*

Proof. This follows from Proposition 2 and Remark 3. $\square$

§5. RELATIONS

In this section, we find several relations satisfied by the elements of $\mathrm{HH}^*(R)$ introduced in the preceding two sections.

First, we produce formulas for the products of the form xy , where $y \in \{f_s, g_s, h_s, p_s\}$.

Remark 6. In the sequel, we assume that either $s_i = 2m_i + l_i(2n - 3)$ or $s_i = 2m_i + 1 + l_i(2n - 3)$, where $0 \leq s_i - l_i(2n - 3) \leq 2n - 4$. In each specific case, it will be clear which version occurs, because of the other assumptions about s_i (cf. Remark 2).

Proposition 3. *The following identities are valid in the algebra $\mathrm{HH}^*(R)$:*

$$(5.1) \quad \left. \begin{aligned} f_{s_1} f_{s_2} &= \begin{cases} f_{s_1+s_2} & \text{if } m_1 + m_2 \leq n - 2, \\ 2(-1)^{n-1} g_{s_1+s_2} & \text{if } m_1 + m_2 \geq n - 1; \end{cases} \\ \varepsilon_1 f_{s_1} &= 0 \text{ if } s_1 = (l+1)(2n-3) - 1, 2 \nmid n; \end{aligned} \right\}$$

$$(5.2) \quad \left. \begin{aligned} g_{s_1} f_{s_2} &= \begin{cases} g_{s_1+s_2} & \text{if } m_1 + m_2 \leq n - 3, \\ \xi_{s_1+s_2} & \text{if } m_1 + m_2 = n - 2, \\ 0 & \text{if } m_1 + m_2 \geq n - 1; \end{cases} \\ g_{s_1} g_{s_2} &= 0; \end{aligned} \right\}$$

$$(5.3) \quad \left. \begin{aligned} p_{s_1} f_{s_2} &= \begin{cases} -g_{s_1+s_2} & \text{if } m_2 \geq 1, \\ p_{s_1+s_2} & \text{if } m_2 = 0; \end{cases} \\ p_{s_1} g_{s_2} &= 0; \quad p_{s_1} p_{s_2} = -\left(\frac{n}{2}\right) g_{s_1+s_2}; \end{aligned} \right\}$$

$$(5.4) \quad \left. \begin{aligned} h_{s_1} f_{s_2} &= \begin{cases} 0 & \text{if } m_2 \geq 1 \text{ and } \mathrm{char} k \neq 2, \\ \varepsilon_1 f_{s_1+s_2-1} & \text{if } m_2 \geq 1 \text{ and } \mathrm{char} k = 2, \\ h_{s_1+s_2} & \text{if } m_2 = 0; \end{cases} \\ h_{s_1} g_{s_2} &= \begin{cases} 0 & \text{if } \mathrm{char} k \neq 2, \\ \varepsilon_1 g_{s_1+s_2-1} & \text{if } m_2 \geq 1 \text{ and } \mathrm{char} k = 2, \\ \chi_{s_1+s_2} & \text{if } m_2 = 0 \text{ and } \mathrm{char} k = 2; \end{cases} \\ h_{s_1} h_{s_2} &= -\left(\frac{n-1}{2}\right) g_{s_1+s_2}; \end{aligned} \right\}$$

$$(5.5) \quad \left. \begin{aligned} \chi_{s_1} f_{s_2} &= \begin{cases} -\varepsilon_1 g_{s_1+s_2-1} & \text{if } m_2 \geq 1, \\ \chi_{s_1+s_2} & \text{if } m_2 = 0; \end{cases} \\ \chi_{s_1} g_{s_2} &= 0; \quad \chi_{s_1} p_{s_2} = 0; \quad \chi_{s_1} h_{s_2} = 0; \end{aligned} \right\}$$

$$(5.6) \quad \xi_{s_1} f_{s_2} = \begin{cases} 0 & \text{if } m_2 \geq 1, \\ \xi_{s_1+s_2} & \text{if } m_2 = 0; \end{cases} \quad \xi_{s_1} g_{s_2} = 0; \quad \xi_{s_1} h_{s_2} = -\varepsilon_1 g_{s_1+s_2-1}.$$

Proof. If the elements f_{s_1} and f_{s_2} are defined and $m_1 + m_2 \leq n - 2$, then it is easy to check that the element f_s is defined, where $s = s_1 + s_2$ (and moreover, we have $m = m_1 + m_2$ and $l = l_1 + l_2$). Using (3.1) and formulas in items 1.1) and 1.5) of Proposition 1, we

obtain $f_{s_1}T_{s_1}(f_{s_2}) = f_s$. If $m_1 + m_2 \geq n - 1$, then the element g_s is defined, where $s = s_1 + s_2$ (and moreover, we have $m = m_1 + m_2 - (n - 1)$ and $l = l_1 + l_2 + 1$). Again, using (3.1) and formulas in items 1.1) and 1.5) of Proposition 1, we see that

$$f_{s_1}T_{s_1}(f_{s_2}) = (-1)^{n-1} \sum_{i=1}^r \left(2 \sum_{j=n-1-m}^{n-2} t_{i,m,j} + u_{i,m,n-1} - u_{i,m,n} + v_{i,m,n-1} - v_{i,m,n} \right).$$

Relation (3.2) shows that we need to prove that the last expression equals

$$2(-1)^{n-1} \sum_{i=1}^r \left(\sum_{j=n-1-m}^{n-2} t_{i,m,j} + u_{i,m,n-1} + v_{i,m,n-1} \right),$$

or equivalently, that $v = \sum_{i=1}^r (u_{i,m,n-1} + u_{i,m,n} + v_{i,m,n-1} + v_{i,m,n}) = 0$ in $\mathrm{HH}^*(R)$. Put $w = \sum_{i=1}^r \sum_{j=1}^{n-2-m} w_{i,m,j} \in \mathrm{Hom}_\Lambda(Q_{s-1}, R)$. It is easily seen that $\delta^{s-1}(w) = v$. Now we assume that $s_1 = (l+1)(2n-3) - 1$ and $2 \nmid n$ and that the element f_{s_1} is defined. In this case, we have $\mathrm{char} k = 2$. Using formulas of items 1.3) and 1.5) of Proposition 1 and the definition of ε_1 , we obtain $\varepsilon_1 T_1(f_{s_1}) = v_{r,n-1} + v_{r,n}$. Hence, using (4.4) and the fact that $\mathrm{char} k = 2$, we get $v_{r,n-1} + v_{r,n} = 0$. This proves identities (5.1).

If the elements g_{s_1} and f_{s_2} are defined and $m_1 + m_2 \leq n - 3$, then it is easy to check that the element g_s is defined, where $s = s_1 + s_2$ (and moreover, we have $m = m_1 + m_2$ and $l = l_1 + l_2$). Using (3.2) and formulas of items 1.2) and 1.5) of Proposition 1, we obtain $g_{s_1}T_{s_1}(f_{s_2}) = g_s$. If $m_1 + m_2 \geq n - 1$, then it is easily seen that $s_1 + s_2$ satisfies none of the conditions in Theorems 2 and 3, i.e., we have $\dim_k \mathrm{HH}^{s_1+s_2}(R) = 0$. Consequently, $g_{s_1}f_{s_2} = 0$ in $\mathrm{HH}^*(R)$. Now we assume $m_1 + m_2 = n - 2$. Then the element ξ_s is defined, where $s = s_1 + s_2$ (and moreover, we have $l = l_1 + l_2 + 1$). Suppose that $\mathrm{char} k \neq 2$. Then $2 \nmid l_1$, $2 \mid l_2$. Since $2 \nmid m_1 + l_1 n$ and $2 \mid m_2 + l_2 n$, we have $2 \nmid m_1 - n$ and $2 \mid m_2$, i.e., $2 \nmid m_1 - n + m_2 = 0$. We arrive at a contradiction, whence we conclude that $\mathrm{char} k = 2$.

Using (3.2) and formulas of parts 1.3) and 1.5) of Proposition 1, we obtain

$$g_{s_1}T_{s_1}(f_{s_2}) = \sum_{i=1}^r \sum_{j=1}^{n-2} t_{i,n-2,j}.$$

Since $\xi_s = t_{r,n-2,1}$, we need to prove that $\sum_{i=1}^r \sum_{j=1}^{n-2} t_{i,n-2,j} - t_{r,n-2,1} \in \mathrm{Im} \delta_{s-1}$. From (4.3) we know that this fact is equivalent to the relation $r(n-2) - 1 = 0$ (in k), or is equivalent to the fact that $2 \nmid r$ and $2 \nmid n$. Since $2 \nmid ln$, we have $2 \nmid n$. Since $2 \nmid l(n-1) - 1$ and $r \mid l(n-1) - 1$, we obtain $2 \nmid r$. Assume that the elements g_{s_1} and g_{s_2} are defined. Then we get $g_{s_1}T_{s_1}(g_{s_2}) = 0$ by analyzing successively the cases where $m_1 + m_2 \leq n - 3$ or $m_1 + m_2 \geq n - 2$, and by using (3.2) and formulas of items 2.2), 2.5), and 2.6) of Proposition 1. Consequently, identities (5.2) are proved.

Assume that the elements p_{s_1} and f_{s_2} are defined and $m_2 \geq 1$. Then it is easily seen that the element g_s is defined, where $s = s_1 + s_2$ (and moreover, we have $m = m_2 - 1$ and $l = l_1 + l_2 + 1$). Now we obtain

$$p_{s_1}T_{s_1}(f_{s_2}) = \sum_{i=1}^r \left(\sum_{j=1}^{n-3-m} (-1)^{j+1} (w_{i,m+1,j} + u_{i,m,n-1} + u_{i,m,n}) \right) - g_s$$

with the help of (3.4) and formulas of items 1.4) and 1.5) of Proposition 1. Consequently, it suffices to prove that $w_{i,m+1,j} + u_{i,m,n-1} + u_{i,m,n} = 0$ in $\mathrm{HH}^*(R)$ for $1 \leq i \leq r$ and $1 \leq j \leq n - 3 - m$. It is easy to check that

$$\delta^{s-1} \left(\sum_{z=j+1}^{n-2-m} w_{i,m,z} \right) = w_{i,m+1,j} + u_{i,m,n-1} + u_{i,m,n}.$$

If $m_2 = 0$, then the element p_s is defined, where $s = s_1 + s_2$ (and moreover, we have $l = l_1 + l_2$). Now we can get the relation $p_{s_1}T_{s_1}(f_{s_2}) = p_s$ with the help of (3.4) and formulas of 1.1) and 1.5) of Proposition 1. If the elements p_{s_1} and g_{s_2} are defined, then, as above, $s_1 + s_2$ satisfies none of the conditions of Theorems 2 and 3, so that $\dim_k \mathrm{HH}^{s_1+s_2}(R) = 0$. Consequently, $p_{s_1}g_{s_2} = 0$ in $\mathrm{HH}^*(R)$. Assume that the elements p_{s_1} and p_{s_2} are defined. Then the element g_s is defined, where $s = s_1 + s_2$ (and moreover, $m = n - 3$ and $l = l_1 + l_2 + 1$). Using formulas of 4.2) and 4.6) of Proposition 1 and formulas (3.2) and (3.4), we obtain $p_{s_1}T_{s_1}(p_{s_2}) = -\left(\frac{n}{2}\right)g_s$. This completes the proof of (5.3).

Assume that the elements h_{s_1} and f_{s_2} are defined and $m_2 \geq 1$. If $\mathrm{char} k \neq 2$, then $s_1 + s_2$ satisfies none of the conditions of Theorems 2 and 3, whence $\dim_k \mathrm{HH}^{s_1+s_2}(R) = 0$. Consequently, $h_{s_1}f_{s_2} = 0$ in $\mathrm{HH}^*(R)$. Now, assume $\mathrm{char} k = 2$. Then the element f_{s-1} is defined, where $s = s_1 + s_2$ (and moreover, $m = m_2 - 1$ and $l = l_1 + l_2 + 1$). The identity

$$h_{s_1}T_{s_1}(f_{s_2}) = \sum_{i=1}^r \left(\sum_{j=1}^{n-3-m} w_{i,m+1,j} + (m+1)(u_{i,m,n-1} + u_{i,m,n}) \right)$$

is verified by using (3.3) and formulas of items 1.4) and 1.5) of Proposition 1. In the proof of part 1) of Lemma 1 we saw that $\varepsilon_1 T_1(f_{s-1}) = u_{1,m,n-1} + u_{1,m,n}$. Hence, we need to prove that in $\mathrm{HH}^*(R)$,

$$v = \sum_{i=1}^r \left(\sum_{j=1}^{n-3-m} w_{i,m+1,j} + (m+1)(u_{i,m,n-1} + u_{i,m,n}) \right) - (u_{1,m,n-1} + u_{1,m,n}) = 0.$$

We define

$$\begin{aligned} w &= \sum_{j=1}^{n-2-m} jw_{1,m,j} + (u_{r,n-1} + u_{r,n}) \\ &\quad + \sum_{i=2}^r \left(\sum_{j=1}^{n-2-m} (i+j)w_{i,m,j} + (i+1)(u_{i-1,n-1} + u_{i-1,n}) \right) \\ &\in \mathrm{Hom}_\Lambda(Q_{s-1}, R). \end{aligned}$$

Since $2 \nmid n$ and $2 \nmid r$, it is easy to check that $\delta^{s-1}(w) = v$. If $m_2 = 0$, then the element h_s is defined, where $s = s_1 + s_2$ (and moreover, $l = l_1 + l_2$). Then we can obtain $h_{s_1}T_{s_1}(f_{s_2}) = h_s$ by using (3.3) and formulas of items 1.1) and 1.5) of Proposition 1. Now, assume that the elements h_{s_1} and g_{s_2} are defined. Assume also that $\mathrm{char} k \neq 2$. It is easily seen that if $\dim_k \mathrm{HH}^s(R) > 0$, then, among the conditions of Theorems 2 and 3, only (2.11) may be fulfilled (and moreover, we have $m = m_2$ and $l = l_1 + l_2 + 1$). Since $2 \mid l$ in this case, we see that $\mathrm{HH}^s(R)$ is generated by f_s . We have $h_{s_1}g_{s_2} - \varkappa f_s = 0$ in $\mathrm{HH}^*(R)$ for some $\varkappa \in k$. Clearly, $h_{s_1}T_{s_1}(g_{s_2})(Q_s) \subset \mathrm{Rad} R$. If $\varkappa \neq 0$, then we obtain $h_{s_1}T_{s_1}(g_{s_2}) - \varkappa f_s \notin \mathrm{Rad} R$, a contradiction. Hence, $\varkappa = 0$, whence $h_{s_1}g_{s_2} = 0$ in $\mathrm{HH}^*(R)$. Now we assume that $\mathrm{char} k = 2$ and $m_2 \geq 1$. Then the element g_{s-1} is defined, where $s = s_1 + s_2$ (and moreover, $m = m_2 - 1$ and $l = l_1 + l_2 + 1$). Using (3.3) and formulas in 2.4) and 2.6) of Proposition 1, we conclude that

$$h_{s_1}T_{s_1}(g_{s_2}) = \sum_{i=1}^r \left(\sum_{j=n-2-m}^{n-2} (t_{i,m+1,j} + t_{i,m+1,n-2-m}) + v_{i,n-1} \right).$$

Since $\varepsilon_1 T_1(g_{s-1}) = v_{r,n-1}$, it suffices to prove that $t_{i,m+1,j} + t_{i,m+1,n-2-m} = 0$ for $1 \leq i \leq r$ and $n-2-m \leq j \leq n-2$, and that $\sum_{i=1}^{r-1} v_{i,n-1} = 0$ in $\mathrm{HH}^*(R)$. It is easy

to verify that

$$\delta^{s-1} \left(\sum_{z=n-1-m}^j t_{i,m,z} \right) = t_{i,m+1,j} + t_{i,m+1,n-2-m}$$

and

$$\delta^{s-1} \left(u_{i,m,n-1} + \sum_{z=n-1-m}^{n-2} t_{i,m,z} + v_{i,m,n-1} \right) = v_{i-1,n-1} + v_{i,n}.$$

The existence of the element h_{s_1} implies $2 \nmid r$, whence we have

$$\sum_{i=1}^{r-1} v_{i,n-1} = \sum_{i=1}^{\frac{r-1}{2}} (v_{2i-1,n-1} + v_{2i,n-1}) = 0$$

in $\mathrm{HH}^*(R)$.

Now we assume that $\mathrm{char} k = 2$ and $m_2 = 0$. Then the element χ_s is defined, where $s = s_1 + s_2$ (and moreover, we have $l = l_1 + l_2 + 1$). Using (3.3) and formulas of items 2.3) and 2.6) of Proposition 1, we obtain $h_{s_1} T_{s_1}(g_{s_2}) = \sum_{i=1}^r v_{i,n-1}$. Since $\chi_s = v_{r,n}$, it suffices to prove that $\sum_{i=1}^r v_{i,n-1} - v_{r,n} = 0$ in $\mathrm{HH}^*(R)$. By (4.4), the last relation is equivalent to the fact that the identity $\sum_{i=1}^r 1 - 1 = r - 1 = 0$ is valid in k . But we have $2 \nmid r$, so that this is true indeed. Now we assume that the elements h_{s_1} and h_{s_2} are defined. Then the element g_s is defined, where $s = s_1 + s_2$ (and moreover, $m = n - 3$ and $l = l_1 + l_2 + 1$). Using formulas in 3.3) and 3.6) of Proposition 1 and formulas (3.2) and (3.3), we see that $h_{s_1} T_{s_1}(h_{s_2}) = -\left(\frac{n-1}{2}\right) g_s$. Consequently, identities (5.4) are proved.

If the elements χ_{s_1} and f_{s_2} are defined and $m_2 \geq 1$, then it is easy to check that the element g_{s-1} is defined, where $s = s_1 + s_2$ (moreover, we have $m = m_2$ and $l = l_1 + l_2$). With the help of formulas of items 1.1) and 1.5) of Proposition 1 and using the definition of χ_{s_1} , we obtain $\chi_{s_1} T_{s_1}(f_{s_2}) = v_{r,n}$. From the proof of the part 2) of Lemma 1, we know that $\varepsilon_1 T_1(g_{s-1}) = v_{r,n-1}$. Because of this we need to prove that $v_{r,n-1} + v_{r,n} = 0$ in $\mathrm{HH}^*(R)$. It is easily seen that $\delta^{s-1}(u_{1,m-1,n-1} + u_{1,m-1,n}) = v_{r,n-1} + v_{r,n}$. If $m_2 = 0$, then the element χ_s is defined, where $s = s_1 + s_2$ (moreover, we have $l = l_1 + l_2$). Hence, using formulas of parts 1.1) and 1.5) of Proposition 1 and the definition of χ_{s_1} , we obtain $\chi_{s_1} T_{s_1}(f_{s_2}) = \chi_s$. Assume that the element χ_{s_1} is defined. If, moreover, the element g_{s_2} (respectively, either h_{s_2} or p_{s_2}) is defined, then, using formulas of items 2.1), 2.6), 3.1), 3.6), 4.1), and 4.6) of Proposition 1 and the definition of χ_{s_1} , we obtain $\chi_{s_1} T_{s_1}(g_{s_2}) = 0$ (respectively, $\chi_{s_1} T_{s_1}(h_{s_2}) = 0$ or $\chi_{s_1} T_{s_1}(p_{s_2}) = 0$). This proves identities (5.5).

Assume that the elements ξ_{s_1} and f_{s_2} are defined. If we assume additionally that $m_2 \geq 1$, then, using formulas of 1.1) and 1.5) of Proposition 1 and the definition of ξ_{s_1} , we obtain $\xi_{s_1} T_{s_1}(f_{s_2}) = 0$. If $m_2 = 0$, then the element ξ_s is defined, where $s = s_1 + s_2$ (moreover, $l = l_1 + l_2$). Using formulas of parts 1.1) and 1.5) of Proposition 1 and the definition of ξ_{s_1} , we get $\xi_{s_1} T_{s_1}(f_{s_2}) = \xi_s$. If the elements ξ_{s_1} and g_{s_2} are defined, then the formulas of items 2.1) and 2.6) of Proposition 1 and the definition of ξ_{s_1} , yield $\xi_{s_1} T_{s_1}(g_{s_2}) = 0$. If the elements ξ_{s_1} and h_{s_2} are defined, then it is easily seen that the element g_{s-1} is defined, where $s = s_1 + s_2$ (moreover, $m = n - 2$ and $l = l_1 + l_2$). Using formulas of items 3.1) and 3.6) of Proposition 1 and the definition of ξ_{s_1} , we get $\xi_{s_1} T_{s_1}(h_{s_2}) = -t_{r,n-2,1}$. From the proof of Lemma 1, we know that $\varepsilon_1 T_1(g_{s-1}) = v_{r,n-1}$. Hence, we need to prove that $t_{r,n-2,1} - v_{r,n-1} = 0$ in $\mathrm{HH}^*(R)$. We put $w = \sum_{j=2}^{n-2} t_{r,n-3,j} + v_{r,n-3,n-1} \in \mathrm{Hom}_\Lambda(Q_{s-1}, R)$. Then it is easy to verify that $\delta^{s-1}(w) = t_{r,n-2,1} - v_{r,n-1}$, i.e., $t_{r,n-2,1} - v_{r,n-1} = 0$ in $\mathrm{HH}^*(R)$. This completes the proof of (5.6). $\square$

Proposition 4. *The product of any two elements of the set $\{\varepsilon_1, \chi_s, \xi_s\}$ is equal to 0 in $\mathrm{HH}^*(R)$.*

Proof. Theorems 2 and 3 show that $\mathrm{HH}^2(R) = 0$, whence $(\varepsilon_1)^2 = 0$. Assume that the element ξ_s is defined. By Theorems 2 and 3, $\mathrm{HH}^{s+1}(R)$ can be nonzero only if $s + 1$ satisfies condition (2.9). Then it is easily seen that $\mathrm{HH}^{s+1}(R)$ is generated by g_{s+1} . If $\varepsilon_1 \xi_s \neq 0$, then $\varepsilon_1 \xi_s = \varkappa g_{s+1}$ with $\varkappa \in k$. Multiplying this by ε_1 , we obtain $0 = \varkappa \varepsilon_1 g_{s+1}$. By Lemma 1, we have $\varepsilon_1 g_{s+1} \neq 0$, whence $\varkappa = 0$.

Assume that the element χ_s is defined. Then, using Theorems 2 and 3, we easily check that $\mathrm{HH}^{s+1}(R)$ can be nonzero only if $s + 1$ satisfies condition (2.10). It is easily seen that $\mathrm{HH}^{s+1}(R)$ is generated by the element $\varepsilon_1 f_s$ in this case. Note that the elements f_4 and f_{s+4} are defined. If we assume, moreover, that $2 \nmid n$, then $2n - 3 \nmid s + 5$, because $2n - 3 \mid s$ and $n \geq 5$. Consequently, $\varepsilon_1 f_{s+4} \neq 0$ by Lemma 1. If $2 \mid n$, then $\varepsilon_1 f_{s+4} \neq 0$ by Lemma 1 and Remark 4. Furthermore, we have $\varepsilon_1 \chi_s = \varkappa \varepsilon_1 f_s$ with $\varkappa \in k$. Multiplying this by f_4 , we obtain $0 = -(\varepsilon_1)^2 g_{s+3} = \varkappa \varepsilon_1 f_{s+4}$. Since $\varepsilon_1 f_{s+4} \neq 0$, we have $\varkappa = 0$.

Assume that the elements χ_{s_1} and χ_{s_2} are defined and suppose that $2 \nmid n$. It is easily seen that $\mathrm{HH}^s(R)$ with $s = s_1 + s_2$ is generated by the elements of the set $\{f_s, \chi_s\}$ that are defined for s . Then we have $\chi_{s_1} \chi_{s_2} = \varkappa_1 f_s + \varkappa_2 \chi_s$ with $\varkappa_1, \varkappa_2 \in k$ (here, if an element is not defined for s , we assume that the corresponding $\varkappa_i$ is zero). Multiplying the last identity by ε_1 , we obtain $0 = \varkappa_1 \varepsilon_1 f_s$. Since $\varepsilon_1 f_s \neq 0$, we have $\varkappa_1 = 0$. Multiplying the same identity by f_{s_1-1} , we get $0 = -\chi_{s_1} \varepsilon_1 g_{s-2} = -\varkappa_2 \varepsilon_1 g_{s_1+s-2}$, using the graded commutativity of the algebra $\mathrm{HH}^*(R)$. Consequently, $\varkappa_2 = 0$.

Again, assume that the elements χ_{s_1} and χ_{s_2} are defined and that $2 \mid n$. It is easily seen that $\mathrm{HH}^s(R)$ with $s = s_1 + s_2$ is generated by the elements of the set $\{f_s, \chi_s, \varepsilon_1 f_{s-1}\}$ that are defined for s . Then $\chi_{s_1} \chi_{s_2} = \varkappa_1 f_s + \varkappa_2 \chi_s + \varkappa_3 \varepsilon_1 f_{s-1}$ with $\varkappa_1, \varkappa_2, \varkappa_3 \in k$ (here, if an element is not defined for s , we assume that the corresponding $\varkappa_i$ is zero). Multiplying the last identity by ε_1 , once again we see that $\varkappa_1 = 0$. Multiplying the resulting relation by p_{s_1-1} and using the graded commutativity of the algebra $\mathrm{HH}^*(R)$, we get $0 = -\varkappa_3 \varepsilon_1 g_{s_1+s-2}$. Consequently, $\varkappa_3 = 0$. Now we multiply the identity obtained by f_{s_1-1} to show that $\varkappa_2 = 0$.

Assume that the elements ξ_{s_1} and ξ_{s_2} are defined. It is easily seen that $\mathrm{HH}^s(R)$ with $s = s_1 + s_2$ is generated by the elements of the set $\{f_s, \chi_s\}$ that are defined for s . Then $\xi_{s_1} \xi_{s_2} = \varkappa_1 f_s + \varkappa_2 \chi_s$ with $\varkappa_1, \varkappa_2 \in k$ (here, if an element is not defined for s , then we assume that the corresponding $\varkappa_i$ is zero). Multiplying this identity by ε_1 and then, in the case where χ_s is defined, by f_{s-1} , we see that $\varkappa_1 = 0$, implying $\varkappa_2 = 0$. Finally, we consider the last possibility, namely, we assume that the elements ξ_{s_1} and χ_{s_2} are defined. Theorems 2 and 3 show that $\mathrm{HH}^s(R)$ can be nonzero for $s = s_1 + s_2$ only if s satisfies condition (2.8). It is easily seen that $\mathrm{HH}^s(R)$ is generated by ξ_s in this case. Then $\xi_{s_1} \chi_{s_2} = \varkappa \xi_s$ with $\varkappa \in k$. Multiplying this by h_{s_1-1} , we obtain $0 = -\varkappa \varepsilon_1 g_{s-2}$, whence $\varkappa = 0$. Proposition 4 is proved, in view of the graded commutativity of the algebra $\mathrm{HH}^*(R)$. $\square$

Corollary 3. *The algebra $\mathrm{HH}^*(R)$ is commutative.*

Proof. Since $\mathrm{HH}^*(R)$ is graded commutative, it is commutative in the case where $\mathrm{char} k = 2$. Assume $\mathrm{char} k \neq 2$. Then, by graded commutativity, it suffices to prove that the product of any two different generators of odd degree is 0. Since $\mathrm{char} k \neq 2$, it is easily seen that the elements f_s, g_s, h_s , and p_s are defined only for even s . Hence, all generators of odd degree are among the elements ε_1, χ_s , and ξ_s . Now, the desired statement follows from Proposition 4. $\square$

It remains to compute the products of the elements $\varepsilon_0^{(j)}$ ($1 \leq j \leq n$) by themselves and by other elements. Since the elements $\varepsilon_0^{(j)}$ are defined only for $r = 1$, we assume that $r = 1$ in the remaining part of this section.

Proposition 5. 1) We have $\varepsilon_0^{(i)} \varepsilon_0^{(j)} = 0$ for $1 \leq i, j \leq n$.

2.1) If $\text{char } k \neq 2$, then $\varepsilon_0^{(j)} \text{HH}^s(R) = 0$ for $s \geq 1$ and $1 \leq j \leq n$.

2.2) Assume that $\text{char } k = 2$ and $2 \nmid n$. Then $\varepsilon_0^{(j)} \text{HH}^s(R) = 0$ for $s \geq 1$ and $1 \leq j \leq n - 2$. Moreover, for $q \in \{n - 1, n\}$ we have

$$\varepsilon_0^{(q)} f_s = \begin{cases} \varepsilon_1 g_{s-1} & \text{if } m \geq 1, \\ \chi_s & \text{if } m = 0 \end{cases} \quad (s \geq 1);$$

$$\varepsilon_0^{(q)} g_s = \varepsilon_0^{(q)} h_s = \varepsilon_0^{(q)} \chi_s = \varepsilon_0^{(q)} \xi_s = \varepsilon_0^{(q)} \varepsilon_1 = 0.$$

2.3) Assume that $\text{char } k = 2$ and $2 \mid n$. Then

$$\varepsilon_0^{(j)} f_s = \begin{cases} 0 & \text{if } m \geq 1 \text{ and } 1 \leq j \leq n - 2, \\ \varepsilon_1 g_{s-1} & \text{if } m \geq 1 \text{ and } j \in \{n - 1, n\}, \\ \chi_s + \chi'_s & \text{if } m = 0 \text{ and } 1 \leq j \leq n - 2 \\ \chi'_s & \text{if } m = 0 \text{ and } j = n - 1, \\ \chi_s & \text{if } m = 0 \text{ and } j = n, \end{cases} \quad (s \geq 1);$$

$$\varepsilon_0^{(j)} p_s = \begin{cases} \varepsilon_1 g_{s-1} & \text{if } 1 \leq j \leq n - 1, \\ 0 & \text{if } j = n; \end{cases}$$

$$\varepsilon_0^{(j)} g_s = \varepsilon_0^{(j)} \chi_s = \varepsilon_0^{(j)} \varepsilon_1 = 0 \quad (1 \leq j \leq n).$$

Proof. 1) Since $\varepsilon_0^{(j)}(Q_0) \subset \text{Rad } R$, we have $T_0(\varepsilon_0^{(j)})(Q_0) \subset \text{Rad } Q_0$ (because $\mu(\theta) \notin \text{Rad } R$ for $\theta \notin \text{Rad } Q_0$). Obviously, $\varepsilon_0^{(i)}(\theta) = 0$ for $\theta \in \text{Rad } Q_0$. Consequently, $\varepsilon_0^{(i)} \varepsilon_0^{(j)} = 0$.

2) Observe the following fact. Assume that $\theta \in \text{HH}^s(R)$ with $s \geq 1$, and suppose $\varepsilon_0^{(j)} f_\lambda \theta = \omega f_\lambda$. Then the relation $\varepsilon_0^{(j)} \theta = \omega$ follows from Remark 3 and the commutativity of $\text{HH}^*(R)$.

2.1) If $\text{char } k \neq 2$, then Theorem 3 shows that $\dim_k \text{HH}^\lambda(R) = 1$, i.e., $\text{HH}^\lambda(R)$ is generated by the element f_λ . Since $\varepsilon_0^{(j)}(Q_0) \subset \text{Rad } R$ and f_λ is an isomorphism, we have $\varepsilon_0^{(j)} f_\lambda = 0$. Then item 2.1) of Proposition 5 follows from the above observation.

2.2) The relations $\varepsilon_0^{(j)} f_\lambda = t_{n-2,j}$ for $1 \leq j \leq n - 2$ and $\varepsilon_0^{(q)} f_\lambda = v_q$ for $q \in \{n - 1, n\}$ are obtained by using formulas of items 1.1) and 1.5) of Proposition 1 (as before, in our notation we omit the first index, which is always equal to 1 for $r = 1$).

Now, suppose that $\text{char } k = 2$ and $2 \nmid n$. It is easily seen that $\text{HH}^\lambda(R)$ is generated by the elements f_λ and χ_λ . From the proof of item 5) of Lemma 1, it follows that $t_{n-2,j} = 0$ for $1 \leq j \leq n - 2$ and $v_{n-1} = v_n = \chi_\lambda$. Then all identities of item 2.2) follow from the observation at the beginning of part 2 of the proof, from Propositions 3 and 4, and from the corollary to Proposition 4.

2.3) Assume that $\text{char } k = 2$ and $2 \mid n$. It is easily seen that $\text{HH}^\lambda(R)$ is generated by the elements f_λ , χ_λ , and χ'_λ . From the proof of part 4) of Lemma 1, it follows that $t_{n-2,j} = \chi_\lambda + \chi'_\lambda$ with $1 \leq j \leq n - 2$, $v_{n-1} = \chi'_\lambda$, and $v_n = \chi_\lambda$. Since (4.2) implies $\chi'_s = \chi_s + \varepsilon_1 f_{s-1}$, all identities in item 2.3) follow from the observation at the beginning of part 2) of the proof, from Propositions 3 and 4, and from the corollary to Proposition 4. $\square$

§6. MAIN RESULT

In this section, we describe the algebra $\text{HH}^*(R)$ with the help of generators and defining relations.

With each element $\theta \neq f_0$ in the set of generators of the algebra $\text{HH}^*(R)$, as described in Corollary 2, we associate a variable $\tilde{\theta}$.

We consider the following sets (in all cases, a variable with index s is included in this set if this variable is defined):

$$\mathcal{A} = \{\tilde{\varepsilon}_1, \tilde{f}_s, \tilde{g}_s\}; \quad \mathcal{B}_1 = \{\tilde{p}_s\}; \quad \mathcal{B}_2 = \{\tilde{h}_s\}; \quad \mathcal{C}_1 = \{\tilde{\chi}_s\}; \quad \mathcal{C}_2 = \{\tilde{\xi}_s\}; \quad \mathcal{D} = \{\tilde{\varepsilon}_0^{(j)}\}_{1 \leq j \leq n}.$$

Next, we define the following sets:

$$\begin{aligned} \mathcal{X}_1 &= \mathcal{A} \cup \mathcal{B}_1; & \mathcal{X}_2 &= \mathcal{X}_3 = \mathcal{X}_1 \cup \mathcal{D}; \\ \mathcal{X}_4 &= \mathcal{X}_1 \cup \mathcal{C}_1; & \mathcal{X}_5 &= \mathcal{X}_4 \cup \mathcal{D}; \\ \mathcal{X}_6 &= \mathcal{A} \cup \mathcal{B}_2; & \mathcal{X}_7 &= \mathcal{X}_6 \cup \mathcal{D}; \\ \mathcal{X}_8 &= \mathcal{X}_6 \cup \mathcal{C}_2; & \mathcal{X}_9 &= \mathcal{X}_8 \cup \mathcal{D}; \\ \mathcal{X}_{10} &= \mathcal{X}_6 \cup \mathcal{C}_1; & \mathcal{X}_{11} &= \mathcal{X}_{10} \cup \mathcal{D}; \\ \mathcal{X}_{12} &= \mathcal{X}_6 \cup \mathcal{C}_1 \cup \mathcal{C}_2; & \mathcal{X}_{13} &= \mathcal{X}_{12} \cup \mathcal{D}. \end{aligned}$$

On $k[\mathcal{X}_i]$ ($1 \leq i \leq 13$), we introduce a grading such that

$$\begin{aligned} \deg \tilde{\varepsilon}_0^{(j)} &= 0, \quad \deg \tilde{\varepsilon}_1 = 1, \quad \deg \tilde{f}_s = s, \quad \deg \tilde{g}_s = s, \\ \deg \tilde{h}_s &= s, \quad \deg \tilde{p}_s = s, \quad \deg \tilde{\chi}_s = s, \quad \deg \tilde{\xi}_s = s. \end{aligned}$$

Let $s = a\lambda + b$ with $a \in \mathbb{N} \cup \{0\}$ and $1 \leq b \leq \lambda$. Then we put

$$\phi(s) = a; \quad \psi(s) = b.$$

Next, let $\mathcal{E}_{\mathcal{A}}$ denote the following set of homogeneous elements of the algebra $k[\mathcal{A}]$:

$$\begin{aligned} &\{(\tilde{\varepsilon}_1)^2\} \cup \{\tilde{f}_{s_1}\tilde{f}_{s_2} - \tilde{f}_{\psi(s_1+s_2)}(\tilde{f}_\lambda)^{\phi(s_1+s_2)} \mid m_1 + m_2 \leq n-2\} \\ &\cup \{\tilde{f}_{s_1}\tilde{f}_{s_2} + 2(-1)^n \tilde{g}_{\psi(s_1+s_2)}(\tilde{f}_\lambda)^{\phi(s_1+s_2)} \mid m_1 + m_2 \geq n-1\} \\ &\cup \{\tilde{g}_{s_1}\tilde{f}_{s_2} - \tilde{g}_{\psi(s_1+s_2)}(\tilde{f}_\lambda)^{\phi(s_1+s_2)} \mid m_1 + m_2 \leq n-3\} \\ &\cup \{\tilde{g}_{s_1}\tilde{f}_{s_2} \mid m_1 + m_2 \geq n-1\} \cup \{\tilde{g}_{s_1}\tilde{g}_{s_2}\}. \end{aligned}$$

Also, $\mathcal{E}_{\mathcal{D}}$ will be the following set of homogeneous elements of the algebra $k[\mathcal{D}]$:

$$\{\tilde{\varepsilon}_0^{(j_1)}\tilde{\varepsilon}_0^{(j_2)} \mid 1 \leq j_1, j_2 \leq n\}.$$

We denote by $\mathcal{E}_{\mathcal{A} \cup \mathcal{B}_2 \cup \mathcal{C}_1}$ the following set of homogeneous elements of the algebra $k[\mathcal{A} \cup \mathcal{B}_2 \cup \mathcal{C}_1]$:

$$\begin{aligned} &\{\tilde{\varepsilon}_1\tilde{f}_s \mid s = (l+1)(2n-3) - 1 \text{ and condition (2.4) is satisfied}\} \\ &\cup \{\tilde{h}_{s_1}\tilde{f}_{s_2} - \tilde{\varepsilon}_1\tilde{f}_{\psi(s_1+s_2-1)}(\tilde{f}_\lambda)^{\phi(s_1+s_2-1)} \mid m_2 \geq 1\} \\ &\cup \{\tilde{h}_{s_1}\tilde{g}_{s_2} - \tilde{\varepsilon}_1\tilde{g}_{\psi(s_1+s_2-1)}(\tilde{f}_\lambda)^{\phi(s_1+s_2-1)} \mid m_2 \geq 1\} \\ &\cup \{\tilde{h}_{s_1}\tilde{g}_{s_2} - \tilde{\chi}_{\psi(s_1+s_2)}(\tilde{f}_\lambda)^{\phi(s_1+s_2)} \mid m_2 = 0\} \\ &\cup \{\tilde{h}_{s_1}\tilde{h}_{s_2} + \left(\frac{n-1}{2}\right)\tilde{g}_{\psi(s_1+s_2)}(\tilde{f}_\lambda)^{\phi(s_1+s_2)}\} \\ &\cup \{\tilde{\chi}_{s_1}\tilde{f}_{s_2} + \tilde{\varepsilon}_1\tilde{g}_{\psi(s_1+s_2-1)}(\tilde{f}_\lambda)^{\phi(s_1+s_2-1)} \mid m_2 \geq 1\} \\ &\cup \{\tilde{\varepsilon}_1\tilde{\chi}_s\} \cup \{\tilde{\chi}_{s_1}\tilde{g}_{s_2}\} \cup \{\tilde{\chi}_{s_1}\tilde{h}_{s_2}\} \cup \{\tilde{\chi}_{s_1}\tilde{\chi}_{s_2}\}. \end{aligned}$$

Now, we pick the following set $\mathcal{E}_i$ of homogeneous elements of the algebra $k[\mathcal{X}_i]$, where $1 \leq i \leq 13$:

$$\begin{aligned} \mathcal{E}_1 &= \mathcal{E}_{\mathcal{A}} \cup \{\tilde{p}_{s_1}\tilde{f}_{s_2} + \tilde{g}_{\psi(s_1+s_2)}(\tilde{f}_\lambda)^{\phi(s_1+s_2)} \mid m_2 \geq 1\} \\ &\cup \{\tilde{p}_{s_1}\tilde{p}_{s_2} + \left(\frac{n}{2}\right)\tilde{g}_{\psi(s_1+s_2)}(\tilde{f}_\lambda)^{\phi(s_1+s_2)}\}; \\ \mathcal{E}_2 &= \mathcal{E}_1 \cup \mathcal{E}_{\mathcal{D}} \cup \{\tilde{\varepsilon}_0^{(j)}\tilde{\varepsilon}_1, \tilde{\varepsilon}_0^{(j)}\tilde{f}_s, \tilde{\varepsilon}_0^{(j)}\tilde{g}_s, \tilde{\varepsilon}_0^{(j)}\tilde{p}_s \mid 1 \leq j \leq n\}; \end{aligned}$$

$$\begin{aligned}
\mathcal{E}_3 &= \mathcal{E}_1 \cup \mathcal{E}_{\mathcal{D}} \cup \{ \tilde{\varepsilon}_0^{(j)} \tilde{f}_s \mid 1 \leq j \leq n-2, s < \lambda \} \\
&\cup \{ \tilde{\varepsilon}_0^{(q)} \tilde{f}_s - \tilde{\varepsilon}_1 \tilde{g}_{s-1} \mid q \in \{n-1, n\}, s < \lambda \} \\
&\cup \{ \tilde{\varepsilon}_0^{(j)} \tilde{f}_\lambda - \tilde{\varepsilon}_1 \tilde{f}_{\lambda-1} \mid 1 \leq j \leq n-2 \} \\
&\cup \{ \tilde{\varepsilon}_0^{(n-1)} \tilde{f}_\lambda - \left(\frac{n-2}{2}\right) \tilde{\varepsilon}_1 \tilde{f}_{\lambda-1} - \tilde{\varepsilon}_1 \tilde{p}_{\lambda-1} \} \\
&\cup \{ \tilde{\varepsilon}_0^{(n)} \tilde{f}_\lambda - \left(\frac{n}{2}\right) \tilde{\varepsilon}_1 \tilde{f}_{\lambda-1} - \tilde{\varepsilon}_1 \tilde{p}_{\lambda-1} \} \\
&\cup \{ \tilde{\varepsilon}_0^{(j)} \tilde{p}_s - \tilde{\varepsilon}_1 \tilde{g}_{s-1} \mid 1 \leq j \leq n-1 \} \cup \{ \tilde{\varepsilon}_0^{(n)} \tilde{p}_s \} \cup \{ \tilde{\varepsilon}_0^{(j)} \tilde{g}_s, \tilde{\varepsilon}_0^{(j)} \tilde{\varepsilon}_1 \mid 1 \leq j \leq n \}; \\
\mathcal{E}_4 &= \mathcal{E}_1 \cup \{ \tilde{\chi}_{s_1} \tilde{f}_{s_2} + \tilde{\varepsilon}_1 \tilde{g}_{\psi(s_1+s_2-1)}(\tilde{f}_\lambda)^{\phi(s_1+s_2-1)} \mid m_2 \geq 1 \} \\
&\cup \{ \tilde{\varepsilon}_1 \tilde{f}_s - 2\tilde{\varepsilon}_1 \tilde{p}_s \} \cup \{ \tilde{\varepsilon}_1 \tilde{\chi}_s \} \cup \{ \tilde{\chi}_{s_1} \tilde{g}_{s_2} \} \cup \{ \tilde{\chi}_{s_1} \tilde{p}_{s_2} \} \cup \{ \tilde{\chi}_{s_1} \tilde{\chi}_{s_2} \}; \\
\mathcal{E}_5 &= \mathcal{E}_4 \cup \mathcal{E}_{\mathcal{D}} \cup \{ \tilde{\varepsilon}_0^{(j)} \tilde{\varepsilon}_1, \tilde{\varepsilon}_0^{(j)} \tilde{f}_s, \tilde{\varepsilon}_0^{(j)} \tilde{g}_s, \tilde{\varepsilon}_0^{(j)} \tilde{p}_s, \tilde{\varepsilon}_0^{(j)} \tilde{\chi}_s \mid 1 \leq j \leq n \}; \\
\mathcal{E}_6 &= \mathcal{E}_A \cup \{ \tilde{h}_{s_1} \tilde{f}_{s_2} \mid m_2 \geq 1 \} \cup \{ \tilde{h}_{s_1} \tilde{g}_{s_2} \} \\
&\cup \{ \tilde{h}_{s_1} \tilde{h}_{s_2} + \left(\frac{n-1}{2}\right) \tilde{g}_{\psi(s_1+s_2)}(\tilde{f}_\lambda)^{\phi(s_1+s_2)} \}; \\
\mathcal{E}_7 &= \mathcal{E}_6 \cup \mathcal{E}_{\mathcal{D}} \cup \{ \tilde{\varepsilon}_0^{(j)} \tilde{\varepsilon}_1, \tilde{\varepsilon}_0^{(j)} \tilde{f}_s, \tilde{\varepsilon}_0^{(j)} \tilde{g}_s, \tilde{\varepsilon}_0^{(j)} \tilde{h}_s \mid 1 \leq j \leq n \}; \\
\mathcal{E}_8 &= \mathcal{E}_6 \cup \{ \tilde{\xi}_{s_1} \tilde{f}_{s_2} \mid m_2 \geq 1 \} \cup \{ \tilde{\varepsilon}_1 \tilde{\xi}_s \} \cup \{ \tilde{\xi}_{s_1} \tilde{g}_{s_2} \} \cup \{ \tilde{\xi}_{s_1} \tilde{\xi}_{s_2} \} \\
&\cup \{ \tilde{\xi}_{s_1} \tilde{h}_{s_2} + \tilde{\varepsilon}_1 \tilde{g}_{\psi(s_1+s_2-1)}(\tilde{f}_\lambda)^{\phi(s_1+s_2-1)} \} \cup \{ \tilde{\varepsilon}_1 \tilde{h}_s \}; \\
\mathcal{E}_9 &= \mathcal{E}_8 \cup \mathcal{E}_{\mathcal{D}} \cup \{ \tilde{\varepsilon}_0^{(j)} \tilde{\varepsilon}_1, \tilde{\varepsilon}_0^{(j)} \tilde{f}_s, \tilde{\varepsilon}_0^{(j)} \tilde{g}_s, \tilde{\varepsilon}_0^{(j)} \tilde{h}_s, \tilde{\varepsilon}_0^{(j)} \tilde{\xi}_s \mid 1 \leq j \leq n \}; \\
\mathcal{E}_{10} &= \mathcal{E}_A \cup \mathcal{E}_{A \cup B_2 \cup C_1} \\
&\cup \{ \tilde{g}_{s_1} \tilde{f}_{s_2} - \left(\frac{n-1}{2}\right)^{-1} \tilde{\varepsilon}_1 \tilde{h}_{\psi(s_1+s_2-1)}(\tilde{f}_\lambda)^{\phi(s_1+s_2-1)} \mid m_1 + m_2 = n-2 \}; \\
\mathcal{E}_{11} &= \mathcal{E}_{10} \cup \mathcal{E}_{\mathcal{D}} \cup \{ \tilde{\varepsilon}_0^{(j)} \tilde{f}_s \mid 1 \leq j \leq n-2 \} \\
&\cup \{ \tilde{\varepsilon}_0^{(q)} \tilde{f}_s - \tilde{\varepsilon}_1 \tilde{g}_{s-1} \mid q \in \{n-1, n\}, s < \lambda \} \\
&\cup \{ \tilde{\varepsilon}_0^{(q)} \tilde{f}_\lambda - \tilde{\chi}_\lambda \mid q \in \{n-1, n\} \} \cup \{ \tilde{\varepsilon}_0^{(j)} \tilde{\varepsilon}_1, \tilde{\varepsilon}_0^{(j)} \tilde{g}_s, \tilde{\varepsilon}_0^{(j)} \tilde{h}_s, \tilde{\varepsilon}_0^{(j)} \tilde{\chi}_s \mid 1 \leq j \leq n \}; \\
\mathcal{E}_{12} &= \mathcal{E}_A \cup \mathcal{E}_{A \cup B_2 \cup C_1} \\
&\cup \{ \tilde{g}_{s_1} \tilde{f}_{s_2} - \tilde{\xi}_{\psi(s_1+s_2)}(\tilde{f}_\lambda)^{\phi(s_1+s_2)} \mid m_1 + m_2 = n-2 \} \\
&\cup \{ \tilde{\xi}_{s_1} \tilde{f}_{s_2} \mid m_2 \geq 1 \} \cup \{ \tilde{\varepsilon}_1 \tilde{\xi}_s \} \cup \{ \tilde{\xi}_{s_1} \tilde{g}_{s_2} \} \cup \{ \tilde{\xi}_{s_1} \tilde{\xi}_{s_2} \} \cup \{ \tilde{\xi}_{s_1} \tilde{\chi}_{s_2} \} \\
&\cup \{ \tilde{\xi}_{s_1} \tilde{h}_{s_2} + \tilde{\varepsilon}_1 \tilde{g}_{\psi(s_1+s_2-1)}(\tilde{f}_\lambda)^{\phi(s_1+s_2-1)} \} \cup \{ \tilde{\varepsilon}_1 \tilde{h}_s \}; \\
\mathcal{E}_{13} &= \mathcal{E}_{12} \cup \mathcal{E}_{\mathcal{D}} \cup \{ \tilde{\varepsilon}_0^{(j)} \tilde{f}_s \mid 1 \leq j \leq n-2 \} \\
&\cup \{ \tilde{\varepsilon}_0^{(q)} \tilde{f}_s - \tilde{\varepsilon}_1 \tilde{g}_{s-1} \mid q \in \{n-1, n\}, s < \lambda \} \\
&\cup \{ \tilde{\varepsilon}_0^{(q)} \tilde{f}_\lambda - \tilde{\chi}_\lambda \mid q \in \{n-1, n\} \} \\
&\cup \{ \tilde{\varepsilon}_0^{(j)} \tilde{\varepsilon}_1, \tilde{\varepsilon}_0^{(j)} \tilde{g}_s, \tilde{\varepsilon}_0^{(j)} \tilde{h}_s, \tilde{\varepsilon}_0^{(j)} \tilde{\chi}_s, \tilde{\varepsilon}_0^{(j)} \tilde{\xi}_s \mid 1 \leq j \leq n \}.
\end{aligned}$$

Finally, we denote by I_i the ideal of the algebra $k[\mathcal{X}_i]$ generated by the set $\mathcal{E}_i$ for all $1 \leq i \leq 13$.

- Theorem 4.** 1) Suppose $2 \mid n$, $\text{char } k \nmid n-1$, and $r > 1$. Then $\text{HH}^*(R) \simeq k[\mathcal{X}_1]/I_1$.
2) Suppose $2 \mid n$, $\text{char } k \nmid n-1$, $\text{char } k \neq 2$, and $r = 1$. Then $\text{HH}^*(R) \simeq k[\mathcal{X}_2]/I_2$.
3) Suppose $2 \mid n$, $\text{char } k = 2$, and $r = 1$. Then $\text{HH}^*(R) \simeq k[\mathcal{X}_3]/I_3$.
4) Suppose $2 \mid n$, $\text{char } k \mid n-1$, and $r > 1$. Then $\text{HH}^*(R) \simeq k[\mathcal{X}_4]/I_4$.
5) Suppose $2 \mid n$, $\text{char } k \mid n-1$, and $r = 1$. Then $\text{HH}^*(R) \simeq k[\mathcal{X}_5]/I_5$.
6) Suppose $2 \nmid n$, $\text{char } k \nmid \left(\frac{n-1}{2}\right)$, $\text{char } k \neq 2$, and $r > 1$. Then $\text{HH}^*(R) \simeq k[\mathcal{X}_6]/I_6$.

- 7) Suppose $2 \nmid n$, $\text{char } k \nmid \left(\frac{n-1}{2}\right)$, $\text{char } k \neq 2$, and $r = 1$. Then $\text{HH}^*(R) \simeq k[\mathcal{X}_7]/I_7$.
- 8) Suppose $2 \nmid n$, $\text{char } k \mid \left(\frac{n-1}{2}\right)$, $\text{char } k \neq 2$, and $r > 1$. Then $\text{HH}^*(R) \simeq k[\mathcal{X}_8]/I_8$.
- 9) Suppose $2 \nmid n$, $\text{char } k \mid \left(\frac{n-1}{2}\right)$, $\text{char } k \neq 2$, and $r = 1$. Then $\text{HH}^*(R) \simeq k[\mathcal{X}_9]/I_9$.
- 10) Suppose $2 \nmid n$, $\text{char } k \nmid \left(\frac{n-1}{2}\right)$, $\text{char } k = 2$, and $r > 1$. Then $\text{HH}^*(R) \simeq k[\mathcal{X}_{10}]/I_{10}$.
- 11) Suppose $2 \nmid n$, $\text{char } k \nmid \left(\frac{n-1}{2}\right)$, $\text{char } k = 2$, and $r = 1$. Then $\text{HH}^*(R) \simeq k[\mathcal{X}_{11}]/I_{11}$.
- 12) Suppose $2 \nmid n$, $\text{char } k \mid \left(\frac{n-1}{2}\right)$, $\text{char } k = 2$, and $r > 1$. Then $\text{HH}^*(R) \simeq k[\mathcal{X}_{12}]/I_{12}$.
- 13) Suppose $2 \nmid n$, $\text{char } k \mid \left(\frac{n-1}{2}\right)$, $\text{char } k = 2$, and $r = 1$. Then $\text{HH}^*(R) \simeq k[\mathcal{X}_{13}]/I_{13}$.

Proof. In each case from 1) to 13), we consider the homomorphism of graded k -algebras $\pi : k[\mathcal{X}_i] \rightarrow \text{HH}^*(R)$ that maps any element $\tilde{\theta}$ of the set $\mathcal{X}_i$ to the corresponding $\theta \in \text{HH}^*(R)$. Corollary 2 shows that π is a surjective homomorphism. From Propositions 3–5 and from (4.1) and (4.2), it follows that $I_i \subset \text{Ker } \pi$ in each case among the 13 cases above. Consequently, there exists a surjective homomorphism of graded k -algebras $\bar{\pi} : k[\mathcal{X}_i]/I_i \rightarrow \text{HH}^*(R)$ that maps the class of the element $\tilde{\theta} \in \mathcal{X}_i$ to θ . It remains to prove that $\bar{\pi}$ is an isomorphism. We denote the algebra $k[\mathcal{X}_i]/I_i$ by A_i . Let $A_i = \bigoplus_{s \geq 0} A_i^s$ be the decomposition of the algebra A_i into homogeneous direct summands. Then it suffices to check that $\dim_k A_i^s \leq \dim_k \text{HH}^s(R)$ for all $s \geq 0$. It is easily seen that any products of the form $\tilde{\theta}_1 \tilde{\theta}_2$, where $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are elements of $\mathcal{X}_i$ different from $\tilde{\varepsilon}_1$ and $\tilde{f}_\lambda$, are equal modulo the ideal I_i either to 0 or to $\varkappa(\tilde{\varepsilon}_1)^{a_1} \tilde{\theta}(\tilde{f}_\lambda)^{a_2}$, where $a_1, a_2 \in \{0, 1\}$, $\varkappa \in k$, and $\tilde{\theta} \in (\mathcal{X}_i \setminus \{\varepsilon_1, f_\lambda\}) \cup \{1\}$. Then we can reduce every nonzero product of the form $\tilde{\theta}_1 \dots \tilde{\theta}_t$, where $\tilde{\theta}_1, \dots, \tilde{\theta}_t \in \mathcal{X}_i$, to $\varkappa(\tilde{\varepsilon}_1)^{a_1} \tilde{\theta}(\tilde{f}_\lambda)^{a_2}$ with $\varkappa \in k$, $a_1 \in \{0, 1\}$, $a_2 \geq 0$, and $\tilde{\theta} \in (\mathcal{X}_i \setminus \{\varepsilon_1, f_\lambda\}) \cup \{1\}$. Since all elements of the set $\mathcal{X}_i$ are of degree at most λ and all elements of degree λ except $\tilde{f}_\lambda$ give 0 in the product with $\tilde{\varepsilon}_1$, it follows that every homogeneous element ρ of degree $s > \lambda$ can be represented in the form $\rho = \varkappa \rho_0(\tilde{f}_\lambda)^a$, where $\varkappa \in k$, $a \geq 1$, and ρ_0 is a homogeneous element of degree $s - a\lambda \leq \lambda$ in A_i . Consequently, if $s > \lambda$, then $\dim_k A_i^s \leq \dim_k A_i^{s-a\lambda}$, where $s - a\lambda \leq \lambda$. In this case, we have $\dim_k \text{HH}^s(R) = \dim_k \text{HH}^{s-a\lambda}(R)$. Hence, it suffices to prove that $\dim_k A_i^s \leq \dim_k \text{HH}^s(R)$ for $0 \leq s \leq \lambda$. From the above it follows that A_i^s is generated by the elements of the form $(\tilde{\varepsilon}_1)^a \tilde{\theta}$, where $a \in \{0, 1\}$ and $\tilde{\theta} \in \mathcal{X}_i \cup \{1\}$. The elements of this form are said to be normal.

Next we should consider the cases where one among the conditions of Theorems 2 or 3, depending on the value of r , is satisfied for s , and also the case where none of the conditions of the corresponding theorem is satisfied for s . We see that the number of normal elements in A_i^s not lying in $\mathcal{E}_i$ is equal to $\dim_k \text{HH}^s(R)$ for all $1 \leq i \leq 13$ and $0 \leq s \leq \lambda$, except for the cases where i equals 4 or 5 and $s = (l+1)(2n-3)$ satisfies either (2.6) or (2.12). Moreover, in the remaining cases, the number of such elements is equal to $\dim_k \text{HH}^s(R) + 1$. Since $\tilde{\varepsilon}_1 \tilde{f}_{s-1} - 2\tilde{\varepsilon}_1 \tilde{p}_{s-1} \in \mathcal{E}_i$ in these cases, the number of linearly independent normal elements in A_i^s does not exceed $\dim_k \text{HH}^s(R)$. Hence, $\dim_k A_i^s \leq \dim_k \text{HH}^s(R)$ for all $s \geq 0$. Thus, the theorem is proved. $\square$

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