

## ON THE MATHEMATICAL WORK OF VLADIMIR SAVEL'EVICH BUSLAEV

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V. S. Buslaev died on March 14, 2012, one month before his 75th anniversary. He was a brilliant scientist, one of the leaders of the modern St. Petersburg mathematical school, and he gained worldwide recognition.

Buslaev was a mathematician of great capacity, and his field of interest was amazingly wide: he obtained principal results concerning quantum and classical trace formulas, diffraction and wave propagation, quantum scattering theory and integrable nonlinear equations, quasiclassical pseudodifferential operators with discontinuous symbols and difference equations with periodic coefficients, soliton stability for nonintegrable nonlinear equations and asymptotic quasiclassical and adiabatic methods. He published more than 150 papers and two books.

Vladimir Buslaev was the discoverer and the first researcher for a series of important topics, and some of his papers became starting points for several directions of investigations in mathematical physics. Many times he gave talks at international conferences and was invited to leading universities and scientific centers. He was a speaker at the International Mathematical Congress in 1983, gave a prestigious series of Coxeter lectures at the Fields Institute (Canada) in 1997, was a plenary speaker at the jubilee session of the German Mathematical Society in 2000; in 2005 he became a Doctor Honoris Causa of the University of Paris-13. In 1975, for a series of papers on diffraction theory, Vladimir Buslaev was awarded the first prize given annually by Leningrad State University for the best scientific investigations. He got a series of honorary titles from the University and the State and was awarded the State Prize of the Russian Federation in Science and Technology (2000).

Buslaev always searched for original analytically rich problems, for which the ways to solve them and the results of investigation were difficult to predict. Like the old masters, he liked to work with formulas, and, for him, the way to general constructions of a theory was through the analysis of nontrivial specific problems. From his point of view, this way of thinking was a distinguishing feature of the St. Petersburg mathematical school. Vladimir Buslaev himself possessed a rare talent for working this way.

A professor who had just begun to teach when Buslaev was a third-year student recalls that “Vladimir Buslaev was a brilliant student”. He continues: “The work of an educator can lead to a professional deformation: if a teacher thinks that he is much better than the students, he is at risk for developing a ‘superiority’ complex which, as any other complex, is a bad thing. The third-year student Vladimir Buslaev quickly liquidated my complex. This is how it happened. According to the curriculum, we were discussing

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the representation of a function (such as the sine) by an infinite product. I propagated the traditional approach: one has to calculate the logarithmic derivative, develop it in the series of algebraic fractions by means of the Cauchy formula, and calculate the exponential function of this series, which gives the answer. And, suddenly, Vladimir came up with another, shorter solution. It turned out that if an entire function does not grow too fast at infinity, then up to an elementary factor (exponential of a linear function) the Weierstrass product gives the representation in question. I had had no idea about this theorem of Hadamard, and, as a result, my ‘superiority’ complex did not develop”.

In 1959, Vladimir Buslaev graduated from the Department of Physics of Leningrad State University, and in 1959–1962 he was a graduate student at the university. His scientific advisers were Olga Ladyzhenskaya and Ludwig Faddeev; Buslaev’s collaboration with them largely formed his mathematical tastes and determined the sphere of his interests for a long time. In 1963 he defended his first thesis [7], and in 1973 he defended the second one [32] and became a “Doctor of Science”. Even his first thesis contained several breakthroughs, and many colleagues undoubtedly estimated the level of his first thesis as actually being the level of a second one. Working on the dissertations, V. S. Buslaev actively and successfully worked on other mathematical problems. He kept this style throughout his life by working simultaneously on problems from different fields.

For 50 years, from 1962 until the end of his life, V. S. Buslaev worked at the Division of Mathematics and Mathematical Physics, as an assistant in the beginning, then as a dozent and, since 1977, as a professor. The last twelve years he was an active and enterprising Head of the Division of Mathematics and Mathematical Physics, possessing a great prestige. Together with L. D. Faddeev and M. S. Birman, he analyzed and organized all the Division’s activity, from the selection of new colleagues to the organization of all the teaching. Simultaneously, he was one of the scientific leaders. Vladimir Buslaev understood very well all the directions of research carried out at the Division, and, when estimating scientific results, he quickly reacted to other colleagues’ ideas, results, talks, was full of emotions, tried to understand everything in detail, to estimate the essence, to look on the work “from a distance” and to find his own understanding. He appreciated original research, ideas and methods, opposing those who worked on numerous problems in one and the same way. It was difficult to meet his high scientific standards.

V. S. Buslaev created a scientific seminar devoted to the problems of the spectral and asymptotic analysis and directed its work during dozens of years. This is how it began. In the late ’70s, Buslaev started to work on nonlinear equations. Simultaneously, V. B. Matveev and A. R. Its organized another group actively working in the same field. This common interest led to the creation of the seminar. At the beginning, it was devoted to questions related to nonlinear equations and inverse problems (of spectral type). Later, it became a seminar on a wide range of analytic problems mostly determined by the scientific interests of Buslaev, and all his students and close colleagues gave talks on their research and results at the seminar. In the late ’90s, the seminar became a Division seminar. All his colleagues and best students reported here on their work. Open discussions of the ideas and results of the speakers by top-level professionals, exchange of ideas, and the creative atmosphere maintained by Buslaev were of great importance for all the participants of the seminar.

Buslaev generously shared his precious scientific experience and gave lecture courses with detailed explanations. For example, from one of his colleague’s reminiscences, Buslaev’s lectures on nonlinear nonintegrable equations given in Vienna, Munich, Banff, and Oberwolfach in 2004–2009 made a great impact on all the participants of the seminars: B. R. Vainberg, A. I. Komech, E. A. Kopylova, D. Stuart (Cambridge) and many others. As a result of these contacts, participants of the seminars proved the soliton asymptotic

stability for a wide range of infinite-dimensional Hamiltonian systems: Klein–Gordon fields, Schrödinger and Dirac equations, Hamiltonian systems related to relativistic particles and nonlinear oscillators [131], and the relativistic nonlinear Ginsburg–Landau equations (A. I. Komech, E. A. Kopylova, 2011).

Buslaev was an adviser to a galaxy of students. Fifteen theses made under his direction were successfully defended, five of his former students, A. A. Fedotov, V. B. Matveev, S. P. Merkur'iev, G. S. Perelman, and M. M. Skriganov, became “Doctors of Science”, many of his students have become well-known scientists working in leading Russian and foreign universities, and academician S. P. Merkur'iev was the president of St. Petersburg State University from 1986–1993. To his students Buslaev demonstrated his affection for mathematics, the mathematical way of thinking, and the mathematical value system. He offered them key problems which would inspire research in new areas. Having a rare mathematical intuition, he picked up problems that well-known specialists were not able to solve and offered them to his pupils. While doing this, he used to say that these problems were “ours”, meaning that these problems can naturally be solved in the framework of our analytic culture. And the problems were solved. He was ready to enter discussions, do calculations, write papers, i.e., was ready to work. Still, however, Buslaev built up his relationships with students in various ways: permanently “accompanying” some of them, he allowed others to carry out research on their own, supporting them with discussions. When discussing analysis, methods, and results with a student, Vladimir Buslaev got emotional and lit up his interlocutor's interest. The process of collaboration with V. S. Buslaev was the cognition of a new, unknown, and beautiful world. He was very demanding to his students and sometimes judged them severely. For him, obtaining decent scientific results was the main criterion to estimate a researcher. At the same time, for his students, he was always a caring person.

Buslaev had the same attitudes about teaching as he did about research and administration: He considered them a personal task and an extremely important part of life. He was a brilliant teacher of high mathematical standards; his lecture courses were always thoroughly thought out and were notable for their coherence. They were filled with interesting material and were popular with students. As in his scientific work, in his teaching Buslaev emphasized the main analytic structures and the links among them. He created and gave a lecture course on mathematical physics, covering complex analysis and special functions, distributions, and calculus of variations. He was also the author of original lecture courses on group theory, theory of Lie algebras and groups, potential and many-particle scattering theory, and quantum mechanics as seen by a mathematician. Buslaev's book [44] is one of the best references on calculus of variations, both encyclopedic and coherent.

V. S. Buslaev was a member of scientific and dissertation councils, a member of the editorial boards of two journals of the Russian Academy of Sciences, the “St. Petersburg Mathematical Journal” and “Functional Analysis and Its Applications”; for many years he was a member of the executive committee of the St. Petersburg Mathematical Society.

Buslaev carried out research and Division administration work up to the end of his life, e.g., he discussed the papers [140] and [139] with A. M. Budylin and S. B. Levin one day before his death. He died while on his way to the Division seminar.

Vladimir Buslaev was an unusually bright personality. His approaches, ideas, and results will be treasured by the mathematical community for many years to come, and his pupils and colleagues will always remember him as a brilliant scientist and a wonderful man.

The authors were lucky to know V. S. Buslaev closely, to participate in many discussions and to collaborate with him. A. M. Budylin, L. A. Dmitrieva, A. A. Fedotov,

S. B. Levin, M. V. Perel, E. A. Rybaina, V. V. Sukhanov were his students. We sincerely thank A. R. Its, B. A. Plamenvskiĭ, and D. R. Yafaev for precious remarks, comments, and advice, and T. A. Suslina for a thorough reading and detailed discussion of the entire text. Below, we present a review of Buslaev's main works, which are listed at the end of this article.

### §1. TRACE FORMULAS IN QUANTUM SCATTERING PROBLEMS

For eigenvalues of a regular Sturm–Liouville problem, I. M. Gelfand and B. M. Levitan got a series of identities that could be interpreted as representations for regularized traces of powers of the operator in terms of the operator itself (its coefficients). In 1960, the paper [2] was published. There, apparently for the first time, the authors rigorously obtained the full set of such identities for a one-dimensional Schrödinger operator with continuous spectrum (it was an operator on the half-line). These were the famous Faddeev–Buslaev trace formulas.

In [3], Vladimir Buslaev extended the result of [2] to the many-dimensional case. This required developing a very nontrivial technique (the complete proofs were published in [10]).

In the trace formulas, the expressions containing spectral objects are regularized moments of the spectral shift function, and it turned out that, in the problems with a continuous spectrum considered, this function could be expressed in terms of the determinant of the scattering matrix. In 1962, in the framework of an abstract approach, M. S. Birman and M. G. Krein derived an elegant formula, giving a general form to this result.

In the early '70s, Vladimir Buslaev came back to trace formulas. This time, they were related to the quantum three-particle problem. It was not simple: the formulas published by other authors were erroneous.

In [19], Buslaev obtained trace formulas for the Friedrichs model; they related the jump along the continuous spectrum of a regularized trace of the resolvent to the determinant of the scattering matrix. In [29], he derived trace formulas for an operator that could be viewed as a “three-particle” analog of the Friedrichs model, and in the paper [30] he generalized the methods of [29] and successfully studied the system of three one-dimensional particles in the cases of Bose, Boltzmann, and Fermi statistics.

Simultaneously, in collaboration with his student S. P. Merkur'iev, in [17, 22, 25] he obtained trace formulas for three three-dimensional particles. In statistical physics, the expansion in degrees of activity for the logarithm of large statistical sums is employed. The expansion coefficients  $b_n$ , called the group integrals, are expressed in terms of the Hamiltonians of  $n$ -particle systems. In [25], the formal expression for the coefficient  $b_3$  got an exact mathematical interpretation. More precisely, using a paper of L. D. Faddeev on three quantum particles, which had appeared not long before (1963), and the papers of M. S. Birman and M. Z. Solomjak on estimates of the singular numbers of integral operators (1967), in [25] the authors proved that  $b_3$  is expressed in terms of the trace of a trace class operator. The problem to relate the expression for  $b_3$  to the scattering data of the corresponding three-particle system and of the pair subsystems turned out to be nontrivial. Vladimir Buslaev showed that the elegant expression obtained by F. A. Berezin should be replaced by a less elegant, but correct, one.

### §2. DIFFRACTION AT A SMOOTH CONVEX BODY

According to Buslaev's recollections, Olga Alexandrovna Ladyzhenskaya was attracted to the investigation of the diffraction of waves at a smooth convex body.

Buslaev's first progress in this direction was related to the so-called Ivanov's error. We mean the following. In 1956, J. B. Keller, an American specialist in mathematical physics, obtained heuristic formulas for the short-wave asymptotics of wave field in the shadow behind a smooth convex body. The question arose about the justification of these formulas, at least at the level of formal asymptotic series. V. I. Ivanov, a mathematician from Moscow, began to work on this problem. To derive Keller's formulas, he used the so-called parabolic equation method. The analysis seemed to be correct, but the result was wrong.

Ivanov was able to describe the quickly changing exponential factor in the leading term, but the slowly varying factor he obtained was not correct. Vladimir Buslaev showed that the origin of the error was simple: to get the correct answer when deriving the parabolic equation, one had to take into account one more term.

Vladimir Buslaev is the author of a remarkably natural justification of the geometrical-optics approximation in the problem of scattering of waves at a compact smooth convex body and in other analogous problems. The main difficulty was to find an *a priori* estimate for the solution of the diffraction problem that remains valid for high frequencies. The integral estimates method works only in the case of the Dirichlet boundary condition. The integral equations of the potential theory cannot be employed, because the norms of integral operators grow with frequency. There are no general methods to get appropriate estimates. Buslaev approached this problem in the following way. First, he described the leading term of the high frequency asymptotics of the field of a point source situated on the surface of the convex body. Then, he skillfully corrected the obtained expression so that, outside the convex body, it satisfied the Helmholtz equation exactly. This allowed him to develop a nontraditional potential theory that led to integral equations with "small" integral operators. With the help of such equations the *a priori* estimate derivation became, in principle, a simple problem. As a result, a very difficult problem of justification of the geometrical-optics approximation for the scattering of waves at a compact smooth convex body (in the lightened domain and in the penumbra), the problem that began to be analyzed by Ursell (1957), was solved in Buslaev's papers in the mid-seventies; see [5, 6, 7, 8, 26, 31].

There is a series of articles closely adjacent to the above papers. They were devoted to the resolvent behavior and asymptotics of spectral objects for large values of the spectral parameter and to trace formulas for elliptic operators in infinite domains [32, 35, 36, 37, 38].

Regarding the problem of diffraction at a smooth convex body, see also the elegant paper [12] on a heuristic derivation of (very nontrivial!) asymptotics of the wave field. The solution of the diffraction problem was obtained from Green's function of a heat equation by using the Laplace transform in the time variable. Green's function can be represented by a Wiener integral. Calculating its small time asymptotics via the stationary phase method ideas, Buslaev obtained a complete asymptotic description of the wave field for the input problem. The content of the papers [9, 28] borders with that of [12].

### §3. QUANTUM SCATTERING PROBLEMS

Buslaev's papers on scattering at long-range potentials and his profound investigations in the analytic theory of many-particle scattering became known world-wide.

For Schrödinger operators with slowly decaying potentials, the limits that define the wave operators in the traditional scattering theory do not exist. For the Schrödinger operator with the Coulomb potential, analogs of the wave operators were constructed by J. D. Dollard. Analyzing his paper, V. S. Buslaev and L. D. Faddeev suggested a

general definition of the wave operators for the case of slowly decaying potentials. In [21], V. S. Buslaev and V. B. Matveev completed the construction of the generalized wave operators and proved their existence. Almost at the same time, in [23], V. S. Buslaev constructed generalized wave operators for the Friedrichs model with a kernel having singularity on the diagonal.

In [33], in collaboration with M. M. Skriganov, Vladimir Buslaev constructed complete asymptotic series satisfying the scattering problem for the Schrödinger equation in  $\mathbb{R}^3$  with a “not too quickly varying” potential that decays at infinity slower than the Coulomb one. The resulting formulas admitted a natural interpretation in terms of the diffraction theory. It can be said that this paper anticipated the diffraction approach to the many-particle quantum scattering problem. We mention that, in [33], the scattering amplitude singularities were described.

Vladimir Buslaev devoted many papers to quantum many-particle scattering problems. In [39, 41], in collaboration with A. F. Vaulenko, and in [55], he described a new approach to constructing the scattering theory for many-particle systems. The idea was to construct a unitary operator, transforming the Hamiltonian of the system in question into an operator resembling the Hamiltonian of a two-particle quantum system, and thus, analyzable by means of general instruments of the well understood two-particle scattering theory. To construct this unitary operator, one has to guess the singularities of the generalized eigenfunctions.

Continuing to work on the quantum many-particle scattering problems, Buslaev developed the diffraction approach, which turned out to be quite fruitful. In collaboration with S. P. Merkur'iev and S. P. Salikov, he studied scattering in the system of three one-dimensional quantum particles with quickly decaying pair potentials. In these papers, up to an outgoing circular wave with a smooth amplitude, the coordinate asymptotics of the solution of the scattering problem was described explicitly, and the scattering matrix kernel singularities were found. Also, the set of pair potentials was described for which scattering is free of diffraction effects.

Somewhat later, in collaboration with N. A. Kaliteevskii, in [73] Buslaev considered scattering in the system of  $n$  one-dimensional quantum particles. For this general case, the authors also obtained explicit expressions for the principal singularities of the scattering matrix and found conditions under which these singularities are free of diffraction effects.

In the 2000s, in collaboration with S. B. Levin, Vladimir Buslaev studied the scattering of three three-dimensional quantum charged particles. For quickly decaying pair potentials, this problem was solved by L. D. Faddeev in 1963. In the case of slowly decaying pair potentials and, in particular, for the system of three charged particles, Faddeev's approach does not work. V. S. Buslaev and S. B. Levin used ideas of the diffraction approach. They made several steps. In [135, 138], they described analytically and numerically the solution of the quantum scattering problem for the system of three one-dimensional particles with quickly decaying pair potentials. In [137], they described the leading term of the asymptotics of the scattering problem solution for three one-dimensional charged particles. Finally, in [139], they described the leading term of the asymptotics of the scattering problem solution for three three-dimensional charged particles. Though the authors constructed only formal asymptotics, the construction turned out to be highly nontrivial. In this construction, an important role was played also by the idea of “near-separation” of variables and by the idea of weak asymptotics, first suggested by Vladimir Buslaev in [10].

For completeness, we note that, for the problem under discussion, the proofs of the asymptotic completeness were already obtained in the framework of the nonstationary approach, but the stationary approach gives more information.

#### §4. TRACE FORMULAS FOR LAGRANGIAN AND HAMILTONIAN SYSTEMS

The papers [13, 50] by Vladimir Buslaev were devoted to the trace formulas describing regularized determinants of differential operators arising in the study of second variations along trajectories of classical Hamiltonian or Lagrangian systems. The trace formulas relate these determinants to the Jacobians of transformations determined by the motion of classical systems in the coordinate space. Buslaev's interest in these formulas was related to the fact that they arise naturally in the study of the quasiclassical asymptotics of propagators in quantum mechanics. The trace formulas reflect the equivalence of the asymptotic formulas obtained when substituting the quasiclassical *Ansatz* into the Schrödinger equation and of those that arise when heuristically calculating the asymptotics of the Feynman path integral representing the propagator. This calculation consists in applying the stationary phase method, and, as a result of that, the differential operator determinant occurring in the trace formula comes out. For Lagrangian systems, the trace formula was described in [13] in geometric terms. In [50], for Hamiltonian systems, V. S. Buslaev and E. A. Rybakina constructed a regularized determinant and obtained the trace formula in two different ways: by a direct calculation of the regularized determinant of the differential operator related to the Jacobi equations along trajectories of the classical system and by studying the quasiclassical asymptotics of the propagator of the quantum system.

The next step consisted of the extension of the trace formulas to the case of general Hamiltonian systems on nonlinear phase spaces and to the case of Lagrangian systems on arbitrary configuration spaces; see [58, 59]. Mainly, Vladimir Buslaev was interested in the influence of the underlying geometric structures on the regularization of the determinants and on the form of the trace formulas. In the Hamiltonian case, regularization required that linear connectedness that agrees with the symplectic structure determining the Hamiltonian system be defined on the tangent bundle of the phase space.

For mechanics systems, the form of the trace formulas is independent of the number of the degrees of freedom. This allowed Vladimir Buslaev to guess that similar formulas are valid also for systems with infinitely many degrees of freedom, i.e., for fields. V. S. Buslaev and E. A. Rybakina constructed a trace formula for the scalar relativistic field described by the nonlinear Klein–Gordon equation [64, 71, 78]. This was done rigorously, and all the necessary regularizations were carried out.

#### §5. COMPLETELY INTEGRABLE NONLINEAR EQUATIONS

At the end of the '70s, Vladimir Buslaev began to work on completely integrable nonlinear equations; for him, it was a new domain of mathematical physics. At that time, the first results on the large-time asymptotics of solutions of Cauchy problems for nonlinear equations of soliton theory appeared. These results were obtained by V. E. Zakharov and S. V. Manakov via the quantum inverse scattering method. They considered the case of quickly decaying initial data and obtained heuristically the asymptotics for the Korteweg–de Vries equation, the nonlinear Schrödinger equation, and the sine–Gordon equation. It became clear that the asymptotic description of the large-time asymptotics was a subtle analytic and asymptotic problem. First, in collaboration with V. V. Sukhanov, Vladimir Buslaev obtained a rigorous justification of the large-time asymptotic behavior of solutions of the Korteweg–de Vries equation [51, 53, 60, 67]. The plan of the proof was based on ideas of the method suggested in the paper of V. E. Zakharov and S. V. Manakov. It

turned out that, to get appropriate estimates, one had to construct at least four leading terms of the asymptotics, whereas the initial hope was that only the principal term would suffice. As a result, the authors constructed the complete asymptotic series. Though the structure of the asymptotics described in the paper of V. E. Zakharov and S. V. Manakov was mostly correct, their final formulas contained a substantial error: the phase of the leading term was wrong (the reason was that V. E. Zakharov and S. V. Manakov made their calculations simultaneously for three models, and some arguments valid for the nonlinear Schrödinger equation were wrong for the Korteweg–de Vries one). In [53], in addition to the asymptotics, wave operators were constructed for the Korteweg–de Vries equation. Vladimir Buslaev managed to extend the notion of wave operators to the case of nonlinear dynamics generated by this equation. As in the scattering theory, the idea was to compare two asymptotic dynamics at infinity. The simple dynamics was defined by the linearized Korteweg–de Vries equation, and the complicated one corresponded to the nonlinear one. The difficulty was that the forms of the leading terms of the asymptotics corresponding to these two dynamics were different. Buslaev’s experience gained while working with the generalized wave operators for Schrödinger operators with slowly decaying potentials proved useful.

The resulting large-time asymptotics turned out to be important for constructing the action-angle variables for the Korteweg–de Vries equation; see [69]. Here, one of the principal ideas suggested and successfully used by V. S. Buslaev, L. D. Faddeev, and L. A. Takhtajan was to use this asymptotics when diagonalizing the symplectic form for the Hamiltonian system corresponding to the Korteweg–de Vries equation. This approach made it possible to simplify significantly the existing calculations and led to an elegant Hamiltonian interpretation of the scattering theory for the Korteweg–de Vries equation.

The calculation of the large-time asymptotics of the solutions of Cauchy problems for integrable equations is a most impressive achievement of the quantum inverse scattering method in the soliton theory. Beginning with the pioneering papers of V. E. Zakharov and S. V. Manakov, the corresponding mathematical theory was developed in the course of twenty years by many mathematicians and achieved its culmination in 1992 in papers by P. Deift and X. Zhou (now, the widely known nonlinear analog of the saddle point method). In the papers of Vladimir Buslaev, for the first time, the asymptotics of the solutions of Cauchy problems for Korteweg–de Vries equation was obtained rigorously, and its operator and Hamiltonian interpretations were understood. These papers have a special place in the history of the modern asymptotic theory of integrable systems.

## §6. INVESTIGATIONS OF SOUND PROPAGATION IN THE OCEAN

In the '80s, at the Division of Mathematics and Mathematical Physics of Leningrad State University, there was a group that, under the direction of V. S. Buslaev and V. S. Buldyrev, studied sound propagation in the ocean.

Mathematically, the problem in question consists of constructing the Green function of a Helmholtz equation in a strip (water) lying on a half-space (bottom). Typically, in the ocean, the sound velocity as a function of depth has a minimum, and near this minimum an underwater sound channel, a waveguide, is formed.

A specific feature of the problem is that there is a large parameter (frequency): the ocean depth (kilometers) is much greater than the wave length (dozens of meters), and the variation of the sound velocity at the distance of order of the wave length is typically small. So, when investigating the underwater sound propagation, one naturally encounters quasiclassical asymptotics. In many cases, the variations of ocean properties



in the horizontal direction either can be neglected or are very slow. This naturally leads to adiabatic constructions.

Experimenters were interested in computer programs that could allow them to compute the sound field very quickly. The leading idea of the group was that, instead of direct sound field computations based upon numerical methods, asymptotic methods should be employed to get new formulas, which should be used as a base for efficient computer programs. The main results were published in [47, 62, 63].

Vladimir Buslaev considered problems that came from acoustics as a source of interesting analytic questions. He and his students published a series of papers devoted to the underwater sound propagation. The investigations of sound fields near the surface of a deep sea and the investigations of the scattering of sound waves at synoptic rings became the most known ones.

In the framework of the ray method (quasiclassical approach), a sound field is the sum of ray (classical trajectories) contributions. In the oceanic waveguide, typically, the rays are periodic curves having one minimum per period. Vladimir Buslaev observed that if the sound source and the receiver are close to the surface, then the set of the rays connecting them is naturally split into groups of four rays. For each of these groups, the acoustic lengths of the rays are close one to another, and all the rays have one and the same number of minimums. Buslaev got a relatively simple formula describing the sum of the contributions of the rays of each group to the sound field [48, 49].

The *four-ray formulas* make it possible to explain the complicated interference structure of the wave field observed experimentally at not too large distances from the source. In the oceanic waveguide, the periods of rays are bounded from above and depend on the rays' slopes at the waveguide axis. There are two natural sets of rays: the rays that are situated in the water and do not reach the bottom and the rays that are periodically reflected from the bottom. The rays staying inside the water have periods of order of dozens of kilometers, and the rays that reach the bottom have smaller periods. The rays staying inside the water make the main contribution to the sound field (the reflection coefficients are small). And thus, near to the sea surface, the sound field is largely the sum of the contribution of the groups of four rays that do not reach the bottom. As a result, at not too large distances from the source (100–200 kilometers), the sound zones and quiet zones alternate, and in each sound zone the wave field approximately equals the contribution of only one group of four rays. In [56], the authors compared the results of computations based on the four-ray formulas and the results of direct numerical computations of sound fields.

At large distances  $r$  from the source, the sound zones overlap, and the field is the sum of the contributions of many four-ray groups. For this case, in [61], V. S. Buslaev and M. V. Perel studied the locally averaged field intensity, which turned out to be asymptotically equal to the sum of a “background” term decaying as  $1/r$  and an additional term responsible for the interference of different four-ray groups and decaying as  $1/r^{3/2}$ . The additional term is the product of a decaying amplitude factor and the sum of a finite number of periodic terms. Even at very large distances, the periodic terms can lead to drastic field intensity increases.

Near to the sea surface, the sound velocity can change so quickly that the quasiclassical approximation cannot be applied. In [66, 70], the authors extended the four-ray formulas to this case.

In [74, 75], V. S. Buslaev and A. A. Fedotov described the scattering of short waves at an adiabatic nonhomogeneity in a plane waveguide with boundaries. This problem models sound scattering at a synoptic ring in the ocean. In the case where the plane “oceanic” waveguide properties are independent of the horizontal variable, the wave field

is represented by a linear combination of normal waves constructed by using separation of variables. If the sound velocity slowly depends on the horizontal variable, then the standard adiabatic approach allows one to construct solutions that asymptotically turn into normal waves outside the adiabatic nonhomogeneity. The situation changes if there is a competing quasiclassical parameter. In this case, for a waveguide without boundaries, thanks to a serious modification of the adiabatic *Ansatz*, V. A. Borovikov and A. V. Popov constructed asymptotic analogs of normal waves. The transition matrix describing the result of transition of normal waves through the adiabatic nonhomogeneity turned out to be diagonal in any order in the adiabatic and quasiclassical parameters. For a waveguide with horizontal boundaries, all the attempts of different authors to modify the adiabatic *Ansatz* appeared to be hopeless: V. S. Buslaev and A. A. Fedotov calculated the asymptotics of the transition matrix coefficients and showed that, in the presence of boundaries, the matrix can become nondiagonal even in the leading order. The problem was solved thanks to using the quasiclassical approach and studying the classical trajectories as trajectories of a Hamiltonian system slowly depending on time. The diagonality of the transition matrix turned out to be related to the preservation of the adiabatic invariant along the trajectories of the Hamiltonian system. For the waveguide without boundaries, the adiabatic invariant is preserved up to terms smaller in order than any power of the small adiabatic parameter, and the singularity of the Hamiltonian system induced by the boundaries leads to nontrivial variations of the adiabatic invariant even in leading terms.

### §7. ADIABATIC PERTURBATIONS OF PERIODIC SCHRÖDINGER EQUATIONS

In the early '80s, Vladimir Buslaev began to develop an original asymptotic approach to the study of solutions of adiabatically perturbed differential equations with periodic coefficients. In particular, a series of papers and talks by V. S. Buslaev, V. S. Buslaev and F. Dmitrieva, and V. S. Buslaev and A. Grigis was devoted to the one-dimensional Schrödinger equation  $-\frac{d^2\psi}{dx^2} + (V(x) + W(\varepsilon x))\psi = E\psi$ , where  $V$  is a periodic function, and  $\varepsilon$  is a small parameter. Such equations arise in various problems of physics. For example, in the solid state physics, they are used as a model to study electrons in a crystal placed into an external electric field (which changes slowly with respect to the internal field).

To explain the nontriviality of the new approach, consider the above equation with  $V = 0$ . Changing the variable  $x \mapsto \xi = \varepsilon x$ , we transform it to  $-\varepsilon^2 \frac{d^2\psi}{d\xi^2} + W(\xi)\psi = E\psi$ . This is an equation with a standard quasiclassical parameter  $\varepsilon$ . The asymptotics of its solutions can be controlled with the help of the standard quasiclassical approach, and it can be said that the standard approach allows one to describe the behavior of solutions of an equation arising when adding the adiabatic perturbation  $W(\varepsilon x)$  to the operator  $-\frac{d^2}{dx^2}$ . Buslaev's method is a nontrivial generalization of the standard one: it leads to the asymptotics of solutions of the equation arising when adding the adiabatic perturbation to the periodic Schrödinger operator  $-\frac{d^2}{dx^2} + V(x)$ .

There are natural parallels between the constructions of the standard quasiclassical approach and those arising in Buslaev's method. All the analytic objects of the new method are defined in terms of the band function  $\mathcal{E}$  of the unperturbed periodic operator. The many-valued function  $\xi \rightarrow k(\xi)$  defined by the relation  $\mathcal{E}(k) + W(\xi) = E$  plays the same role as the complex momentum from the standard approach, and the branch points of  $\mathcal{E}$  play the role of the turning points.

The first paper [57] of the series discussed here was devoted to the above Schrödinger operator. The authors introduced the principal constructions of the method and built the formal asymptotics describing solutions both outside neighborhoods of the branch

points and in these neighborhoods. The latter is done for linear adiabatic perturbations  $W$ .

In [89, 120], both in the case of the Schrödinger operator and in a more general situation, the behavior of solutions near branch points was discussed in more detail.

In [77], V. S. Buslaev and L. A. Dmitrieva constructed asymptotic series satisfying the above Schrödinger equation in a neighborhood of two close branch points and checked that they agree with the asymptotic series satisfying the equation “far” from the branch points. This allowed them to calculate, for a one-dimensional crystal placed in slowly varying external electric field, the asymptotics of the interband transition coefficients and to get a complete mathematical description of the disruption effect from the theory of Bloch electrons; this effect is well known in physics.

In [79] and [80], Buslaev described general constructions of the new method in the many-dimensional case, i.e., for equations of the form  $L(\varepsilon x, x, -i\partial_x) = 0$ ,  $x \in \mathbb{R}^d$ , where the symbol  $L$  is a periodic function of the second variable. Also in those papers, the principal ideas of the method were compared with those of Maslov’s well-known method.

In [79] and [80], for the first time in the theory of adiabatic perturbations of periodic operators, the authors obtained quantization conditions similar to well-known conditions from the traditional quasiclassical analysis. It turned out that, in these quantization conditions, Bohr’s one half is replaced by a functional of the periodic potential. This functional can be expressed in terms of the connectedness of a linear Hermite complex bundle. In [86], in the spirit of the papers by B. Simon and M. Berry on the quantum adiabatic theorem, the Berry phase, and the relationship between the canonical connectedness and quantization conditions, V. S. Buslaev and L. A. Dmitrieva discussed the geometric interpretation of the quantization conditions in more detail.

An important step in the development of the new method consisted in applying the local asymptotics of solutions of the adiabatically perturbed periodic equation to the study of the global behavior of its solutions and, as a result, to the analysis of the spectral properties of the equation. In particular, the authors proved some existing conjectures and got new results on the adiabatic asymptotic properties of the so-called Stark–Wannier ladders arising in the theory of the periodic Schrödinger operator with linear adiabatic perturbation. In [84], [85], and [87], V. S. Buslaev and L. A. Dmitrieva rigorously proved that the Star–Wannier ladders are formed of resonances of this operator, i.e., the poles of the meromorphic continuation of the reflection coefficient across the absolutely continuous spectrum. In the case of finite gap potentials, it was proved that, in a neighborhood of the real axis, there are finitely many ladders (periodic sequences) of resonances. Moreover, it turned out that the real parts of the resonances of the  $n$ th ladder are described by the quantization conditions corresponding to the  $n$ th spectral band of the nonperturbed operator, and the value of their imaginary part is related to the interband transition coefficients corresponding to the spectral gaps situated “above” the  $n$ th spectral band (i.e., the  $n$ th,  $(n + 1)$ st,  $\dots$  gaps). Yet another interesting result was the description of the spectral concentration showing that, when the adiabatic parameter tends to zero, the spectral bands and gaps of the nonperturbed operator “tend to come” from the absolutely continuous spectrum of the perturbed one.

The paper [114] of V. S. Buslaev and A. Grigis was devoted to the description of the asymptotics of the imaginary parts of the resonances.

In [65, 83, 88, 90, 91], Vladimir Buslaev considered spectral problems with other boundary conditions and applied his method to study wave propagation. Later, in the papers by V. S. Buslaev and A. Grigis [120] and by V. S. Buslaev, M. V. Buslaeva, and A. Grigis [127], the authors came back to the asymptotics of solutions of adiabatically perturbed periodic Schrödinger equation; they gave a detailed classification of the branch

points and suggested an appropriate technique for constructing asymptotic series formally satisfying the equation on intervals containing one or two branch points. By using these constructions, the authors got asymptotic formulas for the reflection coefficient of the Stark–Wannier operator. These formulas agree with those obtained earlier; see [85].

Concluding this section, we may say that, in the papers mentioned above, an original and powerful approach was developed, which made it possible to study a wide range of problems arising in physics.

## §8. QUASICLASSICAL PSEUDODIFFERENTIAL OPERATORS WITH DISCONTINUOUS SYMBOLS

In the mid '80s, V. S. Buslaev and A. M. Budylin began to study problems related to quasiclassical pseudodifferential equations with discontinuous symbols. Their interest was stimulated by the problem consisting of calculating the second term of the asymptotics as  $t \rightarrow \infty$  for the trace of a smooth function of a “one-dimensional” integral operator  $(A_t f)(x) = t \int_{-1}^1 A(t(x-y))f(y) dy$  with a smooth quickly decaying kernel. The leading term of the asymptotics was well known. The problem came from the information transmission theory and was reported to V. S. Buslaev by M. S. Birman. The authors were interested in a more general problem for a “many-dimensional” operator of the form  $(A_t f)(x) = t^d \int_{\Omega} A(t(x-y))f(y) dy$ , where  $\Omega \in \mathbb{R}^d$  is a domain with smooth boundary. The growing of  $t$  can be interpreted as a *dilation* of the domain  $\Omega$ . Note that the symbol of the last operator (as a pseudodifferential one) is of the form  $\sigma(\xi, x, y) = \theta(x)a(\xi)\theta(y)$ , where  $a(\xi) = \int A(x)e^{-ix\xi} dx$ , and  $\theta$  is the indicator function of  $\Omega$ . The problem was solved in [68, 94], where, in particular, the authors constructed complete asymptotic power series for the functionals  $\text{tr } \varphi(A_t) \sim \sum_{n \geq 0} t^{d-n} F_n$ , obtained explicit formulas for the first coefficients  $F_0, F_1, F_2$ , and showed that these coefficients depend substantially on the geometrical properties of the domain  $\Omega$  and its boundary. It should be mentioned that, at the same time, an American mathematician, G. Widom, published papers devoted to close questions. He paid special attention to the asymptotics of the above functionals. Widom employed the theory of pseudodifferential operators with operator-valued symbols. Based on the asymptotic analysis of the resolvent of the operator  $A_t$ , Buslaev’s approach is more elementary and direct. As a result, in comparison with Wisdom’s formulas, the formulas found by V. S. Buslaev and A. M. Budylin seem to be more efficient.

Consider the above “one-dimensional” integral operator  $A_t$ . The analysis of the asymptotic behavior of the resolvent of  $A_t$  depends on the smoothness of the symbol  $a(\xi)$ . If the symbol has jumps, the analysis becomes much more difficult. The point is that if the symbol has jumps with respect to the two dual variables, the operator  $A_t$  becomes long range both in the coordinate and in the momentum space representations. The next problem studied for a long time by Vladimir Buslaev and his student A. M. Budylin was that of the asymptotic analysis as  $\varepsilon \rightarrow 0$  of solutions of the equation

$$(A_{\frac{1}{\varepsilon}} f)(x) = g(x),$$

where

$$A_{\frac{1}{\varepsilon}} f(x) = \frac{1}{\varepsilon} \int_{-1}^1 A\left(\frac{x-y}{\varepsilon}, x, y\right) f(y) dy,$$

in the cases where the symbol  $a(\xi, x, y) = \int A(u, x, y) \exp(-iu\xi) du$  has jumps and/or zeros as a function of the variable  $\xi$ . Equations of this type are well known; it suffices to mention the Prandtl equations from aerodynamics, the famous problem concerning a round condenser, which, in statistical physics, is called the Lieb–Lineger problem, the equation with the sine-kernel (also from quantum statistical physics). It is less

known that, in the framework of the quantum inverse scattering method, the analysis of the most difficult questions concerning the large-time asymptotics of solutions of nonlinear equations of mathematical physics, e.g., the nonlinear Schrödinger equation and the Korteweg–de Vries equation, can be reduced to the equations mentioned above. In [92, 93, 101, 109, 107, 111, 117, 118], V. S. Buslaev and A. M. Budylin constructed an asymptotic theory of one-dimensional quasiclassical pseudodifferential equations and applied it to a series of problems and, in particular, to the above nonlinear problems. In these investigations, a generalization of the Schwarz alternating method found by the authors played a substantial role. Note also that, for the asymptotic analysis of nonlinear equations, V. S. Buslaev and A. M. Budylin used the so-called direct approach based on the determinant representations and the stationary phase method. For the first time, the existence of such an approach was indicated in Buslaev's short paper [45].

After the theory of pseudodifferential operators with doubly discontinuous symbols had been largely constructed, it became clear that the developed methods applied to operators of a more general nature, those having the form of Fourier integral operators. As was noted by V. S. Buslaev, the kernel of a Fourier integral operator involving a quickly oscillating exponential function naturally has properties similar to those of long-range kernels of pseudodifferential operators with discontinuous symbols. The first step in the analysis of such operators was the calculation of the Fredholm determinant describing the two-particle correlation function for the Bose gas of one-dimensional particles; see [119, 130]. Similar objects arise in the course of the asymptotic analysis of matrix Riemann–Gilbert problems with quickly oscillating off-diagonal terms. The interest in such Riemann–Gilbert problems is determined by their role in the study of the above-mentioned nonlinear equations of mathematical physics. While working on the large-time asymptotics of solutions of the nonlinear Schrödinger equation and of the modified Korteweg–de Vries equation, P. A. Deift and X. Zhu investigated such Riemann–Gilbert problems. The Deift–Zhou approach can be associated with the saddle point method. It requires solving auxiliary Riemann–Gilbert problems on a complicated collection of contours. V. S. Buslaev suggested a clearer approach based on the asymptotic techniques from the theory of pseudodifferential operators with discontinuous symbols and associated with the stationary phase method. This goal was achieved in one of Buslaev's last papers [140], which unfortunately was posthumous.

## §9. NONLINEAR NONINTEGRABLE EQUATIONS

In 1990–2010, Vladimir Buslaev, in collaboration with Galina Perelman [96, 97, 98, 106] and Catherine Sulem [123, 125], made several important discoveries in the theory of asymptotic stability of solitons (soliton stability with respect to small perturbations) for one-dimensional nonlinear Schrödinger equations with nonlinearities of the form  $F(|u|^2)$  instead of  $|u|^2$ . These investigations were a natural development of the work of M. Vainstein and A. Soffer (1985–1990), who considered nonlinear Schrödinger equations with an external potential (an additional linear term with a coefficient depending on the variable of the equation). Having well understood the Vainstein–Soffer approach geometrical nature related to the symplectic structure of Hamiltonian systems in a Hilbert space, Buslaev and his co-authors managed to generalize their results to a more complicated case of translation invariant Schrödinger equations without an external potential. Namely, they understood that the linearized dynamics turns out to be unstable because, in all the problems related to continuous symmetry groups, e.g.,  $U(1)$ , translation groups, and so on, the linearized equation possesses some purely imaginary discrete spectrum. Therefore, in these cases, the Lyapunov theory required a principal modernization.

The strategy worked out by Buslaev and his co-authors is universal and geometrically clear. First, to overcome the problems caused by the discrete spectrum of the linearized equation, one needs to divide the dynamics into a motion along the soliton manifold and along the direction orthogonal to this manifold with respect to the symplectic form. The longitudinal motion is controlled by modulation equations and is unstable, whereas the transverse motion is asymptotically stable. The proof of the transverse asymptotic stability for a nonautonomous nonlinear equation is the principal achievement of Buslaev's approach.

The authors unveil a deep analytic intuition: the proof is based on a "freezing" of the transverse dynamics at the right end of a long time interval and a skillful balancing between the linearized stability and modulation equations. The method of Poincaré normal forms and asymptotic methods for majorant estimates that invoke the Fermi golden rule were applied extensively and skillfully. Moreover, a spectral theory of nonselfadjoint "Hamiltonian" operators was developed.

### §10. DIFFERENCE EQUATIONS WITH PERIODIC COEFFICIENTS

Vladimir Savel'evich "brought" a problem concerning Harper's equation, the difference Schrödinger equation  $\frac{1}{2}(\psi(x+h) + \psi(x-h)) + \cos x \psi(x) = E\psi(x)$ ,  $x \in \mathbb{R}$ , from France. This equation arises in the study of the Bloch electron in a crystal placed in an external magnetic field. The parameter  $h$  is determined by the value of the magnetic flux. Due to an unusual Cantorian geometry of the spectrum, Harper's equation attracted the attention of both physicists and mathematicians. The first mathematically rigorous constructive description of this geometry was obtained by B. Helffer and J. Sjöstrand with the help of a renormalization procedure based on the quasiclassical pseudodifferential operator theory techniques traditionally used in the study of many-dimensional problems. Influenced by the talks and papers of B. Helffer and J. Sjöstrand, Buslaev became very interested in this problem and suggested to A. A. Fedotov that they work on it. He thought that it should be treated as an analytic one-dimensional problem. The papers of V. S. Buslaev and A. A. Fedotov led to the creation of a whole new direction in the analytic and spectral theory of difference equations. The renormalization approach developed in the course of this work, the monodromization method, was considered by V. S. Buslaev to be a brilliant discovery. Later, the monodromization idea proved to be efficient for the study of difference and differential quasiperiodic equations.

Helffer and Sjöstrand studied the problem in the case where the parameter  $h$  can be represented by a quasiclassical continued fraction (i.e., a continued fraction all the quotients of which are sufficiently large), and the first step of V. S. Buslaev and A. A. Fedotov consisted in the development of an original approach to the study of quasiclassical asymptotics for solutions of difference equations in the complex plane; see [99, 104]. The idea of the approach is similar to that of the classical complex WKB method developed to study solutions of ordinary differential equations in the complex plane. However, the difference equations are nonlocal, and the new method turned out to be quite a nontrivial generalization of the classical one. As the classical method, it is aimed to calculate the asymptotics of exponentially small quantities important for the spectral analysis. To develop this new approach, the authors discovered (and rediscovered) a series of results related to the analytic theory of difference equations in the complex plane.

Simultaneously, the first step was made to extend notions from the classical theory of differential equations with periodic coefficients: the notion of a monodromy matrix was introduced for difference equations. In contrast to the case of ordinary differential equations, the space of solutions of a one-dimensional difference equation is a module over the ring of periodic functions, and a monodromy matrix is a periodic function of the variable

of the equation. The asymptotic (quasiclassical) calculation of a monodromy matrix for Harper's equation (see [102, 99]) has led to two unexpected results: for a solution space basis natural from a quasiclassical point of view, the monodromy matrix turned out to be a first order trigonometric polynomial, and its coefficients have asymptotics such that a difference equation determined naturally by the monodromy matrix is close to Harper's equation (with new parameters).

In [100, 105], the notion of a Bloch solution was introduced for difference equations with periodic coefficients, and the exact meaning of the difference equation determined by the monodromy matrix was explained. This equation, the monodromy equation, is a difference equation with periodic coefficients, and it was shown that, to construct Bloch solutions of the input equation, it suffices to construct Bloch solutions for the monodromy equation. The passage from the input equation to the monodromy equation was called monodromization. To construct Bloch solutions of the monodromy equation, yet another monodromization is needed, and so on. As a result, we arrive at an infinite series of difference equations generated by the monodromization procedure, and it can be said that monodromization gives rise to an infinite-dimensional (in general) dynamical system. In [100], it was shown that, in the case where the shift parameter from Harper's equation is represented by a quasiclassical continued fraction, the monodromizations can be carried out so that all the monodromy matrices generated by Harper's equation belong to a two-parameter family of second order trigonometric polynomials. In this case, the monodromizations lead to a study of a two-dimensional dynamical system. In [105], for the case studied by Helffer and Sjöstrand, this construction was used to explain the Cantorian structure of the spectrum. It became clear that the spectral properties of a difference equation with periodic coefficients are determined by the properties of the dynamical system generated by monodromization.

Many results turned out to be independent of the quasiclassical hypothesis, and, in the subsequent papers, the authors did not impose it. In [108], they discussed general properties and constructions of the Bloch solutions and of the monodromization map for difference equations with periodic coefficients. In [112], for the space of solutions of Harper's equation, they constructed a basis for which the monodromy matrix is a first order trigonometric polynomial. The idea was suggested by the quasiclassical constructions; it consists of constructing bases of minimal entire functions in the space of solutions, where "minimal" means that the function has the slowest growth as the imaginary part of the complex variable tends to infinity. In [113], the authors investigated in detail the analytic properties of the monodromy matrix Fourier coefficients as functions of the spectral parameter. In [122] it was shown that, for systems of two first order difference equations with trigonometric polynomial coefficients, in the generic case, monodromizations can be carried out so that all the monodromy matrices are trigonometric polynomials of one and the same order. In this case, the dynamical system determined by monodromization turns out to be finite-dimensional.

The interesting investigations carried out in these papers require deeper understanding and continued study.

## §11. OTHER PAPERS

Vladimir Buslaev worked on many other interesting analytic problems. Here we briefly discuss some of them.

One can single out the series of papers devoted to quasiclassical asymptotics and quantization.

The papers [15, 16, 24] appeared as a result of the analysis of Maslov's constructions. In those papers, V. S. Buslaev developed a generalization of the WKB method essentially

equivalent to the canonical operator. His starting point was the idea that along with the asymptotic series of the standard WKB method, the series for which the coordinate plane  $Q$  in the phase space  $M$  of the classical system plays a special role, one needs to consider an asymptotic series for which a similar role is played by arbitrary Lagrangian planes obtained from  $Q$  via linear canonical transformations of  $M$ . To construct such series, Vladimir Buslaev used the canonical quantization of the phase space and the induced unitary representation in  $L^2(Q)$  of the group of linear canonical transformations of the phase space, i.e., the nonhomogeneous symplectic group. Combining the above asymptotic series, V. S. Buslaev arrived at their generalizations related to arbitrary Lagrangian manifolds. For such generalized series, he got an invariant representation in terms of a special generating integral over the Lagrangian manifold under consideration. This integral arose as a result of a limiting procedure in the course of which the Lagrangian manifold was approximated at each of its points by the Lagrangian tangent plane. The generating integral, which is actually equivalent to Maslov's canonical operator, gives an invariant and more compact description and can be used in a wide range of problems.

In [20], in collaboration with M. M. Skriganov, V. S. Buslaev formulated a condition under which a continuous linear mapping of  $L^2(M)$ , the space of generalized functionals defined on the phase space  $M$ , into the set of Hilbert–Schmidt operators defined in the Fock space coincides with the Weyl quantization up to a numerical factor.

The paper [116] was related to an intriguing question concerning the spectrum of the one-dimensional Schrödinger operator with a potential equal to the sum of a linear term and a periodic Dirac comb. It was known that if the periodic term is more singular, the spectrum is singular, and if it is sufficiently regular, the spectrum is absolutely continuous. The case of the Dirac comb was viewed as a critical one. Using quasiclassical constructions, V. S. Buslaev showed that there is a nontrivial reduction of the problem to a difference equation for which, in a sense, the behavior of solutions may be compared naturally with that of cubic exponential sums from the number theory.

With the help of computer calculations, in collaboration with V. E. Grikurov, V. S. Buslaev studied solitons for nonlinear nonintegrable equations. They tried to find a range of soliton parameters for which the “collision” of two solitons leads to their nontrivial interaction (i.e., not reducing to scattering). The results of the investigation were published in [121].

In [124], V. S. Buslaev and L. A. Pastur considered ensembles of random large Hermitian matrices related to models of the quantum field theory. For each natural number  $p$ , they singled out a class of polynomial potentials (determining the probability distribution) for which the limiting density of eigenvalues can be described explicitly in terms of elementary functions, and the density support consists of  $p$  intervals (generically disjoint). The authors obtained formulas relating the limiting density to the density of states and the Lyapunov exponent of some  $p$ -periodic Jacobi matrices.

In [132], for a one-dimensional differential Schrödinger operator, V. S. Buslaev and V. Ju. Strazdin obtained a relation similar to that known in the theory of Jacobi matrices: they showed that the generalized eigenfunctions of this operator regarded as functions of the spectral parameter satisfy an explicitly constructible differential equation.

The papers [126, 128, 133] were devoted to investigations of the adiabatic evolution generated by linear operators depending on time. In [126], V. S. Buslaev studied the case of linear operators defined by a two-by-two matrix. This is a standard quasiclassical problem. Assuming that there exist either one or two parabolic turning points, he obtained invariant formulas describing solutions of the problem in terms of solutions of model problems. In [128], V. S. Buslaev and E. A. Grinina suggested a new approach to the study of adiabatic evolution in the case of the Quantum Adiabatic Theorem. Their



approach was similar to the approach described in the book by Ju. A. Daletskiĭ and M. G. Krein but led to projections independent of time in the final formulas. In [133], staying at the level of formal calculations, V. S. Buslaev and K. Sulem studied the adiabatic evolution generated by selfadjoint operators with continuous spectrum.

Under the influence of the work of M. S. Birman and T. A. Suslina, V. S. Buslaev became interested in homogenization theory, which studies solutions of differential equations with quickly oscillating coefficients. In [136], Buslaev, in collaboration with his student A. A. Pozharskiĭ, solved a problem unusual for homogenization theory: they described scattering on a periodic structure. The authors considered the Schrödinger equation  $(-\Delta + q(x/\varepsilon, x) - E)\psi_\varepsilon(x) = f(x)$  in  $\mathbb{R}^d$ , where the (smooth real-valued) potential  $q$  is periodic in the first variable and, as a function of the second variable, has a compact support. As  $\varepsilon \rightarrow 0$ , the authors constructed the asymptotics of the solution of the scattering problem and proved that it is close to the solution of the “averaged” scattering problem with the “effective” potential equal to the potential  $q$  averaged with respect to the first variable. The authors also obtained asymptotics for the scattering amplitude.

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