ELEMENTARY SUBGROUP OF AN ISOTROPIC REDUCTIVE GROUP IS PERFECT

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To our Teacher Nikolai Vavilov, on the occasion of his 60th birthday

ABSTRACT. Let G be an isotropic reductive algebraic group over a commutative ring R . Assume that the elementary subgroup $E(R)$ of the group of points $G(R)$ is well defined. Then $E(R)$ is perfect, except for the well-known case of a split reductive group of type C_2 or G_2 .

INTRODUCTION

Let R be a commutative ring with 1, and let G be an isotropic reductive algebraic group over R . In [1], Victor Petrov and the second author introduced the notion of an elementary subgroup $E(R)$ of the group of points $G(R)$. In this paper we prove that, as in the split case (see, e.g., [11]) and in the field case (see, e.g., [12]), under some natural assumptions the elementary subgroup of a reductive group is perfect.

More precisely, assume that G is isotropic in the following strong sense: it possesses a parabolic subgroup that intersects properly any semisimple normal subgroup of G . Such a parabolic subgroup P is said to be *strictly proper*. Let $E_P(R)$ denote the subgroup of $G(R)$ generated by the R -points of the unipotent radicals of P and of an opposite parabolic subgroup P^- . The main theorem of [1] states that $E_P(R)$ does not depend on the choice of P , as soon as for any maximal ideal M of R all irreducible components of the relative root system of G_{R_M} (see [9, Exp. XXVI, §7] for the definition) are of rank at least 2. Under this assumption, we call $E_P(R)$ the *elementary subgroup* of $G(R)$ and denote it simply by $E(R)$. In particular, $E(R)$ is normal in $G(R)$. This definition of $E(R)$ generalizes the well-known definition of an elementary subgroup of a Chevalley group (or, more generally, of a split reductive group), as well as several other definitions of an elementary subgroup of isotropic classical groups and simple groups over fields; see [5, 12, 13, 14, 6].

By the structure constants of a root system we mean the structure constants of the corresponding semisimple complex Lie group, or, in other words, the constants appearing in the Chevalley commutator formulas for the corresponding Chevalley group. They are among $\pm 1, \pm 2, \pm 3$.

Theorem 1. *Let G be an isotropic reductive algebraic group over a commutative ring R . Assume that for any maximal ideal M of R all irreducible components of the relative root*

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system of G_{R_M} are of rank at least 2, and that, if one of the irreducible components of the (usual) root system of G_{R_M} is of type $B_2 = C_2$ or G_2 , then the residue field R_M/MR_M is not isomorphic to $\mathbb{F}_2$. Then $E(R) = [E(R), E(R)]$.

Observe that the first condition of the theorem ensures that the elementary subgroup $E(R)$ of $G(R)$ is well defined, while the second essentially eliminates the well-known cases where the elementary subgroup of a *split* reductive group is not perfect. Thus, the result is the strongest possible. It should be noted that if we only assume that the rank of relative root systems of G_{R_M} is at most 1, the question as to whether individual subgroups $E_P(R)$ are perfect is of separate interest.

The proof of Theorem 1 is based on the notion of relative root subschemes (with respect to a parabolic subgroup) of an isotropic group, as introduced in [1], the generalized Chevalley commutator formula [1, Lemma 9], and localization in a Quillen–Suslin style [3]. To shorten the proof, we also make use of the classification of Tits indices for isotropic reductive groups over local rings, obtained in [10].

§1. AN ABSTRACT DEFINITION OF RELATIVE ROOTS

In this section we recall the notion of an (abstract) system of relative roots introduced in [1] and prove a technical lemma.

Let Φ be a reduced root system in a Euclidean space with a scalar product $(-, -)$. Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a fixed system of simple roots of Φ ; if Φ is irreducible, we assume that the numbering follows Bourbaki [8]. Let D be the Dynkin diagram of Φ . We identify the nodes of D with the corresponding simple roots in Π . For a subgroup $\Gamma \subseteq \text{Aut}(D)$ and a Γ -invariant subset $J \subseteq \Pi$, consider the projection

$$\pi = \pi_{J,\Gamma}: \mathbb{Z}\Phi \longrightarrow \mathbb{Z}\Phi / \langle \Pi \setminus J; \alpha - \sigma(\alpha), \alpha \in J, \sigma \in \Gamma \rangle.$$

The set $\Phi_{J,\Gamma} = \pi(\Phi) \setminus \{0\}$ is called the system of *relative roots* corresponding to the pair (J, Γ) . The *rank* of $\Phi_{J,\Gamma}$ is the rank of $\pi(\mathbb{Z}\Phi)$ as a free Abelian group.

It is clear that any relative root $A \in \Phi_{J,\Gamma}$ can be represented as a unique linear combination of relative roots in $\pi(\Pi)$. We say that $A \in \Phi_{J,\Gamma}$ is a *positive* (respectively, *negative*) relative root if it is a nonnegative (respectively, nonpositive) linear combination of elements of $\pi(\Pi)$. The sets of positive and negative relative roots will be denoted by $\Phi_{J,\Gamma}^+$ and $\Phi_{J,\Gamma}^-$, respectively. By the *level* $\text{lev}(A)$ of a relative root A we mean the sum of coefficients in its decomposition.

Observe that Γ acts on the set of irreducible components of the root system Φ . If this action is transitive, the system of relative roots $\Phi_{J,\Gamma}$ is *irreducible*. Clearly, any system of relative roots $\Phi_{J,\Gamma}$ is a disjoint union of irreducible ones; we call them the *irreducible components* of $\Phi_{J,\Gamma}$.

We need the following lemma.

Lemma 1. *Let Φ be a root system with a fixed set Π of simple roots, let Γ be a subgroup of $\text{Aut}(D)$ and J a Γ -invariant subset of Π . If a relative root $A \in \Phi_{J,\Gamma}$ lies in an irreducible component of rank at least 2, then there exist noncollinear $B, C \in \Phi_{J,\Gamma}$ such that $A = B + C$ and all relative roots $iB + jC \in \Phi_{J,\Gamma}$, $i, j > 0$, $(i, j) \neq (1, 1)$, have the same sign as A and satisfy $|\text{lev}(iB + jC)| > |\text{lev}(A)|$.*

Proof. We may assume that the root system Φ is irreducible and that A is a positive relative root, that is, $\pi^{-1}(A) \subseteq \Phi^+$.

First, assume that $A = k\pi(\alpha_r)$, where $\alpha_r \in \Pi$ is a simple root and $k > 0$ is a positive integer. Let $\alpha_s \in J$ be a simple root such that the Γ -orbits of α_s and α_r are distinct, and α_s is at the smallest possible distance from α_r on the Dynkin diagram. It is easily seen that for any $\alpha \in \pi^{-1}(A)$ there exists $\beta \in \pi^{-1}(\alpha_s)$ such that $(\alpha, \beta) < 0$ and, consequently,

$\alpha + \beta \in \Phi$. Indeed, since $m_s(\alpha) = 0$ by definition, we can take for β the sum of all simple roots in the Dynkin diagram chain between α_s and the nearest simple root appearing in the decomposition of α . Now we set $B = \pi(\alpha + \beta)$ and $C = \pi(-\beta)$. Clearly, any root in $\pi^{-1}(iB + jC)$, $i, j > 0$, contains the summand $i\alpha_r$ in its decomposition; thus, it is a positive root. Then $iB + jC$ is a positive relative root for any $i, j > 0$. Moreover, $\text{lev}(iB + jC) = \text{lev}(A)$ if and only if $i = j = 1$. Since $\pi(\alpha) = k\pi(\alpha_r)$ and $\pi(-\beta) = -\pi(\alpha_s)$, the roots B and C are noncollinear.

Consider the case where $A \neq k\pi(\alpha_r)$ for any $\alpha_r \in J$. For every $\alpha \in \pi^{-1}(A)$, there exists a sequence of simple roots $\beta_1, \dots, \beta_n \in \Pi$ such that $\alpha = \beta_1 + \dots + \beta_n$ and $\beta_1 + \dots + \beta_i \in \Phi$ for any $1 \leq i \leq n$. Let i be the smallest possible index such that $\beta_{i+1}, \dots, \beta_n \in \Delta$. Then $\beta_i \in J$ and $\pi(\beta_1 + \dots + \beta_{i-1} + \beta_i) = A$. Set $B = \pi(\beta_1 + \dots + \beta_{i-1})$ and $C = \pi(\beta_i)$. Since B and C are positive relative roots, we have $\text{lev}(iB + jC) > \text{lev}(A)$ for any $i, j > 0$ distinct from $i = j = 1$. The relative roots B and C are noncollinear because otherwise we would have had $A = k\pi(\beta_i)$ for some $k > 0$. $\square$

§2. ISOTROPIC REDUCTIVE GROUPS AND RELATIVE ROOT SUBSCHEMES

In this section we recall some basic notions pertaining to reductive groups over rings; see [9, 1, 2] for a more detailed exposition.

Let R be a commutative ring with 1, and let G be a reductive group scheme, or reductive group for short, over R (see [9] for the definition). We denote by G^{ad} and G^{sc} the corresponding adjoint and simply connected semisimple groups, respectively.

Any reductive algebraic group G over R is split locally in the fpqc-topology on $\text{Spec } R$. If G is of constant type over R (that is, the root system of G is the same at any point of $\text{Spec } R$), then G is a twisted form of a split reductive algebraic group G_0 over R , given by a cocycle $\xi \in H_{\text{fpqc}}^1(R, \text{Aut}(G_0))$. Recall that the connected component of $\text{Aut}(G_0)$ is precisely G_0^{ad} . The group G is of *inner type* if ξ is in the image of the natural map $H_{\text{fpqc}}^1(R, G_0^{\text{ad}}) \rightarrow H_{\text{fpqc}}^1(R, \text{Aut}(G_0))$. One can always find a finite Galois extension S of R such that G_S is of *inner type*. The Galois group $\text{Gal}(S/R)$ acts on the Dynkin diagram of each $G_{\overline{k(s)}}$, where $\overline{k(s)}$ is the algebraic closure of the residue field of a point $s \in \text{Spec } R$, via a **-action* (see [10, 2]).

Recall that G is said to be *isotropic* if it contains a proper parabolic subgroup P over R . Recall that a parabolic subgroup P of G is *strictly proper* if it does not contain any semisimple normal subgroup of G . We set

$$E_P(R) = \langle U_P(R), U_{P^-}(R) \rangle,$$

where P^- is any parabolic subgroup of G opposite to P , and U_P and U_{P^-} are the unipotent radicals of P and P^- , respectively. The main theorem of [1] states that $E_P(R)$ does not depend on the choice of a strictly proper parabolic subgroup P , as soon as for any maximal ideal M in R all irreducible components of the relative root system of G_{R_M} are of rank at least 2. Under this assumption, we call $E_P(R)$ the *elementary subgroup* of $G(R)$ and denote it simply by $E(R)$.

Let $P = P^+$ be a parabolic subgroup of G and P^- an opposite parabolic subgroup. Let $L = P^+ \cap P^-$ be their common Levi subgroup. It was shown in [1] that we can represent $\text{Spec}(R)$ as a finite disjoint union

$$\text{Spec}(R) = \coprod_{i=1}^m \text{Spec}(R_i)$$

so that the following conditions hold true for $i = 1, \dots, m$:

- for any $s \in \text{Spec } R_i$, the root system of $G_{\overline{k(s)}}$ is the same;
- for any $s \in \text{Spec } R_i$, the type of the parabolic subgroup $P_{\overline{k(s)}}$ of $G_{\overline{k(s)}}$ is the same;

• if S_i/R_i is a Galois extension of rings such that G_{S_i} is of inner type, then for any $s \in \operatorname{Spec} R_i$ the Galois group $\operatorname{Gal}(S_i/R_i)$ acts on the Dynkin diagram D_i of $G_{\overline{k(s)}}$ via the same subgroup of $\operatorname{Aut}(D_i)$.

From here until the end of this section, we assume that $R = R_i$ for some i (or simply extend the base). Denote by Φ the root system of G , by Π a set of simple roots of Φ , and by D the corresponding Dynkin diagram. Then the $*$ -action on D is determined by a subgroup Γ of $\operatorname{Aut} D$. Let J be the subset of Π such that $\Pi \setminus J$ is the type of $\overline{P_{k(s)}}$ (that is, the set of simple roots of the Levi subgroup $L_{\overline{k(s)}}$). Then J is Γ -invariant. The system of relative roots $\Phi_{J,\Gamma}$ is called the *system of relative roots corresponding to P* and is denoted also by Φ_P . If R is a local ring and P is a minimal parabolic subgroup of G , then Φ_P can be identified with the relative root system of G in the sense of [9, Exp. XXVI, §7] (or [7] for the field case); see also [7, 1, 2].

With any relative root $A \in \Phi_P$, we associate a finitely generated projective R -module V_A and a closed embedding

$$X_A : W(V_A) \rightarrow G,$$

where $W(V_A)$ is the affine group scheme over R defined by V_A and called a *relative root subscheme* of G . These subschemes possess several nice properties similar to those of elementary root subgroups of a split group; see [1, Theorem 2]. In particular, they are subject to certain commutator relations that generalize the Chevalley commutator formula.

More precisely, assume that $A, B \in \Phi_P$ satisfy $mA \neq -kB$ for any $m, k \geq 1$. Then there exists a polynomial map

$$N_{ABij} : V_A \times V_B \rightarrow V_{iA+jB}$$

homogeneous of degree i in the first variable and of degree j in the second variable and such that for any R -algebra R' and any $u \in V_A \otimes_R R'$, $v \in V_B \otimes_R R'$ we have

$$(1) \quad [X_A(u), X_B(v)] = \prod_{i,j>0} X_{iA+jB}(N_{ABij}(u, v))$$

(see [1, Lemma 9]).

In a strict analogy with the split case, for any R -algebra R' we have

$$E(R') = \langle X_A(V_A \otimes_R R'), A \in \Phi_P \rangle$$

(see [1, Lemma 6]).

We shall also use the following statement, which is a slight extension of [1, Lemma 10].

Lemma 2. *Consider $A, B \in \Phi_P$ satisfying $A + B \in \Phi_P$ and $mA \neq -kB$ for any $m, k \geq 1$. Denote by Φ^0 an irreducible component of Φ such that $A, B \in \pi(\Phi^0)$.*

(1) *Suppose that one of the following conditions is fulfilled:*

(a) *the structure constants of Φ^0 are invertible in R (for example, this is so if Φ^0 is simply laced);*

(b) *$A \neq B$ and $A - B \notin \Phi_P$;*

(c) *Φ^0 is of type B_l , C_l , or F_4 , and $\pi^{-1}(A + B)$ consists of short roots;*

(d) *Φ^0 is of type B_l , C_l , or F_4 , and there exist long roots $\alpha \in \pi^{-1}(A)$, $\beta \in \pi^{-1}(B)$ such that $\alpha + \beta$ is a root.*

Then the map $N_{AB11} : V_A \times V_B \rightarrow V_{A+B}$ is surjective.

(2) *If $A - B \in \Phi_P$ and $\Phi^0 \neq G_2$, then*

$$\operatorname{im} N_{AB11} + \operatorname{im} N_{A-B,2B,1,1} + \sum_{v \in V_B} \operatorname{im}(N_{A-B,B,1,2}(-, v)) = V_{A+B},$$

where $\operatorname{im} N_{A-B,2B,1,1} = 0$ if $2B \notin \Phi_P$.

Proof. (1) By [1, Lemma 4], any $\gamma \in \pi^{-1}(A+B)$ decomposes as $\gamma = \alpha + \beta$, $\alpha \in \pi^{-1}(A)$, $\beta \in \pi^{-1}(B)$. Let S_τ be any member of an affine fpqc-covering $\coprod \text{Spec } S_\tau \rightarrow \text{Spec } R$ that splits G . Set

$$\Psi = \{iA + jB \mid i, j > 0, (i, j) \neq (1, 1), iA + jB \in \Phi_P\}.$$

Then, in the notation of [1, Theorem 2], over S_τ the commutator $[X_A(e_\alpha), X_B(e_\beta)]$ computed modulo the subgroup $\langle X_C(V_C), C \in \Psi \rangle$, is of the form $x_\gamma(\pm c) = X_{A+B}(\pm ce_\gamma)$, where $c = \pm 1, \pm 2, \pm 3$ is the corresponding structure constant. If (a) is fulfilled, then c is invertible. If (b), (c), or (d) hold true, then c necessarily equals ± 1 . Indeed, in the only dubious case (d) we should observe that, due to the transitive action of the Weyl group of the Levi subgroup on the roots of the same shape (see [4]), *any* long root $\gamma \in \pi^{-1}(A+B)$ decomposes as a sum of long roots. Hence, c is always invertible. This implies that $\text{im}(N_{AB11})_\tau = V_{A+B} \otimes S_\tau$. Since $\text{im } N_{AB11}$ is a submodule of V_{A+B} defined over the base ring, we have $\text{im } N_{AB11} = V_{A+B}$.

(2) See [1, Lemma 10]. $\square$

Lemma 3. *Suppose that Φ^0 is an irreducible component of Φ such that $\Phi^0 \cong C_l$, $l > 2$, and P is a parabolic subgroup of type $\Pi \setminus J$, where $J \cap \Phi^0 = \{\alpha_i, \alpha_l\}$, $2i = l$ (α_i is short, α_l is long). Set $\pi(\alpha_i) = A_1$ and $\pi(\alpha_l) = A_2$. Then*

$$(2) \quad \text{im}(0, N_{A_1, A_1+A_2, 1, 1}) + \sum_{v \in V_A} \text{im } f_v = V_{A_1+A_2} \oplus V_{2A_1+A_2},$$

where $f_v = (N_{A_1, A_2, 1, 1}(v, -), N_{A_1, A_2, 2, 1}(v, -)): V_{A_2} \rightarrow V_{A_1+A_2} \oplus V_{2A_1+A_2}$.

Proof. Observe that in our case $\pi(\Phi^0)$ can be identified with a root system of type C_2 , so that A_1 is a short simple root and A_2 is a long simple root. Let S_τ be any member of an affine fpqc-covering $\coprod \text{Spec } S_\tau \rightarrow \text{Spec } R$ that splits G . Suppose that $\gamma \in \pi^{-1}(2A_1 + A_2)$ is a short root. We can find short roots $\alpha \in \pi^{-1}(A_1)$ and $\beta \in \pi^{-1}(A_1 + A_2)$ such that $\gamma = \alpha + \beta$. Therefore,

$$[X_{A_1}(e_\alpha), X_{A_1+A_2}(e_\beta)] = x_\gamma(\pm 1) = X_{2A_1+A_2}(\pm e_\gamma).$$

Hence, $e_\gamma \in \text{im}(N_{A_1, A_1+A_2, 1, 1})_\tau$. Now let $\gamma \in \pi^{-1}(2A_1 + A_2)$ be a long root. Take $\alpha \in \pi^{-1}(A_1)$, $\beta \in \pi^{-1}(A_2)$ such that $\beta \neq \alpha_l$ and $\gamma = 2\alpha + \beta$ (note that α is short, β is long). Then

$$[X_{A_1}(e_\alpha), X_{A_2}(e_\beta)] = x_{\alpha+\beta}(\pm 1)x_{2\alpha+\beta}(\pm 1) = X_{A_1+A_2}(\pm e_{\alpha+\beta})X_{2A_1+A_2}(\pm e_\gamma).$$

Next, since any $\delta \in \pi^{-1}(A_1 + A_2)$ is a short root, there exist short roots $\alpha \in \pi^{-1}(A_1)$, $\beta \in \pi^{-1}(A_2)$ such that $\alpha + \beta = \delta$, whence $[X_{A_1}(e_\alpha), X_{A_2}(e_\beta)] = X_{A_1+A_2}(\pm e_\delta)$. Combining these results, we see that (2) is valid over S_τ . Since all our modules are defined over the base ring, fpqc descent shows that the same identity is valid over R . $\square$

§3. PROOF OF THEOREM 1

Let R be a commutative ring with 1, G a reductive group over R , and P a strictly proper parabolic subgroup of G . For any ideal $I \subseteq R$ we write

$$E_P(I) = \langle U_P(I), U_{P^-}(I) \rangle \leq G(R).$$

We denote by $R[Y, Z]$ the ring of polynomials in two variables Z and Y over R .

Lemma 4. *Let G be a reductive group scheme over a commutative ring R , and let P and P' be two strictly proper parabolic subgroups of G such that $P \leq P'$ or $P' \leq P$. Then for any integer $m > 0$ there exists an integer $k > 0$ such that*

$$E_P(Z^k R[Z]) \leq E_{P'}(Z^m R[Z]).$$

Proof. Without loss of generality, we may assume that over R we have two sets of relative root subschemes $X_A(V_A)$, $A \in \Phi_P$, and $X_B(V_B)$, $B \in \Phi_{P'}$, corresponding to P and P' , respectively.

Then, if $P \leq P'$, by [1, Lemma 12] there exists an integer $k > 0$ such that for any $A \in \Phi_P$ and any $v \in V_A$ we can find relative roots $B_i \in \Phi_{P'}$, elements $v_i \in V_{B_i}$, and integers $n_i > 0$ ($1 \leq i \leq m$) such that

$$X_A(Y^k v) = \prod_{i=1}^m X_{B_i}(Y^{n_i} v_i),$$

so that $X_A(Y^k v) \in E_{P'}(YR[Y])$. Replacing R by $R[Z]$ and Y by Z^m , we obtain

$$E_P(Z^k R[Z]) \leq E_{P'}(Z^m R[Z]).$$

But if $P' \leq P$, we have $U_P \leq U_{P'}$. Let $\Psi^\pm \subseteq \Phi^\pm$ be two closed sets of roots corresponding to $U_{P^\pm}$. Then $\pi(\Psi^\pm) \subseteq \Phi_{P'}^\pm$, where $\pi : \Phi \rightarrow \Phi_{P'}$ is the canonical projection. By [1, Lemma 6], the map

$$X_\Psi : W\left(\bigoplus_{A \in \pi(\Psi^\pm)} V_A\right) \rightarrow U_{P^\pm}, \quad (v_A)_A \mapsto \prod_A X_A(v_A),$$

where the product is taken in any fixed order respecting the level of relative roots in $\Phi_{P'}$, is an isomorphism of schemes over R . Therefore, $U_{P^\pm}(Z^k R[Z]) \leq U_{P'^\pm}(Z^m R[Z])$. $\square$

Lemma 5. *Under the conditions of Theorem 1, assume moreover that R is a local ring. Then for any integer $m > 0$ there exists an integer $k > 0$ such that for any R -algebra R' we have*

$$E_P(Z^k R'[Z]) \subseteq [E_P(Z^m R'[Z]), E_P(Z^m R'[Z])].$$

Proof. Let $\text{der}(G)$ be the algebraic derived subgroup of the reductive group scheme G (see [9, Exp. XXII, 6.2]). Then, clearly, $E_P(R) \subseteq \text{der}(G)(R)$. Since $\text{der}(G)$ is a semisimple group, we may assume that G is semisimple. Moreover, since the canonical projection $G^{\text{sc}} \rightarrow G$, where G^{sc} is the simply connected semisimple group corresponding to G , is surjective on $U^\pm(R)$, we may assume that G is simply connected. Any simply connected semisimple group is a direct product of simply connected semisimple groups that cannot be decomposed into a product of smaller semisimple groups. These groups G_i , $i = 1, \dots, n$, are Weil restrictions of certain simple reductive groups G'_i over a finite étale extension S of $R : G_i = R_{S/R}(G'_i)$; see [9, Exp. XIV, Proposition 5.10].

Note that the group of R -points $G_i(R)$ is canonically isomorphic to the group of S -points $G'_i(S)$. This isomorphism also respects the embedding $P_i(R) \rightarrow G_i(R)$, for any parabolic subgroup P_i of G_i . Then, clearly, we may assume from the very beginning that G is a simple reductive group, and the root system Φ of G is irreducible.

Note that Lemma 4 allows us to replace P by any other strictly proper parabolic subgroup P' of G such that $P \leq P'$ or $P' \leq P$. Next, over R we have a set of relative root subschemes $X_A(V_A)$, $A \in \Phi_P$, corresponding to P .

We are going to show by induction on $|\text{lev } A|$ that for any $A \in \Phi_P$ there exists an integer $k = k(A) > 0$ such that for any R -algebra R' and any $v \in V_A \otimes_R R'$ we have

$$(3) \quad X_A(Z^k v) \in [E_P(ZR'[Z]), E_P(ZR'[Z])].$$

Then the claim of the lemma follows by replacing Z by Z^m and R' by $R[Z]$, and taking the final k to be the maximum of all $k(A)$, $A \in \Phi_P$.

Recall that, by Lemma 1, there exist noncollinear relative roots $B, C \in \Phi_P$ such that $A = B + C$ and all roots $iB + jC \in \Phi_{J, \Gamma}$, $i, j > 0$, $(i, j) \neq (1, 1)$, have the same sign as A and satisfy $|\text{lev}(iB + jC)| > |\text{lev}(A)|$. Assume that the map $N_{BC11} : V_B \times V_C \rightarrow V_A$ is

surjective. Using the generalized Chevalley commutator formula (1), for any R -algebra R' and any $v \in V_A \otimes_R R'[Z]$ we can write

$$X_A(Z^k v) = [X_B(Z u_{10}), X_C(Z^{k-1} u_{01})] \cdot \prod_{\substack{i,j>0; \\ (i,j) \neq (1,1)}} X_{iB+jC}(Z^{i+j(k-1)} u_{ij})$$

for some $u_{ij} \in V_{iB+jC} \otimes_R R'[Z]$, $i, j > 0$, $(i, j) \neq (1, 1)$. Then, by the inductive hypothesis, (3) is true if k is sufficiently large.

Now, for any relative root $A \in \Phi_P$ (for a suitable choice of the parabolic subgroup P), we either show that for *any* decomposition $A = B + C$, where B and C are non-collinear, the map N_{BC11} is surjective; or provide an explicit decomposition of $X_A(Z^k v)$, $v \in V_A \otimes_R R'$, into a product of commutators in $E_P(ZR'[Z])$, so that (3) is satisfied for sufficiently large k .

First, assume that all structure constants of the root system Φ of G are invertible in R ; this includes the case where Φ is simply laced. Then, by Lemma 2, the map N_{BC11} is surjective for any decomposition $A = B + C$, where B and C are noncollinear.

Consider the case where $\Phi = \Phi_P = C_2$, so that G is split. Let $\Pi = \{A_1, A_2\}$, $\Phi^+ = \{A_1, A_2, A_1 + A_2, 2A_1 + A_2\}$. Let M be the maximal ideal of R . By the hypotheses of Theorem 1, we have $R/M \not\cong \mathbb{F}_2$; hence, we can take $\varepsilon \in R \setminus M$ such that $\varepsilon^2 - \varepsilon \in R \setminus M = R^*$. If the root $A \in \Phi_P$ is long, we may assume that $A = 2A_1 + A_2$. Let

$$g_1(s, t) = [X_{A_1}(s), X_{A_2}(t)] = X_{A_1+A_2}(st) X_{2A_1+A_2}(s^2 t)$$

and

$$g_2(s, t, u) = [X_{A_2}(u), [X_{A_1+A_2}(s), X_{-A_2}(t)]] = X_{A_1+A_2}(-stu) X_{2A_1+A_2}(-s^2 t^2 u).$$

Here and in similar calculations below, we use the explicit signs of the structure constants in a split group; see, e.g., [11]. Therefore,

$$g_1(Z^2, -Z^{k-4}\varepsilon(\varepsilon^2 - \varepsilon)^{-1}v) \cdot g_2(Z, Z\varepsilon, -Z^{k-4}(\varepsilon^2 - \varepsilon)^{-1}v) = X_{2A_1+A_2}(Z^k v).$$

But if the root $A \in \Phi_P$ is short, we may assume that $A = A_1 + A_2$, whence

$$g_1(Z, Z^{k-1}v) \cdot X_{2A_1+A_2}(-Z^{k+1}v) = X_{A_1+A_2}(Z^k v).$$

This means that (3) holds true for these roots for any $k \geq 5$.

Consider the case where $\Phi = \Phi_P = G_2$, so that G is split. Let $\Pi = \{A_1, A_2\}$, $\Phi^+ = \{A_1, A_2, A_1 + A_2, 2A_1 + A_2, 3A_1 + A_2, 3A_1 + 2A_2\}$. By the hypotheses of Theorem 1, we have $R/M \not\cong \mathbb{F}_2$; hence, we can take $\varepsilon \in R \setminus M$ such that $\varepsilon^2 - \varepsilon \in R \setminus M = R^*$. If the root A is long, we may assume that $A = 3A_1 + 2A_2$. Then

$$[X_{A_2}(Zv), X_{3A_1+A_2}(Z^{k-1})] = X_{3A_1+2A_2}(Z^k v).$$

Therefore, (3) is valid for long roots for any $k \geq 2$. If the root A is short, we may assume that $A = 2A_1 + A_2$. Then

$$[X_{A_1}(s), X_{A_2}(t)] = X_{A_1+A_2}(st) \cdot X_{2A_1+A_2}(s^2 t) \cdot X_{3A_1+A_2}(s^3 t) \cdot X_{3A_1+2A_2}(s^3 t^2).$$

Hence,

$$(4) \quad [X_{A_1}(Z\varepsilon), X_{A_2}(-(\varepsilon^2 - \varepsilon)^{-1}Z^{k-2}v)]^{-1} \cdot [X_{A_1}(Z), X_{A_2}(-\varepsilon(\varepsilon^2 - \varepsilon)^{-1}Z^{k-2}v)] \\ = X_{2A_1+A_2}(Z^k v) X_{3A_1+A_2}((\varepsilon + 1)Z^{k+1}v) X_{3A_1+2A_2}(\varepsilon(\varepsilon^2 - \varepsilon)^{-1}Z^{2k+1}v),$$

and the roots $3A_1 + A_2$ and $3A_1 + 2A_2$ are long. This means that (3) is fulfilled for short roots for any $k \geq 3$.

We are left with the case where Φ is of type B_l , C_l , or F_4 . Recall that, by the hypotheses of Theorem 1, G contains a split torus of rank at least 2. Hence, in the F_4 case the classification of Tits indices over local rings [10] says that G is a split group.

Thus, we may assume that P is a Borel subgroup of G , and $\Phi_P = \Phi$ is a root system of type F_4 . Then if the root $A \in \Phi_P$ is short, Lemma 2 shows that the map N_{BC11} is surjective for any noncollinear $B, C \in \Phi_P$ such that $A = B + C$. If this root is long, then it belongs to the long root subsystem of F_4 , which has type D_4 . Then (for example, by Lemma 1) we can find two long roots $B, C \in \Phi_P$ such that $B + C = A$, and necessarily, $iB + jC$ is not a root for any $i, j > 0$ distinct from $i = j = 1$. Then

$$X_A(Z^k v) = [X_B(Zu_{10}), X_C(Z^{k-1}u_{01})]$$

for some $u_{10} \in V_B \otimes_R R'[Z]$ and $u_{01} \in V_C \otimes_R R'[Z]$.

Consider the case where Φ is of type B_l , $l \geq 3$. By the classification of Tits indices over local rings [10], we may assume that P is a parabolic subgroup of type $\Pi \setminus J$, where Π is a set of simple roots of Φ and $J = \{\alpha_1, \alpha_2\}$. Then Φ_P can be identified with a root system of type B_2 . Using the fact that $l \geq 3$, it is easy to check that for any relative root $A \in \Phi_P$ and any pair $B, C \in \Phi_P$ satisfying $A = B + C$, we can find a pair of long roots $\beta \in \pi^{-1}(B)$, $\gamma \in \pi^{-1}(C)$ such that $\beta + \gamma$ is a root (we may assume that A is one of two simple roots of Φ_P , due to the lifting of the relative Weyl group [9, Exp. XXVI, Theorem 7.13 (ii)]). Then, the map N_{BC11} is surjective by Lemma 2.

It remains to consider the case where Φ is of type C_l , $l \geq 3$. First, assume that P is a parabolic subgroup of type $\Pi \setminus J$, where $J = \{\alpha_i, \alpha_j\}$ for two short simple roots α_i, α_j of Φ . Then Φ_P can be identified with a root system of type BC_2 . It is easily seen that for all extrashort and short relative roots $A \in \Phi_P$ the set $\pi^{-1}(A)$ consists of short roots, and hence, by Lemma 2, the map N_{BC11} is surjective for any decomposition $A = B + C$. Let A be a long root. Let A_1 and A_2 be a short and an extrashort simple roots of Φ_P . There is no loss of generality in assuming that $A = 2A_1 + 2A_2$. Take $k \geq 4$. Then, by Lemma 2 (2) and the generalized Chevalley commutator formula, for any R -algebra R' and any $v \in V_A \otimes_R R'[Z]$ we have

$$X_A(Z^k v) = [X_{A_1}(Zu_1), X_{2A_2}(Z^{k-2}u_2)] \cdot X_{A_1+2A_2}(Z^{k-1}u_3)$$

for some $u_1 \in V_{A_1} \otimes_R R'[Z]$, $u_2 \in V_{2A_2} \otimes_R R'[Z]$, and $u_3 \in V_{A_1+2A_2} \otimes_R R'[Z]$. Next, by Lemma 2 (1) and the generalized Chevalley commutator formula, since $\pi^{-1}(A_1 + 2A_2)$ consists of short roots, we have

$$X_{A_1+2A_2}(Z^{k-1}u_3) = [X_{A_1+A_2}(Zu_4), X_{A_2}(Z^{k-3}u_5)]$$

for some $u_4 \in V_{A_1+A_2} \otimes_R R'[Z]$ and $u_5 \in V_{A_2} \otimes_R R'[Z]$. Consequently, (3) holds true for A for any $k \geq 4$.

The classification of Tits indices over local rings [10] shows that the only remaining case is that where P is a parabolic subgroup of type $\Pi \setminus J$ for $J = \{\alpha_i, \alpha_l\}$, where $l = 2i$. Now α_i is short, α_l is long, and Φ_P can be identified with a root system of type $B_2 = C_2$. As in Lemma 3, we put $A_1 = \pi(\alpha_i)$, $A_2 = \pi(\alpha_l)$. Then if the root $A \in \Phi_P$ is short, by the lifting of the relative Weyl group [9, Exp. XXVI, Theorem 7.13 (ii)], we may assume that $A = A_1 + A_2$. By Lemma 3, for any R -algebra R' and any $v \in V_A \otimes_R R'[Z]$, we have

$$X_A(Z^k v) = \prod_i [X_{A_1}(Zv_i), X_{A_2}(Z^{k-1}u_1)]$$

for some $u_1 \in V_{A_2} \otimes_R R'[Z]$, $v_i \in V_{A_1} \otimes_R R'[Z]$. If the root $A \in \Phi_P$ is long, we may assume that $C = 2A_1 + A_2$. By Lemma 3, for any R -algebra R' and any $v \in V_{2A_1+A_2} \otimes_R R'[Z]$, we have

$$X_A(Z^k v) = [X_{A_1}(Zu_1), X_{A_1+A_2}(Z^{k-1}u_2)] \cdot \prod_i [X_{A_1}(Zv_i), X_{A_2}(Z^{k-2}u_3)]$$

for some $u_1 \in V_{A_1} \otimes_R R'[Z]$, $u_2 \in V_{A_1+A_2} \otimes_R R'[Z]$, $u_3 \in V_{A_2} \otimes_R R'[Z]$, and $v_i \in V_{A_1} \otimes_R R'[Z]$. Hence, (3) is true for A for any $k \geq 3$. $\square$

Proof of Theorem 1. Recall that we can represent $\text{Spec}(R)$ as a finite disjoint union $\text{Spec}(R) = \coprod_{i=1}^n \text{Spec}(R_i)$, so that $E(R) = \prod_{i=1}^n E(R_i)$ and for any $1 \leq i \leq n$ we have

$$E(R_i) = \langle X_A(V_A), A \in \Phi_{P_{R_i}} \rangle$$

for a set of root subschemes X_A , $A \in \Phi_{P_{R_i}}$, over R_i . Hence, we may assume that

$$E(R) = \langle X_A(V_A), A \in \Phi_P \rangle$$

from the very beginning.

We show that $[E(R), E(R)]$ contains any $X_A(v)$, $A \in \Phi_P$, $v \in V_A$, by induction on $|\text{lev } A|$. Take

$$I = \{s \in R \mid X_A(tsv) \in [E(R), E(R)] \text{ for all } t \in R\}.$$

By [1, Theorem 2], for any $u, u' \in V_A$ we have

$$X_A(u + u') = X_A(u)X_A(u') \prod_{i>0} X_{iA}(u_i)$$

for some $u_i \in V_{iA}$. Hence, by the inductive hypothesis, I is an ideal of R . If $I \neq R$, let M be a maximal ideal of R containing I . Let F_M denote both the localization homomorphism $R \rightarrow R_M$ and the induced homomorphism $G(R[Y, Z]) \rightarrow G(R_M[Y, Z])$. By Lemma 5 there is $m > 0$ such that we can represent the element $F_M(X_A(Z^m Y v))$ of $G(R_M[Y, Z])$ as a product

$$F_M(X_A(Z^m Y v)) = \prod_{i=1}^n [h_i(Y, Z), g_i(Y, Z)]$$

for some $h_i(Y, Z), g_i(Y, Z) \in E(ZR_M[Y, Z])$, $1 \leq i \leq n$. By [1, Lemma 15], there exist

$$h'_i(Y, Z), g'_i(Y, Z) \in E_P(R[Y, Z], ZR[Y, Z]) \leq E_P(R[Y, Z]) = E(R[Y, Z])$$

and $s \in R \setminus M$ such that $F_M(h'_i(Y, Z)) = h_i(Y, sZ)$ and $F_M(g'_i(Y, Z)) = g_i(Y, sZ)$ for all i . Consequently,

$$F_M(X_A((sZ)^m Y v)) = F_M\left(\prod_{i=1}^n [h'_i(Y, Z), g'_i(Y, Z)]\right).$$

Using [1, Lemma 14], we see that there exists $t \in R \setminus M$ with

$$X_A((tsZ)^m Y v) = \prod_{i=1}^n [h'_i(Y, tZ), g'_i(Y, tZ)].$$

Replacing Z by 1 and Y by an arbitrary element of R , we conclude that $(ts)^m \in I$. But $(ts)^m \in R \setminus M$, a contradiction. $\square$

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