QUASIISOMETRIC EMBEDDING OF THE FUNDAMENTAL GROUP OF AN ORTHOGONAL GRAPH-MANIFOLD INTO A PRODUCT OF METRIC TREES

A. SMIRNOV

ABSTRACT. In every dimension $n \geq 3$, a class of orthogonal graph-manifolds is introduced. It is proved that the fundamental group of any orthogonal graph-manifold embeds quasiisometrically into a product of n trees. As a consequence, it is shown that the asymptotic dimension and the linearly-controlled asymptotic dimensions of such a group are equal to n .

§1. INTRODUCTION

We introduce a class $\mathcal{O}$ of orthogonally glued higher-dimensional graph-manifolds (called *orthogonal graph-manifolds* throughout this paper; see Subsection 2.2 for the definition). Using the ideas of the paper [7], we generalize the results of that paper to the case of the class $\mathcal{O}$.

Theorem 1. *For every n -dimensional orthogonal graph-manifold, its fundamental group supplied with an arbitrary word metric admits a quasiisometric embedding into a product of n metric trees. As a consequence, the asymptotic and the linearly-controlled asymptotic dimension of such a group is equal to n .*

In the paper [7], this result was obtained in the 3-dimensional case for every graph-manifold in the sense of the definition given in Subsection 2.1. In fact, in accordance with the paper [8], the fundamental group of any 3-dimensional graph-manifold is quasiisometric to the fundamental group of some flip-manifold, which is precisely an orthogonal graph-manifold in dimension 3. Also note that the inequality $\text{asdim } \pi_1(M) \leq n$ for the fundamental group of an orthogonal graph-manifold M follows from the result obtained in [3].

§2. PRELIMINARIES

2.1. Graph-manifolds.

Definition. A higher-dimensional graph-manifold is a closed, n -dimensional with $n \geq 3$, orientable manifold M that is glued from finitely many blocks M_v , $M = \bigcup_{v \in V} M_v$, that satisfy the following conditions (1)–(3).

- (1) Each block M_v is a trivial T^{n-2} -bundle over a compact, orientable surface Φ_v with boundary (this surface should be different from the disk and the annulus), where T^{n-2} is the $(n-2)$ -dimensional torus.
- (2) The manifold M is glued from blocks M_v , $v \in V$, by diffeomorphisms between boundary components (the case of gluing boundary components of one and the same block is not excluded).

2010 *Mathematics Subject Classification.* Primary 57M15.

Key words and phrases. Graph-manifold, quasiisometric embedding.

Supported by RFBR (grant no. 11-01-00302-a).

- (3) The gluing diffeomorphisms do not identify the homotopy classes of the fiber tori.

For brevity, we use the term “graph-manifold” instead of the term “higher-dimensional graph-manifold”.

Let G be a graph dual to the decomposition of M into blocks. The set of blocks of the graph-manifold coincides with the vertex set $V = V(G)$ of the graph G . The set of (nonoriented) edges $E = E(G)$ of G consists of pairs of glued components of blocks. We denote the set of the oriented edges of G by W .

For more information about graph-manifolds, see [4].

2.2. Orthogonal graph-manifolds. In this subsection we define a class of graph-manifolds that admit an orthogonally glued metric of a special form. For brevity, we shall call them *orthogonal graph-manifolds*.

For a fixed graph G and each vertex $v \in V(G)$, consider a surface Φ_v of nonnegative Euler characteristic with $|\partial_v|$ boundary components, where ∂_v is the set of all edges adjacent to the vertex v . Moreover, we assume that there is a bijection between the set of boundary components of the surface Φ_v and the set of all oriented edges adjacent to v . For the block M_v corresponding to a vertex v , we fix a trivialization $M_v = \Phi_v \times S^1 \times \cdots \times S^1$, where Φ_v is the base surface, i.e., we fix simultaneously a trivialization $M_v = \Phi_v \times T^{n-2}$ of M_v and a trivialization $T^{n-2} = S^1 \times \cdots \times S^1$ of the fiber torus. For each block M_v , we fix a coordinate system $(x, x_1, \dots, x_{n-2})$ compatible with this decomposition, where $x \in \Phi_v$ and $x_i \in [0, 1)$ for each $1 \leq i \leq n-2$. For each oriented edge w adjacent to the vertex v , we define a coordinate system (x_0) , $x_0 \in [0, 1)$, on the corresponding component of the boundary $\partial\Phi_v$ of the surface Φ_v . This gives rise to the coordinate system $(x_0, \dots, x_{n-2})$ on the boundary torus T_w of the block M_v . Similarly, for the edge $-w$ inverse to the edge w on the boundary torus T_{-w} , we define a coordinate system $(x'_0, \dots, x'_{n-2})$. For each oriented edge w , we consider a permutation $\mathfrak{s}_w$ of a well-ordered $(n-1)$ -element set $(x_0, \dots, x_{n-2})$ such that $\mathfrak{s}_w(x_0) \neq x_0$. Furthermore, we assume that for mutually inverse edges w and $-w$ the permutations $\mathfrak{s}_w$ and $\mathfrak{s}_{-w}$ are inverse to each other ($\mathfrak{s}_{-w} \circ \mathfrak{s}_w = \text{id}$). We define the gluing map $\eta_w : T_w \rightarrow T_{-w}$ by $\eta_w((x_0, \dots, x_{n-2})) = (\mathfrak{s}_w(x_0), \dots, \mathfrak{s}_w(x_{n-2}))$. Note that this map is a well-defined gluing, because the permutations $\mathfrak{s}_w$ and $\mathfrak{s}_{-w}$ are selected to be mutually inverse. Also, the map η_w does not identify the homotopy classes of fiber tori.

Definition. The graph-manifold described above is called an *orthogonal graph-manifold*.

Remark 1. As was mentioned above, for $n = 3$ the class of all orthogonal graph-manifolds coincides with that of all flip graph-manifolds considered in [8].

2.3. Metric trees. A *tripod* in a geodesic metric space X is a union of three geodesic segments $xt \cup yt \cup zt$ that have only one common point t . A geodesic metric space X is called a *metric tree* if each triangle in it is a tripod (possibly degenerate).

2.4. Finitely generated groups. Let G be a finitely generated group and $S \subset G$ a finite symmetric generating set for G ($S^{-1} = S$). Recall that a *word metric* on the group G (with respect to S) is the left-invariant metric determined by the norm $\|\cdot\|_S$, where for each $g \in G$ its norm $\|g\|_S$ is the smallest number of elements of S whose product is g . It is known that all such metrics for the group G are bi-Lipschitz equivalent (see [2]). In this paper we shall consider only finitely generated groups with a word metric.

2.5. Quasiisometric maps. A map $f : X \rightarrow Y$ is said to be *quasiisometric* if there exist $\lambda \geq 1$, $C \geq 0$ such that

$$\frac{1}{\lambda}|xy| - C \leq |f(x)f(y)| \leq \lambda|xy| + C$$

for each $x, y \in X$. Two metric spaces X and Y are *quasiisometric* if there is a quasiisometric map $f: X \rightarrow Y$ such that $f(X)$ is a net in Y . In this case, f is called a *quasiisometry*.

2.6. Metric on the universal cover. We recall the famous Milnor–Švarc lemma.

Lemma 1. *Let Y be a compact length space, and let X be the universal cover of X with the metric lifted from Y . Then X is quasiisometric to the fundamental group $\pi_1(Y)$ of Y with an arbitrary word metric.*

From this lemma it follows that to prove Theorem 1 it suffices to construct a quasiisometric embedding of the universal cover into a product of n trees.

2.6.1. Metrics of nonpositive curvature. We define a metric on the orthogonal graph-manifold M as follows: for each edge $e \in E(G)$, on its corresponding torus T_e we take a flat metric such that any base circle of the coordinate system described above has length 1 and any two of these circles are perpendicular.

In particular, for each vertex $v \in V(G)$, there is a metric on the boundary surface Φ_v in which every boundary component has length 1. This metric extends to a metric of nonpositive curvature on the surface Φ_v so that its boundary is geodesic. Therefore, from the boundary tori, the metric extends to a metric on the block M_v that is locally a product metric (in general, a metric on the block may fail to be a product metric, and it can have nontrivial holonomy along some loops on the base Φ_v).

In what follows we only consider such metrics on orthogonal graph-manifolds. If we lift the above metric to the universal cover $\widetilde{M}$, then the Reshetnyak gluing theorem (see [2]) shows that the resulting metric space is a nonpositively curved (or Hadamard) space.

We fix an orthogonal graph-manifold M with the metric described above.

2.6.2. The standard hyperbolic surface H_0 with boundary. Consider the hyperbolic plane $\mathbb{H}_\kappa^2$ having a curvature $-\kappa$ ($\kappa > 0$) such that the side of a rectangular equilateral hexagon θ in the plane $\mathbb{H}_\kappa^2$ has length 1. Let ρ be the distance between the middle points of sides that have a common adjacent side, and let δ be the diameter of θ . We mark each second side of θ (so, three sides will be marked) and consider a set H_0 defined as follows. Take the subgroup G_θ of the isometry group of $\mathbb{H}_\kappa^2$ generated by reflections in the (three) marked sides of θ , and let H_0 be the orbit of θ with respect to G_θ . Then H_0 is a convex subset in $\mathbb{H}_\kappa^2$ split into hexagons that are isometric to θ . Furthermore, the boundary of H_0 has infinitely many connected components each of which is a geodesic $\mathbb{H}_\kappa^2$. The graph T_{bin} dual to the decomposition of H_0 into hexagons is the standard binary tree whose vertices all have degree three. Any metric space isometric to H_0 will be called a θ -tree. Given a vertex p of T_{bin} , we denote by θ_p the respective hexagon in H_0 .

Remark 2. In what follows, we view T_{bin} as a metric space with a metric such that the length of each edge is equal to 2ρ . Then these metric spaces are metric trees. We denote the set of vertices in T_{bin} by $V(T_{\text{bin}})$.

2.6.3. Standard metrics and bi-Lipschitz homeomorphisms between bases. Consider a simplicial tree with each vertex of degree 3, and with each edge of length 1. We replace each edge by the rectangle $1 \times 1/3^{100}$ and each vertex by the equilateral Euclidean triangle with each side equal to $1/3^{100}$. Then we glue them in a natural way.

Definition. The resulting metric space is called a *fattened tree* or a *standard surface* and is denoted by X_0 .

Remark 3. Note that the standard surface X_0 is bi-Lipschitz homeomorphic to the θ -tree H_0 . So, we fix an arbitrary bi-Lipschitz homeomorphism $h_0: H_0 \rightarrow X_0$.

Definition. The *standard block* is defined to be a metric product of the θ -tree H_0 and $n - 2$ copies of the Euclidean line $\mathbb{R}$, $B = H_0 \times \mathbb{R} \times \cdots \times \mathbb{R}$.

The partition of the θ -tree into hexagons induces a partition of each boundary component of the θ -tree into unit segments. For each boundary component such a partition is called a *grid* on this component. For each Euclidean factor $\mathbb{R}$ of the standard block, an arbitrary partition by unit segments is called a *grid* on this factor. Finally, for each boundary hyperplane σ of the standard block a partition into unit cubes induced by grids on each factor is called a *grid* on this hyperplane.

Recall the following theorem.

Theorem 2 ([1, Theorem 1.2]). *Let X_0 be as above, with a chosen boundary component $\partial_0 X_0$. Then there exists $K > 0$ and a function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ such that for any $K_0 > 0$ and any K_0 -bi-Lipschitz homeomorphism P_0 from $\partial_0 X_0$ to a boundary component $\partial_1 X_0$, P_0 extends to a $\psi(K_0)$ -bi-Lipschitz homeomorphism $P : X_0 \rightarrow X_0$ that is K -bi-Lipschitz on every other boundary component.*

Corollary 1. *Let $\tilde{\Phi}_v$ be the universal cover of the surface Φ_v supplied with the metric described in Subsection 2.6.1, with a fixed boundary component $\partial_0 \tilde{\Phi}_v$. Then there exists $K > 0$ and a function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ such that for any $K_0 > 0$ and any K_0 -bi-Lipschitz homeomorphism P_0 from $\partial_0 \tilde{\Phi}_v$ to a boundary component $\partial_0 H_0$, P_0 extends to a $\psi(K_0)$ -bi-Lipschitz homeomorphism $P : \tilde{\Phi}_v \rightarrow H_0$ that is K -bi-Lipschitz on every other boundary component.*

Proof. Rename the K_0 , K , and ψ from Theorem 2 to $\bar{K}_0$, $\bar{K}$, and $\bar{\psi}$ respectively. Note that there is a bi-Lipschitz homeomorphism $\psi_v : X_0 \rightarrow \tilde{\Phi}_v$.

$$\begin{array}{ccc} X_0 & \longrightarrow & X_0 \\ \psi_v \downarrow & & \uparrow h_0 \\ \tilde{\Phi}_v & \longrightarrow & H_0 \end{array}$$

Let ψ_v be M_1 -bi-Lipschitz and h_0 M_2 -bi-Lipschitz. We set $M := \max\{M_1, M_2\}$. Consider $P_0 : \partial_0 \tilde{\Phi}_v \rightarrow \partial_0 H_0$ and denote the boundary component $h_0(\partial_0 H_0)$ by $\partial_1 X_0$. Then, if $\partial_0 X_0 = \psi_v^{-1}(\partial_0 \tilde{\Phi}_v)$, the map

$$h_0 \circ P_0 \circ \psi_v : \partial_0 X_0 \rightarrow \partial_1 X_0$$

is an $M^2 \cdot K_0$ -bi-Lipschitz homeomorphism. By Theorem 2, it extends to a $\bar{\psi}(M^2 \cdot K_0)$ -bi-Lipschitz homeomorphism that is $\bar{K}$ -bi-Lipschitz on each remaining boundary component of the space X_0 . Therefore, the homeomorphism $P = h_0^{-1} \circ \bar{P} \circ \psi_v^{-1}$ is $\psi(K_0) = M^2 \bar{\psi}(M^2 \cdot K_0)$ -bi-Lipschitz. Moreover, on every other boundary component it is $K = M^2 \cdot \bar{K}$ -bi-Lipschitz. Therefore, taking $\psi(x) = \bar{\psi}(M^2 \cdot x)$, $x \in \mathbb{R}$, and $K = M^2 \cdot \bar{K}$, we prove Corollary 1. $\square$

Remark 4. Since the graph G is finite, we may assume that the number K and the function ψ are independent of $v \in V(G)$.

2.6.4. The special metric on the universal cover. In this subsection, for each orthogonal graph-manifold M , we inductively construct a special metric on its universal covering $\tilde{M}$ so that $\tilde{M}$ with this metric is quasiisometric to the fundamental group $\pi_1(M)$ of the graph-manifold M . After that it will suffice to construct a quasiisometric embedding of $\tilde{M}$ into a product of n trees.

The decomposition of M into blocks lifts to a decomposition of $\tilde{M}$ into universal cover blocks, see [4]. We denote the tree dual to this decomposition by T_0 . Note that the

degree of every vertex of T_0 is infinite. On T_0 , we consider an intrinsic metric with edges of length 1. Choose a vertex $o \in V(T_0)$ in the tree T_0 and call it the *root*. For each vertex $v \in V(T_0)$, we define its rank $r(v)$ as the distance to o . In particular, $r(o) = 0$. For each vertex $v \in V(T_0)$, we denote by $\widetilde{M}_v$ the corresponding block of the space $\widetilde{M}$. This block is isometric to the product $\widetilde{\Phi}_v \times \mathbb{R}^{n-2}$. Recall that on each boundary hyperplane of the block $\widetilde{M}_v$ we have fixed a coordinate system. We call the axes of this system the *selected axes*.

By induction on the rank of vertices of the tree T_0 , on the space $\widetilde{M}$ we construct a metric of special type that is bi-Lipschitz homeomorphic to the metric lifted from the graph-manifold M .

Base of induction. Let $\psi_o : \widetilde{\Phi}_o \rightarrow H_0$ be a bi-Lipschitz homeomorphism. Consider an isometric copy of the standard block B_o and the map $\psi'_o : \widetilde{M}_o \rightarrow B_o$ that is a direct product of the map $\psi_o : \widetilde{\Phi}_o \rightarrow H_0$ and the identity map $\text{id} : \mathbb{R}^{n-2} \rightarrow \mathbb{R}^{n-2}$.

Inductive step. Suppose that for all vertices $v \in V(T_0)$ such that $r(v) \leq m$ we have built a bi-Lipschitz homeomorphism $\psi'_v : \widetilde{M}_v \rightarrow B_v$, where B_v is an isometric copy of the standard block. Let a vertex $u \in V(T_0)$ be such that $r(u) = m + 1$. There is a unique vertex $v \in V(T_0)$ adjacent to it such that $r(v) = m$. Consider the blocks $\widetilde{M}_v$ and $\widetilde{M}_u$ of the universal cover $\widetilde{M}$ and denote the covering map by $\text{pr} : \widetilde{M} \rightarrow M$. Recall that the gluing of the blocks $\text{pr}(\widetilde{M}_v)$ and $\text{pr}(\widetilde{M}_u)$ is obtained by a permutation $\mathfrak{s}$ of the coordinate system on the torus T_e . Consider an isometric copy B_u of the standard block and glue it to the block B_v by the permutation $\mathfrak{s}^{-1}$ of the coordinates along the corresponding hyperplanes, thus matching the grid on them. By induction, the map $\psi'_v : \widetilde{M}_v \rightarrow B_v$ is the direct product of the map $\widetilde{\Phi}_v \rightarrow H_0$ and $n - 1$ maps $\mathbb{R} \rightarrow \mathbb{R}$. Moreover, the restriction of each of these maps to the intersection with the common boundary hyperplane of the blocks $\widetilde{M}_v$ and $\widetilde{M}_u$ is a K -bi-Lipschitz homeomorphism onto its image. From the orthogonality of the gluing it follows that these restrictions induce a K -bi-Lipschitz homeomorphism $\partial_1 \psi_u : \partial_0 \widetilde{\Phi}_u \rightarrow \partial_0 H_0$ from the boundary component $\partial_0 \widetilde{\psi}_u$ of the surface $\widetilde{\Phi}_u$ adjacent to the block $\widetilde{M}_v$ to the boundary component $\partial_0 H_0$ of the θ -tree adjacent to the block B_v . Also, these restrictions induce a collection of K -bi-Lipschitz homeomorphisms $\partial_i : \mathbb{R} \rightarrow \mathbb{R}$, each of which maps the corresponding $\mathbb{R}$ -factor of the decomposition $\widetilde{M}_u = \widetilde{\Phi}_u \times \mathbb{R} \times \cdots \times \mathbb{R}$ to the corresponding $\mathbb{R}$ -factor of the decomposition $B_u = H_0 \times \mathbb{R} \times \cdots \times \mathbb{R}$.

By Corollary 1, the homeomorphism $\partial_1 \psi_u$ extends to a $\psi(K)$ -bi-Lipschitz homeomorphism $\psi_u : \widetilde{\Phi}_u \rightarrow H_0$ that is K -bi-Lipschitz on every other boundary component. We define the homeomorphism ψ'_u as the direct product of the homeomorphism ψ_u and $n - 3$ homeomorphisms ∂_i ($i = 2, \dots, n - 2$).

Now we construct a map $\psi_M : \widetilde{M} \rightarrow X$, where X is a metric space obtained by gluing the blocks described above. Namely, if the point x lies in the block $\widetilde{M}_v$ for some vertex $v \in V(T_0)$, we define $\psi_M(x) := \psi'_v(x)$. The map ψ_M is well defined, because the maps ψ'_v are compatible with each other.

Proposition 1. *The map constructed above is a bi-Lipschitz homeomorphism.*

Proof. Let $C = \max\{K, \psi(K)\}$. From the construction it follows that the map ψ'_u is C -bi-Lipschitz for each vertex $u \in V(T_0)$. Suppose that $x \in \widetilde{M}_v$ for some vertex $v \in V(T)$ and $y \in \widetilde{M}_u$ for some vertex $u \in V(T)$. Denote $x' := \psi_M(x)$ and $y' := \psi_M(y)$.

Let γ be a geodesic between the vertices v and u in the tree T . Denote its consecutive edges by $e_1, \dots, e_k$. Note that a geodesic $xy \subset \widetilde{M}$ consecutively intersects the hyperplanes $\sigma_1, \dots, \sigma_k$ in the space $\widetilde{M}$ that correspond to these edges. Similarly, a geodesic $x'y' \subset X$ consecutively intersects hyperplanes $\sigma'_1, \dots, \sigma'_k$ in the space X . Moreover, $\sigma'_i = \psi_M(\sigma_i)$.

Let z_i be an intersection point of the geodesic xy and the hyperplane σ_i . (We assume that $z_0 = x$, $z_{k+1} = y$.)

Let $z'_i = \psi_M(z_i)$. Since for each vertex v the restriction of the map ψ_M to the block $\widetilde{M}_v$ is C -bi-Lipschitz and the points z_i and z_{i+1} ($i = 0, \dots, k$) lie in the same block, we have $|z'_i z'_{i+1}| \leq C|z_i z_{i+1}|$. Combining all these inequalities, we see that $\sum_{i=0}^k |z'_i z'_{i+1}| \leq C|xy|$. On the other hand, by the triangle inequality we have $|x'y'| \leq \sum_{i=0}^k |z'_i z'_{i+1}|$. This implies that $|x'y'| \leq C|xy|$. Similarly, $|xy| \leq C|x'y'|$. $\square$

Thus, a special type metric can be defined on the universal cover of the orthogonal graph-manifold. Such a metric has nonpositive curvature in the sense of Alexandrov. Moreover, for every vertex $v \in V(T_0)$ the corresponding block $\widetilde{M}_v$ is isometric to the direct product of the θ -tree H_0 and $n - 2$ factors $\mathbb{R}$.

We introduce some technical notation that will be required later. For each vertex v and the corresponding block of $\widetilde{M}_v$, we denote by X_v a copy of the corresponding θ -tree. Moreover, denote by T_v a copy of the tree T_{bin} naturally (isometrically) embedded in the surface X_v and considered with the metric described above. Let G'_θ be the isometry group of the θ -tree. Note that for each vertex v there exists a 2δ -Lipschitz retraction $r_v : X_v \rightarrow T_v$ equivariant under the action of G'_θ .

Recall that the block $\widetilde{M}_v$ is a product $X_v \times \mathbb{R} \times \dots \times \mathbb{R}$. Denote the projections to the corresponding factors by $p_v^1, \dots, p_v^{n-1}$.

For each point $x \in \widetilde{M}_v$, consider the map given by

$$\pi_x(y) := (y, p_v^2(x), \dots, p_v^{n-1}(x)) \text{ for every point } y \in X_v.$$

We call an orthogonal graph-manifold *irreducible* if its universal cover $\widetilde{M}$ is not a product of a Euclidean space and the universal cover of a orthogonal graph-manifold of lower dimension. It suffices to prove Theorem 1 for the irreducible case.

§3. TREES T_c AND MAPS TO THEM

3.1. Construction of trees. Let $\gamma = w_1 \dots w_k$ be an oriented path in the tree T_0 . Denote by $\mathfrak{s}_\gamma$ the permutation $\mathfrak{s}_{w_k} \circ \dots \circ \mathfrak{s}_{w_1}$ of a well-ordered $(n - 1)$ -element set.

Suppose that vertices $u, v \in T_0$ are connected by two oriented paths γ_1 and γ_2 . Note that $\mathfrak{s}_{\gamma_1} = \mathfrak{s}_{\gamma_2}$; therefore, we can define the permutation $\mathfrak{s}_{uv}$ as the permutation $\mathfrak{s}_\gamma$ along any path γ between u and v . Furthermore, $\mathfrak{s}_{v_1 v_3} = \mathfrak{s}_{v_2 v_3} \circ \mathfrak{s}_{v_1 v_2}$ and $\mathfrak{s}_{uv} = \mathfrak{s}_{vu}^{-1}$.

We define a relation $\sim$ on the set of vertices of T_0 as follows: $u \sim v$ if and only if the permutation $\mathfrak{s}_{uv}$ fixes the smallest element. It is easy to check that the relation $\sim$ is an equivalence relation.

We prove that the relation $\sim$ splits the set $V(T_0)$ into at most $n - 1$ equivalence classes. Indeed, if this fails, then we can choose n pairwise nonequivalent vertices $v_1, \dots, v_n$. Then, for some different $2 \leq i, j \leq n$, we have $\mathfrak{s}_{v_1 v_i}(x_0) = \mathfrak{s}_{v_1 v_j}(x_0)$, where x_0 is the smallest element. Then, since $\mathfrak{s}_{v_i v_j}(x_0) = \mathfrak{s}_{v_1 v_j} \circ \mathfrak{s}_{v_i v_1}(x_0) = x_0$, we have $v_i \sim v_j$, a contradiction.

Fix a vertex u in the tree T_0 . Since the manifold M is irreducible, the set of permutations $\{\mathfrak{s}_{vu} \mid u, v \in V(T_0)\}$ is transitive. That is, for each element x of a well-ordered $(n - 1)$ -element set, there is a vertex $v \in V(T_0)$ such that $\mathfrak{s}_{vu}(x_0) = x$. It follows that there are at least $n - 1$ different equivalence classes. Hence, there are exactly $n - 1$ of them.

Denote by $\mathcal{C}$ the set of all these classes. We have shown that $|\mathcal{C}| = n - 1$. Given $c \in \mathcal{C}$, note that if $u, v \in c$, then for any vertex $v' \in V(T_0)$ we have $\mathfrak{s}_{uv'}(x_0) = \mathfrak{s}_{vv'}(x_0)$.

Fix a vertex $v \in V(T_0)$ and an equivalence class $c \in \mathcal{C}$. We construct a tree $T_{v,c}$ as follows. If v belongs to c , then we set $T_{v,c} := T_v$, see the end of §2. Otherwise, we set $T_{v,c} := \mathbb{R}$.

Also, for each vertex $v \in V(T_0)$ we construct a map $r_{v,c} : \widetilde{M}_v \rightarrow T_{v,c}$. If v belongs to c , then we set $r_{v,c} := r_v \circ p_v^1$. Otherwise, if v belongs to some class $c' \neq c$, then we set $r_{v,c} = p_v^k$, where $k = \mathfrak{s}_{uv}(x_0)$, $u \in c$, and x_0 is the smallest element.

For each class $c \in \mathcal{C}$, we construct a tree of T_c as follows. For each pair of adjacent vertices $u, v \in V(T_0)$, we say that a point $x \in T_{u,c}$ and a point $y \in T_{v,c}$ are $\sim_c$ -equivalent if there exists a point $z \in \widetilde{M}_u \cap \widetilde{M}_v$ such that $x = r_{u,c}(z) = r_{v,c}(z) = y$.

This relation is well defined. Indeed, for every point $x \in T_{u,c}$, the preimage $r_{u,c}^{-1}(x) \cap \widetilde{M}_u \cap \widetilde{M}_v$ is an $(n-2)$ -dimensional subspace orthogonal to the coordinate $\mathfrak{s}_{v'u}(x_0)$, where the vertex v' belongs to c . Similarly, for each point $y \in T_{v,c}$, the preimage $r_{v,c}^{-1}(y) \cap \widetilde{M}_u \cap \widetilde{M}_v$ is an $(n-2)$ -dimensional subspace orthogonal to the coordinate $\mathfrak{s}_{v'u}(x_0)$. But by the definition of the coordinates $\mathfrak{s}_{v'u}(x_0)$ and $\mathfrak{s}_{v'v}(x_0)$, any two such subspaces either are disjoint or coincide. This implies the following lemma.

Lemma 2. *Let $u, v \in V(T_0)$ be a pair of adjacent vertices, and let $c \in \mathcal{C}$ be an equivalence class. If the points $x, x' \in T_{u,c}$ and $y \in T_{v,c}$ are such that $x \sim_c y$ and $x' \sim_c y$, then $x = x'$.*

Extend the relation $\sim_c$ by transitivity. This means that we set $x \sim_c y$ if and only if there exists a chain $x = x_0, \dots, x_l = y$ such that $x_i \sim_c x_{i+1}$ and $x_i \in T_{v_i,c}$ for each $0 \leq i \leq l-1$, and the vertices v_i and v_{i+1} are adjacent in the tree T_0 . From Lemma 2, it follows that the relation $\sim_c$ is an equivalence relation.

Lemma 3. *Let $c \in \mathcal{C}$ be an equivalence class. Fix any pair of vertices $u, v \in c$ and consider points $x, y \in T_{u,c}$ and points $x', y' \in T_{v,c}$ such that $x \sim_c x'$ and $y \sim_c y'$. Then $|xy| = |x'y'|$.*

Proof. Indeed, let $u = v_0, \dots, v_l = v$ be consecutive vertices of the geodesic between two vertices u and v in the tree T_0 . Note that it suffices to consider the case where the vertices $v_1, \dots, v_{l-1}$ are not in the class c . For each $0 \leq i \leq l-1$, denote by σ_i the common hyperplane of the blocks $\widetilde{M}_{v_i}$ and $\widetilde{M}_{v_{i+1}}$. Let ∂X_u be the boundary component of the θ -tree X_u corresponding to the hyperplane σ_0 , and let ∂X_v be the boundary component of the θ -tree X_v corresponding to the hyperplane σ_{l-1} . By the construction of the metric on the universal cover $\widetilde{M}$, for each interval I_0 of the grid on the boundary component ∂X_u , there are segments $I_1, \dots, I_{l-2}$ of the grids on the corresponding $\mathbb{R}$ -factors and the segment I_{l-1} of the grid on the boundary component ∂X_v such that for each $1 \leq i \leq l-1$ we have $p_{v_i}^j(I_{i-1}) = p_{v_i}^j(I_i)$, where $j = \mathfrak{s}_{uv_i}(x_0)$. Note that it suffices to prove the lemma for arbitrary points $x, y \in r_{u,c}(I_0) \subset T_{u,c}$. But for such points we have $x', y' \in r_{v,c}(I_{l-1})$; hence, by the equivariance of the retractions r_u and r_v , the required identity $|xy| = |x'y'|$ is satisfied. $\square$

Fix $c \in \mathcal{C}$. Define the space T_c as the factor $\{\bigsqcup T_{v,c} \mid v \in V(T_0)\} / \sim_c$. From Lemma 3, it follows that the resulting space is a metric tree. Moreover, for each vertex $v \in c$ the natural embedding $proj_{v,c} : T_{v,c} \rightarrow T_c$ is isometric.

Thus, for each $c \in \mathcal{C}$ we have constructed a tree T_c , which is naturally divided into blocks $T_{v,c}$.

3.2. Maps to trees. In the remaining part of this paper, we regard the product of $|\mathcal{C}| + 1 = n$ constructed trees $T_0 \times \prod_{c \in \mathcal{C}} T_c$ as a metric space with the sum metric. This means that the distance between two points $x, y \in T_0 \times \prod_{c \in \mathcal{C}} T_c$ is defined as the sum of

the distances between their projections to the trees T_0, T_c , for each $c \in \mathcal{C}$:

$$|xy| = |x_0y_0|_{T_0} + \sum_{c \in \mathcal{C}} |x_cy_c|_{T_c}.$$

We define a map $\phi_c : \widetilde{M} \rightarrow T_c$ by the formula

$$\phi_c(x) := (\text{proj}_{v,c} \circ r_{v,c})(x),$$

where $x \in \widetilde{M}_v$ and $\text{proj}_{v,c}$ is the natural embedding of the tree $T_{v,c}$ in the tree T_c . The definition of the maps $r_{v,c}$ and the construction of the tree T_c show that the map ϕ_c is well defined on the intersection $\widetilde{M}_u \cap \widetilde{M}_v$ of each pair of adjacent blocks $\widetilde{M}_u$ and $\widetilde{M}_v$.

The definition of the map $r_{v,c}$ implies that ϕ_c is 2δ -Lipschitz. To prove this, it suffices to show that

$$|\phi_c(x)\phi_c(y)| \leq 2\delta|xy|,$$

where x and y belong to the same block $\widetilde{M}_v$. This fact follows from the definition of ϕ_c .

Define a map $\phi_0 : \widetilde{M} \rightarrow T_0$ as follows. If $x \in \widetilde{M}_v$ and $x \notin \widetilde{M}_u$ for any other vertex $u \in V(T_0)$, then we set $\phi_0(x) = v$. Otherwise, if $x \in \widetilde{M}_u \cap \widetilde{M}_v$ and $r(u) < r(v)$, we set $\phi_0(x) = u$. It is clear that $|\phi_0(x)\phi_0(y)| \leq |xy| + 1$.

Let a map $\phi : \widetilde{M} \rightarrow T_0 \times \prod_{c \in \mathcal{C}} T_c$ be defined by $\phi := \phi_0 \times \prod_{c \in \mathcal{C}} \phi_c$.

Then

$$|\phi(x)\phi(y)| = |\phi_0(x)\phi_0(y)| + \sum_{c \in \mathcal{C}} |\phi_c(x)\phi_c(y)| \leq (2\delta(n-1) + 1)|xy| + 1.$$

§4. SPECIAL CURVES

For a curve γ in a metric space X we denote its length by $|\gamma|$. We need the following lemma.

Lemma 4. *Let $x, y \in \widetilde{M}$ be a pair of points. Then there exists a curve $\gamma \subset \widetilde{M}$ between them such that $|\gamma| \leq (2\delta + 1)|\phi(x)\phi(y)| + 2\delta$.*

Proof. Consider vertices $u, v \in V(T_0)$ such that $x \in \widetilde{M}_v$, $y \in \widetilde{M}_u$. We prove the lemma by induction on the length of the path $|uv|_{T_0}$.

Base of induction. There exists a vertex $v \in V(T_0)$ such that $x, y \in \widetilde{M}_v$. In this case, the geodesic xy does the job. Indeed,

$$|xy| = \sqrt{|p_1^v(x)p_1^v(y)|^2 + \cdots + |p_{n-1}^v(x)p_{n-1}^v(y)|^2},$$

which does not exceed

$$2\delta + \sum_{c \in \mathcal{C}} |\phi_c(x)\phi_c(y)| \leq (2\delta + 1)|\phi(x)\phi(y)| + 2\delta.$$

Inductive step. Fix vertices $u, v \in V(T_0)$. Let η_0 be a geodesic between u and v in the tree T_0 . Choose vertices $u', v' \in \eta_0$ such that the vertices u and u' as well as the vertices v and v' are adjacent. Note that either $r(u) > r(u')$ or $r(v) > r(v')$. Without loss of generality, assume that $r(v) > r(v')$. We may also assume that the point x does not belong to the block $\widetilde{M}_{v'}$. In this case $\phi_0(x) = v$, and for any point $x' \in \widetilde{M}_v \cap \widetilde{M}_{v'}$ we have $\phi_0(x') = v'$. It follows that $|\phi_0(x)\phi_0(x')| = 1$.

Assume that the vertex v belongs to a class $c \in \mathcal{C}$. Consider a geodesic η between $\phi(x)$ and $\phi(y)$ and denote by η_c its projection to the tree. Note that the curve η_c is a geodesic between $\phi_c(x)$ and $\phi_c(y)$. We divide the curve η_c into two parts $\eta_c^1 = \eta_c \cap T_{v,c}$ and $\eta_c^2 = \eta_c \setminus \eta_c^1$. Let z be the end of the curve η_c^1 different from $\phi_c(x)$.

Recall that $\pi_x : X_v \rightarrow \widetilde{M}_v$ is a horizontal embedding such that the image contains the point x . Next, we assume that the tree $T_{v,c}$ is naturally embedded in the θ -tree X_v .

Define a map π'_x as the composition of this embedding and the map π_x . Let γ_c denote the image of the curve η_c^1 under the map π'_x . We set $x_0 := \pi'_x(\phi_c(x))$ and $x_1 := \pi'_x(z)$. Then $|xx_0| < \delta$, and there exists a point $x_2 \in \widetilde{M}_v \cap \widetilde{M}_{v'}$ such that $x_2 \in \pi_x(X_v)$, $\phi_c(x_2) = z$, and for any $c \neq c' \in \mathcal{C}$ we have $\phi_{c'}(x_2) = \phi_{c'}(x)$. Note that $|x_1x_2| < \delta$ and

$$|\phi_{c'}(x)\phi_{c'}(x_2)| + |\phi_{c'}(x_2)\phi_{c'}(y)| = |\phi_{c'}(x)\phi_{c'}(y)|.$$

On the other hand, since $\phi_c(x_2) = z$, we have

$$|\phi_c(x)\phi_c(x_2)| + |\phi_c(x_2)\phi_c(y)| = |\phi_c(x)z| + |z\phi_c(y)| = |\phi_c(x)\phi_c(y)|.$$

Finally, we note that the point $\phi_0(x_2)$ belongs to the geodesic between $\phi_0(x)$ and $\phi_0(y)$ in the tree T_0 . It follows that

$$|\phi_0(x)\phi_0(x_2)| + |\phi_0(x_2)\phi_0(y)| = |\phi_0(x)\phi_0(y)|.$$

This means that

$$|\phi(x)\phi(x_2)| + |\phi(x_2)\phi(y)| = |\phi(x)\phi(y)|,$$

and $x_2 \in \widetilde{M}_{v'}$. By induction, there exists a curve γ' between x_2 and y with

$$|\gamma'| \leq (2\delta + 1)|\phi(x_2)\phi(y)| + 2\delta.$$

Consider the curve γ that is the union of the geodesic xx_0 , the curve γ_c , the geodesic x_1x_2 , and the curve γ' . We have

$$|\gamma| = |xx_0| + |\gamma_c| + |x_1x_2| + |\gamma'| \leq 2\delta + |\gamma_c| + |\gamma'| \leq 2\delta + |\phi_c(x)\phi_c(x_2)| + |\gamma'|,$$

which by induction does not exceed

$$2\delta + |\phi(x)\phi(x_2)| + (2\delta + 1)|\phi(x_2)\phi(y)| + 2\delta.$$

We have shown that $|\phi_0(x)\phi_0(x_2)| = 1$; therefore,

$$2\delta + |\phi(x)\phi(x_2)| = 2\delta|\phi_0(x)\phi_0(x_2)| + |\phi(x)\phi(x_2)| \leq (2\delta + 1)|\phi(x)\phi(x_2)|.$$

So,

$$\begin{aligned} |\gamma| &\leq 2\delta + |\phi(x)\phi(x_2)| + (2\delta + 1)|\phi(x_2)\phi(y)| + 2\delta \\ &\leq (2\delta + 1)(|\phi(x)\phi(x_2)| + |\phi(x_2)\phi(y)|) + 2\delta, \end{aligned}$$

whence $|\gamma| \leq (2\delta + 1)(|\phi(x)\phi(y)|) + 2\delta$. □

Corollary 2. *For any points $x, y \in \widetilde{M}$ we have $|xy| \leq (2\delta + 1)|\phi(x)\phi(y)| + 2\delta$.*

Applying the above inequalities, we obtain

$$|xy|/(2\delta + 1) - 2\delta/(2\delta + 1) \leq |\phi(x)\phi(y)| \leq (2\delta(n - 1) + 1)|xy| + 1,$$

which yields

$$1/C|xy| - 1 \leq |\phi(x)\phi(y)| \leq C|xy| + 1$$

with $C = \max\{2\delta + 1, 2\delta(n - 1) + 1\}$. This completes the proof of Theorem 1.

§5. ASYMPTOTIC DIMENSIONS

5.1. Definitions. We recall some basic definitions and notation. Let X be a metric space. We denote by $|xy|$ the distance between $x, y \in X$, and

$$d(U, V) := \inf\{|uv| \mid u \in U, v \in V\}$$

is the distance between $U, V \subset X$.

We say that a family $\mathcal{U}$ of subsets of X is a *covering* if for each point $x \in X$ there is a subset $U \in \mathcal{U}$ such that $x \in U$. A family $\mathcal{U}$ of sets is *disjoint* if each two sets $U, V \in \mathcal{U}$ are disjoint. The union $\mathcal{U} = \bigcup\{U^\alpha \mid \alpha \in \mathcal{A}\}$ of disjoint families U^α is said to be *n-colored*, where $n = |\mathcal{A}|$ is the cardinality of $\mathcal{A}$.

Also, recall that a family $\mathcal{U}$ is *D-bounded* if the diameter of every $U \in \mathcal{U}$ does not exceed D , $\text{diam } U \leq D$. An *n-colored* family of sets $\mathcal{U}$ is *r-disjoint* if for every color $\alpha \in \mathcal{A}$ and each two sets $U, V \in \mathcal{U}^\alpha$ we have $d(U, V) \geq r$.

The linearly-controlled asymptotic dimension is a version of the Gromov asymptotic dimension asdim .

Definition (Gromov [6]). The *asymptotic dimension* $\text{asdim } X$ of a metric space X is the smallest integer n such that for each sufficiently large real R there exists an $(n+1)$ -colored, R -disjoint, D -bounded covering of the space X , where the number $D > 0$ is independent of R .

Definition (Roe [9]). The *linearly-controlled asymptotic dimension* $\ell\text{-asdim } X$ of a metric space X is the smallest integer n such that for each sufficiently large real R there exists an $(n+1)$ -colored, R -disjoint, CR -bounded covering of the space X , where the number $C > 0$ is independent of R .

From the definition it follows that $\text{asdim } X \leq \ell\text{-asdim } X$ for any metric space X .

Now we show that the fundamental group of an orthogonal graph-manifold satisfies $n \leq \text{asdim } \pi_1(M) \leq \ell\text{-asdim } \pi_1(M) \leq n$.

5.2. Upper and lower bounds. Recall some properties of the above dimensions.

Let X and Y be metric spaces. If X is quasiisometric to Y , then $\text{asdim } X = \text{asdim } Y$ and $\ell\text{-asdim } X = \ell\text{-asdim } Y$. If $X \subset Y$, then $\ell\text{-asdim } X \leq \ell\text{-asdim } Y$. Also, $\ell\text{-asdim } X \times Y \leq \ell\text{-asdim } X + \ell\text{-asdim } Y$. Let T be a metric tree; then $\ell\text{-asdim } T \leq 1$. The above properties show that

$$\text{asdim } \pi_1(M) = \text{asdim } \widetilde{M} \leq \ell\text{-asdim } \widetilde{M} \leq \ell\text{-asdim}(T_0 \times \prod_{c \in \mathcal{C}} T_c) \leq n.$$

On the other hand, since the space $\widetilde{M}$ is a Hadamard manifold, we have $\text{asdim } \widetilde{M} \geq n$, see [5, Theorem 10.1.1].

REFERENCES

- [1] J. A. Behrstock and W. D. Neumann, *Quasi-isometric classification of graph manifold groups*, Duke Math. J. **141** (2008), no. 2, 217–240. MR2376814 (2009c:20070)
- [2] D. Yu. Burago, Yu. D. Burago, and S. V. Ivanov, *A course in metric geometry*, Inst. Komp'yuter. Issled., Moscow–Izhevsk, 2005; English variant, Grad. Stud. Math., vol. 33, Amer. Math. Soc., Providence, RI, 2001. MR1835418 (2002e:53053)
- [3] G. Bell and A. Dranishnikov, *On Asymptotic Dimension of Groups Acting on Trees*, Geometriae Dedicata, **103** (2004), no. 1, 89–101. MR2034954 (2005b:20078)
- [4] S. V. Buyalo and V. L. Kobelskiĭ, *Generalized graphmanifolds of nonpositive curvature*, Algebra i Analiz **11** (1999), no. 2, 64–87; English transl., St. Petersburg Math. J. **11** (2000), no. 2, 251–268. MR1702579 (2001f:53062)
- [5] S. V. Buyalo and V. Schroeder, *Elements of asymptotic geometry*, EMS Monogr. in Math., Europ. Math. Soc., Zurich, 2007. MR2327160 (2009a:53068)

- [6] M. Gromov., *Asymptotic invariants of infinite groups*, Geometric Group Theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295. MR1253544 (95m:20041)
- [7] D. Hume and A. Sisto, *Embedding universal covers of graph manifolds in products of trees*, Preprint arXiv:math.GT/1112.0263.
- [8] M. Kapovich and B. Leeb, *3-manifold groups and nonpositive curvature*, Geom. Funct. Anal. **8** (1998), 841–852. MR1650098 (2000a:57040)
- [9] J. Roe, *Lectures on coarse geometry*, Univ. Lecture Ser., vol. 31, Amer. Math. Soc., Providence, RI, 2003. MR2007488 (2004g:53050)

ST. PETERSBURG BRANCH, STEKLOV MATHEMATICAL INSTITUTE, RUSSIAN ACADEMY OF SCIENCES,
FONTANKA 27, ST. PETERSBURG 191023, RUSSIA

E-mail address: `smirnov@pdmi.ras.ru`

Received 14/DEC/2011

Translated by THE AUTHOR