Алгебра и анализ
Том 24 (2012), № 3

St. Petersburg Math. J.
Vol. 24 (2013), No. 3, Pages 387–430
S 1061-0022(2013)01245-3
Article electronically published on March 21, 2013

LONG ROOT TORI IN CHEVALLEY GROUPS

N. A. VAVILOV AND A. A. SEMENOV

To Nikolaĭ Gordeev, a remarkable mathematician, a dear friend, and a generous colleague

Abstract. In this paper we study some remarkable semisimple elements of an (extended) Chevalley group that are diagonalizable over the ground field — the 'weight elements'. In particular, we calculate the Bruhat decomposition of microweight elements. Results of the present paper are crucial for the description of overgroups of split maximal tori in Chevalley groups.

The paper is devoted to a detailed study of the most important and, in general, the simplest semisimple elements in Chevalley groups $G = G(\Phi, K)$, namely, of the long root type elements $gh_\alpha(\varepsilon)g^{-1}$, where the root α is long, $\varepsilon \in K^*$, and $g \in G$. We furnish detailed proofs of all previously announced results related to these elements. Let $Q = \{gh_\alpha(\varepsilon)g^{-1}, \varepsilon \in K^*\}$, where $g \in G$, be a long root torus. Let us fix a Borel subgroup $B = B(\Phi, K)$ and let $U = U(\Phi, K)$ be its unipotent radical. We prove a strong version of reduction to D_4, asserting that there exists $u \in U$ such that uQu^{-1} is contained in a Chevalley subgroup $G(\Delta, K)$ of type $\Delta \leq \Phi$, where Δ is a twisted subsystem of D_4. It turns out that all elements $gh_\alpha(\varepsilon)g^{-1}$, $\varepsilon \in K^*$, apart from the identity element, and at most two further ones, lie in the same typical Bruhat cell Bw_0B. In other words, there exist at most one element $\theta \neq 1$ such that $gh_\alpha(\theta)g^{-1} \in BwB$ and $gh_\alpha(\theta^{-1})g^{-1} \in Bw^{-1}B$ for some $w \neq w_0$. Further, we reproduce — hitherto unpublished! — complete proofs of the results from the Thesis of the second author, on the number and depth of degenerations. In particular, we prove all results announced in [28, 80], and, in fact, get sharper results, producing explicit lists of possible degenerations. Before, such lists were only available for the two simplest cases where $\Phi = A_l$ and $\Phi = C_l$. These results are instrumental in the work of the first author and Vladimir Nesterov, devoted to the description of orbits of Chevalley groups on pairs of long root tori.

Introduction

In the present paper, which is a sequel of [17], we prove all results on Bruhat decomposition of long root tori in Chevalley groups over a field that were announced in [28, 83].

2010 *Mathematics Subject Classification.* Primary 20G15, 20G40, 20G35.

Key words and phrases. Chevalley groups, semisimple root elements, Bruhat decomposition, Borel orbits, parabolic subgroups with extraspecial unipotent radical.

For the first author, the main motivation to finalize the present paper came from the research within the framework of the RFBR project 10-01-90016 "The study of the structure of forms of reductive groups, and behavior of small unipotent elements in representations of algebraic groups" (SPbGU). Apart from that, at the final stage his work was supported by the RFBR projects 09-01-00762 (Siberian Federal University), 09-01-00784 (POMI RAS), 09-01-00878 (SPbGU), 09-01-91333 (POMI RAS), 09-01-90304 (SPbGU), 10-01-92651 (SPbGU), and 11-01-00756 (RGPU). The work of both authors was supported by the Presidential Grant NSh-5282.2010.1 "Motives, cohomologies, algebraic groups, representations, reciprocity laws, lower and upper bounds of scheme complexity of Boolean functions" and by the State Financed Research Task 6.38.74.2011 at the Saint Petersburg State University "Structure theory and geometry of algebraic groups and their applications in representation theory and algebraic K-theory".

Here we produce first complete proofs of reduction to D_4, as well as theorems on the number, types, and depth of degenerations.

Generally speaking, the simply connected Chevalley group $G = G(\Phi, K)$ does not have semisimple elements smaller than semisimple long root elements $xh_\alpha(\varepsilon)x^{-1}$, where $\alpha \in \Phi$ is a long root, $\varepsilon \in K^*$, $x \in G$. On the other hand, semisimple long root elements generate the split maximal torus $T = T(\Phi, K)$ of this group, and thus inevitably occur in the study of all questions related to this torus. This circumstance is especially manifest for the types $\Phi = E_8, F_4$, and G_2, because for these types there are no microweights.

For other types, the maximal torus of the adjoint group is generated by microweight elements, which may be even structurally easier than semisimple root elements. Nevertheless, even in these cases the use of semisimple root elements in the proofs usually renders better estimates on the order of the ground field, compared with the use of microweight elements; see, in particular, [7, 11, 12, 89].

In [8, 9, 10] and [27, 44][1] we studied semisimple long root elements in some detail. As opposed to the case of microweight elements, considered in [5, 6, 17], here it is quite false that all elements $xh_\alpha(\varepsilon)x^{-1}$, $\varepsilon \neq 0, 1$, must lie in the same Bruhat cell. However, all of them lie in some cells BwB coming from a subsystem $\Delta \subseteq \Phi$ that is a subsystem of D_4, or of its twisting.

Set

$$Q_x = \left\{ xh_\delta(\varepsilon)x^{-1} \mid \varepsilon \in K^* \right\},$$

where δ is the maximal root of the root system Φ, and $x \in G(\Phi, K)$. The following reduction theorem is the main tool in the proofs of all other results of both the present paper, and its sequel [25] by Vladimir Nesterov and the first author, devoted to *pairs* of root tori.

Theorem 1. *Let $G = G(\Phi, K)$ be a Chevalley group, and let Q_x, $x \in G$, be a long root torus in it. Then there exists a subsystem $\Delta \subset \Phi$ isomorphic to a twisted subsystem of D_4 and an element $u \in U(\Phi, K)$ such that*

$$Q_x \leq uG(\Delta, K)u^{-1}.$$

Observe that Theorem 1 implies, in particular, that description of *reductive parts* of subgroups of $\langle X, Y \rangle$ generated by two long root tori X, Y reduces to the case of D_4. The possibilities for Δ are listed below:

$$\Delta = A_1,\ A_2,\ A_3,\ B_2,\ B_3,\ D_4,\ G_2.$$

A preliminary form of this reduction, at the level of individual root elements, can be found in the papers [8, 10] of the first author. The above slightly more precise statement, pertaining to the long root tori themselves, was formulated without proof in our joint papers [27, 28, 83]. Soon thereafter, a similar reduction to D_4 was rediscovered by Gerhard Röhrle [82]. However, Röhrle considered the orbits of a Levi subgroup itself, whereas we need orbits of its Borel subgroup, and a detailed proof of Theorem 1 itself has never been published before. In [8], we published a proof of a somewhat weaker statement; unfortunately, it was not translated into English. On the other hand, in [27] we reproduced merely a sketch of the proof, skipping most of the actual calculations.

Next, today the exposition in [8, 27] can be improved on several counts. For instance, now we can replace circumlocution of the form "the case ... was *essentially* considered in ..." or "studying the reduction on pages ... of ..., we can observe that ...", by *explicit* references to the corresponding places of [17] and the present paper. Incidentally,

[1]As a scientometric curiosity, let us mention that the English translation, and thus also all usual databases, erroneously indicate A. V. Yakovlev as the author of [44].

we correct a couple of petty inaccuracies[2] in the analysis of the cases where $\Phi = \mathrm{B}_l, \mathrm{F}_4$. Therefore, we opted for inclusion of all details of calculations. Actually, one could write the proof in a more uniform way, skipping *some* of the case by case analysis, but a thorough understanding of the arising unipotent radicals is interesting in itself and is of vital importance for future applications, in particular, in the forthcoming papers by the first author and Vladimir Nesterov. Therefore, we decided to present all details for each case.

From a technical viewpoint, Theorem 1 amounts to describing the Borel orbits of the Levi factor L_Q of a parabolic subgroup Q with the extraspecial unipotent radical U_Q, under the conjugation action on $U_Q/[U_Q, U_Q]$. The orbits of the Levi factor itself, in these representations, naturally occur in a variety of contexts, and their classification was considered systematically in many important works, starting with [76, 65, 51, 66, 52], [79, 80, 81, 82]. However, our results are somewhat more general, in the two following respects: first, we describe not the orbits of the Levi factor L_Q itself, but rather the orbits of its Borel subgroup B_Q; second, we do not assume the field K to be algebraically closed.

Observe that in the last years there was a striking revival of the interest in the study of these orbits, in various contexts, such as higher composition laws, Jordan algebras, etc. (see, in particular, [58, 68, 69]). Among other things, Sergei Krutelevich [69] studied the orbits of the actions

- $(\mathrm{SL}(6,\mathbb{Z}), \bigwedge^3(\mathbb{Z}^6))$,
- $(\mathrm{Spin}(12,\mathbb{Z}), \text{half-spin}_{\mathbb{Z}})$,
- $(\mathrm{G}(\mathrm{E}_7,\mathbb{Z}), V(\varpi_7)_{\mathbb{Z}})$.

These are *precisely* the cases that occur in our analysis of long root tori in exceptional groups of types E_6, E_7, and E_8, respectively. However, we made no attempt to update our exposition, to incorporate the methods of these recent papers, or possible arithmetic applications, but rather unaffectedly reproduce, as is, *all* details of our elementary proofs, mostly elaborated some 15–20 years ago.

Theorem 4 of [17] asserts that for *any* weight $\omega \in P(\Phi^\vee)$ and for any element $x \in G(\Phi, K)$ all weight elements $xh_\omega(\varepsilon)x^{-1}$, $\varepsilon \in K^*$, apart from a finite number, sit in the same Bruhat cell Bw_0B, where w_0 is an involution. Now, Theorem 1 of the present paper implies, in particular, that a typical element of a long root torus falls into a Bruhat cell corresponding to pairwise strongly orthogonal roots in a subsystem descending from the root system of type D_4.

Theorem 2. *For a fixed $x \in G$, all elements of the long root torus Q_x, except for finitely many of them, fall in the same typical Bruhat cell*

$$Bw_0B = Bw_{\gamma_1} \dots w_{\gamma_{r+s}}B,$$

where $\gamma_1, \dots, \gamma_{r+s}$ are pairwise distinct strictly orthogonal roots. If, moreover, $\gamma_1, \dots, \gamma_r$ are long, whereas $\gamma_{r+1}, \dots, \gamma_{r+s}$ are short, we have $r + 2s \leq m = m(\Phi)$.

The precise values of m for various root systems are listed below:

$$\begin{aligned}
&m = 1 \quad \mathrm{A}_1, \mathrm{A}_2,\\
&m = 2 \quad \mathrm{A}_l,\ l \geq 3, \quad \mathrm{C}_l,\ l \geq 2,\\
&m = 3 \quad \mathrm{G}_2,\\
&m = 4 \quad \mathrm{B}_l,\ l \geq 3, \quad \mathrm{D}_l,\ l \geq 4, \quad \mathrm{E}_6,\ \mathrm{E}_7,\ \mathrm{E}_8,\ \mathrm{F}_4.
\end{aligned}$$

[2]The lack of a proviso concerning characteristic 2; the omission of the root β^* in the case where α is short whereas β is long.

From the above, it is clear that in the linear and symplectic case semisimple long root elements have *distinctly* easier structure than in all other cases. Incidentally, this is precisely why the proofs of all results of the present paper for types A_l and C_l are *substantially* shorter and were published already in [9, 44]. On the other hand, for all other types, starting with rank 4, the complexity of long root elements is the same, and is identical with that in $\mathrm{SO}(8, K)$.

It is extremely instructive to compare this result with the above-cited Theorem 1 of [17]. That theorem asserts that *all* microweight elements $xh_\omega(\varepsilon)x^{-1}$, $\varepsilon \neq 0, 1$, lie in the same Bruhat cell

$$Bw_0B = Bw_{\gamma_1} \dots w_{\gamma_{r+s}}B.$$

For all microweights, the specific values of m such that $r + 2s \leq m$ are listed in Table 1 of [17]. It turns out that, for the simplest microweight elements, $m = 1$ for type A_l, $m = 2$ for types B_l and D_l, as well as for type E_6, and, finally, $m = 3$ for E_7.

Thus, as far as the simplest semisimple elements are concerned, type A_l is easier than all other classical cases. In their turn, all classical cases are easier than the exceptional cases. In this respect, the junior exceptional case is that of E_6. For the simplest semisimple elements in groups of types E_6, E_7, and E_8, the estimate m takes the consecutive values 2, 3, 4.

As a matter of fact, at about the same time, in the Thesis of the second author [45], written under the supervision of the first author — for which Nikolaĭ Leonidovich Gordeev was an *external assessor*[3] — we obtained precise results concerning the number and *types* of possible degenerations. These results are extremely important in themselves, as also to identify the spans of pairs of tori, which we started to carry out in [25], or to get sharp estimates on the order of the ground field in the results on overgroups of tori and many other similar problems; see [11]. The theorem specifying the *number* of degenerations was essentially established in [27], where we considered the senior case. However, even for this result a proper proof has never been published before.

In its turn, a specific *taxonomy* of degenerations was announced in [28] and, with somewhat more details, in [83]. However, apart from the simplest case of $\Phi = \mathrm{A}_l$ considered in [44], detailed proofs have never been published before. One of the reasons was that the proofs in the Thesis [45] of the second author heavily relied on direct computer calculations. Another reason was that at that time we could not complete the project of constructing geometry of root tori. This is why we have not seasonably published the results we perceived as secondary and subordinate to this principal objective.

Theorem 1 of [17] asserts, in particular, that for microweights, *all* elements $h_\omega(\varepsilon)$ with $\varepsilon \neq 0, 1$, belong to the same Bruhat cell. As a next step, we verify that for root tori, one can come up with a *substantially* stronger version of the above result, as compared with the general case. Quite amazingly, it turns out that, apart from the identity element, there are *at most* two further elements of a root torus, outside of the typical cell.

Theorem 3. *In the notation of Theorem 2, there is at most one $\theta \neq 1$ such that*

$$xh_\alpha(\theta)x^{-1} \in BwB, \quad xh_\alpha(\theta^{-1})x^{-1} \in Bw^{-1}B,$$

for some $w = w_\theta \neq w_0$.

Incidentally, as we established in [44], a unique value for this θ in the case of $\Phi = \mathrm{C}_l$ is $\theta = -1$. Actually, setting $\theta = -1$ does indeed result in a *multiple* degeneration. This is precisely what explains the dominant role of long root involutions in the calculations of [18]. Subsequently, the same consideration was used again in [16]. It was only much

[3]In Russian, this expression is *slightly alarmingly* rendered as an *official opponent*.

later, and recoursing to considerably more delicate kind of arguments, that Elisaveta Dybkova [29, 30] succeeded in dropping the assumption $\mathrm{char}(K) \neq 2$ in the above results.

From our prospective, Theorem 3 is somewhat unexpected, as far as the number of degenerations is concerned, because in a minimal representation of a Chevalley group, almost always a long root element $h_\alpha(\varepsilon)$ has three distinct eigenvalues. The only exception is the adjoint representation of the group of type E_8, where it has five distinct eigenvalues. It seems that, since in this last case the polynomials responsible for degeneration have larger degree, more degenerations should occur in the case of E_8 than for all other root systems. Actually, this is not so!

The following result was announced in [28] and [83] and was provided with a complete proof in the Thesis of the second author [45]. However, that proof was never published before either, with the only exception of the two easy cases A_l and C_l; see [9] and [44]. Herewith, we present such a proof for all types. Essentially, this proof follows Chapter 3 of the Thesis [45]. However, it is mounted differently, which allows us to somewhat shorten case by case analysis, completely eliminate any reference to computer calculations, and redo everything by hand. Actually, this helped us to restore some possibilities for multiple degeneration, which were missing in [45].

Theorem 4. *Only the following elements of the Weyl group $W = W(\Phi)$ may occur as nontrivial Weyl factors in the Bruhat decomposition of semisimple long root elements*:

• *all Coxeter elements of subsystems of type* A_1, $\widetilde{\mathrm{A}}_1$, $2\mathrm{A}_1$, $\mathrm{A}_1 + \widetilde{\mathrm{A}}_1$, $2\widetilde{\mathrm{A}}_1 \subset \Phi$, *and those* $3\mathrm{A}_1$, $4\mathrm{A}_1$ *that actually lie in some subsystem of type* D_4;

• *all Coxeter elements of subsystems of type* $\mathrm{A}_2 \subseteq \Phi$;

• *some Coxeter elements of subsystems of type* $\mathrm{A}_3, \mathrm{D}_4, \mathrm{B}_3, \mathrm{G}_2 \subseteq \Phi$.

The Coxeter elements of subsystems of type B_2 *do not occur.*

Observe that subsystems of types $3\mathrm{A}_1$ and $4\mathrm{A}_1$ that are actually contained in some D_4 are precisely the subsystems of conjugacy classes $\mathrm{D}_2+\mathrm{A}_1$, $\mathrm{D}_2+\widetilde{\mathrm{A}}_1$, and $2\mathrm{D}_2$ in B_l and D_l, and subsystems of conjugacy classes $3\mathrm{A}_1''$ and $4\mathrm{A}_1''$ in E_7 and E_8. It should be mentioned that the answer to the question as to whether an element $w \in W$ actually appears as the Weyl factor in a Bruhat decomposition of some semisimple long root element, does not depend on the conjugacy class of w alone.

Initially, we intended to merely reproduce the analysis of some several most difficult cases in the Thesis [45], and annex Tables 3 and 4, listing all possible degenerations in type D_4. However, later on we noticed that the table of multiple degenerations on pages 61–62 of that Thesis contains several omissions, some of which sprang out of two such omissions in the paper [44], while other missing involutions were discussed in the text of the proof on pages 44–59, but were not enrolled in the forementioned table.

This forced us to repeat all calculations anew, by hand, restoring all such missing cases. It should be observed, however, that though the tables of multiple degenerations in [45] are incomplete, this does not affect the main results of that work, *not in the least*! In particular, this applies to Theorem 4, since already in [8] it was observed that all allowed involutions do indeed occur! All new cases in Theorem 4 arise for *simple* degenerations, and the table of simple degenerations on pages 60–61 of the above Thesis is perfectly accurate.

We record the following corollary to Theorem 4, due to the fact that outer automorphisms of D_4 can be effected by elements of the Weyl group $W(\mathrm{E}_6)$. Thus, only elements of 8 conjugacy classes of the Weyl group $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$ occur in a Bruhat decomposition of semisimple long root elements. Recall that the total number of conjugacy classes of $W(\mathrm{E}_6)$, $W(\mathrm{E}_7)$, and $W(\mathrm{E}_8)$ equals 25, 61, and 109, respectively; see [56].

Finally, we state the theorem on the depth of degenerations. Fixing a long root torus $Q = Q_x$, $x \in G(\Phi, K)$, we consider the smallest root system $\Delta \subseteq \Phi$ such that

$Q \le uG(\Delta, K)u^{-1}$ for some $u \in U(\Phi, K)$. Further, let J be a fundamental root system of Δ contained in Φ^+. Denote by $l_J(w)$ the length of a Weyl group element $w \in W(\Delta)$ with respect to that fundamental root system.

The following result was announced in [28] and [83], and proved in the Thesis [45] of the second author. It follows immediately from Table 3 of simple degenerations, constructed in the process of the proof of Theorem 4.

Theorem 5. *In the case of a simple degeneration, in other words, when $\theta \neq -1$, and also in the case where $\theta = -1$ while w_0 is either a fundamental reflection or an involution of type $4\mathrm{A}_1$, we have $l_J(w_\theta) = l_J(w_0) - 1$.*

In all other cases where $\theta = -1$, multiple degeneration occurs, in other words, $l_J(w_{-1}) \le l_J(w_0) - 2$.

Generally speaking, for multiple degeneration the length of w_{-1} *typically* drops by 2, but it may well drop by more than 2. Thus, one cannot exclude that $l_J(w_{-1}) < l_J(w_0) - 2$. Of course, in any case, w_{-1} is an involution.

The rest of the paper is devoted to detailed proofs of the results stated above, and is organized as follows. In §1 we very briefly recall basic notation pertaining to Chevalley groups. In §2 we discuss some general facts pertaining to extra-special unipotent radicals, and in §3 we classify the orbits of unipotent radicals of Borel subgroups in the resulting internal Chevalley modules. In §4 we apply this classification to prove Theorems 1 and 2. After that, all other calculations are performed directly in the Chevalley group of type D_4, and in §5 we recall some background facts related to this group itself and to its Weyl group. In §6 we prove Theorem 3 for Chevalley groups of type D_4 by straightforward matrix calculations, and in §7 we completely describe the arising degenerations. In §8 we summarize the results of the two preceding sections and draw up the tables of possible degenerations, thus proving Theorems 4 and 5. Finally, in §9 we state some further related unsolved problems.

Partially, the contents of the present paper was announced in [28, 83]. In our papers [14, 17, 20, 26], one can find a much broader discussion of the general context, the reasons why we returned to the study of this type of problems, and many further related references. We mention two of such recent motivations here.

• On the one hand, we wished to set stage for the work by the first author and Vladimir Nesterov (see [21]–[25]), where we took the very first steps towards the geometry of microweight tori and long root tori. In particular, the results of the present paper are instrumental in the description of orbits of Chevalley groups, acting by simultaneous conjugation on pairs of long root tori; see [25].

• On the other hand, as became clear lately, our results are naturally incorporated into the immense project to describe intersections of conjugacy classes and Bruhat cells in Chevalley groups, as started by the work by Erich Ellers and Nikolaĭ Gordeev; see [61]–[63]. Recently, for *unipotent* elements there was amazing progress in this direction, in the works by George Lusztig [72]–[75]. The present paper, as also [17], is one of the very few examples where a complete answer is obtained for some [simplest classes of] *semisimple* elements.

§1. Basic notation

As general background references on Chevalley groups one could cite [1, 44, 55].

1°. Basic notation. Let Φ be a reduced irreducible root system in an l-dimensional Euclidean space V, and let $(\ ,\ )$ be the inner product in V. For two roots $\alpha, \beta \in \Phi$, we denote by $\langle \beta, \alpha \rangle = 2(\beta, \alpha)/(\alpha, \alpha) = (\beta, \alpha^\vee)$ the corresponding Cartan number, where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ is the dual root. Let $\Phi^\vee = \{\alpha^\vee, \alpha \in \Phi\}$ denote the dual root system.

Further, $Q(\Phi)$ is the root lattice, generated by all $\alpha \in \Phi$, and $P(\Phi)$ is the weight lattice, consisting of all $\omega \in V$ such that $(\alpha^\vee, \omega) \in \mathbb{Z}$ for all $\alpha^\vee \in \Phi^\vee$. We fix a fundamental root system $\Pi = \{\alpha_1, \dots, \alpha_l\}$ in Φ. Let Φ^+ and Φ^- be the corresponding sets of positive and negative roots, respectively. The choice of Π gives rise to the following partial order on $P(\Phi)$: we set $\lambda \succeq \mu$ if $\lambda - \mu = \sum m_i \alpha_i$, where $m_i \geq 0$.

Next, let K be a field, let P be a lattice lying between $Q(\Phi)$ and $P(\Phi)$, let $G = G(\Phi, K)$ be the Chevalley group of type Φ, P over K, and let $T = T(\Phi, K)$ be a split maximal torus in G. In the case where $P = P(\Phi)$, the group $G = G_{\mathrm{sc}}$ is called simply connected, and in the case where $P = Q(\Phi)$, the group $G = G_{\mathrm{ad}}$ is called adjoint. Usually, we assume that the group G is simply connected. For a root $\alpha \in \Phi$ and an element $\xi \in K$, we denote by $x_\alpha(\xi)$ the corresponding elementary root unipotent in G. For a given α, the set of all $x_\alpha(\xi)$ forms the elementary unipotent root subgroup $X_\alpha = \{x_\alpha(\xi),\ \xi \in K\}$.

2°. Weight elements. Let $\omega \in P(\Phi^\vee)$. Then, by the very definition, $(\alpha, \omega) \in \mathbb{Z}$ for all $\alpha \in \Phi$. Thus, for $\varepsilon \in K^*$ one can define the K-character $\chi = \chi_{\omega,\varepsilon}$ of the root lattice $Q(\Phi)$ by setting $\chi_{\omega,\varepsilon}(\alpha) = \varepsilon^{(\alpha,\omega)}$. Now, we can consider the diagonal automorphism determined by this character; see [5, 11, 17]. We may think of this diagonal automorphism as the conjugation by a certain semisimple element $h_\omega(\varepsilon)$. In other words, this element commutes with all elements of T and is subject to the following commutator relation:

$$h_\omega(\varepsilon) x_\alpha(\xi) h_\omega(\varepsilon)^{-1} = x_\alpha(\varepsilon^{(\alpha,\omega)}\xi)$$

for all $\alpha \in \Phi$, $\xi \in K$. Generally speaking, these elements belong not to the group G itself, but rather to its diagonal extension $\bar{G}$, the *extended Chevalley group*, which stands in the same relation to the usual Chevalley group as the general linear group $\mathrm{GL}(n, K)$ to the special linear group $\mathrm{SL}(n, K)$, or for that matter, as the general symplectic group $\mathrm{GSp}(2l, K)$ to the symplectic group $\mathrm{Sp}(2l, K)$. We call an element h of the group $\bar{G}$ a *weight element* of type ω if it is conjugate to some $h_\omega(\varepsilon)$.

Theorem 2 leans upon the following result, which is essentially Theorem 4 of [17].

Lemma 1. *Let $\omega \in P(\Phi^\vee)$, and let $x \in G$. Then for all $\varepsilon \in K^*$ apart from finitely many of them, the weight elements $x h_\omega(\varepsilon) x^{-1}$ lie in the same typical Bruhat cell $BwB = Bw_0B$. Moreover, w_0 is an involution.*

Actually, in [17] this result was established for extended Chevalley groups. But since its proof only involves the existence of an appropriate embedding in $\mathrm{GL}(n, K)$, *a fortiori*, the corresponding fact holds for the usual Chevalley groups.

3°. Root subsystems. In the sequel we repeatedly use the classification of subsystems in root systems. Such a classification was obtained in the late 1940s by Borel and de Siebenthal, and by Dynkin. We do not reproduce the resulting answer, which can be found, for instance, in [53] or [61]

Nevertheless, we make the following observation, which is extremely important for our purposes. Two isomorphic root subsystems are *almost* always conjugate. It is easy to list all possible exceptions.

• If Φ has roots of different lengths, while all roots of Δ have the same length, then, generally speaking, Δ can be embedded in Φ in two entirely different ways: on long roots, and on short roots. In such cases, we write $\widetilde{\Delta} \leq \Phi$ for the short root embedding, and $\Delta \leq \Phi$ for the long root embedding.

• In the root systems B_l and D_l, one should distinguish $2\mathrm{A}_1 = \{\pm(e_1 - e_2), \pm(e_3 - e_4)\}$ from $\mathrm{D}_2 = \{\pm e_1 \pm e_2\}$, and $\mathrm{A}_3 = \langle 2\mathrm{A}_1, e_2 - e_3 \rangle$ from $\mathrm{D}_3 = \langle \mathrm{D}_2, e_2 - e_3 \rangle$.

• In the root systems D_l, E_7, and E_8, some subsystems Δ of type $\mathrm{A}_{l_1} + \dots + \mathrm{A}_{l_r}$, where all l_i are odd, fall into two conjugacy classes denoted by Δ' and Δ'', respectively. In the case of D_l these two classes are fused by an outer automorphism.

Thus, for instance, subsystems D_2 and $2A_1$, as also subsystems D_3 and A_3, are isomorphic, but not conjugate. Below, talking of subsystems of a certain type, we always view them *up to conjugacy*. For instance, when we say about a pair of roots that they generate a subsystem of type D_2, we do not merely mean that these roots are orthogonal. What we mean is that they indeed generate a subsystem of type D_2, rather than $2A_1$, in other words, that they can be transformed to the pair $e_1 + e_2$, $e_1 - e_2$ by an element of the Weyl group.

§2. Extraspecial unipotent radicals

In this and the next sections we study the Borel orbits for Levi subgroups of parabolic subgroups with extraspecial unipotent radical, under the conjugation action on these radicals. These easy results are of technical nature, but they constitute a key step in the proof of the main results of the present paper. A similar analysis for Levi subgroups of parabolic subgroups with Abelian unipotent radical can be found in §§10–12 of [17].

Let δ be the maximal root of the root system Φ. In all cases apart from $\Phi = A_l$, the root δ is orthogonal to all fundamental root except exactly one, denoted by α_k in the sequel. More precisely, $\alpha_k = \alpha_2$ for $\Phi = B_l, D_l, E_6$ and G_2; $\alpha_k = \alpha_1$ for $\Phi = C_l, F_4$ and E_7; and finally, $\alpha_k = \alpha_8$ for $\Phi = E_8$. In other words, α_k is a unique fundamental root connected to δ in the extended Dynkin diagram. In the exceptional case of $\Phi = A_l$, $l \geq 2$, the root δ is connected to two fundamental roots, namely α_1 and α_l. By the way, this is yet another reason to view the system A_1 as belonging to the series C_l, in other words, to think of the group SL_2 as symplectic, rather than linear.

Denote by Σ the set of all positive roots of Φ that are distinct from δ and not orthogonal to δ. In other words, typically, the set Σ consists of the roots whose linear expansion with respect to fundamental roots involves α_k with coefficient 1:

$$\Sigma = \Big\{\alpha = \sum m_i\alpha_i \in \Phi \mid m_k = 1\Big\}.$$

In the exceptional case of $\Phi = A_l$, this set consists of those roots in the expansion of which *either* α_1 occurs with coefficient 1, *or* α_l occurs with coefficient 1, but not both. In fact, δ itself is a unique root whose expansion involves α_k with coefficient 2 or, respectively, involves *both* α_1 *and* α_l with coefficient 1.

The set Σ has the following structure: the sum of any two roots of Σ is not a root, apart from the pairs $(\alpha, \delta - \alpha)$, whose sum equals δ. In other words,

$$E(\Sigma \cup \{\delta\}, K) = X_\delta \prod_{\alpha\in\Sigma} X_\alpha$$

is isomorphic to the Heisenberg group. In particular, its center equals its commutator subgroup, and coincides with the root subgroup X_δ.

Further, let Δ be the closure of the set $\Pi \setminus \{\alpha_k\}$, or, in the exceptional case of $\Phi = A_l$, of the set $\Pi \setminus \{\alpha_1, \alpha_l\}$. In other words, $\Delta = \{\alpha \in \Phi \mid (\alpha, \delta) = 0\}$. If the difference of two roots in Σ is a root, it must belong to Δ. The Hasse diagram of the set Σ coincides with the weight diagram of some representation of the Chevalley group $G(\Delta, K)$ of type Δ; see references in [82, 75, 84, 87].

For all systems Φ the type $\Delta = \Delta_\omega$, the highest weight ρ of the corresponding representation and its dimension n are listed in Table 1.

We view the set Σ as a partially ordered set with respect to the usual order determined by the choice of a fundamental root system. In other words, $\alpha \succeq \beta$ if and only if $\alpha - \beta = \sum m_i\alpha_i$, where $m_i \geq 0$ for all $1 \leq i \leq l$. Consider the *Hasse diagram* of this partially ordered set. The vertices of this diagram correspond to the roots of Σ. Two vertices α and β are joined by an edge if and only if their difference is a fundamental

Table 1. Representations in extra-special radicals.

Φ	α_k	Δ	ρ	n
A_l	α_1, α_l	A_{l-2}	$\varpi_1 \oplus \varpi_{l-2}$	$2l-2$
B_l	α_2	$\mathrm{B}_{l-2} + \mathrm{A}_1$	(ϖ_1, ϖ_1)	$4l-6$
C_l	α_1	C_{l-1}	ϖ_1	$2l-2$
D_l	α_2	$\mathrm{D}_{l-2} + \mathrm{A}_1$	(ϖ_1, ϖ_1)	$4l-8$
E_6	α_2	A_5	ϖ_3	20
E_7	α_1	D_6	ϖ_6	32
E_8	α_8	E_7	ϖ_7	56
F_4	α_4	C_3	ϖ_3	14
G_2	α_2	A_1	$3\varpi_1$	4

root α_i. In this case we equip this edge with label i. As in many other papers where such diagrams are used, we read them *right to left* and *bottom up*. Thus, the edge with label i, when read from left to right, expresses *subtraction* of the fundamental root α_i, rather than its *addition*. Also, we use a common shortcut, omitting labels on some of the edges, where they can easily be recovered by using the fact that the labels on two opposite edges of a parallelogram coincide.

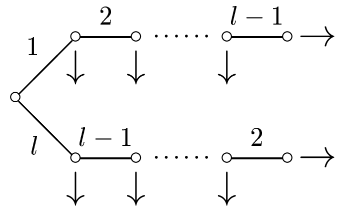
§3. Orbits of the unipotent radical of a Borel subgroup

Now, we proceed to a detailed study of Borel orbits in the representations listed in Table 1. We start with a description of orbits of the unipotent radical $U(\Delta, K)$. These are precisely the cases we need to prove Theorem 1.

Proposition. *Let (V, ρ) be one of the representations listed in Table* 1*. Assume additionally that* $\mathrm{char} K \neq 2$ *when* $\Phi = \mathrm{B}_l, \mathrm{F}_4$. *Then each orbit of $U = U(\Delta, K)$ on V contains a vector $v = \sum a_\gamma v_\gamma$ whose support $S(v) = \{\gamma \in \Sigma \mid a_\gamma \neq 0\}$ together with the root δ generates a subsystem contained in* D_4 *or in its twisting.*

Proof. As in [17], we consider all arising possibilities case by case. Also as in [17], to simplify the notation somewhat, actually we study the orbits of the unipotent radical of the *opposite* Borel subgroup $U^- = U^-(\Delta, K)$. However, the symmetry between the positive and negative roots ensures that the resulting answer is valid also for the unipotent radical $U = U(\Delta, K)$ of the standard Borel subgroup itself.

• **The case of $\Phi = \mathrm{A}_l$.** For that case, the diagram of the set $\Sigma \cup \{\delta\}$ looks like this. Arrows represent edges with labels 1 or l, which show how Σ is glued to Δ. More precisely, the vertical arrows coming from the upper component should have label l, while the vertical arrows coming from the lower component should have label 1. The horizontal arrows lead to the zero weight.



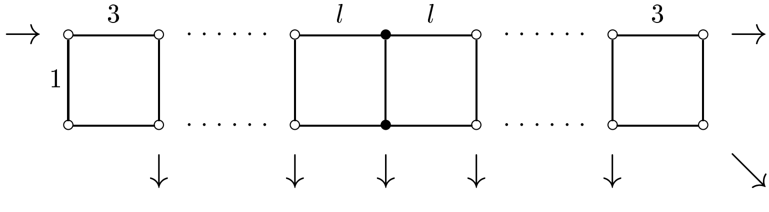
Consider the vector $v = \sum a_\gamma v_\gamma$, where $\gamma \in \Sigma$, $a_\gamma \in K$. Let α be the largest root in the upper chain such that $a_\alpha \neq 0$. Multiplying v by a matrix in $U^-(\mathrm{A}_{l-2}, K)$, namely, by $u_1 = \prod x_{\gamma-\alpha}(\mp a_\gamma a_\alpha^{-1}) \in U^-(\Delta, K)$, where the product is taken over all roots $\gamma \neq \alpha$ from the upper chain that are smaller than α, we can kill all coordinates corresponding to the roots of the upper chain, apart from a_α itself.

Remark. In view of Bruhat decomposition, this calculation is equivalent to the obvious fact that any nonzero row is the first row of a matrix from the special linear group. The further cases below can be interpreted similarly.

Next, consider the largest root β in the lower chain, *distinct from* $\delta - \alpha$ and such that in the vector $u_1 v$ obtained after the first step, the coordinate at the position β is distinct from 0. Now, multiplying the vector $u_1 v$ by the matrix $u_2 = \prod x_{\gamma-\beta}(\mp a_\gamma a_\beta^{-1}) \in U^-(\Delta, K)$, where the product is taken over all roots $\gamma \neq \beta$ from the lower chain that are smaller than β, we can kill all coordinates corresponding to the roots of the lower chain, apart from $a_{\delta-\alpha}$ and a_β. Since $\beta \neq \delta - \alpha$ by assumption, $\alpha + (\gamma - \beta)$ is not a root for all such γ, so that there are no nontrivial additions in the upper chain, which could affect the corresponding coordinates.

This means that in the vector $u_2 u_1 v$ there are at most three nonzero coordinates, namely, a_α, $a_{\delta-\alpha}$ and a_β. Clearly, together with δ the roots α, β, $\delta - \alpha$ generate the subsystem $\mathrm{A}_3 = \langle \alpha, \delta, \beta \rangle$, which finishes the analysis of the case A_l.

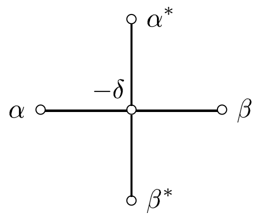
• **The case of $\Phi = \mathrm{B}_l$.** This is the most complicated of the classical cases. In the following diagram of the set Σ, arrows represent edges with label α_2, and the vertical arrows show how Σ is glued to B_{l-2}, the left horizontal arrow comes from δ, the right horizontal arrow leads to A_1, and finally, the diagonal arrow leads to the zero weight, in the diagram of the adjoint representation of type B_l.



Consider a vector $v = \sum a_\gamma v_\gamma$, where $\gamma \in \Sigma$, $a_\gamma \in K$. First, assume that the senior nonzero coordinate a_α corresponds to a long root $\alpha \in \Sigma$. Then the roots of Σ orthogonal to α form the following configuration: a chain Ω of length $2l - 3$ and another root α^*, which together with α generates a subsystem of type D_2. Multiplying v by the matrix $u_1 = \prod x_{\gamma-\alpha}(\mp a_\gamma a_\alpha^{-1}) \in U^-(\Delta, K)$, where the product is taken over all roots smaller than α and *not orthogonal* to α, apart from $\delta-\alpha$, we can kill all coordinates corresponding to the roots not orthogonal to α, except, maybe, the coordinate corresponding to $\delta - \alpha$.

Next, consider the senior nonzero coordinate a_β corresponding to a root $\beta \in \Omega$. First, assume that the root β is also long. Then there is a unique long root in Ω not orthogonal to β, namely β^*. Here the asterisk is used in the same sense as above. Namely, it is a unique positive root that, together with β, generates a subsystem of type D_2. Multiplying u_1v by the matrix $u_2 = \prod x_{\gamma-\beta}(\mp a_\gamma a_\beta^{-1}) \in U^-(\Delta, K)$, where the product is taken over all $\gamma \in \Omega$ smaller than β and distinct from β^*, we can ensure that the coordinates of the column u_2u_1v, except maybe for the coordinates corresponding to α, α^*, $\delta - \alpha$, β, β^*, are all zero.

It remains to find out what type of subsystem is generated by these roots together with δ. Clearly, $\delta - \alpha$ can be dropped, while the remaining four roots together with $-\delta$ form the diagram

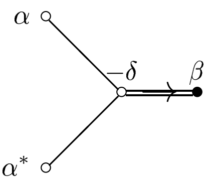


which is precisely the extended Dynkin diagram of type D_4.

Next, we continue to assume that the root α is long, but let now the root β be short. In this case, in Ω there are no roots orthogonal to β, and multiplying by

$$u_2 = \prod x_{\gamma-\beta}(\mp a_\gamma a_\beta^{-1}/2) \in U^-(\Delta, K),$$

we can kill all coordinates of v apart from the coordinates corresponding to α, α^*, $\delta - \alpha$, β. Again, inserting $-\delta$ and dropping $\delta - \alpha$, which is a linear combination of the remaining roots, we obtain the diagram



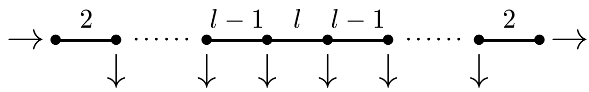
which is precisely the extended Dynkin diagram of type B_3.

It only remains to consider the case where the senior coordinate of the vector v corresponds to a short root α. Setting

$$u_1 = \prod x_{\gamma-\alpha}(\mp a_\gamma a_\alpha^{-1}/2) \in U^-(\Delta, K),$$

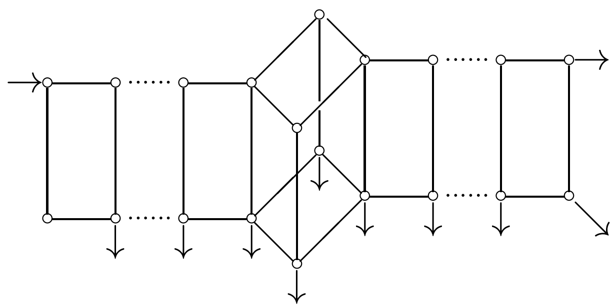
where the product is taken over all γ smaller than α and such that $\gamma - \alpha$ is a root, we can kill all coordinates of the vector u_1v corresponding to such γ. Now, let $\beta \neq \alpha$ be the largest among the remaining roots for which the corresponding coordinates are nonzero. Since the difference of any two linearly independent short roots is a root, it follows that β must be long. Now, multiplying u_1v by the same matrix $u_2 \in U^-(\Delta, K)$ as in the case where both roots α, β are long, we see that we can kill all coordinates apart from those corresponding to α, β, β^*. This means that we again obtain precisely the same Dynkin diagram as in the case where α was long and β was short, but with α and β interchanged. In other words, once again we get a root system of type B_3. This concludes the analysis of the case B_l, under the assumption $\mathrm{char}(K) \neq 2$.

• **The case of $\Phi = C_l$.** In this case the diagram of the set Σ is a chain consisting entirely of short roots, whose pairwise differences are roots of Δ. Here, arrows refer to the root α_1, the vertical arrows show how Σ is glued to Δ, while the horizontal ones lead from δ and to the zero weight, respectively.



The same procedure as in the case of $\Phi = \mathrm{A}_l$ allows us to kill all coordinates apart from exactly one of them.

• **The case of** $\Phi = \mathrm{D}_l$**.** The diagram of the set Σ is reproduced in the following picture. Here, arrows correspond to the root α_2, and their direction has precisely the same meaning as in the case of $\Phi = \mathrm{B}_l$.

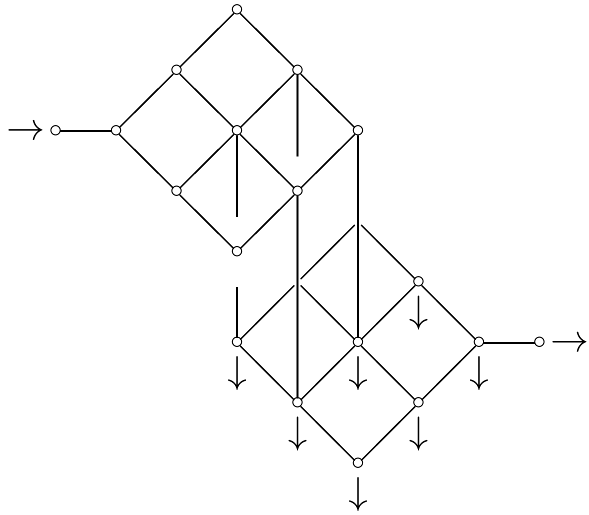


The roots $\gamma \in \Sigma$ orthogonal to one root $\alpha \in \Sigma$ are precisely the root α^* (which, together with α, generates a root system of type D_2) and the set Ω, which has the same structure as the representation of the group of type $\mathrm{D}_{l-3} = \langle \alpha_4, \dots, \alpha_l \rangle$ with highest weight ϖ_4.

As above, let α be the largest root such that the corresponding coordinate of the vector $v = \sum a_\gamma v_\gamma$, where $\gamma \in \Sigma$, $a_\gamma \in K$, is nonzero. Multiplying v by the same matrix as in the case where $\Phi = \mathrm{B}_l$ and the root α is long, we can kill all coordinates of the vector $u_1 v$, apart maybe from the coordinates corresponding to the roots α, α^*, $\delta - \alpha$ and the roots in Ω.

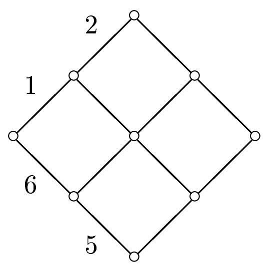
As always, we can replace v by $u_1 v$. If $\beta \in \Omega$ is the largest among all roots $\gamma \in \Omega$ such that $a_\gamma \neq 0$, then multiplying v by the same matrix u_2 as in the case where $\Phi = \mathrm{B}_l$ and both roots α and β are long, we can kill all coordinates of the vector $u_2 u_1 v$, apart from the coordinates corresponding to the roots α, α^*, $\delta - \alpha$, β, β^*. However, as we already know from the analysis of the case B_l, these roots together with the root δ generate a subsystem of type D_4, which concludes the analysis of this case.

• **The case of** $\Phi = \mathrm{E}_6$**.** The diagram of the set Σ is reproduced below. It is precisely the diagram of the fundamental representation ϖ_4 of the root system $\mathrm{A}_5 = \langle \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \rangle$. Here, the arrow refers to the root α_2.



Let α be the largest among the roots in Σ such that $a_\alpha \neq 0$. As always, multiplying v by a matrix $u \in U^-(\mathrm{A}_5, K)$ from the very start we may assume that all coordinates a_γ, apart maybe from the coordinates corresponding to the roots α, $\delta - \alpha$ and the roots orthogonal to α, are equal to zero.

The roots in Σ orthogonal to one of them form a set Ω of the shape

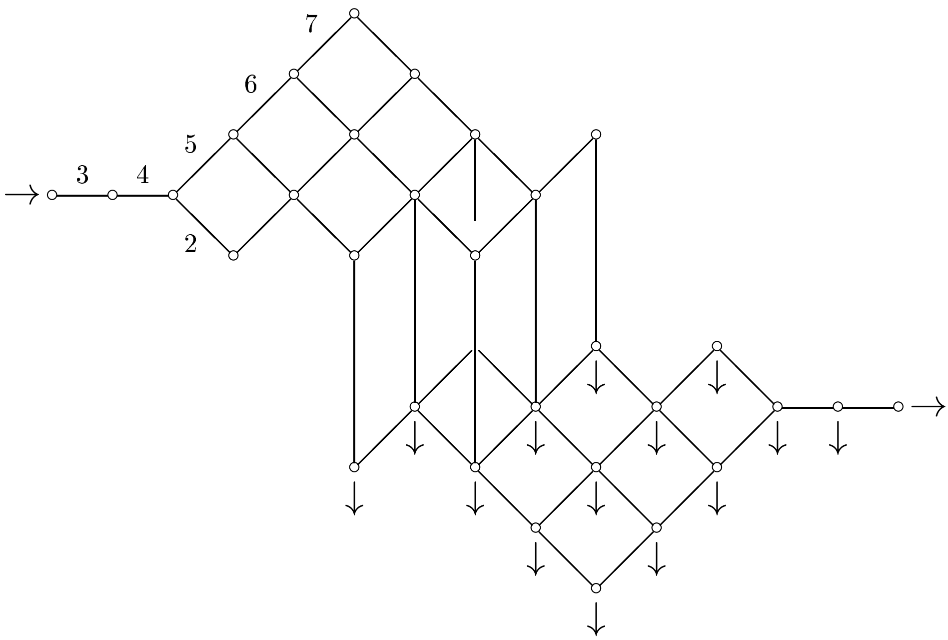


(the numbering in this particular case corresponds to the choice $\alpha = \delta - \alpha_4$). Now, recall that the orbits of the unitriangular group of type $2\mathrm{A}_2$ on the set Ω were already described in [17], in the process of the analysis of microweight elements of type $h_{\varpi_3}(\varepsilon)$ for the root system A_5, in the usual numbering.

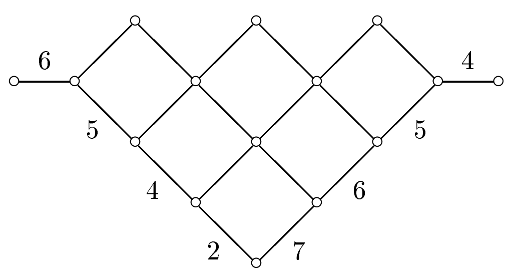
It turns out that the orbit of any such vector $v = \sum a_\gamma c_\gamma$, $\gamma \in \Omega$, under the action of $U^-(2\mathrm{A}_2, K)$ contains a vector with at most three nonzero coordinates corresponding to pairwise orthogonal roots $\beta_1, \beta_2, \beta_3 \in \Omega$.

Again, the roots δ, α, $\delta - \alpha$, β_1, β_2, β_3 generate a subsystem of type D_4, which concludes the analysis of this case.

• **The case of $\Phi = \mathrm{E}_7$.** For the group of type E_7, the set Σ has the diagram depicted below. Here, the arrow refers to the root α_1. Eliminating arrows, one arrives at the diagram of the half-spin representation of the group of type D_6.

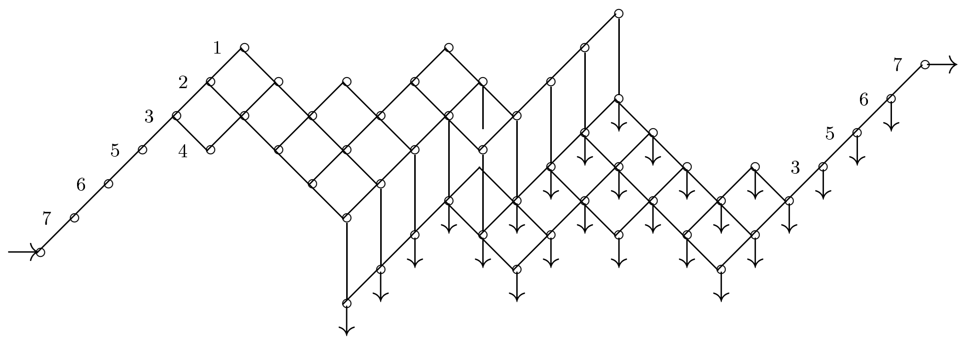


The roots of Σ orthogonal to one root $\alpha \in \Sigma$ form a set Ω, which has the same diagram as the representation with highest weight ϖ_6 of the group of type $A_5 = \langle \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \rangle$. We reproduce the diagram of this set, with the numbering corresponding to the choice $\alpha = \delta - \alpha_2$:

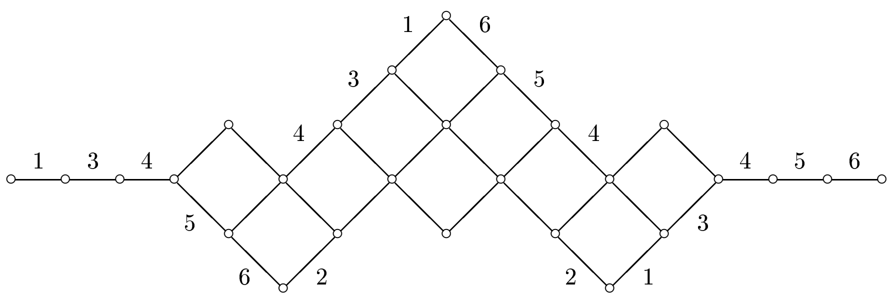


This case was alredy analysed in the proof of Proposition 3 in [17] and leads to at most three pairwise orthogonal roots β_1, β_2, β_3, each of them orthogonal to α.

• **The case of $\Phi = E_8$.** For the group of type E_8, the set Σ has the following diagram. This is the fanciest case, in the sense that not only the system Φ itself, but also the system Δ is exceptional. Here, the arrows refer to the root α_8, the leftmost horizontal arrow comes from the maximal root δ, the rightmost horizontal arrow leads from the root α_8 to the zero weight, and the 27 vertical arrows glue the unipotent radical of the parabolic subgroup P_8 with its Levi factor of type E_7. The vertical lines correspond to the root α_7 and glue together the representations (E_6, ϖ_6) and (E_6, ϖ_1), by a ten-dimensional representation of D_5.

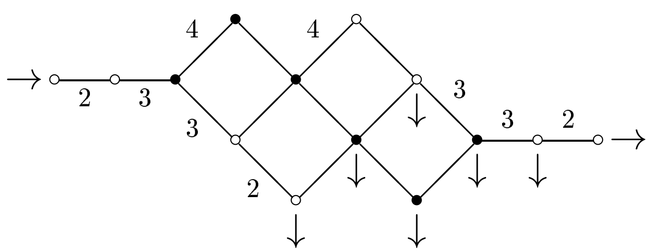


The roots $\gamma \in \Sigma$ orthogonal to one of them form a set Ω with the same structure as the representation of the group of type E_6 with the highest weight ϖ_1. The diagram of this set is reproduced below, in the numbering that corresponds to the choice $\alpha = \delta - \alpha_8$.

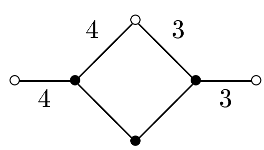


This case too was already analyzed in the proof of Proposition 3 of [17] and leads to at most three pairwise orthogonal roots β_1, β_2, β_3, each of them orthogonal to α.

• **The case of $\Phi = F_4$.** For the group of type F_4 the set Σ has the following diagram. Here, the arrows correspond to the root α_1.



If the largest root α for which the corresponding coordinate of the vector v is nonzero is long, we can kill all coordinates corresponding to the roots not orthogonal to α, apart maybe from the coordinate corresponding to $\delta - \alpha$. The roots orthogonal to α form a set Ω with the following diagram:



The numbering here corresponds to the case where $\alpha = \delta - \alpha_1$.

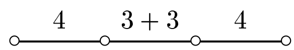
Now, if the coordinate corresponding to the senior root β of Ω is nonzero, then, multiplying the vector v by appropriate root elements

$$x_{\alpha_4}(\mp a_{\beta-\alpha_4} a_\beta^{-1}) \quad \text{and} \quad x_{\alpha_3+\alpha_4}(\mp a_{\beta-\alpha_3-\alpha_4} a_\beta^{-1}),$$

we can kill the coordinates at two out of the three short roots in Ω. If in the resulting vector the coordinate at the next long root γ is zero, then, multiplying by $x_{\alpha_3}(\mp a_{\gamma-\alpha_3} a_\gamma^{-1})$, in exactly the same way we can kill also the last remaining coordinate at a short root. This leaves us with at most four nonzero coordinates at pairwise orthogonal long roots, plus a nonzero coordinate at $\delta-\alpha$. Together with δ, these five roots generate a subsystem of type D_4.

On the other hand, if the coordinate a_β is zero or if we get zero coordinate a_γ after the first reduction step, then each time — provided that 2 is invertible! — apart from the nonzero coordinates at α and $\delta-\alpha$, we are left with two more nonzero coordinates, at one further long root, and at one short root. This means that in both cases we end up in B_3. This completes the analysis of the case where the senior nonzero coordinate corresponds to a long root.

It remains to consider the case where the senior nonzero coordinate corresponds to a short root. Since the difference of any two linearly independent short roots is a long root, multiplying v by root elements of the form $x_{\beta-\alpha}(\mp a_\beta a_\alpha^{-1})$, we can kill all other coordinates at short roots. The long roots orthogonal to α form a set with the diagram



which gives us two orthogonal long roots in the worst case. Thus, once again we encounter the same configuration of roots as in the analysis of the case B_l, which again leads to B_3.

• **The case of $\Phi = \mathrm{G}_2$.** The group of type G_2 is itself a twisting of D_4, so that there is nothing to prove in this case. □

§4. Proof of Theorems 1 and 2

Consider a long root torus Q_x. Let $x = uwvd$, where $u \in U$, $w \in W$, $v \in U_w^-$, $d \in T$, be the Bruhat decomposition of an element x. Let $v \in \prod x_\alpha(v_\alpha)$, where $v_\alpha \in K$, and the product is taken over all positive roots, in the following order: first all factors from $E(\Sigma \cup \{\delta\})$, in an arbitrary order, then all factors from $G(\Delta, K)$, also in an arbitrary order. Since $v \in U_w^-$, the parameter v_α can be distinct from 0 only for roots $\alpha \in \Phi^+ \cap w^{-1}\Phi^-$.

First, we observe that d commutes with $h_\delta(\varepsilon)$ and does not play any role in the subsequent calculation. From the outset, we may assume that $d = e$. Next, recall that $G(\Delta, K)$ commutes with $h_\delta(\varepsilon)$, whereas all $x_\gamma(*)$, $\gamma \in \Sigma$, commute pairwise, modulo X_δ. Thus, in view of our agreement on the order of roots in the expression of v, the commutator relations imply that

$$vh_\delta(\varepsilon)v^{-1} = \prod x_\gamma\big((1-\varepsilon)v_\gamma\big) x_\delta(*) h_\delta(\varepsilon),$$

where the product is taken over all $\gamma \in \Sigma$. The parameter of $x_\delta(*)$ depends quadratically on ε, and can be calculated explicitly, after specifying the order of the factors $x_\gamma(v_\gamma)$, $\gamma \in \Sigma$. However, we do not attempt to do this here, because the specific value of this parameter does not play any role in the proof of Theorem 1.

Now, set

$$z(\varepsilon) = wvh_\delta(\varepsilon)v^{-1}w^{-1} = \prod x_{w\gamma}\big(\pm(1-\varepsilon)v_\gamma\big) x_{w\delta}(*) h_{w\delta}(\varepsilon).$$

Expressing the element w in the form $w = w'w''$, where $w'' \in W(\Delta)$, whereas $w'(\Delta \cap \Pi) \subseteq \Phi^+$, we see that w'' stabilizes δ and permutes the roots from Σ. Thus, it does not change the shape of $vh_\delta(\varepsilon)v^{-1}$, it merely permutes the parameters of $x_\gamma(*)$ and, possibly, changes their signs. Thus, without loss of generality, we may assume that $w'' = e$, $w = w'$.

Further, since $uz(\varepsilon)u^{-1}$ lies in the same Bruhat cell as $z(\varepsilon)$, $u \in U(\Phi, K)$, and we are only interested in the shape of Q_x under conjugation by an element of $u \in U(\Phi, K)$, we can also discard the factor u right from the outset, thus assuming that

$$Q_x = \{z(\varepsilon), \varepsilon \in K^*\}.$$

The set of roots $w\Sigma$ has the same structure as the set Σ itself. In particular, for all $\Phi \neq \mathrm{A}_l$, in the set $w\Sigma$ there is a unique smallest root such that all other roots are obtained from it, by adding some of the roots $w(\alpha)$, where $\alpha \in \Delta \cap \Pi$. These roots may fail to be fundamental, but by the condition we imposed on the element $w = w'$, all of them are positive. We fix an isomorphism of $E(w\Sigma \cup \{w\delta\})/X_{w\delta}$ with V mapping $x_{w\gamma}(1)X_{w\delta}$ to v_γ.

This means that, applying the proposition in the preceding section to the image of the vector $z(\varepsilon)$ under this isomorphism, we can conclude that for *any* $\varepsilon \in K^*$ there exists a matrix $g \in U(\Phi, K)$ such that

$$gz(\varepsilon)g^{-1} = \prod x_{-\gamma}\big((\varepsilon - 1)a_\gamma\big)x_{w\delta}(*)h_{w\delta}(\varepsilon),$$

where the product is taken over some subset Θ of *positive* roots contained in $-w\Sigma$ and such that the roots in Θ together with $w\delta$ generate a subsystem which can be obtained from D_4.

However, now we need to prove that such a reduction can be implemented for all $\varepsilon \in K^*$ *simultaneously*. For this, we observe that, under the above isomorphism, the elements $z(\varepsilon)$, where $\varepsilon \neq 0, 1$, are mapped to proportional vectors. In the proof of the above proposition, all transformations were implemented via multiplication by root unipotents $x_{\beta-\alpha}(\mp a_\beta/a_\alpha)$, and in the exceptional case of $\Phi = \mathrm{B}_l$, also by the root unipotents $x_{\beta-\alpha}(\mp a_\beta/(2a_\alpha))$. Therefore, all of these transformations are precisely the same for any two proportional vectors.

In other words, killing nonzero coefficients at roots $\gamma \in w\Sigma$ by conjugation by an element of $U(\Delta, K)$ for *some* element $z(\varepsilon)$, $\varepsilon \neq 0, 1$, we simultaneously kill the coefficients at the same positions in all other such elements. The positions of the remaining nonzero coefficients do not depend on the choice of $\varepsilon \neq 0, 1$, and from the preceding section we know that together with $w\delta$ they generate a subsystem that can be obtained from D_4. This finishes the proof of Theorem 1.

In its turn, Theorem 2 follows immediately from Theorem 1, Lemma 1 $\approx$ Theorem 4 of [17], and from the classical description of involutions in the Weyl groups; see [31, 56].

Remark. It should be emphasized that, despite an apparent similarity, this calculation is radically different from the calculation in §14 of [17]. There, we calculated the Bruhat decomposition of microweight tori, and the space V was isomorphic to the unipotent radical itself. Here, the unipotent radical is non-Abelian, so that V is isomorphic to its abelianization, in this case the quotient modulo X_δ. Thus, taking into account the parameter of $z(\varepsilon)$ at the root δ, which depends on ε quadratically, and not linearly, as the parameters at all other roots, we see that the picture changes grossly. In particular, the Weyl factor occurring in the Bruhat decomposition of $z(\varepsilon)$ may, in general, depend on the choice of $\varepsilon \neq 0, 1$.

§5. The group of type D_4

Theorem 1 focuses our attention on the root tori in the Chevalley group of type D_4. In fact, all further calculations in the present paper can be performed in any group of type D_4, for instance in the special orthogonal group $SO(8, K)$. Here, we very briefly recall some of the most important notation and facts pertaining to this group.

As always, $G = GL(n, K)$ and $SL(n, K)$ denote the general linear group and the special linear group of degree n over K, respectively. As usual, for a matrix $x \in G$ we denote by x_{ij} its matrix entry at the position (i, j), so that $x = (x_{ij})$, $1 \leq i, j \leq n$. Next, $x^{-1} = (x'_{ij})$ denotes the inverse of x, while x^t denotes its transpose. As usual, e is the identity matrix of order n, and e_{ij} is the standard matrix unit, i.e., the matrix whose entry at the position (i, j) equals 1, while all other entries are zeros.

We are interested in the case where $n = 2l$. As in [55], we number the rows and the columns of matrices in G as follows: $1, \dots, l, -l, \dots, -1$. Denote by $f = f_n$ the per-identity matrix of degree n, whose entries at the skew diagonal are equal to 1, while all remaining entries are equal to 0. In other words, the entries of the matrix $f = (f_{ij})$ are described by the condition $f_{ij} = \delta_{i,-j}$.

The *split special orthogonal group* $SO(2l, R)$ consists of those matrices x in the special linear group $SL(2l, R)$ that preserve the bilinear form with the Gram matrix g, coinciding with $f_{2\lambda}$. In other words, $SO(2l, R)$ consists of the matrices $x \in G$ such that $\det x = 1$ and $x f_{2\lambda} x^t = f_{2l}$. This amounts to saying that $SO(2l, R)$ preserves the quadratic form $x_1 x_{-1} + \cdots + x_l x_{-l}$. Usually we invoke the condition that a matrix $x = (x_{ij})$ is orthogonal in the following handy form: $x'_{ij} = x_{-j,-i}$.

Viewed from the prospective of the theory of algebraic groups, the orthogonal group $SO(2l, K)$ is an [intermediate] Chevalley group of type D_l. Since we do not encounter other forms of the orthogonal group, in the sequel we routinely omit the modifier "split", and speak simply of the even orthogonal groups.

An advantage of the above numbering is that, with this numbering, the split maximal torus of $SO(2l, K)$ coincides with its subgroup of diagonal matrices, the standard Borel subgroup consists of all upper triangular matrices contained in $SO(2l, K)$, etc.

In the even orthogonal group $SO(2l, K)$, the elementary unipotent root elements $x_\alpha(\xi)$ are precisely the *elementary orthogonal transvections*, in other words, the matrices of the form

$$T_{ij}(\xi) = T_{-j,-i}(-\xi) = e + \xi e_{ij} - \xi e_{-j,-i},$$

where $1 \leq i, j \leq -1$, $i \neq \pm j$, $\xi \in K$. In accordance with that, the [elementary] semisimple root elements $h_\alpha(\varepsilon)$ in $SO(2l, K)$ have the form

$$D_{ij}(\varepsilon) = e + (\varepsilon - 1)e_{ii} + (\varepsilon^{-1} - 1)e_{jj} + (\varepsilon - 1)e_{-j,-j} + (\varepsilon^{-1} - 1)e_{-i,-i}$$

for $i \neq \pm j$ and an invertible $\varepsilon \in K^*$. For instance, the root element in $SO(8, K)$ corresponding to the maximal root equals

$$h_\delta(\varepsilon) = \mathrm{diag}(\varepsilon, \varepsilon, 1, 1, 1, 1, \varepsilon^{-1}, \varepsilon^{-1}),$$

this is precisely the semisimple element that will be used throughout the calculations for the rest of this paper.

Next, we describe the structure of the Weyl group $W(D_l)$. Recall that this group is an index 2 subgroup of the octahedral group Oct_l, isomorphic to the Weyl group $W(B_l) = W(C_l)$. The easiest way to define the octahedral group Oct_l is to describe it as the permutation group of the set $\{1, \dots, l, -l, \dots, -1\}$ that consists of all permutations $w \in S_{2l}$ such that $w(-i) = -w(i)$. The subgroup $W(D_l)$ consists of the permutations changing an even number of signs.

Thus, Oct_l is the permutation group of the signed orthonormal base $\pm e_1, \dots, \pm e_l$ of the Euclidean space $\mathbb{R}^l$, whereas $W(D_l)$ is its index 2 subgroup consisting of permutations that change an even number of signs. Take a base vector e_i and consider the smallest r such that $w^r(e_i) = \pm e_i$. One faces the following alternative: either $w^r(e_i) = e_i$ or $w^r(e_i) = -e_i$.

• In the first case we say that e_i belongs to an *even* signed cycle of type r. Obviously, in this case, together with the cycle $\sigma = \big(e_i, w(e_i), \dots, w^{r-1}(e_i)\big)$, the permutation w contains also the *opposite* cycle

$$\overline{\sigma} = \big(-e_i, -w(e_i), \dots, -w^{r-1}(e_i)\big),$$

because the cycles σ and $\overline{\sigma}$ are independent and their product $\sigma\overline{\sigma}$ is *even*, regardless of the parity of r.

• In the second case we say that e_i belongs to an *odd* signed cycle of type $-r$. Clearly, an odd cycle has the form

$$\sigma = \big(e_i, w(e_i), \dots, w^{r-1}(e_i), -e_i, -w(e_i), \dots, -w^{r-1}(e_i)\big),$$

so that its support is symmetric with respect to the change of sign, and the cycle σ itself is *odd*, regardless of the parity of r.

Each element $w \in \mathrm{Oct}_l \le S_{2l}$ can be expanded into a product of independent cycles. Moreover, cycles of odd type occur in pairs, each cycle necessarily appears together with its opposite. The cycles of odd type occur individually, but since their supports are symmetric with respect to the change of signs, the length of an odd type cycle is even. To a permutation w, one can ascribe its *signed cycle type*, which lists:

• the length of *one* of each pair of cycles of even type,

• *half* of the length of each cycle of odd type, with a "$-$" sign.

For instance, $(1, -1)$ is a cycle of type -1, whereas $(1, 2)(-1, -2)$ is the product of two opposite cycles of type 2, and thus contributes 2 to the cycle type. At the same time, $(1, 2, -1, -2)$ has cycle type -2.

Clearly, if $\nu = (n_1, \dots, n_s)$ is the cycle type of an element of Oct_l, then $|n_1| + \dots + |n_2| = l$. In this notation, $W(\mathrm{D}_l)$ consists precisely of the permutations whose cycle decomposition involves an *even* number of cycles of odd type.

The signed cycle type of a permutation w *almost* uniquely prescribes its conjugacy class. The only exception to that rule can be conveyed as follows: permutations of cycle type $(n_1, \dots, n_s)$, where all $n_i > 0$ are *even* and $n_1 + \dots + n_s = l$, fall into *two* conjugacy classes in $W(\mathrm{D}_l)$. Obviously, these two classes are merged in $W(\mathrm{B}_l) = \mathrm{Oct}_l$.

Apart from the signed cycle type, the conjugacy classes of the Weyl group are sometimes alternatively described by their Carter graphs. For a Coxeter element of a subsystem $\Delta \subseteq \Phi$ — no other conjugacy classes occur in our main results — its Carter graph coincides with the Dynkin diagram of the root system Δ. These are precisely Carter graphs without cycles, compare [56] for a general definition. Oftentimes, Carter graphs are more suggestive than cycle types. For convenience, in the following tables we indicate both the cycle type and the Carter graph of all occurring elements.

Apart from that, to shorten the notation in the tables we usually render $-r$ simply as $\bar{r}$. In Table 2 we indicate the signed cycle type ν of a permutation $w \in W(\mathrm{D}_4)$, the Carter graph C of its conjugacy class, the order m of the permutation w, and the number n of its conjugates.

§6. Proof of Theorem 3

Theorem 1 implies that the number of degenerations in a Chevalley group of any type does not exceed that in the Chevalley group of type D_4. Since the number of degenerations does not depend on the choice of a group in a given isogeny class either, in the sequel we can limit ourselves to the Bruhat decomposition of long root tori in the [split] special orthogonal group $\mathrm{SO}(8, K)$.

Consider an element $xh_\delta(\varepsilon)x^{-1}$ of a long root torus Q_x. Let $x = uwvd$, where $u \in U$, $w \in W$, $v \in U_w$, $d \in T$, be the right reduced Bruhat decomposition of x.

TABLE 2. Conjugacy classes of the group $W(\mathrm{D}_4)$.

ν	w	C	m	n
1111	e	$\varnothing$	1	1
$11-1-1$	$(1,-1)(2,-2)$	D_2	2	6
$-1-1-1-1$	$(1,-1)(2,-2)(3,-3)(4,-4)$	$2\mathrm{D}_2$	2	1
211	$(12)(-2,-1)$	A_1	2	12
$2-1-1$	$(12)(-2,-1)(3,-3)(4,-4)$	$\mathrm{A}_1+\mathrm{D}_2$	2	12
22	$(12)(-2,-1)(3,4)(-4,-3)$	$(2\mathrm{A}_1)'$	2	6
22	$(12)(-2,-1)(3,-4)(4,-3)$	$(2\mathrm{A}_1)''$	2	6
31	$(123)(-3,-1,-2)$	A_2	3	32
$-21-1$	$(1,-2,-1,2)(3,-3)$	D_3	4	24
4	$(1234)(-3,-4,-1,-2)$	A_3'	4	24
4	$(1,2,3,-4)(-3,4,-1,-2)$	A_3''	4	24
$-2-2$	$(1,-2,-1,2)(3,-4,-3,4)$	$\mathrm{D}_4(a_1)$	4	12
$-3-1$	$(1,2,-3,-1,-2,3)(4,-4)$	D_4	6	32

Then $xh_\delta(\varepsilon)x^{-1}$ lies in the same Bruhat cell as $wvh_\delta(\varepsilon)v^{-1}w^{-1}$. Furthermore, since $h_\delta^{-1}(\varepsilon)w^{-1} = w^{-1}h_{w(\delta)}(\varepsilon)^{-1}$, the element

$$z(\varepsilon) = w[v, h_\delta(\varepsilon)]w^{-1}$$

lies in the same cell. We calculate this element.

To this end, we recall that the elements $v^{-1} = (v'_{ij})$ are related to the elements $v = (v_{ij})$ by the formula $v'_{ij} = -v_{-j,-i}$. Since v is orthogonal, we have $v_{ij} = -v_{-j,-i}$, for $i \neq j$, $(i,j) \neq (1,-2),(2,-1)$. Thus,

$$\begin{aligned} v(\varepsilon)_{ij} &= -v(\varepsilon)_{-j,-i} = -(\varepsilon-1)v_{ij}, \qquad i=1,2, \quad j=\pm3,\pm4,\\ v(\varepsilon)_{1,-2} &= (\varepsilon-1)(\varepsilon v_{2,-1} - v_{1,-2}),\\ v(\varepsilon)_{2,-1} &= (\varepsilon-1)(\varepsilon v_{1,-2} - v_{2,-1}),\\ v(\varepsilon)_{1,-1} &= (\varepsilon-1)^2 v_{1,-1},\\ v(\varepsilon)_{2,-2} &= (\varepsilon-1)^2 v_{2,-2}, \end{aligned}$$

and $v(\varepsilon)_{ij} = \delta_{ij}$ in all other cases.

We emphasize that most entries of the matrix $v(\varepsilon)$ coincide with the corresponding entries of the identity matrix, or otherwise are linear in ε and thus have a unique root, $\varepsilon = 1$. At the same time, the following four entries are quadratic in ε.

• The entries $v(\varepsilon)_{1,-2}$ and $v(\varepsilon)_{1,-2}$. Apart from the root $\varepsilon = 1$, they may have one more root, $\theta = v_{1,-2}/v_{2,-1}$ and θ^{-1}, respectively. In what follows, these entries will be called *ambiguous*.

• The entries $v(\varepsilon)_{1,-1}$ and $v(\varepsilon)_{2,-2}$ have a multiple root $\varepsilon = 1$.

We introduce another notation, which will be used in the proof of Theorem 3. Namely, recall the algorithm to calculate the Bruhat decomposition of an element z. This algorithm is described, for instance, in §13 of [17] and is stated in terms of vanishing or nonvanishing of the minors of the matrix z, starting with the South-West corner.

Denote the minor of a matrix z at the intersection of the rows with indices $i_1, \dots, i_s$ and columns with indices $j_1, \dots, j_s$ by $M_{i_1 \dots i_s}^{j_1 \dots j_s}(z)$. Recall that throughout this and the next sections we deal exclusively with matrices in $\mathrm{SO}(8, K)$, whose columns and rows are indexed as follows: $1, 2, 3, 4, -4, -3, -2, -1$.

Now, we define a sequence of the horizontal pivotal minors of a matrix z as follows.

• $\Delta^1(z) = M_{-1}^{j_1}(z)$ if $M_{-1}^{j_1}(z) = z_{-1,j_1} \neq 0$, whereas $M_{-1}^{j}(z) = 0$ for all j smaller than j_1.

• $\Delta^2(z) = M_{-1,-2}^{j_1 j_2}(z)$ if $M_{-1,-2}^{j_1 j_2}(z) \neq 0$, whereas $M_{-1,-2}^{j_1 j}(z) = 0$ for all j smaller than j_2.

• $\Delta^3(z) = M_{-1,-2,-3}^{j_1 j_2 j_3}(z)$ if $M_{-1,-2,-3}^{j_1 j_2 j_3}(z) \neq 0$, whereas $M_{-1,-2,-3}^{j_1 j_2 j}(z) = 0$ for all j smaller than j_3.

• $\Delta^4(z) = M_{-1,-2,-3,-4}^{j_1,j_2,j_3,j_4}(z)$ if $M_{-1,-2,-3,-4}^{j_1,j_2,j_3,j_4}(z) \neq 0$, whereas $M_{-1,-2,-3,-4}^{j_1,j_2,j_3,j}(z) = 0$ for all j smaller than j_4.

Observe that, by the orthogonality of the matrix z, among j_1, j_2, j_3, j_4 there are no opposite indices.

The sequence of vertical pivotal minors

$$\Delta_1(z) = M_{i_1}^{1}(z), \quad \Delta_2(z) = M_{i_1 i_2}^{12}(z), \quad \Delta_3(z) = M_{i_1 i_2 i_3}^{123}(z), \quad \Delta_4(z) = M_{i_1 i_2 i_3 i_4}^{1234}(z)$$

is defined similarly.

Now we are all set to start the proof of Theorem 3. First, observe that we have no need to consider the cases where the images of ambiguous elements under the action of w lie above the principal diagonal. There are two easy arguments to this effect.

• On the one hand, in these cases the pivotal minors do not depend on the ambiguous elements at all, so that

$$\Delta^s(z(\varepsilon))/(\varepsilon - 1)^s = \Delta^s(z(\eta))/(\eta - 1)^s.$$

In particular, the pivotal minors are nonzero for all $\varepsilon \neq 1$.

• On the other hand, in these cases $z(\varepsilon)$ is contained in a regularly embedded subgroup of type A_3 or D_3, so that Theorem 3 was already established in [9].

Now, we state another obvious general fact that dramatically simplifies our analysis of degenerations.

Lemma 2. *The pivotal minors*

$$M_{-1,-2,-3,-4}^{j_1 j_2 j_3 j_4}(z(\varepsilon)), \qquad M_{i_1 i_2 i_3 i_4}^{1234}(z(\varepsilon)),$$

do not have any zeros distinct from the zeros of the pivotal minors of smaller orders.

Proof. Let, for instance, η be a root of the pivotal minor $M_{-1,-2,-3,-4}^{j_1 j_2 j_3 j_4}(z(\varepsilon))$ that is not a root of the pivotal minors $M_{-1}^{j_1}(z(\varepsilon))$, $M_{-1,-2}^{j_1 j_2}(z(\varepsilon))$, $M_{-1,-2,-3}^{j_1 j_2 j_3}(z(\varepsilon))$. Let w_0 be the element of the Weyl group occurring in the Bruhat decomposition of the element $z(\varepsilon)$ in general position. Then, since $M_{-1,-2,-3,-4}^{j_1 j_2 j_3 j_4}(z(\varepsilon))$ and $M_{-1,-2,-3,-4}^{j_1 j_2 j_3,-j_4}(z(\varepsilon))$ cannot vanish simultaneously, our torus would contain the element

$$z(\eta) \in B w_0 w_{j_4,-j_4} B.$$

But this is clearly impossible, because that element has different parity as compared to w_0, and thus cannot belong to the Weyl group $W(\mathrm{D}_4)$. ☐

Now, we are in a position to start the case by case analysis of all situations when the images of the ambiguous entries under the action of w fall below the principal diagonal. In the discussion of these cases $*$ denotes the entries of the matrix $z(\varepsilon)$ that depend on ε linearly, while $\bullet$ denotes the entries that depend on ε quadratically, and thus can beget new roots in the pivotal minors involving these entries. Not to encumber the resulting matrices, we replace zero entries by dots.

• We start with the two junior patterns cited as pattern A and pattern B in the proofs of Theorems 4 and 5:

$$z(\varepsilon)=\begin{pmatrix} 1 & . & * & . & * & . & . & . \\ . & 1 & * & . & * & . & . & . \\ . & . & 1 & . & . & . & . & . \\ * & * & \bullet & 1 & \bullet & . & * & * \\ . & . & . & . & 1 & . & . & . \\ * & * & \bullet & . & \bullet & 1 & * & * \\ . & . & * & . & * & . & 1 & . \\ . & . & * & . & * & . & . & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & . & * & * & . & . & . & . \\ . & 1 & * & * & . & . & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . \\ * & * & \bullet & \bullet & 1 & . & * & * \\ * & * & \bullet & \bullet & . & 1 & * & * \\ . & . & * & * & . & . & 1 & . \\ . & . & * & * & . & . & . & 1 \end{pmatrix}.$$

As a model, we give a minute analysis of the first pattern, the second can be treated in a completely similar fashion. In the sequel, we usually do not list *all* occurring subpatterns, and only indicate those that actually can lead to degeneration.

The first of the above patterns can lead to the following subpatterns.

∘ If $\Delta^1(z(\varepsilon))=M^{3}_{-1}(z(\varepsilon))$, $\Delta^2(z(\varepsilon))=M^{3,-4}_{-1,-2}(z(\varepsilon))$, then the third pivotal minor equals $\Delta^3(z(\varepsilon))=M^{3,-4,1}_{-1,-2,-3}(z(\varepsilon))$.

∘ If $\Delta^1(z(\varepsilon))=M^{-4}_{-1}(z(\varepsilon))$, $\Delta^2(z(\varepsilon))=M^{-4,3}_{-1,-2}(z(\varepsilon))$, then the third pivotal minor equals $\Delta^3(z(\varepsilon))=M^{-4,3,2}_{-1,-2,-3}(z(\varepsilon))$.

∘ If $\Delta^1(z(\varepsilon))=M^{-1}_{-1}(z(\varepsilon))$, $\Delta^2(z(\varepsilon))=M^{-1,3}_{-1,-2}(z(\varepsilon))$, then the third pivotal minor equals $\Delta^3(z(\varepsilon))=M^{-1,3,2}_{-1,-2,-3}(z(\varepsilon))$.

∘ If $\Delta^1(z(\varepsilon))=M^{-1}_{-1}(z(\varepsilon))$, $\Delta^2(z(\varepsilon))=M^{-1,-4}_{-1,-2}(z(\varepsilon))$, then the third pivotal minor equals $\Delta^3(z(\varepsilon))=M^{-1,-4,3}_{-1,-2,-3}(z(\varepsilon))$ or $\Delta^3(z(\varepsilon))=M^{-1,-4,-3}_{-1,-2,-3}(z(\varepsilon))$.

In all these subpatterns, the resulting pivotal minors of the third order have no nontrivial roots, so that no degeneration can occur. Thus, we are left with the following subpattern.

∘ If $\Delta^1(z(\varepsilon))=M^{-1}_{-1}(z(\varepsilon))$, $\Delta^2(z(\varepsilon))=M^{-1,-2}_{-1,-2}(z(\varepsilon))$, then the third pivotal minor equals $\Delta^3(z(\varepsilon))=M^{-1,-2,-3}_{-1,-2,-3}(z(\varepsilon))$ or $\Delta^3(z(\varepsilon))=M^{-1,-2,-4}_{-1,-2,-3}(z(\varepsilon))$.

The second of the above minors can have nontrivial roots. Similarly, the vertical pivotal minor $\Delta_3(z(\varepsilon))=M^{123}_{-1,-2,-4}(z(\varepsilon))$ can have nontrivial roots. Since no further degenerations can occur, these two roots must be mutually inverse.

Also for the second pattern, among all occurring possibilities, only the pivotal minors $\Delta^3(z(\varepsilon))=M^{-1,-2,-3}_{-1,-2,-3}(z(\varepsilon))$ and $\Delta_3(z(\varepsilon))=M^{123}_{-1,-2,-3}(z(\varepsilon))$ can possibly have nontrivial roots, which again leads to at most two degenerations.

• The next three patterns, following by height, will be cited as C, D, and E in the sequel. In these cases, the matrix $z(\varepsilon)$ looks like this:

$$\begin{pmatrix} 1 & \ast & . & . & . & \ast & . & . \\ . & 1 & . & . & . & . & . & . \\ \ast & \bullet & 1 & \ast & \ast & \bullet & . & \ast \\ . & \ast & . & 1 & . & \ast & . & . \\ . & \ast & . & . & 1 & \ast & . & . \\ . & . & . & . & . & 1 & . & . \\ \ast & \bullet & . & \ast & \ast & \bullet & 1 & \ast \\ . & \ast & . & . & . & \ast & . & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \ast & . & . & \ast & . & . & . \\ . & 1 & . & . & . & . & . & . \\ . & \ast & 1 & . & \ast & . & . & . \\ \ast & \bullet & \ast & 1 & \bullet & \ast & . & \ast \\ . & . & . & . & 1 & . & . & . \\ . & \ast & . & . & \ast & 1 & . & . \\ \ast & \bullet & \ast & . & \bullet & . & 1 & \ast \\ . & \ast & . & . & \ast & . & . & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \ast & . & \ast & . & . & . & . \\ . & 1 & . & . & . & . & . & . \\ . & \ast & 1 & \ast & . & . & . & . \\ . & . & . & 1 & . & . & . & . \\ \ast & \bullet & \ast & \bullet & 1 & \ast & . & \ast \\ . & \ast & . & \ast & . & 1 & . & . \\ \ast & \bullet & \ast & \bullet & . & \ast & 1 & \ast \\ . & \ast & . & \ast & . & . & . & 1 \end{pmatrix}.$$

In any of these patterns, no nontrivial degenerations can occur. As a sample, we verify this for the first of them.

∘ If $\Delta^1(z(\varepsilon)) = M^{2}_{-1}(z(\varepsilon))$, then $\Delta^2(z(\varepsilon)) = M^{21}_{-1,-2}(z(\varepsilon))$, so that none of the third order pivotal minors $\Delta^3(z(\varepsilon)) = M^{21j}_{-1,-2,-3}(z(\varepsilon))$ has nontrivial roots.

∘ On the other hand, if $\Delta^1(z(\varepsilon)) = M^{-3}_{-1}(z(\varepsilon))$, then one of the following possibilities arises:

$$\Delta^2(z(\varepsilon)) = M^{-3,4}_{-1,-2}(z(\varepsilon)), M^{-3,-4}_{-1,-2}(z(\varepsilon)), M^{-3,-2}_{-1,-2}(z(\varepsilon)).$$

None of these minors, as also none of the third order pivotal minors involving them, can have nontrivial roots.

For the second pattern, if $\Delta^1(z(\varepsilon)) = M^{2}_{-1}(z(\varepsilon))$, then necessarily

$$\Delta^2(z(\varepsilon)) = M^{21}_{-1,-2}(z(\varepsilon)),$$

but

$$\Delta^3(z(\varepsilon)) = M^{21,-4}_{-1,-2,-3}(z(\varepsilon)), \qquad \Delta^3(z(\varepsilon)) = M^{21,-4}_{-1,-2,-3}(z(\varepsilon))$$

do not have nontrivial roots. The third pattern is completely similar.

• In Pattern F, the matrix $z(\varepsilon)$ looks like this:

$$\begin{pmatrix} 1 & \ast & \ast & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . \\ . & \ast & \ast & 1 & . & . & . & . \\ . & \ast & \ast & . & 1 & . & . & . \\ \ast & \bullet & \bullet & \ast & \ast & 1 & . & \ast \\ \ast & \bullet & \bullet & \ast & \ast & . & 1 & \ast \\ . & \ast & \ast & . & . & . & . & 1 \end{pmatrix},$$

and no degeneration can occur either. This can be verified as follows. If $\Delta^1(z(\varepsilon)) = M^{2}_{-1}(z(\varepsilon))$, then $\Delta^2(z(\varepsilon)) = M^{21}_{-1,-2}(z(\varepsilon))$. Clearly, there are no nontrivial roots in either of the following subpatterns:

$$\Delta^3(z(\varepsilon)) = M^{214}_{-1,-2,-3}(z(\varepsilon)), M^{21,-4}_{-1,-2,-3}(z(\varepsilon)), M^{21,-3}_{-1,-2,-3}(z(\varepsilon)).$$

It only remains to consider the subpattern where

$$\Delta^3(z(\varepsilon)) = M^{213}_{-1,-2,-3}(z(\varepsilon)).$$

An easy calculation shows that the suspicious elements appear in this pivotal minor with opposite coefficients, proportional to $(\varepsilon-1)^3$. Thus, once again it is of the form $(\varepsilon-1)^4\lambda$ for some $\lambda \in K^*$.

• In Pattern G, the matrix $z(\varepsilon)$ looks like this:

$$\begin{pmatrix}
1 & . & . & . & . & . & . & . \\
\bullet & 1 & \ast & \ast & \ast & \ast & \bullet & . \\
\ast & . & 1 & . & . & . & \ast & . \\
\ast & . & . & 1 & . & . & \ast & . \\
\ast & . & . & . & 1 & . & \ast & . \\
\ast & . & . & . & . & 1 & \ast & . \\
. & . & . & . & . & . & 1 & . \\
\bullet & . & \ast & \ast & \ast & \ast & \bullet & 1
\end{pmatrix}.$$

Here, all degenerations are already visible at the level of vanishing of individual entries. Indeed, if $\Delta^1 = M^k_{-1}(z(\varepsilon))$, where k precedes -2, then both pivotal minors $\Delta^2 = M^{k,-2}_{-1,-2}(z(\varepsilon))$ and

$$\Delta^3 = M^{k,-2,1}_{-1,-2,-3}(z(\varepsilon)), M^{k,-2,-3}_{-1,-2,-3}(z(\varepsilon))$$

do not have nontrivial roots. On the other hand, the minor $\Delta^1 = M^{-2}_{-1}(z(\varepsilon))$ can have one nontrivial root, which is then inverse to the nontrivial root of $\Delta_1 = M^1_2(z(\varepsilon))$. In this last case, neither $\Delta^2 = M^{-2,-1}_{-1,-2}(z(\varepsilon))$, nor $\Delta^3 = M^{-2,-1,-3}_{-1,-2,-3}(z(\varepsilon))$ can have nontrivial roots.

• The following three patterns, in the sequel denoted by H, I, J (respectively),

$$\begin{pmatrix}
1 & . & . & . & . & . & . & . \\
\ast & 1 & . & . & . & \ast & . & . \\
\bullet & \ast & 1 & \ast & \ast & \bullet & \ast & . \\
\ast & . & . & 1 & . & \ast & . & . \\
\ast & . & . & . & 1 & \ast & . & . \\
. & . & . & . & . & 1 & . & . \\
\ast & . & . & . & . & \ast & 1 & . \\
\bullet & \ast & . & \ast & \ast & \bullet & \ast & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & . & . & . & . & . & . & . \\
\ast & 1 & . & . & \ast & . & . & . \\
\ast & . & 1 & . & \ast & . & . & . \\
\bullet & \ast & \ast & 1 & \bullet & \ast & \ast & . \\
. & . & . & . & 1 & . & . & . \\
\ast & . & . & . & \ast & 1 & . & . \\
\ast & . & . & . & \ast & . & 1 & . \\
\bullet & \ast & \ast & . & \bullet & \ast & \ast & 1
\end{pmatrix},$$

$$\begin{pmatrix}
1 & . & . & . & . & . & . & \\
\ast & 1 & . & \ast & . & . & . & . \\
\ast & . & 1 & \ast & . & . & . & . \\
. & . & . & 1 & . & . & . & . \\
\bullet & \ast & \ast & \bullet & 1 & \ast & \ast & . \\
\ast & . & . & \ast & . & 1 & . & . \\
\ast & . & . & \ast & . & . & 1 & . \\
\bullet & \ast & \ast & \bullet & . & \ast & \ast & 1
\end{pmatrix},$$

are already somewhat harder.

Let us illustrate the new phenomenon, encountered for these patterns, for the first of them. The minor $\Delta^1 = M^{-3}_{-1}(z(\varepsilon))$ can have nontrivial roots. Clearly, the pivotal minors $\Delta^2 = M^{-3,-1}_{-1,-2}(z(\varepsilon))$ and $\Delta^3 = M^{-3,-1,-2}_{-1,-2,-3}(z(\varepsilon))$ have no nontrivial roots. There is another possibility though, which looks problematic. Namely, at first glance, it seems that the pivotal minor $\Delta^2 = M^{-3,-2}_{-1,-2}(z(\varepsilon))$ can have a nontrivial root η distinct from the root of the pivotal minor $\Delta^1 = M^{-3}_{-1}(z(\varepsilon))$. However, an easy computation involving the orthogonality of the matrix $z(\varepsilon)$ shows that this root must coincide with the nontrivial root of $\Delta_1 = M^1_{-3}(z(\varepsilon))$, so that we get at most two degenerations, once again.

The analysis of the two other patterns is completely similar.

It only remains to consider the two senior cases, which are by far the hardest.

• In the subdominant pattern K we have

$$z(\varepsilon) = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ * & 1 & * & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . \\ * & . & * & 1 & . & . & . & . \\ * & . & * & . & 1 & . & . & . \\ \bullet & * & \bullet & * & * & 1 & * & . \\ * & . & * & . & . & . & 1 & . \\ \bullet & * & \bullet & * & * & . & * & 1 \end{pmatrix}.$$

We list the resulting possibilities.

◦ If $\Delta^1 = M^1_{-1}(z(\varepsilon))$, and $\Delta^2 = M^{12}_{-1,-2}(z(\varepsilon))$, $\Delta^3 = M^{123}_{-1,-2,-3}(z(\varepsilon))$, then it is obvious that the first two of these minors have no nontrivial roots, the third does not have them either, because ambiguous entries appear with opposite coefficients.

◦ If $\Delta^1 = M^1_{-1}(z(\varepsilon))$, $\Delta^2 = M^{12}_{-1,-2}(z(\varepsilon))$, and $\Delta^3 = M^{124}_{-1,-2,-3}(z(\varepsilon))$ or $\Delta^3 = M^{12,-4}_{-1,-2,-3}(z(\varepsilon))$, then the third order pivotal minor has at most one nontrivial root, which is then inverse of the nontrivial root of the vertical pivotal minor $\Delta_3 = M^{123}_{12?}(z(\varepsilon))$ or $\Delta_3 = M^{123}_{12?}(z(\varepsilon))$, respectively. Thus, here again at most two degenerations occur.

◦ If $\Delta^1 = M^1_{-1}(z(\varepsilon))$, $\Delta^2 = M^{12}_{-1,-2}(z(\varepsilon))$, and $\Delta^3 = M^{12,-3}_{-1,-2,-3}(z(\varepsilon))$, then there are no nontrivial degenerations.

◦ If $\Delta^1 = M^1_{-1}(z(\varepsilon))$, $\Delta^2 = M^{13}_{-1,-2}(z(\varepsilon))$, then $\Delta^3 = M^{142}_{-1,-2,-3}(z(\varepsilon))$ and again there are no nontrivial degenerations.

◦ If $\Delta^1 = M^2_{-1}(z(\varepsilon))$, then $\Delta^2 = M^{21}_{-1,-2}(z(\varepsilon))$. Clearly,

$$\Delta^3 = M^{213}_{-1,-2,-3}(z(\varepsilon)), M^{21,-3}_{-1,-2,-3}(z(\varepsilon))$$

does not have nontrivial roots. On the other hand, in the subpatterns

$$\Delta^3 = M^{214}_{-1,-2,-3}(z(\varepsilon)), M^{21,-4}_{-1,-2,-3}(z(\varepsilon)),$$

there are, as usual, at most two degenerations.

◦ Now, let $\Delta^1 = M^3_{-1}(z(\varepsilon))$, where $\Delta^1(\eta) = 0$. Clearly, then $\Delta_3(\eta^{-1}) = 0$. It is easily seen that neither $\Delta^2 = M^{34}_{-1,-2}(z(\varepsilon))$ nor $\Delta^2 = M^{3,-4}_{-1,-2}(z(\varepsilon))$ have nontrivial roots. On the other hand, $\Delta^3 = M^{341}_{-1,-2,-3}(z(\varepsilon))$, or, respectively, $\Delta^3 = M^{3,-4,1}_{-1,-2,-3}(z(\varepsilon))$ can have a nontrivial root. However, again a straightforward calculation based on the orthogonality of the matrix $z(\varepsilon)$ shows that this root equals η^{-1}. This means that also for this subpattern at most two nontrivial degenerations may occur.

◦ Finally, let $\Delta^1 = M^3_{-1}(z(\varepsilon))$ and $\Delta^2 = M^{3,-2}_{-1,-2}(z(\varepsilon))$, where $\Delta^1(\eta) = 0$ and $\Delta^2(\zeta) = 0$. It is easy to check that in this case $\Delta^3 = M^{3,-2,1}_{-1,-2,-3}(z(\varepsilon))$ does not have new roots. Next, the nonzero minor of the matrix $z(\zeta)$ in the -1st and the -2nd rows is $M^{3,-1}_{-1,-2}(z(\zeta))$. Thus, $M^1_3(\zeta) = 0$, so that $\zeta = \eta^{-1}$, and again at most two nontrivial degenerations may occur. This concludes the analysis of the subdominant pattern.

• It remains to consider the general position pattern L, where

$$z(\varepsilon)=\begin{pmatrix}1&.&.&.&.&.&.&.\\ .&1&.&.&.&.&.&.\\ *&*&1&.&.&.&.&.\\ *&*&.&1&.&.&.&.\\ *&*&.&.&1&.&.&.\\ *&*&.&.&.&1&.&.\\ \bullet&\bullet&*&*&*&*&1&.\\ \bullet&\bullet&*&*&*&*&.&1\end{pmatrix}.$$

Actually, this pattern, as the hardest one, was analyzed completely in [27], where it was claimed that all other patterns are considered similarly, and are easier. For completeness sake, we reproduce the arguments from [27].

∘ First, let $\Delta^1=M^1_{-1}(z(\varepsilon))$. We verify that $\Delta^2=M^{12}_{-1,-2}(z(\varepsilon))$ has at most two nontrivial roots. Indeed,

$$M^{12}_{-1,-2}(\varepsilon)=\begin{vmatrix}(\varepsilon-1)(\varepsilon v_{2,-1}-v_{1,-2}) & (\varepsilon-1)^2v_{2,-2}\\ (\varepsilon-1)^2v_{1,-1} & (\varepsilon-1)(\varepsilon v_{1,-2}-v_{2,-1})\end{vmatrix}.$$

A straightforward calculation shows that $\Delta^{12}(\varepsilon)=(\varepsilon-1)^2(a\varepsilon^2+b\varepsilon+a)$, so that the nontrivial roots of this minor are mutually inverse.

Next, if $\Delta^3=M^{123}_{-1,-2,-3}(z(\varepsilon))$, this minor has no new roots, and there are no further degenerations.

On the other hand, if $\Delta^3=M^{124}_{-1,-2,-3}(z(\varepsilon))$, then at least one of the entries $z(\varepsilon)_{-3,1}$ or $z(\varepsilon)_{-3,2}$ is distinct from 0. Indeed, if $z(\varepsilon)_{-3,1}=0$, then $M^{123}_{-1,-2,-3}(z(\varepsilon))\neq 0$, so that we may assume that $z(\varepsilon)_{-3,1}\neq 0$. In this last case we can argue as follows. Conjugating $z(\varepsilon)$ by an elementary orthogonal transvection $T_{12}(z(\varepsilon)_{-3,2}/z(\varepsilon)_{-3,1})$, which does not affect the above minors, in the South-West corner we get

$$\begin{pmatrix}*&.&.&.\\ \bullet&.&.&*\\ \bullet&\bullet&*&*\end{pmatrix}.$$

Observe that the presence of zero at the position $(-2,2)$ follows from the fact that $M^{123}_{-1,-2,-3}=0$. Thus, the pivotal minor $M^{124}_{-1,-2,-3}(z(\varepsilon))$ does not have any roots apart from the roots of $\Delta^2=M^{12}_{-1,-2}(z(\varepsilon))$.

∘ The subpattern $\Delta^1=M^1_{-1}(z(\varepsilon))$, $\Delta^2=M^{12}_{-1,-2}(z(\varepsilon))$, $\Delta^3=M^{12,-4}_{-1,-2,-3}(z(\varepsilon))$ can be considered in a completely similar fashion.

∘ Now, let $\Delta^1=M^1_{-1}(z(\varepsilon))$, $\Delta^2=M^{12}_{-1,-2}(z(\varepsilon))$, and $\Delta^3=M^{12,-3}_{-1,-2,-3}(z(\varepsilon))$. If $\Delta^3(\zeta)=0$, but $\Delta^2(\zeta)\neq 0$, then by orthogonality none of the minors $M^{12k}_{-1,-2,-3}(z(\zeta))$, $k=3,4,-4,-3$, may be nonzero, which contradicts the invertibility of $z(\zeta)$.

∘ Finally, let $\Delta^1=M^2_{-1}(z(\varepsilon))$ have a nontrivial root η. Then $M^1_2(z(\eta^{-1}))=0$. Exactly the same argument as used in the preceding subpattern shows that here too no nontrivial degenerations apart from $\varepsilon=\eta,\eta^{-1}$ may occur.

This concludes the analysis of the senior pattern, and thus also the proof Theorem 3.

§7. A complete description of degenerations in D_4

Theorem 3 addresses the following problem: *how many* Bruhat cells BwB does a given long root torus intersect? Now, we are going to explore precisely *what* cells it can intersect.

For this, we recall that all nontrivial elements $y(\varepsilon)=xh_\delta(\varepsilon)x^{-1}$, apart from at most two of them, lie in the same typical cell Bw_0B, $w_0^2=e$. More precisely, all $y(\varepsilon)$ for which

ε is not a root of any of the pivotal minors are generic, in this sense. It only remains to elucidate what precisely happens when $\varepsilon = \theta, \theta^{-1}$, where θ and θ^{-1} are the roots of some pivotal minors. In other words, we should reconsider the above proof of Theorem 3 and trace which other minors vanish automatically at the roots of the pivotal minors, in the respective cases. This will give us a complete picture of degenerations, and serve as a proof of Theorems 4 and 5.

For the first time, this was done on pages 44–64 of the Thesis of the second author [42], using the tables on pages 73–115. The results are summarized in the next section, in the form of Tables 3 and 4, listing all simple and multiple degenerations, respectively, in the Chevalley group of type D_4. Here, we come up with a complete analysis of degenerations in D_4, which served to construct these tables. Essentially, this analysis follows the same general outline as designed in [42], but it is organized somewhat differently, which allows us to immediately exclude most of the situations where degenerations definitely do not occur.

We return to the notation from the proof of Theorem 3. Namely, let $x \in G(\mathrm{D}_4, K)$, and let $Q_x = \{xh_\delta(\varepsilon)x^{-1}, \varepsilon \in K^*\}$ be the corresponding long root torus. Further, let $x = uwvd$, where $u \in U$, $w \in W$, $v \in U_w$, and $d \in T$, be the reduced Bruhat decomposition of the element x. As in the proof of Theorem 3, in the sequel we consider not the element $y(\varepsilon) = xh_\delta(\varepsilon)x^{-1}$ itself, but rather the element $z = z(\varepsilon) = w[v, h_\delta(\varepsilon)]w^{-1}$, lying in the same Bruhat cell as y.

Suppose that a generic element of the torus Q_x sits in the Bruhat cell Bw_0B. For which w can this happen? Actually, we are interested not in the element w itself, but only in its class modulo $W(3\mathrm{A}_1)$, for the subsystem $3\mathrm{A}_1$ generated by $\alpha_1, \alpha_3, \alpha_4$. The number of such classes is equal to $192/8 = 24$. However, half of them, namely those for which ambiguous entries lie over the principal diagonal, are not relevant for us at all, because in these cases no additional degenerations may occur for $\varepsilon \neq 0, 1$.

This leaves us with the analysis of $24/2 = 12$ possibilities for w, which correspond to the 12 patterns A–L for the image of the unipotent radical U_2, considered in the proof of Theorem 3. Now, we scan all 44 elements w_0 of order at most 2 of the Weyl group $W(\mathrm{D}_4)$. Since $w_0 = e$ cannot degenerate any further, in the sequel we consider only involutions.

Actually, the existence of an embedding $\mathrm{A}_3 \subseteq \mathrm{D}_4$ implies that, essentially, a significant part of this analysis, namely, the study of degenerations occurring for involutions of types A_1, D_2, and $2\mathrm{A}_1$, was already carried through in [9, 41]. However, the existence of three nonconjugate embeddings of A_3 in D_4, namely, of D_3, A_3', and A_3'', suggests that a considerable effort is requisite to transcribe these results in the setting of D_4. This is why we eventually decided to treat all involutions in D_4 uniformly, in the context of calculations of the preceding section. On the other hand, the analysis of the cases $\mathrm{A}_1 + \mathrm{D}_2$ and $2\mathrm{D}_2$, which lead to the appearance of new conjugacy classes in Item (3) of Theorem 4, was done only in [45] and was never published before.

Involutions of class $\mathbf{A}_1$. There are 12 involutions of the form $(ij)(-j,-i)$, where $j \neq \pm i$. The first four of them are fundamental reflections. To match the numbering of rows in Tables 3 and 4, we do not include these four cases in Table 4.

• $w_0 = (34)(-4,-3)$ arises in Patterns A, D, F, I, and K. It is clear, though, that only for Pattern A ambiguous entries can influence the type of $z = z(\varepsilon)$, and even in this case only when all other entries of y below them and to the left of them are zeros. It follows that $v_{1,-2} = -v_{2,-1}$, so that the nontrivial roots of both ambiguous entries are equal to -1. Since the length of w_0 equals 1, it can only degenerate to e, so that $z(-1) \in B$.

• $w_0 = (3, -4)(4, -3)$ arises in Patterns B, E, F, J, and K. Exactly the same argument as in the preceding case shows that degeneration occurs only for Pattern B, at $\varepsilon = -1$, so that again $z(-1) \in B$.

• $w_0 = (23)(-3, -2)$ arises in Patterns C, D, E, H, L. Again, the same argument as in the first two cases shows that degeneration may occur only for Pattern C for $\varepsilon = -1$, and, again, $z(-1) \in B$.

• $w_0 = (12)(-2, -1)$ arises in Patterns G, H, I, J, K. Again the same argument as above shows that degeneration occurs only for Pattern G at $\varepsilon = -1$, and that $z(-1) \in B$, as usual.

So far, we have only seen the lack of degenerations, or multiple degenerations. The analysis of the following case is somewhat subtler, because, depending on vanishing of certain entries, both simple and multiple degenerations may occur.

• $w_0 = (24)(-4, -2)$ arises in Patterns A, C, D, F, I, L. Clearly, ambiguous entries can only influence the type of $z = z(\varepsilon)$ for Pattern D and only when $z(\varepsilon)_{41} = 0$. Denote by η the nontrivial root of the horizontal pivotal minor $\Delta^2(\varepsilon) = M^{-1,-4}_{-1,-2}(z)$. As we know from the proof of Theorem 3, in this case the nontrivial root of the vertical pivotal minor $\Delta_2(\varepsilon) = M^{12}_{14}(z)$ automatically equals η^{-1}. In the sequel, we usually do not explicitly mention such obvious facts.

∘ If $z(\varepsilon)_{-2,-3} \neq 0$ and $z(\varepsilon)_{-3,-4} \neq 0$, then the horizontal pivotal minors of $z = z(\eta)$ are $M^{-1}_{-1}(z)$, $M^{-1,-3}_{-1,-2}(z)$, and $M^{-1,-3,-4}_{-1,-2,-3}(z)$, so that $z \in BwB$, where $w = (243)(-4, -3, -2)$. Then $z^{-1} = z(\eta^{-1})$ lies in the Bruhat cell $Bw^{-1}B$, where $w^{-1} = (234)(-4, -2, -3)$. This gives us Row 1 of Table 3.

∘ On the other hand, if at least one of the entries $z(\varepsilon)_{-2,-3}$ or $z(\varepsilon)_{-3,-4}$ equals 0, then $\eta = \eta^{-1}$ by orthogonality, so that degeneration should be multiple. The type of the resulting involutions depends on whether only one of $z(\varepsilon)_{-2,-3}$ or $z(\varepsilon)_{-3,-4}$ vanishes, and which one, or they both vanish. This leads to the three possibilities listed in Row 1 of Table 4.

• $w_0 = (2, -4)(4, -2)$ arises in Patterns B, C, E, F, J, L, but degenerations can only occur for Pattern E, and their analysis is completely similar to that in the previous case. Again only the pivotal minors $\Delta^2(\varepsilon)$ and $\Delta_2(\varepsilon)$ may have nontrivial roots.

∘ If $z(\varepsilon)_{-4,3} \neq 0$ and $z(\varepsilon)_{-3,4} \neq 0$, then a simple degeneration occurs, described in Row 2 of Table 3.

∘ On the other hand, the vanishing of one of the entries $z(\varepsilon)_{-4,3}$ or $z(\varepsilon)_{-3,4}$, or of both of them, gives us three cases of multiple degeneration, listed in Row 2 of Table 4.

• $w_0 = (1, 3)(-3, -1)$ arises in Patterns C, G, H, I, J, L, but degeneration may only occur for Pattern H. Again exactly the same analysis of zeros of the pivotal minors shows that one of the following two possibilities arises.

∘ If $\eta \neq \eta^{-1}$, then a simple degeneration occurs, described in Row 3 of Table 3.

∘ On the other hand, if $\eta = \eta^{-1} = -1$, then, depending on whether the entries $z(\varepsilon)_{-1,-2}$ or $z(\varepsilon)_{-2,-3}$ vanish or do not vanish, we get one of the three cases of multiple degeneration, as listed in Row 3 of Table 4.

The analysis of the next case is again considerably harder than before, because here, depending on the vanishing of certain entries, one can get *different* simple degenerations, as well as *six* distinct patterns of multiple degeneration.

• $w_0 = (2, -3)(3, -2)$ arises in Patterns A, B, D, E, F, K, L, but degeneration may only occur for Pattern F. As above, denote by η and η^{-1} the roots of the pivotal minors $\Delta^2(\varepsilon)$ and $\Delta_2(\varepsilon)$.

∘ If $\eta \neq \eta^{-1}$ and $z(\eta)_{-2,4} \neq 0$, then we get the first subcase in Row 4 of Table 3. On the other hand, if $z(\eta)_{-2,4} = 0$ but $z(\eta)_{-2,-4} \neq 0$, then we get the second such subcase.

∘ If $\eta=\eta^{-1}=-1$, but at least one of the entries $z(\eta)_{-2,4}$ or $z(\eta)_{-2,-4}=0$ is distinct from 0, then $z(\eta)_{-3,3}=0$, for otherwise $\Delta^{234}_{-2,-3,-4}(z)\neq 0$ would contradict the fact that $w\in W(\mathrm{D}_4)$. Depending on whether $z(\eta)_{-2,4}\neq 0$ or $z(\eta)_{-2,4}=0$, we get the first and the second subcases in Row 4 of Table 4, respectively.

∘ On the other hand, if $z(\eta)_{-2,4}=z(\eta)_{-2,-4}=0$, then automatically $\eta=\eta^{-1}=-1$, and then the type of the element $z(-1)$ depends on which among the entries $z_{-3,3}$, $z_{-3,4}$, and $z_{-3,-4}$, if any, does not vanish. This gives us — in the correct order! — the remaining four cases in Row 4 of Table 4. Obviously, the last one of them, $w=e$, corresponds to the situation where $z_{-3,3}=z_{-3,4}=z_{-3,-4}=0$.

• $w_0=(14)(-4,-1)$ arises in Patterns A, D, G, H, I, K, L, but degeneration may only occur for Pattern I when the ambiguous entries $z(\varepsilon)_{-1,-4}$ and $z(\varepsilon)_{41}$ vanish. Let η and η^{-1} be the corresponding roots.

∘ First, let $\eta\neq\eta^{-1}$. In this case the degeneration of $z(\eta)$ — if it actually occurs! — is controlled by the vanishing of the entries $z(\varepsilon)_{-1,-3}$ and $z(\varepsilon)_{-2,-4}$. The situation where $z(\varepsilon)_{-2,-4}=0$ and $z(\eta)_{-1,-3},z(\eta)_{-3,-4}\neq 0$ leads to the first subcase, whereas the situation where $z(\eta)_{-1,-3}\neq 0$ and $z(\eta)_{-2,-4},z(\eta)_{-1,-2}\neq 0$ leads to the second subcase in Row 5 of Table 3.

∘ On the other hand, in the case where $\eta=\eta^{-1}=-1$, many possibilities arise, depending on whether the entries $z(-1)_{-1,-3}$, $z(-1)_{-2,-4}$, $z(-1)_{-3,-4}$, and $z(-1)_{-1,-2}$ vanish or do not vanish. All these possibilities are listed in Row 5 of Table 4. The first two subcases correspond to the situation where exactly one of the entries $z(-1)_{-1,-3}$ or $z(-1)_{-2,-4}$ does not vanish. Observe that since $M^{-3,-4}_{-1,-2}=0$, they cannot be distinct from 0 simultaneously, which excludes degeneration to the involution $(13)(2,-4)$ $(-4,2)(-3,-1)$. In the sequel, we usually omit such obvious considerations, which exclude various possibilities, and concentrate on less trivial aspects. The following three subcases correspond to the situation where $z(-1)_{-1,-3}=z(-1)_{-2,-4}=0$, but at least one of the entries $z(-1)_{-3,-4}$ or $z(-1)_{-1,-2}$ is distinct from 0. Naturally, $w=e$ presents itself when all these entries are 0.

• $w_0=(1,-4)(4,-1)$ arises in Patterns B, E, G, H, J, K, L, but degeneration only occurs for Pattern J when the ambiguous entries $z(\varepsilon)_{-1,4}$ or $z(\varepsilon)_{-4,1}$ vanish. Let η and η^{-1} be the corresponding roots.

∘ In the case where $\eta\neq\eta^{-1}$, the degeneration of $z(\eta)$ is controlled by the vanishing of entries $z(\varepsilon)_{-1,-3}$ and $z(\varepsilon)_{-2,4}$. Observe that these entries cannot vanish simultaneously, because otherwise orthogonality implies that $\eta=\eta^{-1}=\pm 1$. Now, exactly the same calculation as in the preceding case shows that we get precisely the cases listed in Row 6 of Table 3.

∘ The analysis of the multiple degeneration $\eta=\eta^{-1}=-1$ is not any different from the preceding case as well. The resulting possibilities are listed in Row 6 of Table 4.

• $w_0=(1,-3)(3,-1)$ arises in Patterns A, B, F, G, I, J, K, L, but degeneration may only occur for Pattern K when the ambiguous entries $z(\varepsilon)_{-1,3}$ or $z(\varepsilon)_{-3,1}$ vanish. Let η and η^{-1} be the corresponding roots.

∘ In the case where $\eta\neq\eta^{-1}$, the type of degeneration is determined by which of the entries $z(\eta)_{-1,4}$, $z(\eta)_{-1,-4}$, and $z(\eta)_{-1,-2}$ does not vanish first. In accordance with that, we get the three subcases listed in Row 10 of Table 3.

∘ As usual, reflections with respect to one root give the largest number of multiple degenerations. Several possibilities for w_{-1} are excluded by the vanishing of pivotal minors. For instance,

$$w_{-1}=(1,-4)(2,-3)(3,-4)(4,-1)$$

contradicts the relation $\Delta^{1234}_{-1,-2,-3,-4}(z)=0$, etc. The remaining possibilities are listed in Row 10 of Table 4.

Thus, we are left with the analysis of the senior element of class A_1. As could be expected, this is by far the hardest case, among all involutions of this class.

• $w_0 = (1,-2)(2,-1)$ arises in Patterns C, D, F, H, I, J, K, L, but degenerations can only occur for Pattern L when the entry $z(\varepsilon)_{-1,1}$ is zero, or else when the ambiguous entries $z(\varepsilon)_{-1,2}$ or $z(\varepsilon)_{-2,1}$ vanish. Let η and η^{-1} be the corresponding roots.

∘ In the case where $\eta \neq \eta^{-1}$, the type of degeneration is determined by which of the entries $z(\eta)_{-1,3}$, $z(\eta)_{-1,4}$, $z(\eta)_{-1,-4}$, and $z(\eta)_{-1,-3}$ does not vanish first. Observe that for simple degenerations these entries cannot vanish simultaneously, because in that case orthogonality would imply that the first row of the matrix $z(\eta)$ should be proportional to the first row of identity matrix, so that, in particular, $z(\eta)_{-2,1} = 0$, contrary to the assumption $\eta \neq \eta^{-1}$. Thus, we get the four possibilities of simple degenerations listed in Row 14 of Table 3.

∘ This case is responsible for the largest number of distinct multiple degenerations, the arising possibilities are described by the vanishing of the entries $z(\varepsilon)_{-2,2}$, $z(\varepsilon)_{-1,3}$, $z(\varepsilon)_{-2,3}$, $z(\varepsilon)_{-1,4}$, $z(\varepsilon)_{-2,4}$, $z(\varepsilon)_{-1,-4}$, $z(\varepsilon)_{-2,-4}$, $z(\varepsilon)_{-1,-3}$, and $z(\varepsilon)_{-2,-3}$. As always, many possibilities for w_{-1} are actually excluded by the vanishing of appropriate pivotal minors. For instance, $w_{-1} = (1,-3)(2,-2)(3,-1)(4,-4)$ contradicts the fact that $\Delta^{123}_{-1,-2,-3}(z) = 0$, whereas $w_{-1} = (1,-4)(2,-3)(3,-2)(4,-1)$ contradicts the fact that $\Delta^{123}_{-1,-2,-3}(z) = 0$, etc. The remaining 15 possibilities are listed in Row 14 of Table 4.

Involutions of class $\mathbf{D_2}$. There are six involutions of the form $(i,-i)(j,-j)$, where $j \neq \pm i$.

• $w_0 = (3,-3)(4,-4)$ arises in Patterns F, K, but no degenerations may occur here. At first glance, it seems that the element w_0 could arise also in Pattern B, but this is not the case. Indeed, if $z_{-3,1} = z_{-3,2} = 0$, then, by orthogonality, we have $z_{-3,3} = 0$, so that y cannot fall into the cell Bw_0B.

• $w_0 = (2,-2)(4,-4)$ arises in Patterns C, E, F, but for the only ambiguous Pattern E we have $z(\varepsilon)_{-2,3} = 0$, so that the pivotal minors of orders 2 and 3 cannot have any nontrivial roots, and there are no degenerations in this case either.

In the next case degenerations do occur, and, since the degeneration type depends on roots of a *third* order pivotal minor, this case requires a much more careful scrutiny.

• $w_0 = (2,-2)(3,-3)$ arises in Patterns D, E, F, L, but degenerations may only occur for Pattern F, and their type depends on the vanishing of the pivotal minor $M^{2,3,-1}_{-1,-2,-3}(z)$. As we know from the proof of Theorem 2, this minor has at most two nontrivial roots θ and θ^{-1}. Note that, since in this case automatically $z(\theta)_{-2,2} \neq 0$, there are exactly three possible degenerations, to $(2,-2)(4,-4)$, $(2,-2)(3,-4,-3,4)$, or $(2,-2)(3,4,-3,-4)$, and it remains to ascertain when they actually arise.

∘ If $\theta \neq \theta^{-1}$, then the type of the element $z = z(\theta)$ is determined by the vanishing of the minors $M^{2,4,-1}_{-1,-2,-3}(z)$ and $M^{123}_{1,-2,-4}(z)$. An easy calculation shows that these minors are of the form

$$\begin{aligned} M^{2,4,-1}_{-1,-2,-3}(z) &= M^{24}_{-2,-3}(z) = (\theta-1)^2\big((ad-be)\theta + (-bd+ce)\big) \\ M^{123}_{-1,-2,-4}(z) &= M^{-2,-3}_{24}(z) = (\theta-1)^2\big((ce-bd)\theta + (ad-be)\big), \end{aligned}$$

for some a,b,c,d,e. Now, since $z(\theta)_{-2,2} \neq 0$, by orthogonality also $d = z(\theta)_{-2,4} \neq 0$. Moreover, since the minor $M^{2,3,-1}_{-1,-2,-3}(z)$ has nontrivial roots, we have $a,b \neq 0$ and $a \neq b$. But then these minors cannot vanish simultaneously for $\theta \neq -1$. If one of them vanishes for $\varepsilon = \theta$, then the second should vanish for $\varepsilon = \theta^{-1}$. On the other hand, since $M^{2,3,4}_{-1,-2,-3}(z) = 0$, these minors cannot be simultaneously distinct from 0. This means that we fall into the situation described in Row 7 of Table 3.

∘ On the other hand, if $\theta = \theta^{-1} = -1$, then these minors vanish or do not vanish simultaneously. When they both vanish, we fall into the situation described in Row 7 of Table 4.

• $w_0 = (1,-1)(4,-4)$ arises in Patterns G, H, I, J, K, L, but it is easily seen that this w_0 only arises when $z_{-1,1} \neq 0$, and, moreover, $M^{14}_{-1,-4}(z) \neq 0$ for the first three patterns, and $M^{1234}_{-1,-2,-3,-4}(z) \neq 0$ for the remaining three patterns, whereas $M^{12}_{-1,-2}(z) = 0$ and $M^{123}_{-1,-2,-3}(z) = 0$. In the first case, the second order pivotal minor does not depend on ambiguous entries at all. In the second case, looking at the structure of the above fourth order pivotal minor, we see that it does not depend on ambiguous entries at all, and thus has no nontrivial roots. Thus, no degenerations are possible in this case.

• $w_0 = (1,-1)(3,-3)$ arises in Patterns G, I, J, K, L, but degeneration can only occur for Pattern K when the pivotal minor $M^{13}_{-1,-3}(z)$ vanishes. Since $w(1) = -1$ and $w(2) = 2$, in the group $W(\mathrm{D}_4)$ there are exactly three contenders for the role of w, namely, $(1,-1)(3,-4,-3,4)$, $(1,-1)(3,4,-3,-4)$, and $w = (1,-1)(4,-4)$. The first two of them occur indeed in the case of simple degeneration, whereas the third occurs in the case of multiple degeneration, which gives us Rows 13 of Tables 3 and 4, respectively.

As always, the senior case is by far the hardest one.

• $w_0 = (1,-1)(2,-2)$ arises in Patterns H, I, J, K, L, but degeneration may only occur for Pattern L when the minor $M^{12}_{-1,-2}(z)$ vanishes. Let η and η^{-1} be the roots of this minor.

∘ If the entries $z(\varepsilon)_{-1,3}$ and $z(\varepsilon)_{-2,3}$ do not vanish simultaneously, then the relation $M^{12m}_{-1,-2,-3}(z) = M^{123}_{-1,-2,-m}(z) = 0$ for $m = 3,4,-4$ shows that the pieces of the last two rows at positions $3,4,-4$ are proportional. For the same reason, also the pieces of the first two columns at positions $4,-4,-3$ should be proportional. It follows that if $M^{13}_{-1,-2}(z(\eta)) = 0$, then also $M^{14}_{-1,-2}(z(\eta)) = M^{1,-4}_{-1,-2}(z(\eta)) = 0$. On the other hand, since $M^{123}_{-1,-2,-3}(z) = 0$, the minors $M^{13}_{-1,-2}(z(\eta))$ and $M^{12}_{-1,-3}(z(\eta))$ cannot vanish simultaneously. This leads to the first possibility in Row 17 of Table 3 and to the first two possibilities in Row 17 of Table 4.

∘ On the other hand, if $z(\varepsilon)_{-1,3} = z(\varepsilon)_{-2,3} = 0$, then $w(3) = 3$; as in the preceding case, this leaves us exactly three contenders for the role of w, namely, $(1,-1)(2,-4,-2,4)$, $(1,-1)(2,4,-2,-4)$, and $w = (1,-1)(4,-4)$. These are precisely the remaining subcases in the above rows.

Involutions of classes $(2\mathbf{A}_1)'$ and $(2\mathbf{A}_1)''$. There are 12 involutions of the form

$$(i,j)(h,k)(-k,-h)(-j,-i), \qquad \{\pm i, \pm j, \pm h, \pm k\} = \{\pm 1, \pm 2, \pm 3, \pm 4\},$$

which split into two conjugacy classes in the group $W(\mathrm{D}_4)$, denoted by $(2\mathrm{A}_1)'$ and $(2\mathrm{A}_1)''$, according to parity.

• $w_0 = (12)(34)(-4,-3)(-2,-1)$ only occurs in Patterns I, K; clearly, there are no degenerations.

• $w_0 = (12)(3,-4)(4,-3)(-2,-1)$ only occurs in Patterns J, K; clearly, there are no degenerations.

• $w_0 = (13)(24)(-4,-2)(-3,-1)$ only occurs in Patterns I, L; there are no degenerations.

• $w_0 = (13)(2,-4)(4,-2)(-3,-1)$ only occurs in Patterns J, L; there are no degenerations.

• $w_0 = (14)(23)(-4,-1)(-3,-2)$ occurs in Patterns A, H, I, L, of which only Pattern H may potentially produce degenerations. When the nontrivial roots of $z(\varepsilon)_{-1,-4}$ and $z(\varepsilon)_{41}$ are distinct, we fall into the situation described in Row 8 of Table 3, and when they coincide and are equal to -1 we fall into the situation described in Row 8 of Table 4.

• $w_0 = (14)(2,-3)(3,-2)(-4,-1)$ only occurs in Patterns A, D, K, L; there are no degenerations.

• $w_0 = (1,-4)(23)(4,-1)(-3,-2)$ only occurs in Patterns E, H, J, L, of which only Pattern J may potentially produce degenerations. Since the second order horizontal pivotal minor of $z = z(\varepsilon)$ equals $M^{4,-3}_{-1,-2}(z)$, automatically $z(\varepsilon)_{-1,-3}z(\varepsilon)_{-2,4} \neq 0$. It follows that, if the nontrivial roots $z(\varepsilon)_{-1,-4}$ and $z(\varepsilon)_{41}$ are distinct, we fall into the situation described in Row 9 of Table 3. On the other hand, if they coincide and are equal to -1, we fall into the situation described in Row 9 of Table 4.

• $w_0 = (1,-4)(2,-3)(3,-2)(4,-1)$ only occurs in Patterns B, E, K, L; there are no degenerations.

• $w_0 = (1,-3)(2,4)(3,-1)(-4,-2)$ occurs in Patterns A, F, I, K, L, but degenerations may only occur in Pattern K and only when $z(\varepsilon)_{-1,4} = 0$. Clearly, we can get the above w_0 only when the entries $z(\varepsilon)_{-1,-4}$, $z(\varepsilon)_{-2,3}$, $z(\varepsilon)_{-3,2}$ and $z(\varepsilon)_{4,1}$ are all distinct from 0. In this case, a unique possible simple degeneration consists in the vanishing of one of the entries $z(\varepsilon)_{-1,3}$ or $z(\varepsilon)_{-3,1}$ and gives us Row 11 of Table 3. By the same token, a unique possible multiple degeneration consists in the vanishing of the two entries, and gives us Row 11 of Table 4.

• $w_0 = (1,-3)(2,-4)(3,-1)(4,-2)$ only occurs in Patterns B, F, J, K, L, and we get this w_0 only when the entries $z(\varepsilon)_{-1,4}$, $z(\varepsilon)_{-2,3}$, $z(\varepsilon)_{-3,2}$ and $z(\varepsilon)_{-4,1}$ are all distinct from 0. Now exactly the same argument as in the preceding case gives us Row 12 of Tables 3 and 4.

So far, all cases in this class were extremely easy. It remains to consider two senior, much harder cases.

• $w_0 = (1,-2)(2,-1)(3,4)(-4,-3)$ only occurs in Patterns D, F, K, L, and degenerations are possible only in Pattern L, at the roots of ambiguous entries.

∘ Observe that the entries $z(\varepsilon)_{-1,3}$ and $z(\varepsilon)_{-2,3}$ cannot vanish simultaneously. Indeed, should it be the case, orthogonality would imply that $z(\varepsilon)_{-3,1} = z(\varepsilon)_{-3,2} = 0$, which is impossible because $M^{1,2,-4}_{-1,-2,-3} \neq 0$. On the other hand, the minors $M^{34}_{-1,-2}(z)$ and $M^{12}_{-3,-4}(z)$ are both equal to zero, for otherwise $M^{1234}_{-1,-2,-3,-4}(z) \neq 0$, which is impossible for our w_0. But then $M^{124}_{-1,-2,-3}(z) \neq 0$, which also contradicts our choice of w_0. Thus, the only possibility of simple degeneration is the situation described in Row 15 of Table 3.

∘ An argument similar to the above in the case of simple degenerations, shows that in the typical case the entries $z(\varepsilon)_{-1,3}$ and $z(\varepsilon)_{-1,4}$ cannot vanish simultaneously. This leaves us with three possibilities for multiple degenerations, as listed in Row 15 of Table 4.

• $w_0 = (1,-2)(2,-1)(3,-4)(4,-3)$ only occurs in Patterns E, F, K, L, and again degenerations are only possible in Pattern L at roots of ambiguous entries. Since the pivotal minor $M^{1234}_{-1,-2,-3,-4}(z)$ is distinct from 0, automatically $M^{34}_{-1,-2}(z) \neq 0$. This means that at the root η of an entry $z(\varepsilon)_{-2,1}$ we have exactly two possibilities for w, depending on whether the entry $z(\varepsilon)_{-2,3}$ does not vanish, or does vanish. As we have observed, if it equals 0, then automatically $z(\varepsilon)_{-2,4} \neq 0$. This gives us two possibilities listed in Row 16 of Table 3. Clearly, the possibilities for multiple degenerations only differ from those obtained in the preceding case by switching 4 and -4.

Involutions of classes $\mathbf{A}_1 + \mathbf{D}_2$ and $2\mathbf{D}_2$. There are 12 involutions of the form

$$(i,j)(h,-h)(k,-k)(-j,-i), \qquad \{\pm i, \pm j, \pm h, \pm k\} = \{\pm 1, \pm 2, \pm 3, \pm 4\}.$$

of class $\mathrm{A}_1 + \mathrm{D}_2$, and one involution $(1,-1)(2,-2)(3,-3)(4,-4)$ of class $2\mathrm{D}_2$.

Since in the following five cases the pivotal minors do not depend on ambiguous entries at all, there is no room for degenerations.

• $w_0 = (12)(3,-3)(4,-4)(-2,-1)$ only occurs in Pattern K.

• $w_0 = (13)(2,-2)(4,-4)(-3,-1)$ only occurs in Patterns C, K.

• $w_0 = (14)(2,-2)(3,-3)(-4,-1)$ only occurs in Patterns D, K.
• $w_0 = (1,-4)(2,-2)(3,-3)(4,-1)$ only occurs in Patterns E, L.
• $w_0 = (1,-3)(2,-2)(3,-1)(4,-4)$ only occurs in Patterns F, L.

So far, there were no degenerations for involutions of this class; the following case is the first when degenerations actually arise. These degenerations are not complicated, but they are *new*, since before we have never seen Coxeter elements of the root system D_4 itself, as Weyl factors in Bruhat decomposition of semisimple long root elements. Clearly, they could not have occurred either, because all involutions w_0 we considered so far were coming from subsystems of type A_3, in various embeddings. The two senior involutions of this class will also produce *new* types of simple degenerations.

• $w_0 = (1,-2)(2,-1)(3,-3)(4,-4)$ only occurs in Patterns F, K, L; obviously, degenerations are only possible for Pattern L. The element w_0 can occur when $z_{-1,1} = 0$, whereas $M^{123}_{-1,-2,-3}(z) \neq 0$. Since this third order minor does not have nontrivial roots, the only degenerations arise when the entries $z(\varepsilon)_{-1,2}$ and/or $z(\varepsilon)_{-2,1}$ vanish. When one of them vanishes, we get possibilities listed in Row 18 of Table 3. When they both vanish, we get the possibility listed in Row 18 of Table 4.

In the following four cases the pivotal minors do not depend on ambiguous entries either, so that, again, there is no room for degeneration.

• $w_0 = (1,-1)(2,3)(3,2)(4,-4)$ only occurs in Patterns H, L.
• $w_0 = (1,-1)(2,4)(4,2)(3,-3)$ only occurs in Patterns I, L.
• $w_0 = (1,-1)(2,-4)(3,-3)(4,-2)$ only occurs in Patterns J, L.
• $w_0 = (1,-1)(2,-3)(3,-2)(4,-4)$ only occurs in Patterns K, L.

It is fairly easy to describe degenerations for the two senior involutions of this class, all such degenerations arise from the vanishing of the second order pivotal minor $M^{12}_{-1,-2}(z)$. In fact, the analysis of these two cases does not differ from the analysis of degenerations of the involution $w_0 = (1,-1)(2,-2)$, senior in the class of type D_2. Actually, now it is even slightly easier, because the presence of the additional factors $(3,4)(-4,-3)$ or $(3,-4)(4,-3)$, limiting w from below, prohibits one value of m in the above argument, namely, either $m = -4$, or $m = 4$, respectively.

• $w_0 = (1,-1)(2,-2)(3,4)(-4,-3)$ only occurs in Patterns I, L, but degenerations only arise in Pattern L. The answer is reproduced in Row 19 of Tables 3 and 4.

• $w_0 = (1,-1)(2,-2)(3,-4)(4,-3)$ only occurs in Patterns J, K, L, but degenerations only arise in Pattern L. The answer is reproduced in Row 20 of Tables 3 and 4.

It only remains to consider the longest element of the Weyl group, of type $2D_2$. In this case we encounter another extremely interesting new phenomenon, namely, a simple degeneration to the Bruhat cell BwB, where w is also an involution. In other words, the values of the parameter, η and η^{-1}, at which degeneration occurs, are distinct, but nevertheless, $z(\eta)$ and $z(\eta^{-1})$ sit in the same Bruhat cell BwB. The reason is that, to implement this typical w_0, it is necessary that $z(\varepsilon)_{-1,1} \neq 0$ but then, of course, degeneration can only arise due to the vanishing of the second order pivotal minor $M^{12}_{-1,-2}(z)$.

• $w_0 = (1,-1)(2,-2)(3,-3)(-4,-4)$ only occurs in Patterns K, L, but degeneration only arises in Pattern L. Since the pivotal minors $M^{123}_{-1,-2,-3}(z)$ and $M^{123}_{-1,-2,-3}(z)$ of orders 3 and 4 must be nonzero, and do not have nontrivial roots, the vanishing of the second order pivotal minor $M^{12}_{-1,-2}(z(\eta))$ implies that $M^{13}_{-1,-2}(z(\eta)) \neq 0$ and $M^{12}_{-1,-3}(z(\eta)) \neq 0$ automatically, regardless of whether $\eta = \eta^{-1}$ or not. This shows that all possible degenerations are listed in Row 21 of Tables 3 and 4.

§8. Proof of Theorems 4 and 5

Now, we bring together the results obtained in the preceding section. Table 3 below lists the simple degenerations; as a matter of fact, only such degenerations are relevant for the proof of Theorem 4. In its turn, Table 4 lists all possible degenerations at $\varepsilon = -1$. All degenerations in Table 4, apart from the very last one, are indeed *multiple*. In other words, they drop the length of the Weyl group factor at least by 2.

Table 3 indicates the signed cycle type ν_0 of the permutation $w_0 \in W(\mathrm{D}_4)$, its length $l(w_0)$, the Carter graph C_0 of its conjugacy class, the signed cycle types ν of all permutations $w \in W(\mathrm{D}_4)$ that come up as degenerations of w_0, and their Carter graphs C. On top of that, Table 4 indicates the length $l(w)$ — in Table 3 this was not necessary, because there $l(w) = l(w_0) - 1$, in all cases.

TABLE 3. SIMPLE DEGENERATIONS.

N	w_0	$l(w_0)$	C_0	w	C
1	$(24)(\bar4\bar2)$	3	A_1	$(243)(\bar4\bar3\bar2)$, $(234)(\bar4\bar2\bar3)$	A_2
2	$(2\bar4)(4\bar2)$	3	A_1	$(2\bar43)(4\bar3\bar2)$, $(23\bar4)(4\bar2\bar3)$	A_2
3	$(13)(\bar3\bar1)$	3	A_1	$(132)(\bar3\bar2\bar1)$, $(123)(\bar3\bar1\bar2)$	A_2
4	$(2\bar3)(3\bar2)$	5	A_1	$(2\bar34)(3\bar4\bar2)$, $(24\bar3)(3\bar2\bar4)$	A_2
				$(2\bar3\bar4)(34\bar2)$, $(2\bar4\bar3)(3\bar24)$	A_2
5	$(14)(\bar4\bar1)$	5	A_1	$(143)(\bar4\bar3\bar1)$, $(134)(\bar4\bar1\bar3)$	A_2
				$(142)(\bar4\bar2\bar1)$, $(124)(\bar4\bar1\bar2)$	A_2
6	$(1\bar4)(4\bar1)$	5	A_1	$(1\bar43)(4\bar3\bar1)$, $(13\bar4)(4\bar1\bar3)$	A_2
				$(1\bar42)(4\bar2\bar1)$, $(12\bar4)(4\bar1\bar2)$	A_2
7	$(2\bar2)(3\bar3)$	6	D_2	$(2\bar2)(3\bar4\bar34)$, $(2\bar2)(34\bar3\bar4)$	D_3
8	$(14)(23)(\bar4\bar1)(\bar3\bar2)$	6	$(2\mathrm{A}_1)'$	$(1423)(\bar4\bar2\bar3\bar1)$, $(1324)(\bar4\bar1\bar3\bar2)$	$(\mathrm{A}_3)'$
9	$(1\bar4)(23)(4\bar1)(\bar3\bar2)$	6	$(2\mathrm{A}_1)''$	$(1\bar423)(4\bar2\bar3\bar1)$, $(132\bar4)(4\bar1\bar3\bar2)$	$(\mathrm{A}_3)''$
10	$(1\bar3)(3\bar1)$	7	A_1	$(1\bar3\bar4)(34\bar1)$, $(1\bar4\bar3)(3\bar14)$	A_2
				$(1\bar34)(3\bar4\bar1)$, $(14\bar3)(3\bar1\bar4)$	A_2
				$(1\bar32)(3\bar2\bar1)$, $(12\bar3)(3\bar1\bar2)$	A_2
11	$(1\bar3)(24)(3\bar1)(\bar4\bar2)$	8	$(2\mathrm{A}_1)''$	$(1\bar324)(3\bar2\bar4\bar1)$, $(142\bar3)(3\bar1\bar4\bar2)$	$(\mathrm{A}_3)''$
12	$(1\bar3)(2\bar4)(3\bar1)(4\bar2)$	8	$(2\mathrm{A}_1)'$	$(1\bar32\bar4)(3\bar24\bar1)$, $(1\bar42\bar3)(3\bar14\bar2)$	$(\mathrm{A}_3)'$
13	$(1\bar1)(3\bar3)$	8	D_2	$(1\bar1)(3\bar4\bar34)$, $(1\bar1)(34\bar3\bar4)$	D_3
14	$(1\bar2)(2\bar1)$	9	A_1	$(1\bar2\bar3)(23\bar1)$, $(1\bar3\bar2)(3\bar13)$	A_2
				$(1\bar2\bar4)(24\bar1)$, $(1\bar4\bar2)(3\bar14)$	A_2
				$(1\bar24)(2\bar4\bar1)$, $(14\bar2)(2\bar1\bar4)$	A_2
				$(1\bar23)(2\bar3\bar1)$, $(13\bar2)(2\bar1\bar3)$	A_2
15	$(1\bar2)(2\bar1)(34)(\bar4\bar3)$	10	$(2\mathrm{A}_1)''$	$(1\bar2\bar4\bar3)(243\bar1)$, $(1\bar3\bar4\bar2)(2\bar134)$	$(\mathrm{A}_3)''$
				$(1\bar234)(2\bar3\bar4\bar1)$, $(143\bar2)(2\bar1\bar4\bar3)$	$(\mathrm{A}_3)''$
16	$(1\bar2)(2\bar1)(3\bar4)(4\bar3)$	10	$(2\mathrm{A}_1)'$	$(1\bar24\bar3)(2\bar43\bar1)$, $(1\bar34\bar2)(2\bar13\bar4)$	$(\mathrm{A}_3)'$

				$(1\bar{2}3\bar{4})(2\bar{3}4\bar{1})$, $(1\bar{4}3\bar{2})(2\bar{1}4\bar{3})$	$(\mathrm{A}_3)'$
17	$(1\bar{1})(2\bar{2})$	10	D_2	$(1\bar{1})(2\bar{3}\,\bar{2}3)$, $(1\bar{1})(23\bar{2}\,\bar{3})$	D_3
				$(1\bar{1})(2\bar{4}\,\bar{2}4)$, $(1\bar{1})(24\bar{2}\,\bar{4})$	D_3
18	$(1\bar{2})(2\bar{1})(3\bar{3})(4\bar{4})$	11	$\mathrm{A}_1 + \mathrm{D}_2$	$(1\bar{2}3\bar{1}2\bar{3})(4\bar{4})$, $(1\bar{3}2\bar{1}3\bar{2})(4\bar{4})$	D_4
19	$(1\bar{1})(2\bar{2})(34)(\bar{4}\,\bar{3})$	11	$\mathrm{A}_1 + \mathrm{D}_2$	$(1\bar{1})(2\bar{3}\,\bar{4}\,\bar{2}34)$, $(1\bar{1})(243\bar{2}\,\bar{4}\,\bar{3})$	D_4
20	$(1\bar{1})(2\bar{2})(3\bar{4})(4\bar{3})$	11	$\mathrm{A}_1 + \mathrm{D}_2$	$(1\bar{1})(2\bar{3}4\bar{2}3\bar{4})$, $(1\bar{1})(2\bar{4}3\bar{2}4\bar{3})$	D_4
21	$(1\bar{1})(2\bar{2})(3\bar{3})(4\bar{4})$	12	$2\mathrm{D}_2$	$(1\bar{1})(2\bar{3})(3\bar{2})(4\bar{4})$,	$\mathrm{A}_1 + \mathrm{D}_2$

To match the numbering of rows in the following table with that in Table 3, we have omitted degenerations of the fundamental reflections in $W(\mathrm{D}_4)$, namely, the rows representing degenerations of $(34)(\bar{4}\,\bar{3})$, $(3\bar{4})(4\bar{3})$, $(23)(\bar{3}\,\bar{2})$, and $(12)(\bar{2}\,\bar{1})$ to e.

Table 4. Multiple degenerations.

N	w_0	$l(w_0)$	C_0	w	$l(w)$	C
1	$(24)(\bar{4}\,\bar{2})$	3	A_1	$(34)(\bar{4}\,\bar{3})$	1	A_1
				$(23)(\bar{3}\,\bar{2})$	1	A_1
				e	0	$\varnothing$
2	$(2\bar{4})(4\bar{2})$	3	A_1	$(3\bar{4})(4\bar{3})$	1	A_1
				$(23)(\bar{3}\,\bar{2})$	1	A_1
				e	0	$\varnothing$
3	$(13)(\bar{3}\,\bar{1})$	3	A_1	$(23)(\bar{3}\,\bar{2})$	1	A_1
				$(12)(\bar{2}\,\bar{1})$	1	A_1
				e	0	$\varnothing$
4	$(2\bar{3})(3\bar{2})$	5	A_1	$(24)(\bar{4}\,\bar{2})$	3	A_1
				$(2\bar{4})(4\bar{2})$	3	A_1
				$(3\bar{3})(4\bar{4})$	2	D_2
				$(34)(\overline{42})$	1	A_1
				$(3\bar{4})(4\bar{3})$	1	A_1
				e	0	$\varnothing$
5	$(14)(\bar{4}\,\bar{1})$	5	A_1	$(13)(\bar{3}\,\bar{1})$	3	A_1
				$(24)(\bar{2}\,\bar{1})$	3	A_1
				$(12)(34)(\bar{4}\,\bar{2})(\bar{2}\,\bar{1})$	2	$(2\mathrm{A}_1)'$
				$(12)(\bar{2}\,\bar{1})$	1	A_1
				$(34)(\bar{2}\,\bar{1})$	1	A_1
				e	0	$\varnothing$
6	$(1\bar{4})(4\bar{1})$	5	A_1	$(13)(\bar{3}\,\bar{1})$	3	A_1

				$(2\bar{4})(4\bar{2})$	3	A_1
				$(12)(3\bar{4})(4\bar{3})(\bar{2}\,\bar{1})$	2	$(2A_1)''$
				$(12)(\bar{2}\,\bar{1})$	1	A_1
				$(3\bar{4})(\bar{2}\,\bar{1})$	1	A_1
				e	0	$\varnothing$
7	$(2\bar{2})(3\bar{3})$	6	D_2	$(2\bar{2})(4\bar{4})$	4	D_2
8	$(14)(23)(\bar{4}\,\bar{1})(\bar{3}\,\bar{2})$	6	$(2A_1)'$	$(13)(24)(\bar{4}\,\bar{2})(\bar{3}\,\bar{1})$	4	$(2A_1)'$
9	$(1\bar{4})(23)(4\bar{1})(\bar{3}\,\bar{2})$	6	$(2A_1)''$	$(13)(2\bar{4})(4\bar{2})(\bar{3}\,\bar{1})$	4	$(2A_1)''$
10	$(1\bar{3})(3\bar{1})$	7	A_1	$(14)(\bar{4}\,\bar{1})$	5	A_1
				$(1\bar{4})(4\bar{1})$	5	A_1
				$(2\bar{3})(3\bar{2})$	5	A_1
				$(12)(34)(\bar{4}\,\bar{3})(\bar{2}\,\bar{1})$	2	$(2A_1)'$
				$(12)(3\bar{4})(4\bar{3})(\bar{2}\,\bar{1})$	2	$(2A_1)''$
				$(3\bar{3})(4\bar{4})$	2	D_2
				$(12)(\bar{2}\,\bar{1})$	1	A_1
				$(34)(\bar{4}\,\bar{3})$	1	A_1
				$(3\bar{4})(4\bar{3})$	1	A_1
				e	0	$\varnothing$
11	$(1\bar{3})(24)(3\bar{1})(\bar{4}\,\bar{2})$	8	$(2A_1)''$	$(14)(2\bar{3})(3\bar{2})(\bar{4}\,\bar{1})$	6	$(2A_1)''$
12	$(1\bar{3})(2\bar{4})(3\bar{1})(4\bar{2})$	8	$(2A_1)'$	$(1\bar{4})(2\bar{3})(3\bar{2})(4\bar{1})$	6	$(2A_1)'$
13	$(1\bar{1})(3\bar{3})$	8	D_2	$(1\bar{1})(4\bar{4})$	6	D_2
14	$(1\bar{2})(2\bar{1})$	9	A_1	$(1\bar{3})(3\bar{1})$	7	A_1
				$(1\bar{4})(23)(4\bar{1})(\bar{3}\,\bar{2})$	6	$(2A_1)''$
				$(14)(23)(\bar{4}\,\bar{1})(\bar{3}\,\bar{2})$	6	$(2A_1)'$
				$(2\bar{2})(3\bar{3})$	6	D_2
				$(14)(\bar{4}\,\bar{1})$	5	A_1
				$(1\bar{4})(4\bar{1})$	5	A_1
				$(2\bar{3})(3\bar{2})$	5	A_1
				$(13)(24)(\bar{4}\,\bar{2})(\bar{3}\,\bar{1})$	4	$(2A_1)'$
				$(13)(2\bar{4})(4\bar{2})(\bar{3}\,\bar{1})$	4	$(2A_1)''$
				$(2\bar{2})(4\bar{4})$	4	D_2
				$(24)(\bar{4}\,\bar{2})$	3	A_1
				$(2\bar{4})(4\bar{2})$	3	A_1
				$(13)(\bar{3}\,\bar{1})$	3	A_1

				$(23)(\bar 3\bar 2)$	1	A_1
				e	0	$\varnothing$
15	$(1\bar 2)(2\bar 1)(34)(\bar 4\bar 3)$	10	$(2A_1)''$	$(1\bar 3)(24)(3\bar 1)(\bar 4\bar 2)$	8	$(2A_1)''$
				$(14)(2\bar 2)(3\bar 3)(\bar 4\bar 1)$	7	A_1+D_2
				$(14)(2\bar 3)(3\bar 2)(\bar 4\bar 1)$	6	$(2A_1)''$
16	$(1\bar 2)(2\bar 1)(3\bar 4)(4\bar 3)$	10	$(2A_1)'$	$(1\bar 3)(2\bar 4)(3\bar 1)(4\bar 2)$	8	$(2A_1)'$
				$(1\bar 4)(2\bar 2)(3\bar 3)(4\bar 1)$	7	A_1+D_2
				$(1\bar 4)(2\bar 3)(3\bar 2)(4\bar 1)$	6	$(2A_1)'$
17	$(1\bar 1)(2\bar 2)$	10	D_2	$(1\bar 1)(3\bar 3)$	8	D_2
				$(1\bar 1)(23)(4\bar 4)(\bar 3\bar 2)$	7	A_1+D_2
				$(1\bar 1)(4\bar 4)$	6	D_2
18	$(1\bar 2)(2\bar 1)(3\bar 3)(4\bar 4)$	11	A_1+D_2	$(1\bar 3)(2\bar 2)(3\bar 1)(4\bar 4)$	9	A_1+D_2
19	$(1\bar 1)(2\bar 2)(34)(\bar 4\bar 3)$	11	A_1+D_2	$(1\bar 1)(24)(3\bar 3)(\bar 4\bar 2)$	9	A_1+D_2
20	$(1\bar 1)(2\bar 2)(3\bar 4)(4\bar 3)$	11	A_1+D_2	$(1\bar 1)(2\bar 4)(3\bar 3)(4\bar 2)$	9	A_1+D_2
21	$(1\bar 1)(2\bar 2)(3\bar 3)(4\bar 4)$	12	$2D_2$	$(1\bar 1)(2\bar 3)(3\bar 2)(4\bar 4)$	11	A_1+D_2

As could be expected, these tables possess inner symmetry reflecting triality. Any row corresponding to one of the classes D_2, $(2A_1)'$, $(2A_1)''$ is necessarily accompanied by two matching rows corresponding to the other two classes whose involutions have the same length. The same applies, of course, also to the occurrence of these classes, as also the classes D_3, A_3', A_3'', as part of an answer. The fact that the above tables agree with triality, while their rows have been calculated independently, is an additional evidence in favor of their completeness.

Now, we are in a position to conclude the proofs of Theorems 4 and 5. Theorem 5 immediately follows by comparison of the lengths of the typical and degenerate elements of the Weyl group, in Tables 3 and 4. For the proof of Theorem 4, we make the following observations.

• Obviously, the involutions of the described types are realized as Weyl factors in Bruhat decomposition of semisimple long root elements, even generically; this was already noticed in [8].

• Direct inspection shows that Coxeter elements of *all* 16 subsystems of type A_2 in D_4 appear in Table 3.

• Direct inspection shows that 4 out of 24 Coxeter elements of each of the three conjugacy classes of subsystems of type A_3 in D_4, namely, D_3, A_3', and A_3'', appear in Table 3.

• Direct inspection shows that *some* Coxeter elements of the system D_4 itself appear in Table 3. Coxeter elements of subsystems of type B_3 and G_2 appear upon twisting of D_4.

• Coxeter elements of subsystems of type $B_2 = C_2$ could only appear as a result of multiple degeneration. However, this is impossible, because the system of type A_3 for which B_2 is a twisted form only appears in Table 3, but not in Table 4.

This concludes the proof of Theorem 4.

§9. Final remarks

The main results of the present paper arose in the context of the following problem, which is still not completely solved.

Problem 1. In the nonsymplectic case, improve the estimate on the cardinality of the ground fields to $|K| \geq 7$, in the results of [11, 12] on overgroups of split maximal tori in Chevalley groups.

However, presently we have returned to this subject in connection with the following problem.

Problem 2. Describe the orbits of the Chevalley group $G(\Phi, K)$ acting by simultaneous conjugation on pairs of long root tori,

$$X, Y \sim \{h_\alpha(\varepsilon) \mid \varepsilon \in K^*\}, \qquad \alpha \in \Phi_l,$$

and the corresponding spans.

As can be seen already from the papers [14, 59, 20, 21, 22, 23, 24], where similar results were obtained for analogous, but much easier problems concerning pairs of microweight tori, a complete solution of this problem would require *enormous* effort.

Fortunately, most of the common applications do not depend on such a complete answer. Usually, it suffices to be able to reduce the description of the subgroup $\langle X, Y \rangle$ generated by two long root tori X and Y to groups of smaller rank, and to answer the following problem, which would then make it possible to reduce many important questions to the classical case of unipotent elements.

Problem 3. Prove that if the field K is not too small, then the subgroup $\langle X, Y \rangle$ generated by two noncommuting long root tori X, Y in the Chevalley group $G(\Phi, K)$ contains unipotent elements of small rank.

From the main results of the present paper, it is apparent that long root tori in linear and symplectic groups have a *much* easier structure than in all other groups. Thus, it would be natural to start an attack to Problem 2 from the following special case.

Problem 4. Describe the orbits of the group $\mathrm{SL}(n, K)$ acting by simultaneous conjugation of pairs of long root tori, and the corresponding spans.

However, we are convinced that getting a complete answer even in this special case would be *extremely* challenging. Indeed, already the analysis of the much easier case of 2-tori consisting of matrices with *one* nontrivial eigenvalue of multiplicity 2, required dramatical technical effort [21].

The proofs of the main results of the present paper depend on the case by case analysis. They are far from what a good proof should be, which addresses the essence and explains *why* the result is true. Instead, the proofs here are not actual mathematical proofs, but rather *verifications*, not much different in nature and style from the initial computer verifications.

Actually, for Theorem 1 we could propose a conceptual proof, minimizing case by case analysis, in the style of Richardson, Röhrle, and Steinberg [76] or Krutelevich [66]. It was a deliberate decision not to do so, because the details of calculations in §2 will be needed in our joint paper with Vladimir Nesterov [25], to study the unipotent parts of subgroups generated by two long root tori.

At the same time, for Theorems 3–5 we are not aware of any such conceptual proof. What is worse, we do not have any idea what such a proof would look like. In fact, devising such a conceptual proof would be an important step towards the solution of the following problem, whose systematic study was initiated by Erich Ellers and Nikolai Gordeev [58, 59, 60].

Problem 5. Describe intersections of conjugacy classes of Chevalley groups with Bruhat cells.

To illustrate the magnitude and complexity of this problem, we mention that as of today, even the following minor subproblem is not solved for any Chevalley group other than $\mathrm{SL}(n, K)$.

Problem 6. For a given $x \in G(\Phi, K)$, find all $w \in W(\Phi)$ such that $x^G \cap BwB \neq \varnothing$.

Until very recently, an answer to this problem was only known for some very special conjugacy classes and/or Bruhat cells, usually either *extremely* large, such as regular classes and the open Bruhat cell, or for very small ones, such as classes of microweight elements or quadratic unipotent elements, and the cells defined by small-rank involutions or Coxeter elements. Even for some of these cases the explicit answers were only known for the case of $\mathrm{SL}(n, K)$ and/or are still not published.

We mention two paradigmatic classical results in this direction. Noriaki Kawanaka [64] calculated the number of *unipotent* elements in a given Bruhat cell BwB. In particular, his results imply that a regular unipotent class in any Chevalley group intersects *all* Bruhat cells BwB, $w \in W$. On the other hand, Robert Steinberg [83] proved that, among all unipotent classes, regular classes are *the only* ones that intersect the Bruhat cells BwB, where w is a Coxeter element. Recently, George Lusztig obtained very important progress in this direction, for arbitrary unipotent elements [72]–[75].

The point is that, for an *individual* element, the problem of listing the Bruhat cells intersected by its conjugacy class — let alone the precise description of these intersections — is *extremely* difficult. This is precisely why in most applications one considers *typical* elements, such as, in the context of the present paper, typical elements of one-parameter subgroups.

A large part of this difficulty is explained by the fact that the question as to whether a given conjugacy class C intersects a Bruhat cell BwB cannot be answered in terms of the conjugacy class of w in the Weyl group *alone*. Actually, this became apparent already in the study of two-dimensional transformations in $\mathrm{SL}(n, K)$, where some — but not all! — 4-cycles occur as possibilities for w; see [9, 41]. In the early 1990s, the authors obtained similar results also for some other types of transformations in $\mathrm{SL}(n, K)$, over an arbitrary field.

Recently, Erich Ellers and Nikolaĭ Gordeev proposed an *algorithm* which, in the case of an algebraically closed field, allows one to calculate, for each conjugacy class of the group $\mathrm{SL}(n, K)$, what Bruhat cells it intersects in terms of its Jordan form [59]. For semisimple elements this can be refined to an explicit answer, given in terms of multiplicities of eigenvalues.

Intersections of conjugacy classes with Bruhat cells, and other related problems were also addressed in recent papers by Nicoletta Cantarini, Giovanna Carnovale, and Mauro Costantini [53, 54, 60]. In particular, [54] completely describes the intersections $C \cap BwB$ for *spherical* conjugacy classes. Denote by W_C the set of all $w \in W$ such that $C \cap BwB \neq \varnothing$. Provided that the characteristic of the ground field is distinct from 2 and from the bad primes for G, the class C is spherical if and only if W_C consists entirely of involutions. Thus, in particular, the results of Carnovale [54] generalize results by the first author [17], but not the results of the present paper.

On the other hand, in [57] Chan Key Yuen, Lu Jiang-Hua, and To Kai Ming studied intersections of conjugacy classes with Birkhoff cells $C \cap BwB^-$. A typical result of that paper asserts that for each conjugacy class C there is a unique largest element $w_C \in W$

such that $C \cap Bw_C B^- \neq \varnothing$. In other words, if $C \cap BwB^- \neq \varnothing$ for some $w \in W$, then $w \leq w_C$, with respect to the Bruhat order.

We state another subproblem of Problem 4, somewhat harder than Problem 5. This problem was addressed in the recent paper by George Lusztig [73].

Problem 7. For given $x \in G(\Phi, K)$ and $w \in W(\Phi)$, calculate the dimension of the intersection $x^G \cap BwB$.

The authors dedicate this paper to Nikolaĭ Gordeev, who is our close colleague and personal friend for many decades. From the very start, we shared our interest to this subject with him, and for the last 25 years we discussed with him all aspects of this work, and the field at large.

We are very grateful to our colleagues, especially to Ernest Borisovich Vinberg, Alexander Efimovich Zalesskiĭ, Gary Seitz, and Roger Carter for many stimulating discussions. At the early stage, one of the most important challenges for us was to give a definitive form to the results contained in the remarkable preprint by Alexander Zalesskiĭ [32] which, unfortunately, stayed unpublished. Zalesskiĭ and Seitz suggested a simpler approach to the proof of the results of [11]. The present paper is a part of the implementation of this approach.

The main part of this work was executed in the late 1980s and in the early 1990s, when the second author worked in Berlin for some time, while Nikolaĭ Gordeev and the first author worked in Bielefeld. We are very grateful to our German colleagues, and especially to Herbert Abels, Anthony Bak, Ulf Rehmann, and Gernot Stroth, for their unfailing friendly support.

Unfortunately, at that time we could not finalize the proofs of our results on pairs of microweight tori and pairs of long root tori. Consequently, we never properly published what we perceived as auxiliary steps to this major goal. This program was recommenced in our recent joint papers with Vladimir Nesterov [20]–[25], powered by his new methods to describe such orbits; see [36]–[40].

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Department of Mathematics and Mechanics, St. Petersburg State University, Petrodvorets, St. Petersburg 198904, Russia
E-mail address: nikolai-vavilov@yandex.ru

Department of Mathematics and Mechanics, St. Petersburg State University, Petrodvorets, St. Petersburg 198904, Russia
E-mail address: semenov@math.spbu.ru

Received 9/SEP/2011

Translated by N. A. VAVILOV