INTEGER POINTS IN DOMAINS AND ADIABATIC LIMITS

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ABSTRACT. An asymptotic formula is proved for the number of integral points in a family of bounded domains with smooth boundary in Euclidean space; these domains remain unchanged along some linear subspace and expand in the directions orthogonal to this subspace. A sharper estimate for the remainder is obtained in the case where the domains are strictly convex. These results make it possible to improve the remainder estimate in the adiabatic limit formula (due to the first author) for the eigenvalue distribution function of the Laplace operator associated with a bundle-like metric on a compact manifold equipped with a Riemannian foliation in the particular case where the foliation is a linear foliation on the torus and the metric is the standard Euclidean metric on the torus.

§1. STATEMENT OF THE PROBLEM AND THE MAIN RESULTS

The classical problem on integer point distributions is to study the asymptotic behavior of the number of points of the integer lattice $\mathbb{Z}^n$ in a family of homothetic domains in $\mathbb{R}^n$. This problem originated with the Gauss problem on the number of integer points in a disk (directly related to the arithmetic problem on the number of representations of an integer as a sum of two squares) and is studied sufficiently well (see, e.g., the books [3, 2, 4, 8] and the references therein).

In this paper we investigate the much less studied problem on counting the integer points in a family of anisotropically expanding domains. More precisely, let F be a p -dimensional linear subspace of $\mathbb{R}^n$ and $H = F^\perp$ the q -dimensional orthogonal complement of F with respect to the standard inner product $(\cdot, \cdot)$ in $\mathbb{R}^n$, $p + q = n$. For any $\varepsilon > 0$, consider the following linear transformation $T_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n$:

$$T_\varepsilon(x) = \begin{cases} x & \text{if } x \in F, \\ \varepsilon^{-1}x & \text{if } x \in H. \end{cases}$$

For any bounded domain S in $\mathbb{R}^n$ with smooth boundary, we put

$$(1.1) \quad n_\varepsilon(S) = \#(T_\varepsilon(S) \cap \mathbb{Z}^n), \quad \varepsilon > 0.$$

Our main goal in this paper is to study the asymptotic behavior of the function $n_\varepsilon(S)$ as $\varepsilon \rightarrow 0$. Before stating the main results of the paper, we introduce some auxiliary notions.

Let $\Gamma = \mathbb{Z}^n \cap F$; Γ is a free Abelian group. We denote by $r = \text{rank } \Gamma \leq p$ the rank of Γ . For $r \geq 1$, denote by $(\ell_1, \ell_2, \dots, \ell_r)$ some basis in Γ . Let V be the r -dimensional subspace of $\mathbb{R}^n$ spanned by the vectors $(\ell_1, \ell_2, \dots, \ell_r)$. Observe that Γ is a lattice in V . Denote by $Q \subset V$ the parallelepiped spanned by the vectors $(\ell_1, \ell_2, \dots, \ell_r)$ and by $|Q|$

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its r -dimensional Euclidean volume,

$$|Q| = \text{vol}_r(\ell_1, \ell_2, \dots, \ell_r) = \text{vol}(V/\Gamma).$$

Let Γ^* denote the lattice in V dual to the lattice Γ :

$$\Gamma^* = \{\gamma^* \in V : (\gamma^*, \Gamma) \subset \mathbb{Z}\}.$$

For $r = 0$, the groups Γ and Γ^* are trivial, and it is natural to put $|Q| = 1$.

For any $x \in V$, we denote by P_x the $(n-r)$ -dimensional affine subspace of $\mathbb{R}^n$ passing through x and orthogonal to V .

Theorem 1. *For any bounded open set S in $\mathbb{R}^n$ with smooth boundary, we have*

$$(1.2) \quad n_\varepsilon(S) = \frac{\varepsilon^{-q}}{|Q|} \sum_{\gamma^* \in \Gamma^*} \text{vol}_{n-r}(P_{\gamma^*} \cap S) + O(\varepsilon^{\frac{1}{p-r+1}-q}), \quad \varepsilon \rightarrow 0.$$

Note that, in the general case, the leading term in the asymptotic formula for $n_\varepsilon(S)$ as $\varepsilon \rightarrow 0$ has not been known before.

Remark 2. In the case when F is trivial, we have $p = r = 0$, $q = n$. The problem reduces to the classical problem on the asymptotics of the number of integer points in a family of homothetic domains in $\mathbb{R}^n$. Formula (1.2) reduces to the classical formula dating back to Gauss:

$$n_\varepsilon(S) = \varepsilon^{-n} \text{vol}_n(S) + O(\varepsilon^{1-n}), \quad \varepsilon \rightarrow 0.$$

Remark 3. If Γ is trivial, we have $r = 0$, and formula (1.2) takes the form

$$n_\varepsilon(S) = \varepsilon^{-q} \text{vol}_n(S) + O(\varepsilon^{\frac{1}{p+1}-q}), \quad \varepsilon \rightarrow 0.$$

In order to obtain a sharper remainder estimate, we need to impose some restrictions on the boundary of the domain S . We give only one result of this type.

Theorem 4. *If a bounded open set S in $\mathbb{R}^n$ has smooth boundary and is such that, for any $x \in F$, the intersection $S \cap \{x + H\}$ is strictly convex, then*

$$(1.3) \quad n_\varepsilon(S) = \frac{\varepsilon^{-q}}{|Q|} \sum_{\gamma^* \in \Gamma^*} \text{vol}_{n-r}(P_{\gamma^*} \cap S) + O(\varepsilon^{k-q}), \quad \varepsilon \rightarrow 0,$$

where

$$k = \begin{cases} \frac{q+1}{2(p-r+1)} & \text{if } \frac{q-1}{2} \leq p-r, \\ \frac{2q}{q+1+2(p-r)} & \text{if } \frac{q-1}{2} > p-r. \end{cases}$$

Remark 5. In the case where F is trivial, we get $k = 2n/(n+1)$, and formula (1.2) reduces to the following:

$$n_\varepsilon(S) = \varepsilon^{-n} \text{vol}_n(S) + O(\varepsilon^{-n+2-\frac{2}{n+1}}), \quad \varepsilon \rightarrow 0.$$

This formula was proved by Randol [12, 13].

Example 1. As the simplest nontrivial example, we consider the case where $n = 2$ and $p = 1$. Thus, let F be the one-dimensional linear subspace of $\mathbb{R}^2$ spanned by $(1, \alpha) \in \mathbb{R}^2$. Then its orthogonal complement H is spanned by $(-\alpha, 1) \in \mathbb{R}^2$. Two cases can be distinguished.

Case 1 ($\alpha \notin \mathbb{Q}$). In this case Γ is trivial, so that $r = 0$. Moreover, Γ^* is trivial, and, as has been mentioned above, it is natural to put $|Q| = 1$. For any bounded domain S in $\mathbb{R}^2$ with smooth boundary, we get

$$n_\varepsilon(S) = \varepsilon^{-1} \text{area}(S) + O(\varepsilon^{-1/2}), \quad \varepsilon \rightarrow 0.$$

Case 2 ($\alpha \in \mathbb{Q}$). Write $\alpha = \frac{p}{q}$, where p and q are coprime numbers. Then Γ is generated by $\ell_1 = (q, p)$. Therefore,

$$|Q| = |\ell_1| = \sqrt{p^2 + q^2}.$$

The dual lattice Γ^* is generated by $\frac{1}{p^2+q^2}\ell_1$. An arbitrary element of Γ^* has the form

$$\gamma^* = \frac{1}{p^2 + q^2} k \ell_1, \quad k \in \mathbb{Z}.$$

The corresponding subspace P_{γ^*} is the line L_k on the plane $\mathbb{R}^2$ given by the equation $qx + py - k = 0$. Therefore, for any bounded domain S in $\mathbb{R}^2$ with smooth boundary, we get

$$n_\varepsilon(S) = \varepsilon^{-1} \frac{1}{\sqrt{p^2 + q^2}} \sum_{k \in \mathbb{Z}} |S \cap L_k| + O(1), \quad \varepsilon \rightarrow 0.$$

In a slightly different context, the problem on counting the integer points in anisotropically expanding domains was studied in considerable detail in [14, 15, 10, 11]. More precisely, those papers were devoted to estimates of the number $N(S, \Gamma)$ of points of a lattice Γ in $\mathbb{R}^n$ in a bounded domain S for special domains and lattices. We briefly describe some results of those papers and indicate how they can be applied to the problem under consideration.

Let k be a totally real algebraic number field of degree n , σ the canonical embedding of k in the Euclidean space $\mathbb{R}^n$, $M \subset k$ an arbitrary $\mathbb{Z}$ -module of rank n , and $\Gamma_M = \sigma(M)$ the corresponding algebraic lattice in $\mathbb{R}^n$. Let $\Pi \subset \mathbb{R}^n$ be an n -dimensional parallelepiped centered at the origin and with edges parallel to coordinated axes:

$$\Pi = \prod_{j=1}^n (-a_j, a_j).$$

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, we put

$$\text{Nm } \lambda = \prod_{j=1}^n \lambda_j.$$

Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Any vector $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ gives rise to a linear transformation of $\mathbb{R}^n$:

$$\lambda \cdot e_j = \lambda_j e_j, \quad j \in \{1, \dots, n\}.$$

By [15, Theorem 1.1] (see also [14, 11]), for any $\lambda \in \mathbb{R}^n$ we have the estimate

$$(1.4) \quad \left| N(\lambda \cdot \Pi, \Gamma_M) - |\text{Nm } \lambda| \frac{\text{vol}_n(\Pi)}{d(\Gamma_M)} \right| < C [\ln(2 + |\text{Nm } \lambda|)]^{n-1},$$

where $d(\Gamma_M)$ is the volume of a fundamental domain of the lattice Γ_M and $C > 0$ is a constant independent of λ .

In [10, 11], this result was extended to the case of algebraic lattices associated with an arbitrary algebraic number field, as well as to a wider class of domains. As an application, the authors obtained nontrivial remainder estimates in the asymptotic formula due to Lang [9] for the number of elements of an algebraic number field in a parallelotope determined by the canonical system of valuations.

The investigations carried out in [14, 15, 10, 11] show that the remainder estimate in the asymptotic formula for $N(\lambda \cdot S, \Gamma)$ turns out to be very sensitive to the Diophantine properties of the lattice Γ and to the geometric properties of the domain S .

In order to apply the results described above to the problems considered in this paper, we fix $p \in \{1, \dots, n\}$ and use formula (1.4) with $\lambda = \lambda(\varepsilon) \in \mathbb{R}^n$ of the form

$$\lambda_j = \begin{cases} 1 & \text{if } 1 \leq j \leq p, \\ \varepsilon^{-1} & \text{if } p+1 \leq j \leq n, \end{cases}$$

for some $\varepsilon > 0$. We get

$$\left| N(\lambda(\varepsilon) \cdot \Pi, \Gamma_M) - \varepsilon^{-q} \frac{\text{vol}_n(\Pi)}{d(\Gamma_M)} \right| < C_1 |\ln \varepsilon|^{n-1}, \quad \varepsilon \rightarrow 0,$$

where $C_1 > 0$ is a constant independent of ε , and $p+q=n$.

On the other hand, the lattice Γ_M can be represented as $\Gamma_M = A(\mathbb{Z}^n)$ with some linear isomorphism A of $\mathbb{R}^n$. Therefore,

$$N(\Pi, \Gamma_M) = N(\Pi, A(\mathbb{Z}^n)) = N(A^{-1}(\Pi), \mathbb{Z}^n).$$

From this, it is not difficult to deduce that

$$N(\lambda(\varepsilon) \cdot \Pi, \Gamma_M) = n_\varepsilon(A^{-1}(\Pi)),$$

where the right-hand side is defined by (1.1) with the subspace $F \subset \mathbb{R}^n$ spanned by $A^{-1}(e_1), \dots, A^{-1}(e_p)$ and with $H \subset \mathbb{R}^n$ spanned by $A^{-1}(e_{p+1}), \dots, A^{-1}(e_n)$. Observe that $\mathbb{Z}^n \cap F = \{0\}$.

Thus, finally we get

$$n_\varepsilon(A^{-1}(\Pi)) = \varepsilon^{-q} \text{vol}_n(A^{-1}(\Pi)) + O(|\ln \varepsilon|^{n-1}), \quad \varepsilon \rightarrow 0,$$

which is formula (1.2) with a sharper remainder estimate in the case where $S = A^{-1}\Pi$ is a parallelepiped centered at the origin and with edges parallel to the vectors $A^{-1}(e_1), \dots, A^{-1}(e_n)$. Note that, in this case, the subspaces F and H have rather special form and, in general, are not orthogonal.

Similarly, the results of [10, 11] can be used for obtaining sharper remainder estimates in (1.2) for some special subspaces F and H and domains S .

It would be interesting to continue the study of remainder estimates in the asymptotic formula (1.2), depending on the geometry of S and the properties of F and H .

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§2. APPLICATIONS TO ADIABATIC LIMITS

It is well known that the Gauss problem on counting the integer points in a disk is equivalent to the problem on the asymptotic behavior of the eigenvalue distribution function of a certain elliptic differential operator on a compact manifold, namely, of the Laplace operator on a torus. In the case under consideration, there is also an equivalent asymptotic spectral problem, namely, that on the asymptotic behavior of the eigenvalue distribution function for the Laplace operator on a torus in the adiabatic limit associated with a linear foliation.

Consider the n -dimensional torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$. Let $\mathcal{F}$ be a linear foliation on $\mathbb{T}^n$: the leaf L_x of $\mathcal{F}$ through $x \in \mathbb{T}^n$ has the form

$$L_x = x + F \pmod{\mathbb{Z}^n}.$$

The decomposition of $\mathbb{R}^n$ into the direct sum of subspaces,

$$\mathbb{R}^n = F \oplus H,$$

induces the corresponding decomposition $g = g_F + g_H$ of the Euclidean metric into the sum of the tangential and transversal components. Define a one-parameter family g_ε of Euclidean metrics on $\mathbb{R}^n$ by

$$g_\varepsilon = g_F + \varepsilon^{-2} g_H, \quad \varepsilon > 0.$$

We shall also view the metrics g_ε as Riemannian metrics on $\mathbb{T}^n$.

Let $A = (a_1, \dots, a_n) \in \mathbb{R}^n$ be some point. We define a 1-form $\mathbf{A}$ on $\mathbb{T}^n$ by

$$\mathbf{A} = \sum_{j=1}^n a_j dx_j.$$

Consider the operator $d - 2\pi i \mathbf{A}$ acting from $C^\infty(\mathbb{T}^n)$ to the space $\Omega^1(\mathbb{T}^n)$ of smooth 1-forms on $\mathbb{T}^n$, where d is the de Rham differential, and $\mathbf{A}$ is the operator of multiplication by $\mathbf{A}$. Let $(d - 2\pi i \mathbf{A})_{g_\varepsilon}^* : \Omega^1(\mathbb{T}^n) \rightarrow C^\infty(\mathbb{T}^n)$ be the adjoint to $d - 2\pi i \mathbf{A}$ with respect to the inner products in $C^\infty(\mathbb{T}^n)$ and $\Omega^1(\mathbb{T}^n)$ determined by g_ε .

For any $\varepsilon > 0$, consider the operator H_ε in $C^\infty(\mathbb{T}^n)$ defined by

$$H_\varepsilon = (d - 2\pi i \mathbf{A})_{g_\varepsilon}^* (d - 2\pi i \mathbf{A}).$$

In the local coordinates $(x_1, x_2, \dots, x_n)$ of $\mathbb{R}^n$, the operator H_ε looks like this:

$$H_\varepsilon = \sum_{j,\ell=1}^n g_\varepsilon^{j\ell} \left(\frac{\partial}{\partial x_j} - 2\pi i a_j \right) \left(\frac{\partial}{\partial x_\ell} - 2\pi i a_\ell \right),$$

where the $g_\varepsilon^{j\ell}$ are the elements of the inverse matrix of g_ε .

The operator H_ε can be regarded as the magnetic Schrödinger operator on $\mathbb{T}^n$ associated with the metric g_ε , with the constant magnetic potential $\mathbf{A}$. It has a complete orthogonal systems of eigenfunctions

$$U_k(x) = e^{2\pi i(k,x)}, \quad x \in \mathbb{R}^n, \quad k \in \mathbb{Z}^n,$$

with the corresponding eigenvalues

$$\lambda_k = (2\pi)^2 \|k - A\|_{g_\varepsilon}^2 = (2\pi)^2 \sum_{j,\ell=1}^n g_\varepsilon^{j\ell} (k_j - a_j) (k_\ell - a_\ell).$$

Let $N_\varepsilon(\lambda)$ denote the eigenvalue distribution function of H_ε :

$$N_\varepsilon(\lambda) = \#\{k \in \mathbb{Z}^n : \lambda_k < \lambda\}, \quad \lambda \in \mathbb{R}.$$

It is easily seen that

$$n_\varepsilon(B_{\sqrt{\lambda}}(A)) = N_\varepsilon(4\pi^2\lambda), \quad \lambda \in \mathbb{R}.$$

Thus, the problem on the asymptotic behavior of the number $n_\varepsilon(B_{\sqrt{\lambda}}(A))$ of integer points in the ellipsoid $T_\varepsilon(B_{\sqrt{\lambda}}(A))$ as $\varepsilon \rightarrow 0$ is equivalent to that on the asymptotic behavior of the eigenvalue distribution function $N_\varepsilon(\lambda)$ as $\varepsilon \rightarrow 0$. The limiting procedure $\varepsilon \rightarrow 0$ will be called the adiabatic limit. This notion was introduced by Witten in 1985 in the study of global anomalies in string theory. We refer the reader to a survey paper [7] for some historic remarks and references.

In [5] (see also [6]), the first author computed the leading term of the asymptotics of the eigenvalue distribution function for the Laplace operator associated with a bundle-like metric on a compact manifold equipped with a Riemannian foliation, in the adiabatic limit. The linear foliation on the torus is a Riemannian foliation, and a Euclidean metric on the torus is bundle-like. As a straightforward consequence of Theorem 4, in this particular case we obtain a sharper estimate for the remainder in the asymptotic formula of [5] (see also [16]).

Theorem 6. *For $\lambda > 0$, the following asymptotic formula is valid as $\varepsilon \rightarrow 0$:*

$$N_\varepsilon(\lambda) = \varepsilon^{-q} \frac{\omega_{n-r}}{|Q|} \sum_{\gamma^* \in \Gamma^*} \left(\frac{\lambda}{4\pi^2} - |\gamma^* - A|^2 \right)^{(n-r)/2} + O(\varepsilon^{k-q}),$$

where ω_{n-r} is the volume of the unit ball in $\mathbb{R}^{n-r}$, and

$$k = \begin{cases} \frac{q+1}{2(p-r+1)} & \text{if } \frac{q-1}{2} \leq p-r, \\ \frac{2q}{q+1+2(p-r)} & \text{if } \frac{q-1}{2} > p-r. \end{cases}$$

§3. PROOFS OF THE MAIN RESULTS

We shall use a method based on the Poisson summation formula and the stationary phase method (Van der Corput, Randol [12, 13], Colin de Verdière [1]). First, we observe that

$$(3.1) \quad \mathbb{Z}^n \subset \bigsqcup_{\gamma^* \in \Gamma^*} P_{\gamma^*}.$$

Indeed, let $k \in \mathbb{Z}^n$. Denote by $\pi_V : \mathbb{R}^n \rightarrow V$ the orthogonal projection to V . Then for any $\gamma \in \Gamma$, we have

$$(\pi_V(k), \gamma) = (k, \gamma) \in \mathbb{Z},$$

because $\gamma \in \mathbb{Z}^n$ and $\gamma \in F \subset V$. Hence, $\pi_V(k) \in \Gamma^*$, which immediately implies (3.1).

For any $\gamma^* \in \Gamma^*$, denote

$$\mathbb{Z}_{\gamma^*}^n = \mathbb{Z}^n \cap P_{\gamma^*} = \{k \in \mathbb{Z}^n : \pi_V(k) = \gamma^*\}.$$

We identify the affine subspace P_{γ^*} with the linear space $V^\perp$, fixing an arbitrary point $k_{\gamma^*} \in \mathbb{Z}_{\gamma^*}^n$:

$$P_{\gamma^*} = k_{\gamma^*} + V^\perp.$$

It is easily seen that

$$\mathbb{Z}_{\gamma^*}^n = k_{\gamma^*} + \Gamma^\perp,$$

where

$$\Gamma^\perp = \mathbb{Z}^n \cap V^\perp$$

is a lattice in $V^\perp$.

We observe the following relation, which will be needed later:

$$(3.2) \quad \text{vol}(V^\perp/\Gamma^\perp) = |Q|.$$

For its proof, we choose some basis $(\ell_1, \dots, \ell_r)$ in Γ and denote by $(\ell_1^*, \dots, \ell_r^*)$ the dual basis in Γ^* : $(\ell_i, \ell_j^*) = \delta_{ij}$ for any $i, j = 1, \dots, r$. For any $i = 1, \dots, r$, choose some $k_i^* \in \mathbb{Z}^n$ such that $\pi_V(k_i^*) = \ell_i^*$. Let $(k_1^\perp, \dots, k_{n-r}^\perp)$ be a basis in $\Gamma^\perp$. Using (3.1), it is easy to show that the system $(k_1^*, \dots, k_r^*, k_1^\perp, \dots, k_{n-r}^\perp)$ is a basis in $\mathbb{Z}^n$. Therefore, for the volume of the parallelepiped spanned by $(k_1^*, \dots, k_r^*, k_1^\perp, \dots, k_{n-r}^\perp)$ we have

$$\text{vol}_n(k_1^*, \dots, k_r^*, k_1^\perp, \dots, k_{n-r}^\perp) = \text{vol}(\mathbb{R}^n/\mathbb{Z}^n) = 1.$$

On the other hand, since $k_i^* - \ell_i^* \in V^\perp$ for any $i = 1, \dots, r$, and the systems $(\ell_1^*, \dots, \ell_r^*)$ and $(k_1^\perp, \dots, k_{n-r}^\perp)$ are mutually orthogonal, we get

$$\begin{aligned} \text{vol}_n(k_1^*, \dots, k_r^*, k_1^\perp, \dots, k_{n-r}^\perp) &= \text{vol}_n(\ell_1^*, \dots, \ell_r^*, k_1^\perp, \dots, k_{n-r}^\perp) \\ &= \text{vol}_r(\ell_1^*, \dots, \ell_r^*) \text{vol}_{n-r}(k_1^\perp, \dots, k_{n-r}^\perp) \\ &= \text{vol}(V/\Gamma^*) \text{vol}(V^\perp/\Gamma^\perp). \end{aligned}$$

Therefore,

$$\text{vol}(V^\perp/\Gamma^\perp) = \frac{1}{\text{vol}(V/\Gamma^*)}.$$

In order to complete the proof of (3.2), it remains to apply the well-known relation

$$\text{vol}(V/\Gamma) \text{vol}(V/\Gamma^*) = 1.$$

Thus, we can write

$$(3.3) \quad n_\varepsilon(S) = \sum_{\gamma^* \in \Gamma^*} n_\varepsilon(S, \gamma^*),$$

where

$$n_\varepsilon(S, \gamma^*) = \#(T_\varepsilon(S) \cap \mathbb{Z}_{\gamma^*}^n).$$

Note that, since S is bounded, the sum on the right-hand side of (3.3) has finitely many nonzero terms.

Fix $\gamma^* \in \Gamma^*$. Let $\chi_{S_{\gamma^*}}$ be the indicator of the set $S_{\gamma^*} = S \cap P_{\gamma^*}$. It is easily seen that

$$\begin{aligned} n_\varepsilon(S, \gamma^*) &= \sum_{k \in \mathbb{Z}_{\gamma^*}^n} \chi_{T_\varepsilon(S_{\gamma^*})}(k) = \sum_{\gamma \in \Gamma^\perp} \chi_{T_\varepsilon(S_{\gamma^*})}(k_{\gamma^*} + \gamma) = \sum_{\gamma \in \Gamma^\perp} \chi_{S_{\gamma^*}}(T_{\varepsilon^{-1}}(k_{\gamma^*} + \gamma)) \\ &= \sum_{\gamma \in \Gamma^\perp} \chi_{S_{\gamma^*}}(k_{\gamma^*} + (T_{\varepsilon^{-1}}(k_{\gamma^*}) - k_{\gamma^*}) + T_{\varepsilon^{-1}}(\gamma)). \end{aligned}$$

The space $V^\perp$ is represented as the direct sum

$$(3.4) \quad V^\perp = F_V \oplus H,$$

where $F_V = F \cap V^\perp$. We write the decomposition of $x \in V^\perp$ corresponding to (3.4) as follows:

$$x = x_F + x_H, \quad x_F \in F_V, x_H \in H.$$

Note that

$$T_\varepsilon(x) = x_F + \varepsilon^{-1}x_H.$$

Let $\rho \in C_0^\infty(\mathbb{R})$ be an even function such that $0 \leq \rho(x) \leq 1$ for any $x \in \mathbb{R}$ and $\text{supp } \rho \subset (-1, 1)$. For any $t_F > 0$ and $t_H > 0$, we define a function $\rho_{t_F, t_H} \in C_0^\infty(V^\perp)$ by

$$\rho_{t_F, t_H}(x) = \frac{c}{t_F^{p-r} t_H^q} \rho\left((t_F^{-2} x_F^2 + t_H^{-2} x_H^2)^{1/2}\right), \quad x \in V^\perp,$$

where the constant $c > 0$ is chosen so that $\int_{V^\perp} \rho_{1,1}(x) dx = 1$. The function ρ_{t_F, t_H} is supported in the ellipsoid

$$B(0, t_F, t_H) = \left\{ x \in V^\perp : \frac{x_F^2}{t_F^2} + \frac{x_H^2}{t_H^2} < 1 \right\}.$$

We define a function $n_{\varepsilon, t_F, t_H}(S, \gamma^*)$ by

$$n_{\varepsilon, t_F, t_H}(S, \gamma^*) = \sum_{k \in \mathbb{Z}_{\gamma^*}^n} (\chi_{T_\varepsilon(S_{\gamma^*})} * \rho_{t_F, t_H})(k),$$

where $\chi_{T_\varepsilon(S_{\gamma^*})} * \rho_{t_F, t_H} \in C_0^\infty(P_{\gamma^*})$ is defined by

$$(\chi_{T_\varepsilon(S_{\gamma^*})} * \rho_{t_F, t_H})(y) = \int_{V^\perp} \chi_{T_\varepsilon(S_{\gamma^*})}(y - x) \rho_{t_F, t_H}(x) dx, \quad y \in P_{\gamma^*}.$$

For any domain $S \subset P_{\gamma^*}$ and any $t_F > 0$ and $t_H > 0$, we put

$$S_{t_F, t_H} = \bigcup_{x \in S} (x + B(0, t_F, t_H)),$$

and

$$S_{-t_F, -t_H} = P_{\gamma^*} \setminus (P_{\gamma^*} \setminus S)_{t_F, t_H}.$$

It is easily seen that

$$T_\varepsilon(S_{t_F, \varepsilon t_H}) = (T_\varepsilon(S))_{t_F, t_H}$$

for any $\varepsilon > 0$, $t_F > 0$, and $t_H > 0$.

Lemma 7. For any $\varepsilon > 0$, $t_F > 0$, and $t_H > 0$, we have

$$n_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{-t_F, -\varepsilon t_H}, \gamma^*) \leq n_{\varepsilon}(S, \gamma^*) \leq n_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*).$$

Proof. Suppose that $k \in \mathbb{Z}^n \cap T_{\varepsilon}(S_{\gamma^*})$. For any $x \in B(0, t_F, t_H)$, the point $k - x$ belongs to $(T_{\varepsilon}(S_{\gamma^*}))_{t_F, t_H}$. Therefore, $\chi_{T_{\varepsilon}((S_{\gamma^*})_{t_F, \varepsilon t_H})}(k - x) = 1$ and

$$(\chi_{T_{\varepsilon}((S_{\gamma^*})_{t_F, \varepsilon t_H})} * \rho_{t_F, t_H})(k) = \int_{V^{\perp}} \rho_{t_F, t_H}(x) dx = 1 = \chi_{T_{\varepsilon}(S_{\gamma^*})}(k).$$

If $k \notin \mathbb{Z}^n \cap T_{\varepsilon}(S_{\gamma^*})$, then

$$\chi_{T_{\varepsilon}(S_{\gamma^*})}(k) = 0 \leq (\chi_{T_{\varepsilon}((S_{\gamma^*})_{t_F, \varepsilon t_H})} * \rho_{t_F, t_H})(k).$$

We get

$$\begin{aligned} n_{\varepsilon}(S, \gamma^*) &= \sum_{k \in \mathbb{Z}_{\gamma^*}^n} \chi_{T_{\varepsilon}(S_{\gamma^*})}(k) \\ &\leq \sum_{k \in \mathbb{Z}_{\gamma^*}^n} (\chi_{T_{\varepsilon}((S_{\gamma^*})_{t_F, \varepsilon t_H})} * \rho_{t_F, t_H})(k) = n_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*). \end{aligned}$$

The right inequality in the lemma follows from the one proved above if we apply it to the domain $B \setminus S$, where B is a sufficiently large ball containing S . $\square$

For any $f \in \mathcal{S}(V^{\perp})$, we define its Fourier transform $\hat{f} \in \mathcal{S}(V^{\perp})$ by

$$\hat{f}(\xi) = \int_{V^{\perp}} e^{-2\pi i(\xi, x)} f(x) dx.$$

Recall the Poisson summation formula

$$(3.5) \quad \sum_{k \in \Gamma^{\perp}} f(k) = \frac{1}{|Q|} \sum_{k^* \in \Gamma^{\perp*}} \hat{f}(k^*), \quad f \in \mathcal{S}(V^{\perp}),$$

where $\Gamma^{\perp*} \subset V^{\perp}$ is the dual lattice of $\Gamma^{\perp}$ (we have used (3.2)).

We apply (3.5) to the function

$$(3.6) \quad f(x) = (\chi_{T_{\varepsilon}((S_{\gamma^*})_{t_F, \varepsilon t_H})} * \rho_{t_F, t_H})(k_{\gamma^*} + x), \quad x \in V^{\perp}.$$

Formula (3.5) applies, because, for any $N > 0$, we have the estimate

$$(3.7) \quad |\hat{\rho}_{t_F, t_H}(\xi)| \leq C_N \frac{1}{1 + t_F^N |\xi_F|^N + t_H^N |\xi_H|^N}, \quad \xi \in V^{\perp}.$$

Observe that

$$\hat{\chi}_{T_{\varepsilon}((S_{\gamma^*})_{t_F, \varepsilon t_H})}(\xi) = \varepsilon^{-q} e^{2\pi i(\xi, (1-T_{\varepsilon})(k_{\gamma^*}))} \hat{\chi}_{(S_{\gamma^*})_{t_F, \varepsilon t_H}}(T_{\varepsilon}(\xi)).$$

Indeed, for any subset $S \subset V^{\perp}$, we have

$$\begin{aligned} \hat{\chi}_{T_{\varepsilon}(S)}(\xi) &= \int_{V^{\perp}} e^{-2\pi i(\xi, x)} \chi_{T_{\varepsilon}(S)}(k_{\gamma^*} + x) dx \\ &= \int_{V^{\perp}} e^{-2\pi i(\xi, x)} \chi_S(k_{\gamma^*} + (T_{\varepsilon}^{-1}(k_{\gamma^*}) - k_{\gamma^*}) + T_{\varepsilon}^{-1}(x)) dx \\ &= \varepsilon^{-q} \int_{V^{\perp}} e^{-2\pi i(\xi, T_{\varepsilon}(x') + (T_{\varepsilon}(k_{\gamma^*}) - k_{\gamma^*}))} \chi_S(k_{\gamma^*} + x') dx \\ &= \varepsilon^{-q} e^{2\pi i(\xi, (1-T_{\varepsilon})(k_{\gamma^*}))} \hat{\chi}_S(T_{\varepsilon}(\xi)). \end{aligned}$$

Also,

$$\hat{\rho}_{t_F, t_H}(\xi) = \hat{\rho}_{1,1}(t_F \xi_F + t_H \xi_H), \quad \xi \in V^{\perp}.$$

These relations show that the Poisson formula (3.5) applied to the function f as in (3.6) looks like this:

$$(3.8) \quad \begin{aligned} & n_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*) \\ &= \frac{\varepsilon^{-q}}{|Q|} \sum_{k \in \Gamma^{\perp*}} e^{2\pi i(k^*, (1-T_\varepsilon)(k_{\gamma^*}))} \hat{\chi}_{(S_{\gamma^*})_{\varepsilon, t_F, t_H}}(T_\varepsilon(k)) \hat{\rho}_{1,1}(t_F k_F + t_H k_H). \end{aligned}$$

The series on the right-hand side of (3.8) converges uniformly by (3.7).

We write

$$n_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*) = n'_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*) + n''_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*),$$

where

$$n'_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*) = \frac{\varepsilon^{-q}}{|Q|} \sum_{k \in \Gamma^{\perp*}, k_H=0} \hat{\chi}_{(S_{\gamma^*})_{t_F, \varepsilon t_H}}(T_\varepsilon(k)) \hat{\rho}_{1,1}(t_F k_F + t_H k_H)$$

and

$$\begin{aligned} & n''_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*) \\ &= \frac{\varepsilon^{-q}}{|Q|} \sum_{k \in \Gamma^{\perp*}, k_H \neq 0} e^{2\pi i(1-\varepsilon^{-1})(k_H, k_{\gamma^*})} \hat{\chi}_{(S_{\gamma^*})_{t_F, \varepsilon t_H}}(T_\varepsilon(k)) \hat{\rho}_{1,1}(t_F k_F + t_H k_H). \end{aligned}$$

Let $k \in \Gamma^{\perp*}$ be such that $k_H = 0$. Then $k \in F_V$. Since $\Gamma^{\perp*} \subset \mathbb{Q}^n$ and $F_V \cap \mathbb{Q}^n = \{0\}$, we get $k = 0$. Thus,

$$\begin{aligned} n'_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*) &= \frac{\varepsilon^{-q}}{|Q|} \hat{\chi}_{(S_{\gamma^*})_{t_F, \varepsilon t_H}}(0) = \frac{\varepsilon^{-q}}{|Q|} \text{vol}_{n-r}((S_{\gamma^*})_{t_F, \varepsilon t_H}) \\ &= \frac{\varepsilon^{-q}}{|Q|} \text{vol}_{n-r}(P_{\gamma^*} \cap S) + \frac{\varepsilon^{-q}}{|Q|} \text{vol}_{n-r}((S_{\gamma^*})_{t_F, \varepsilon t_H} \setminus S_{\gamma^*}). \end{aligned}$$

Since

$$\text{vol}_{n-r}((S_{\gamma^*})_{t_F, \varepsilon t_H} \setminus S_{\gamma^*}) \leq C(t_F + t_H \varepsilon),$$

we see that

$$(3.9) \quad n'_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*) = \frac{\varepsilon^{-q}}{|Q|} \text{vol}_{n-r}(P_{\gamma^*} \cap S) + O(t_F \varepsilon^{-q} + t_H \varepsilon^{1-q}).$$

Consider the case where $k \in \Gamma^{\perp*}$ and $k_H \neq 0$. For any $t \in F_V$ and any domain $D \subset P_{\gamma^*}$, we denote

$$D(t) = \{x_H \in H : k_{\gamma^*} + t + x_H \in D\} \subset H.$$

For $\phi \in \mathcal{S}(H)$, we denote by $F_H(\phi) \in \mathcal{S}(H)$ its Fourier transform:

$$[F_H(\phi)](\xi_H) = \int_H \phi(x_H) e^{-2\pi i(\xi_H, x_H)} dx_H, \quad \xi_H \in H.$$

It is easily seen that

$$[F_H(\chi_{D(t)})](\xi_H) = \int_H \chi_D(k_{\gamma^*} + t + x_H) e^{-2\pi i(\xi_H, x_H)} dx_H, \quad \xi_H \in H.$$

Therefore,

$$\begin{aligned}\widehat{\chi}_{(S_{\gamma^*})_{t_F, \varepsilon t_H}}(T_\varepsilon(k)) &= \int_{V^\perp} \chi_{(S_{\gamma^*})_{t_F, \varepsilon t_H}}(k_{\gamma^*} + x) e^{-2\pi i(T_\varepsilon(k), x)} dx \\ &= \int_{F_V} \int_H \chi_{(S_{\gamma^*})_{t_F, \varepsilon t_H}}(k_{\gamma^*} + x_F + x_H) e^{-2\pi i((k_F, x_F) + \varepsilon^{-1}(k_H, x_H))} dx_F dx_H \\ &= \int_{F_V} e^{-2\pi i(k_F, x_F)} F_H[\chi_{(S_{\gamma^*})_{t_F, \varepsilon t_H}}(x_F)](\varepsilon^{-1}k_H) dx_F,\end{aligned}$$

which implies the estimate

$$(3.10) \quad |\widehat{\chi}_{(S_{\gamma^*})_{t_F, \varepsilon t_H}}(T_\varepsilon(k))| \leq \int_{F_V} |F_H[\chi_{(S_{\gamma^*})_{t_F, \varepsilon t_H}}(x_F)](\varepsilon^{-1}k_H)| dx_F.$$

Hence, our considerations reduce to the sufficiently well-understood problem of estimating the Fourier transform of the indicator of a domain, and we can apply known results.

Proof of Theorem 1. For any sufficiently small $\varepsilon > 0$, $t_F > 0$, and $t_H > 0$, the domain $(S_{\gamma^*})_{t_F, \varepsilon t_H}(x_F)$ has smooth boundary. The Stokes formula allows us to write the Fourier transform $F_H[\chi_{(S_{\gamma^*})_{t_F, \varepsilon t_H}}(x_F)]$ as an oscillating integral over the boundary of $(S_{\gamma^*})_{t_F, \varepsilon t_H}(x_F)$:

$$(3.11) \quad F_H[\chi_{(S_{\gamma^*})_{t_F, \varepsilon t_H}}(x_F)](\tau\omega) = \frac{1}{\tau} \int_{\partial[(S_{\gamma^*})_{t_F, \varepsilon t_H}(x_F)]} e^{-i\tau(\omega, x)} i_\omega(dx_1 \wedge \cdots \wedge dx_q),$$

which yields

$$(3.12) \quad |F_H[\chi_{(S_{\gamma^*})_{t_F, \varepsilon t_H}}(x_F)](\xi)| = O(|\xi|^{-1}), \quad |\xi| \rightarrow \infty.$$

Hence, using (3.10), (3.12), and (3.7), we get

$$\begin{aligned}(3.13) \quad |n''_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*)| &\leq C\varepsilon^{-q} \sum_{k \in \Gamma^{\perp*}, k_H \neq 0} \varepsilon |k_H|^{-1} \frac{1}{1 + t_F^N |k_F|^N + t_H^N |k_H|^N} \\ &\leq C\varepsilon^{1-q} \int_{V^\perp} |x_H|^{-1} \frac{dx_F dx_H}{1 + t_F^N |x_F|^N + t_H^N |x_H|^N} \leq C\varepsilon^{1-q} t_F^{-(p-r)} t_H^{1-q}.\end{aligned}$$

Letting $t_H = c_H$ be a constant independent of $\varepsilon > 0$, putting $t_F = c_F \varepsilon^\alpha$ with

$$\alpha = \frac{1}{p-r+1},$$

and using (3.9) and (3.13), we arrive at the claim of the theorem. $\square$

Proof of Theorem 4. By assumption, the domain $S_{\gamma^*}(x_F) = S \cap \{\gamma^* + x_F + H\}$ is strictly convex. Therefore, for any sufficiently small $\varepsilon > 0$, $t_F > 0$, and $t_H > 0$, the domain $(S_{\gamma^*})_{t_F, \varepsilon t_H}(x_F)$ is strictly convex. By the stationary phase method, the representation (3.11) implies the following estimate:

$$(3.14) \quad |F_H[\chi_{(S_{\gamma^*})_{t_F, \varepsilon t_H}}(x_F)](\xi)| = O(|\xi|^{-(q+1)/2}), \quad |\xi| \rightarrow \infty.$$

Thus, using (3.10), (3.14), and (3.7), we see that

$$\begin{aligned}(3.15) \quad |n''_{\varepsilon, t_F, t_H}((S_{\gamma^*})_{t_F, \varepsilon t_H}, \gamma^*)| &\leq C\varepsilon^{-q} \sum_{k \in \Gamma^{\perp*}, k_H \neq 0} \varepsilon^{(q+1)/2} |k_H|^{-(q+1)/2} \frac{1}{1 + t_F^N |k_F|^N + t_H^N |k_H|^N} \\ &\leq C\varepsilon^{-q} \varepsilon^{(q+1)/2} \int_{V^\perp} |x_H|^{-(q+1)/2} \frac{dx_F dx_H}{1 + t_F^N |x_F|^N + t_H^N |x_H|^N} \\ &\leq C\varepsilon^{-(q-1)/2} t_F^{-(p-r)} t_H^{-(q-1)/2}.\end{aligned}$$

Now we put $t_F = \varepsilon^{\alpha_F}$, $t_H = \varepsilon^{\alpha_H}$, where $\alpha_F \geq 0$ and $\alpha_H \geq 0$ are chosen as follows. If $\frac{q-1}{2} > p-r$, then

$$\alpha_F = \frac{2q}{q+1+2(p-r)}, \quad \alpha_H = \frac{q-1-2(p-r)}{q+1+2(p-r)}.$$

If $\frac{q-1}{2} \leq p-r$, then

$$\alpha_F = \frac{q+1}{2(p-r+1)}, \quad \alpha_H = 0.$$

Using (3.9) and (3.15), we immediately conclude the proof. $\square$

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