QUASICLASSICAL ASYMPTOTICS FOR SOLUTIONS OF THE MATRIX CONJUGATION PROBLEM WITH RAPID OSCILLATION OF OFF-DIAGONAL ENTRIES

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*To blessed memory of my Teacher
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ABSTRACT. The (2×2) -matrix conjugation problem (Riemann–Hilbert problem) with rapidly oscillating off-diagonal entries is considered, along with its applications to nonlinear problems of mathematical physics. The phase function that determines oscillation is assumed to have finitely many simple stationary points and to admit power-like growth at infinity. Quasiclassical asymptotics are constructed for solutions of such a problem in the class of Hölder functions, under appropriate restrictions on the entries of the conjugation matrix. It is proved that, after separation of a certain background, the stationary points of the phase function contribute to the asymptotics additively. Along with the M. G. Kreĭn theory, the justification of the resulting asymptotic solutions employs the stationary phase method and the Schwarz alternating method.

§1. PRELIMINARY REMARKS

The investigation presented in this paper was conceived by V. S. Buslaev. I am deeply grateful to him for numerous talks and advice.

Setting of the problem. We generalize some results of [1] to the case of more general matrix conjugation problems. Recall that by the conjugation problem (or the Riemann–Hilbert problem) we mean the following analytic factorization problem for a function on the real axis: given a function V of the real argument, construct functions H_+, H_- that admit analytic extensions to the upper and lower (respectively) half-planes of the complex plane and satisfy the identity

$$(1.1) \quad H_+ = H_- V$$

on the real axis.

In the case of the *matrix* conjugation problem, all function involved are assumed to be matrix-valued. We restrict ourselves to square matrices of size 2×2 . We are interested in the conjugation matrices V of the form

$$(1.2) \quad V(x) = \begin{pmatrix} a(x) & b(x) e^{-it\Phi(x)} \\ c(x) e^{it\Phi(x)} & d(x) \end{pmatrix},$$

where the phase function Φ is a real-valued function with finitely many simple stationary points and with power-like behavior at infinity with exponent at least 1. The functions

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$(a-1), b, c, (d-1)$ are assumed to be complex-valued and to belong to the Schwartz class, so that the matrix V turns into the identity one as $x \rightarrow \infty$. Viewing the factorization problem for the identity matrix as elementary, we see that the restrictions imposed above are of fairly general nature. In fact, we shall need to impose some additional requirements on the functions a, b, c, d , which will allow us to avoid certain specific issues concerning factorization of functions.

First, we assume that the condition

$$(1.3) \quad \Delta(x) \equiv a(x)d(x) - b(x)c(x) \neq 0$$

is fulfilled on the entire axis, and that the index of the function Δ on the closed axis is zero.

Next, denote by k_j the stationary points of Φ , put $Q = 1 - P$, and define P to be the indicator function of the union of the intervals on which Φ grows. Assuming that the function d does not vanish on the support of P , we complement the graph of the discontinuous function $aQ + \Delta d^{-1}P$ by line segments connecting the points $a(k_j)$ and $D(k_j)d^{-1}(k_j)$. It is assumed that the resulting continuous curve also has index zero (in particular, the function a does not vanish on the support of Q). Recall that the index of the continuous curve described above is meant when we talk about the index of the function $aQ + \Delta d^{-1}P$.

The variable t will play the role of a large positive parameter. We are interested in the asymptotic behavior of the factorizing functions $H_{\pm}$ as $t \rightarrow +\infty$, under the natural normalization condition

$$(1.4) \quad H_{\pm}(x) \xrightarrow{x \rightarrow \infty} I,$$

where I is the identity matrix.

We emphasize once again that, with V as above and with techniques to be described below, the only substantial property is the presence of rapid oscillations in the off-diagonal entries. The case of multiple stationary points of the phase function makes no exclusion. Of course, this latter case would require exact solution of some model problems distinct from those treated in the present paper.

In the paper [1] mentioned above, the role of the phase function was played by $\Phi(x) = \frac{x^2}{2}$. That case serves as a base for the present considerations.

For completeness of presentation, some results of [1] will be stated below. However, we refer the reader to [1] for the history of the problem. Now we only mention the papers [7, 8], also devoted to the problem discussed here, but with substantially different line of arguments.

General characterization of the answer. The main contribution to the asymptotics as $t \rightarrow \infty$ of the factors $H_{\pm}$ is made by the stationary points of the phase function Φ , i.e., the roots of the equation

$$\Phi'(x) = 0.$$

After separation of some background, these contributions sum asymptotically additively. Moreover, for large x the impact of the background disappears at all.

It is natural to distinguish the case of only one simple stationary point first. Importantly, this is not only the simplest case, but also a model for analysing the general case of several simple stationary points. For this simplest case, we can write the leading term of the solution's asymptotics explicitly, and also can construct asymptotic expansions complete in power orders (but not power-like), see [1].

In the case of a general phase function, we restrict ourselves to obtaining the leading asymptotic term in the space $H^{\mu,1}(\mathbb{R})$, where $\mu > 0$ is sufficiently small. Like in [1], here

by $H^{\mu,\theta}(\mathbb{R})$ we mean the Banach space of functions with the norm

$$\|f\|_{\mu,\theta} = \sup_{x,h \in \mathbb{R}} (1 + |x|)^\theta \left(|f(x)| + \frac{|f(x+h) - f(x)|}{|h|^\mu} \right).$$

The precise description of the asymptotics mentioned requires some special constructions and is postponed till Theorem 3.1.

The result obtained will be used at once for the calculation of solution asymptotics for the modified Kortevég–de Vries equation at large times.

§2. CONJUGATION PROBLEM WITH QUADRATIC OSCILLATION

In this section, for the case of the phase function $\Phi(x) = \frac{x^2}{2}$, we state the principal results of [1] needed in what follows. Also, once again we describe the main idea that leads to the extraction of the leading term of the asymptotics: this idea will work also in the general case.

So, at this stage we are interested in analytic factorization of a matrix V ,

$$(2.1) \quad H_+(x) = H_-(x)V(x), \quad V(x) = \begin{pmatrix} a & bT^{-1} \\ cT & d \end{pmatrix}, \quad T = e^{it\frac{x^2}{2}}, \quad T^{-1} = e^{-it\frac{x^2}{2}},$$

for large values of the parameter t , under the condition $H_\pm(x) \rightarrow I$ as $x \rightarrow \infty$.

Heuristic arguments. The most important motivating idea can be described as follows. It is well known that if f is an arbitrary smooth function decaying sufficiently fast at infinity, then the function

$$g(x) = f(x) e^{itx}$$

is almost analytic in $\mathbb{C}_+$ as $t \rightarrow +\infty$, i.e., we have the asymptotic relation

$$pg \underset{t \rightarrow +\infty}{\sim} g,$$

where

$$(pg)(x) = \frac{-1}{2\pi i} \int_{\mathbb{R}} \frac{g(y) dy}{x - y + i0},$$

so that the operator p is the orthogonal projection in $L_2(\mathbb{R})$ onto the subspace of functions analytic in the upper half-plane.

Similarly, the function g is almost analytic in $\mathbb{C}_-$ as $t \rightarrow -\infty$:

$$qg \underset{t \rightarrow -\infty}{\sim} g,$$

where

$$(qg)(x) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{g(y) dy}{x - y - i0}.$$

The operator q is the orthogonal projection onto the subspace of functions in $L_2(\mathbb{R})$ that extend analytically to the lower half-plane. It is well known that $p + q = I$.

Application to quadratic exponentials. In the case where rapid oscillation comes from a quadratic exponential, the above considerations need modification. Namely, for the function $g = fT$ we still have

$$pg \underset{t \rightarrow +\infty}{\sim} g$$

only if the smooth function f has support on the half-axis $(0, +\infty)$, and in this case we have

$$qg \underset{t \rightarrow -\infty}{\sim} g.$$

Conversely,

$$pg \underset{t \rightarrow -\infty}{\sim} g \quad \text{and} \quad qg \underset{t \rightarrow +\infty}{\sim} g$$

whenever the support of f lies on the negative half-axis.

Clearly, the reason is that the quadratic exponential $T = e^{it\frac{x^2}{2}}$ extends up to a monotone decreasing function of the variable x to the first and third quarters of the complex plane if $t > 0$, and to the second and fourth quarters if $t < 0$.

These remarks allow us to easily obtain the analytic factorization of our conjugation matrix for the first-fourth and the second-third quarters of the complex plane, respectively. To solve the factorization problem on the entire axis, we need to sew these procedures together, and such sewing requires a certain model problem, the solution of which is based on well-known aspects of analytic theory of differential equations.

Efficient factorization on half-axes. We pass from the conjugation matrix V to the matrix

$$(2.2) \quad W = \begin{pmatrix} 1 & b_1 T^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_1 T & 1 \end{pmatrix}.$$

In accordance with the above considerations, we shall treat this representation as an asymptotically efficient analytic factorization

$$W = W_- W_+$$

for $x > 0$, as $t \rightarrow +\infty$. Note that the quadratic exponentials on the right in (2.2) admit analytic extension up to bounded functions to the first and fourth quarters of the complex plane, respectively.

The passage from V to W can be done via multiplication of V by functions analytic in the corresponding half-planes:

$$(2.3) \quad W = \Theta_- V \Theta_+,$$

where

$$(2.4) \quad \Theta_- = \begin{pmatrix} \alpha_- & 0 \\ 0 & \beta_- \end{pmatrix}, \quad \Theta_+ = \begin{pmatrix} \alpha_+ & 0 \\ 0 & \beta_+ \end{pmatrix},$$

and $\alpha_{\pm}, \beta_{\pm}$ are functions that are determined uniquely (under due assumptions) by the analytic factorization relations

$$\beta_+ \beta_- d = 1, \quad \alpha_+ \alpha_- (ad - bc) = d$$

and by the unit asymptotics at infinity. Indeed, it is easily seen that W will take the form

$$W = \begin{pmatrix} 1 + b_1 c_1 & b_1 T^{-1} \\ c_1 T & 1 \end{pmatrix}, \quad b_1 = \alpha_- \beta_+ b, \quad c_1 = \alpha_+ \beta_- c,$$

and can be written as in (2.2).

To present a similar formula for $x < 0$, we should pass from V to a matrix W of the form

$$W = \begin{pmatrix} 1 & 0 \\ c_2 T & 1 \end{pmatrix} \begin{pmatrix} 1 & b_2 T^{-1} \\ 0 & 1 \end{pmatrix},$$

which, for the same reasons, we may view as an asymptotically efficient analytic factorization, but now for $x < 0$.

For this passage, again it suffices to multiply V as in (2.3), (2.4), if we redefine $\alpha_{\pm}, \beta_{\pm}$ by putting

$$\alpha_+ \alpha_- a = 1, \quad \beta_+ \beta_- (ad - bc) = a,$$

with the unit asymptotics at infinity as before. Then

$$c_2 = \alpha_+ \beta_- c, \quad b_2 = \alpha_- \beta_+ b.$$

Auxiliary scalar conjugation problems. In order to join the two cases, we define the functions $\alpha_{\pm}, \beta_{\pm}$ as solutions of the following scalar Riemann problems

$$(2.5) \quad \begin{aligned} \alpha_+ \alpha_- &= d \Delta^{-1} P + a^{-1} Q, & \beta_+ \beta_- &= d^{-1} P + a \Delta^{-1} Q, \\ \Delta &= ad - bc, & \alpha_{\pm}(x), \beta_{\pm}(x) &\xrightarrow{x \rightarrow \infty} 1, \end{aligned}$$

where P and Q are the indicator functions of the half-axes $(0, \infty)$ and $(-\infty, 0)$, respectively. Recall that, by assumption, the functions Δ and $aQ + \Delta d^{-1}P$ have zero index on the axis. Then the conjugation problems (2.5) are uniquely solvable (see, e.g., [3]). Note that the identity

$$(\alpha_+ \beta_+)(\alpha_- \beta_-) = \Delta^{-1}$$

yields an analytic factorization on the axis of the smooth function Δ^{-1} . On the union of the half-axes we have

$$(2.6) \quad W = W_l W_r, \quad W_l = \begin{pmatrix} 1 & \gamma T^{-1} P \\ \delta T Q & 1 \end{pmatrix}, \quad W_r = \begin{pmatrix} 1 & \gamma T^{-1} Q \\ \delta T P & 1 \end{pmatrix},$$

where $\gamma = \alpha_- \beta_+ b$, $\delta = \alpha_+ \beta_- c$.

Clearly, at the point zero this representation of the matrix W cannot serve as an approximation to analytic factorization, because the conjugation functions for the auxiliary scalar problems have a jump at zero. To correct this expansion, from each factor W_l, W_r in (2.6) we extract its asymptotics at zero (for $x = 0$), and then find a precise analytic factorization $Y_- Y_+$ for the product W_0 of these two asymptotics. The remaining additional embordering factors, call them L and R , turn out to be efficiently analytic in the corresponding half-planes. Schematically, this procedure is presented by the passages

$$(2.7) \quad V \mapsto W = W_l W_r = L W_0 R = L(Y_- Y_+) R = (L Y_-)(Y_+ R).$$

The asymptotics of W_l, W_r . Putting

$$(2.8) \quad \begin{aligned} \nu_0 &= \frac{-1}{2\pi} \ln(a_0 d_0 \Delta_0^{-1}), & a_0 &= a(0), & b_0 &= b(0), \\ c_0 &= c(0), & d_0 &= d(0), & \Delta_0 &= \Delta(0), \end{aligned}$$

we arrive at the following asymptotic expansions:

$$(2.9) \quad \begin{aligned} \ln \alpha_+(x) &= \frac{-1}{2\pi i} \int_{-\infty}^0 \frac{-\ln a(y) dy}{x - y + i0} + \frac{-1}{2\pi i} \int_0^{\infty} \frac{\ln[d(y) \Delta^{-1}(y)] dy}{x - y + i0} \\ &\underset{x \rightarrow 0}{\sim} -i\nu_0 \ln(x + i0) + A_+ + \sum_{\substack{k \geq 1 \\ 0 \leq j \leq k}} A_{kj}^+ x^k \ln^j(x + i0). \end{aligned}$$

Here $\ln(x + i0)$ is the principal branch of the logarithm with cut along the positive semiaxis. Similarly,

$$\begin{aligned} \ln \alpha_-(x) &= \frac{1}{2\pi i} \int_{-\infty}^0 \frac{-\ln a(y) dy}{x - y - i0} + \frac{1}{2\pi i} \int_0^{\infty} \frac{\ln[d(y) \Delta^{-1}(y)] dy}{x - y - i0} \\ &\underset{x \rightarrow 0}{\sim} i\nu_0 \ln(x - i0) + A_- + \sum_{\substack{k \geq 1 \\ 0 \leq j \leq k}} A_{kj}^- x^k \ln^j(x - i0) \end{aligned}$$

and

$$\begin{aligned}\ln \beta_+(x) &\underset{x \rightarrow 0}{\sim} i\nu_0 \ln(x + i0) + B_+ + \sum_{\substack{k \geq 1 \\ 0 \leq j \leq k}} B_{kj}^+ x^k \ln^j(x + i0), \\ \ln \beta_-(x) &\underset{x \rightarrow 0}{\sim} -i\nu_0 \ln(x + i0) + B_- + \sum_{\substack{k \geq 1 \\ 0 \leq j \leq k}} B_{kj}^- x^k \ln^j(x - i0).\end{aligned}$$

Thus,

$$\alpha_{\pm}(x) \underset{x \rightarrow 0}{\sim} \sigma_{\pm}(x) = (x \pm i0)^{\mp i\nu_0} e^{A_{\pm}}, \quad \beta_{\pm}(x) \underset{x \rightarrow 0}{\sim} \tau_{\pm}(x) = (x \pm i0)^{\pm i\nu_0} e^{B_{\pm}},$$

where $(x \pm i0)^{\pm i\nu_0} = e^{\pm i\nu_0 \ln(x \pm i0)}$, and the functions $\ln(x \pm i0)$ are the principal branches of the logarithm with cut along the positive semiaxis. Observe that

$$\sigma_+ \sigma_- = d_0 \Delta_0^{-1} P + a_0^{-1} Q, \quad \tau_+ \tau_- = d_0^{-1} P + a_0 \Delta_0^{-1} Q.$$

As a consequence, $\gamma(x) \underset{x \rightarrow 0}{\sim} \gamma_0(x)$, where

$$\begin{aligned}\gamma_0 &= b_0 \sigma_- \tau_+ = b_0 (x - i0)^{i\nu_0} (x + i0)^{i\nu_0} e^{A_- + B_+}, \\ \delta_0 &= c_0 \sigma_+ \tau_- = b_0 (x - i0)^{-i\nu_0} (x + i0)^{-i\nu_0} e^{A_+ + B_-}.\end{aligned}$$

Thus, the asymptotics of W at zero has the form

$$\begin{aligned}(2.10) \quad W_0 &= \begin{pmatrix} 1 & \gamma_0 T^{-1} P \\ \delta_0 T Q & 1 \end{pmatrix} \begin{pmatrix} 1 & \gamma_0 T^{-1} Q \\ \delta_0 T P & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 + \gamma_0 \delta_0 P & \gamma_0 T^{-1} \\ \delta_0 T & 1 + \gamma_0 \delta_0 Q \end{pmatrix} = \Omega_- V_0 \Omega_+, \end{aligned}$$

where

$$\Omega_{\pm} = \begin{pmatrix} \sigma_{\pm} & 0 \\ 0 & \tau_{\pm} \end{pmatrix}, \quad V_0 = \begin{pmatrix} a_0 & b_0 T^{-1} \\ c_0 T & d_0 \end{pmatrix}.$$

This representation leads us to the model matrix conjugation problem with the conjugation function V_0 . This model problem involves no slow dependence, and should admit exact solution.

Solution of the model problem. Our model problem has the form

$$G_+ = G_- V_0.$$

Its solutions are well known due to its relationship with the ordinary differential equation for the harmonic oscillator. We must only select the solutions with proper asymptotic behavior. Referring to [1] for the details, here we write the solution for $t = 1$ in terms of the standard functions of the parabolic cylinder (Weber functions):

$$(G_-)_{12} = e^{-i\frac{x^2}{4}} (C_1 D_{\nu}(e^{i\frac{\pi}{4}} x) + C_2 D_{\nu}(-e^{i\frac{\pi}{4}} x)).$$

Here

$$\nu = i\nu_0, \quad C_2 = -C_1 e^{-i\pi\nu}, \quad C_1 = -\frac{a_0 d_0}{c_0 \Delta_0} t^{-i\frac{\nu_0}{2}} e^{-A_+ + \frac{\pi}{4}\nu_0}.$$

The function G_{22} is determined by the relations

$$G'_{12} + ixG_{12} = \lambda G_{22}, \quad \lambda = \frac{a_0 d_0 \sqrt{2\pi}}{c_0 \Delta_0 \Gamma(-i\nu_0)} e^{i\frac{\pi}{4} - \frac{\pi}{2}\nu_0 - A_+ - B_-}.$$

Similarly,

$$(G_-)_{21} = e^{i\frac{x^2}{4}} E_1 D_{\nu-1}(e^{i\frac{\pi}{4}} x), \quad E_1 = -\frac{c_0 \Gamma(1 - i\nu_0)}{\Delta_0 \sqrt{2\pi}} t^{\frac{\nu_0}{2}} e^{\frac{3\pi}{4}\nu_0 - B_+},$$

and G_{11} is defined by

$$G'_{21} - ixG_{21} = \mu G_{11}, \quad \lambda\mu = \nu_0.$$

Embordering factors. Concerning the differences

$$\gamma_\infty = \gamma - \gamma_0, \quad \delta_\infty = \delta - \delta_0,$$

the following estimates were proved in [1]:

$$(2.11) \quad p(\gamma_\infty T^{-1}P) = O\left(\frac{t^{-i\nu_0} \ln t}{\sqrt{t}(1+|x|\sqrt{t})}\right), \quad p(\delta_\infty TQ) = O\left(\frac{t^{-i\nu_0} \ln t}{\sqrt{t}(1+|x|\sqrt{t})}\right).$$

Similarly,

$$(2.12) \quad q(\gamma_\infty T^{-1}Q) = O\left(\frac{t^{i\nu_0} \ln t}{\sqrt{t}(1+|x|\sqrt{t})}\right), \quad q(\delta_\infty TP) = O\left(\frac{t^{i\nu_0} \ln t}{\sqrt{t}(1+|x|\sqrt{t})}\right).$$

Put

$$(2.13) \quad L = \begin{pmatrix} 1 & \gamma_\infty T^{-1}P \\ \delta_\infty TQ & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & \gamma_\infty T^{-1}Q \\ \delta_\infty TP & 1 \end{pmatrix},$$

$$L_- = q(L), \quad R_+ = p(R).$$

Estimates (2.11) and (2.12) show that the matrix L is almost analytic in $\mathbb{C}_-$, and R is almost analytic in $\mathbb{C}_+$; therefore, we have

$$(2.14) \quad L \approx L_-, \quad R \approx R_+$$

up to quantities of order

$$O\left(\frac{t^{\pm i\nu_0} \ln t}{\sqrt{t}(1+|x|\sqrt{t})}\right)$$

Properties of triangular matrices allow us to write W_l and W_r in the form

$$W_l = L \begin{pmatrix} 1 & \gamma_0 T^{-1}P \\ \delta_0 TQ & 1 \end{pmatrix}, \quad W_r = \begin{pmatrix} 1 & \gamma_0 T^{-1}Q \\ \delta_0 TP & 1 \end{pmatrix} R,$$

in full agreement with (2.7), (2.10).

If we replace the almost analytic factors L and R with the corresponding analytic functions L_- and R_+ , then, by (2.11) and (2.12), the error will be of order

$$O\left(\frac{t^{|\operatorname{Re}(i\nu_0)|-1/2} \ln t}{1+\sqrt{t}|x|}\right).$$

This error can be leveled by the discrepancy method, see [1].

Asymptotics of H . Now, the asymptotics as $t \rightarrow +\infty$ of the matrix H is given by

$$(2.15) \quad H_- \sim G_- \begin{pmatrix} \sigma_-^{-1} & 0 \\ 0 & \tau_-^{-1} \end{pmatrix} L_-^{-1} \begin{pmatrix} \alpha_- & 0 \\ 0 & \beta_- \end{pmatrix}.$$

For example, for $|x|$ separated away from zero, we find

$$(2.16) \quad H_- \sim \begin{pmatrix} \alpha_- + \alpha_- \tau_-^{-1} G_{12} \delta_0 TQ & \beta_- \tau_-^{-1} G_{12} + \beta_- \gamma_0 T^{-1}P \\ \alpha_- \sigma_-^{-1} G_{21} + \alpha_- \delta_0 TQ & \beta_- + \beta_- \sigma_-^{-1} G_{21} \gamma_0 T^{-1}P \end{pmatrix}.$$

We write the leading term of this asymptotics for large $|x|$ for the entry $(H_-)_{12}$. Precisely this asymptotics is most important in applications to nonlinear equations of mathematical physics. We have

$$(2.17) \quad (H_-)_{12} \sim \frac{a_0 d_0}{c_0 \Delta_0} \frac{\sqrt{2\pi}}{\Gamma(-i\nu_0)} t^{-\frac{1}{2}-i\nu_0} e^{-A_+-B_-} e^{-i\frac{\pi}{4}} e^{-\frac{\pi\nu_0}{2}} \cdot \frac{1}{x}.$$

So, the following statement is one of the results of [1].

Theorem 2.1. *Under the conditions imposed above on the entries of the conjugation matrix, for any sufficiently large $t > 0$ the solution of the conjugation problem (2.1) exists and has asymptotics determined by relations (2.15), (2.16), and (2.17) with discrepancy $O(t^{|\operatorname{Im} \nu_0| - 1 + \varepsilon})$ for an arbitrarily small $\varepsilon > 0$ relative to $H^{\mu, 1}(\mathbb{R})$, where $\mu > 0$ is sufficiently small.*

For what follows we note that, to justify the results in [1], a relationship was employed between the transposed conjugation problem

$$H_+^T = V^T H_-^T, \quad H_\pm^T \underset{x \rightarrow \infty}{\sim} I,$$

and the inversion problem for the singular operator $I - F^T q$, where F is the operator of multiplication by the matrix-valued function $F(x) = I - V$, and F^T is the operator of multiplication by the transpose matrix.

Namely, if we define a matrix-valued function K^T as a solution of the equation

$$(I - F^T q)K^T = F^T,$$

then the transposed conjugation problem has a solution of the form

$$H_+^T = I - p(K^T), \quad H_-^T = I + q(K^T).$$

As a consequence, we find

$$(2.18) \quad H_+ = I - p(K), \quad H_- = I + q(K).$$

If a pair of factors $S_\pm$ determines a solution K_0^T of the equation

$$(2.19) \quad (I - F_0^T q)K_0^T = F_0^T, \quad F_0 = S_+^{-1} S_-,$$

then for the perturbed conjugation matrix $V = I + F_0 + \mathcal{E}$ we get an equation for K^T :

$$(2.20) \quad (I + F_0^T q + \mathcal{E}^T q)K^T = F_0^T + \mathcal{E}^T,$$

which can be solved by iteration provided the perturbation $\mathcal{E}$ is appropriately small.

§3. ASYMPTOTICS OF SOLUTIONS OF THE CONJUGATION PROBLEM IN THE GENERAL CASE

3.1. Preparing separation of contributions.

Idea of generalization. We turn to the general case of analytic factorization of the matrix-valued function

$$(3.1) \quad V(x) = \begin{pmatrix} a(x) & b(x) e^{-it\Phi(x)} \\ c(x) e^{it\Phi(x)} & d(x) \end{pmatrix}.$$

The preceding considerations generalize to the case of an arbitrary smooth phase function Φ that has finitely many simple stationary points and a power-like growth at infinity with an exponent at least 1. Here, the central question will be about separation of the contributions of individual stationary points.

We split the procedure of separating the contributions into several steps.

First, we solve an auxiliary scalar problem, arriving at an efficient (in the same sense as in (2.6)) analytic factorization outside of the stationary points of the phase function.

At the next step, we find the asymptotic behavior of the efficient factorizing functions (denoting them U_l and U_r this time) at the stationary points. However, unlike the case of the quadratic phase function, now we preserve the local nature of these asymptotic formulas by involving standard cut-off functions in order to exclude interference of contributions of stationary points. Like in the case of the quadratic phase function, the

supplementary factors will be described as the embordering functions L and R (cf. (2.7)). These factors will be efficiently analytic in the corresponding half-planes.

The third step is specific for separation of contributions. It consists in analytic factorization of the diagonal of the conjugation matrix obtained at the preceding step. It is aimed at reducing the problem to a form adjusted to application of the alternating Schwarz method, which is the standard algebraic procedure for summation of contributions.

At the fourth step, we use the Schwarz method to justify the summation of contributions made by the stationary points of the phase function. However, to obtain explicit formulas we shall need yet another step (a step back, a sort of “defactorization” of the diagonal of the conjugation matrix), because the procedure of summation of contributions and that of analytic factorization of the model problem with the quadratic phase function are done for matrices of different structures, which requires an elementary recalculation.

Schematically, all this process is presented by the passages

$$\begin{aligned} V \mapsto \Theta_- V \Theta_+ &= U_l U_r = L \prod V_i R = L \Psi_- W \Psi_+ R \\ &\cong L \Psi_- \left(I - \sum N_{i-} \right) \left(I + \sum N_{i+} \right) \Psi_+ R \cong U_- U_+ \end{aligned}$$

and is described below in more detail.

Step 1: the auxiliary scalar conjugation problem. We denote by k_i the stationary points of the phase function Φ and by P the indicator function for the union of the intervals on which Φ is monotone increasing. Put $Q = 1 - P$.

Following the line described above, we introduce auxiliary scalar analytic factorizations

$$\begin{aligned} (3.2) \quad \alpha_+ \alpha_- &= d \Delta^{-1} P + a^{-1} Q, \quad \beta_+ \beta_- = d^{-1} P + a \Delta^{-1} Q, \\ \Delta &= ad - bc, \quad \alpha_{\pm}(x), \beta_{\pm}(x) \xrightarrow{x \rightarrow \infty} 1. \end{aligned}$$

Again, we observe the identity

$$(\alpha_+ \beta_+)(\alpha_- \beta_-) = \Delta^{-1}.$$

As before,

$$\begin{aligned} (3.3) \quad W &= \Theta_- V \Theta_+ = \begin{pmatrix} 1 & \gamma T^{-1} P \\ \delta T Q & 1 \end{pmatrix} \begin{pmatrix} 1 & \gamma T^{-1} Q \\ \delta T P & 1 \end{pmatrix}, \\ \gamma &= \alpha_- \beta_+ b, \quad \delta = \alpha_+ \beta_- c, \end{aligned}$$

where $\Theta_{\pm}$ are the matrices defined in (2.4). We put

$$\begin{aligned} (3.4) \quad \nu_j &= \frac{-1}{2\pi} \ln(a_j d_j \Delta_j^{-1}), \quad a_j = a(k_j), \quad b_j = b(k_j), \\ c_j &= c(k_j), \quad d_j = d(k_j), \quad \Delta_j = \Delta(k_j). \end{aligned}$$

We assume additionally that the parameters ν_j are real. This assumption is not principal, but will allow us to treat the contributions of different stationary points as quantities of one order of smallness. Precisely this case is most important in applications.

The set $Q(\mathbb{R})$ on which the phase function is monotone decreasing is a union of disjoint intervals (including, possibly, infinite intervals). The points k_j are ends of these intervals. If a point k_j is the right end of an interval in the system $Q(\mathbb{R})$, i.e., k_i is a point of

minimum, then, as before, we put

$$\begin{aligned} \ln \alpha_+(x) &= \frac{-1}{2\pi i} \int_{\mathbb{R}} \frac{-\ln a(y)Q(y) dy}{x-y+i0} + \frac{-1}{2\pi i} \int_{\mathbb{R}} \frac{\ln[d(y)\Delta^{-1}(y)P(y)] dy}{x-y+i0} \\ &\underset{x \rightarrow k_j}{\sim} -i\nu_j \ln(x - k_j + i0) + A_{j+}. \end{aligned}$$

Here, again, $\ln(x+i0)$ is the principal branch of the logarithm with cut along the positive semiaxis. Similarly,

$$\begin{aligned} \ln \alpha_-(x) &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{-\ln a(y)Q(y) dy}{x-y-i0} + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\ln[d(y)\Delta^{-1}(y)P(y)] dy}{x-y-i0} \\ &\underset{x \rightarrow k_j}{\sim} i\nu_j \ln(x - k_j - i0) + A_{j-}. \end{aligned}$$

Also, we have

$$\ln \beta_+(x) \underset{x \rightarrow k_j}{\sim} i\nu_j \ln(x - k_j + i0) + B_{j+}, \quad \ln \beta_-(x) \underset{x \rightarrow k_j}{\sim} -i\nu_j \ln(x - k_j - i0) + B_{j-}.$$

If k_j is the left end of an interval in the system $Q(\mathbb{R})$, i.e., k_j is a point of maximum, then, for the same choice of a branch the logarithm, we have

$$\ln \alpha_+(x) \underset{x \rightarrow k_j}{\sim} i\nu_j \ln(x - k_j + i0) + A_{j+}, \quad \ln \alpha_-(x) \underset{x \rightarrow k_j}{\sim} -i\nu_j \ln(x - k_j - i0) + A_{j-},$$

and, similarly,

$$\ln \beta_+(x) \underset{x \rightarrow k_j}{\sim} -i\nu_j \ln(x - k_j + i0) + B_{j+}, \quad \ln \beta_-(x) \underset{x \rightarrow k_j}{\sim} i\nu_j \ln(x - k_j - i0) + B_{j-}.$$

We shall not present precise expressions for $A_{j\pm}$ and $B_{j\pm}$, viewing the above asymptotic relations as the definitions of these constants.

In what follows, we call the points k_j the left and right turning points.

Step 2: finding the asymptotic behavior at the stationary points. Our goal is to modify the expansion (3.3) in a neighborhood of the stationary points of the phase function Φ so as to ensure the applicability of the discrepancy method for obtaining the exact analytic factorization.

So, in the case of a minimum point k_j we have

$$\begin{aligned} \alpha_{j\pm}(x) &\underset{x \rightarrow k_j}{\sim} \sigma_{j\pm}(x) = (x - k_j + i0)^{\mp i\nu_j} e^{A_{j\pm}}, \\ \beta_{j\pm}(x) &\underset{x \rightarrow k_j}{\sim} \tau_{j\pm}(x) = (x - k_j - i0)^{\pm i\nu_j} e^{B_{j\pm}}, \end{aligned}$$

where, as before, $(x \pm i0)^{\mp i\nu_j} = e^{\mp i\nu_j \ln(x \pm i0)}$, and the functions $\ln(x \pm i0)$ are the principal branches of the logarithm with cut along the positive semiaxis. In the case of a point of maximum,

$$\begin{aligned} \alpha_{j\pm}(x) &\underset{x \rightarrow k_j}{\sim} \sigma_{j\pm}(x) = (x - k_j + i0)^{\pm i\nu_j} e^{A_{j\pm}}, \\ \beta_{j\pm}(x) &\underset{x \rightarrow k_j}{\sim} \tau_{j\pm}(x) = (x - k_j - i0)^{\mp i\nu_j} e^{B_{j\pm}}. \end{aligned}$$

In any case,

$$\sigma_{j+}\sigma_{j-} = d_j\Delta_j^{-1}P_j + a_j^{-1}Q_j, \quad \tau_{j+}\tau_{j-} = d_j^{-1}P_j + a_j\Delta_j^{-1}Q_j,$$

where $P_j + Q_j = 1$, and P_j is the indicator function of the real semiaxis ending at k_j and coinciding with P near k_j .

It follows that $\gamma(x) \underset{x \rightarrow k_j}{\sim} \gamma_j(x)$ and $\delta(x) \underset{x \rightarrow k_j}{\sim} \delta_j(x)$, where

$$\gamma_j = b_j\sigma_{j-}\tau_{j+}, \quad \delta_j = c_j\sigma_{j+}\tau_{j-}.$$

Now, we additionally split every factor on the right in (3.3), extracting the asymptotic behavior in neighborhoods of the stationary points:

$$(3.5) \quad \begin{pmatrix} 1 & \gamma T^{-1}P \\ \delta TQ & 1 \end{pmatrix} \underset{k \rightarrow k_j}{\sim} V_j^- = \begin{pmatrix} 1 & \gamma_j \chi_j T_j^{-1}P \\ \delta_j \chi_j T_j Q & 1 \end{pmatrix}$$

and

$$(3.6) \quad \begin{pmatrix} 1 & \gamma T^{-1}Q \\ \delta TP & 1 \end{pmatrix} \underset{k \rightarrow k_j}{\sim} V_j^+ = \begin{pmatrix} 1 & \gamma_j \chi_j T_j^{-1}Q \\ \delta_j \chi_j T_j P & 1 \end{pmatrix}.$$

Here, in a neighborhood of a turning point k_j we put

$$T \sim T_j,$$

where

$$(3.7) \quad T_j(x) = e^{it\Phi(k_j)} e^{it \frac{\Phi''(k_j)}{2}(x-k_j)^2},$$

and χ_j denotes a smooth cut-off function equal to 1 near k_j . The supports of χ_j are assumed to be disjoint, which implies that the matrices V_j commute, as well as the matrices V_j^+ . Also, the matrices V_j^- and V_k^+ commute for different indices j and k .

We obtain the expansions

$$(3.8) \quad U_l = \begin{pmatrix} 1 & \gamma T^{-1}P \\ \delta TQ & 1 \end{pmatrix} = L \prod_j \begin{pmatrix} 1 & \gamma_j \chi_j T_j^{-1}P \\ \delta_j \chi_j T_j Q & 1 \end{pmatrix}$$

and

$$(3.9) \quad U_r = \begin{pmatrix} 1 & \gamma T^{-1}Q \\ \delta TP & 1 \end{pmatrix} = \prod_j \begin{pmatrix} 1 & \gamma_j \chi_j T_j^{-1}Q \\ \delta_j \chi_j T_j P & 1 \end{pmatrix} R.$$

Like in the case of one turning point, the factors L and R are almost analytic, respectively, in $\mathbb{C}_-$ and $\mathbb{C}_+$, are close to 1 with an error of $O(t^{-1/2} \ln t(1 + \sqrt{t}|x|)^{-1})$ (cf. (2.14)), and can be reshaped into strictly analytic functions with the help of the discrepancy method (see Subsection 2.5 in [1]). These facts can be proved precisely as it was done in the proof of Theorem 2.2 in [1].

Step 3: factorization of the diagonal. At the preceding step we arrived at the following factorization:

$$\Theta_- V \Theta_+ = L \prod V_j R,$$

where

$$V_j = V_j^- V_j^+ = \begin{pmatrix} 1 + \gamma_j \delta_j \chi_j^2 P & \gamma_j \chi_j T_j^{-1} \\ \delta_j \chi_j T_j & 1 + \gamma_j \delta_j \chi_j^2 Q \end{pmatrix}$$

and

$$\prod V_j = \begin{pmatrix} 1 + \sum \gamma_j \delta_j \chi_j^2 P & \sum \gamma_j \chi_j T_j^{-1} \\ \sum \delta_j \chi_j T_j & 1 + \sum \gamma_j \delta_j \chi_j^2 Q \end{pmatrix}.$$

It should be noted that the asymptotics of the factorizing functions of the matrix $\prod V_j$ depends, in the senior order, only on the asymptotic behavior of the functions γ_j, δ_j at the stationary points of the phase function Φ , which can be seen from the presence of the cut-off functions χ_j in the definition of $\prod V_j$.

Factorization of the diagonal of the matrix $\prod V_j$ will lead us to a conjugation problem in which the contributions of the stationary points sum asymptotically additively.

To show this, we need some auxiliary calculations. Namely, we introduce the scalar analytic factorizations $\varphi_{j-} \varphi_{j+} = 1 + \gamma_j \delta_j \chi_j^2 P$ and $\psi_{j-} \psi_{j+} = 1 + \gamma_j \delta_j \chi_j^2 Q$. The functions

$\varphi_{j\pm}(x)$, $\psi_{j\pm}(x)$ are normalized by the requirement that they tend to 1 as $x \rightarrow \infty$. We put

$$\varphi_{\pm} = \prod \varphi_{j\pm}, \quad \psi_{\pm} = \prod \psi_{j\pm}$$

and

$$\Psi_- = \begin{pmatrix} \varphi_- & 0 \\ 0 & \psi_- \end{pmatrix}, \quad \Psi_+ = \begin{pmatrix} \varphi_+ & 0 \\ 0 & \psi_+ \end{pmatrix}.$$

We have the identity

$$\Psi_- \Psi_+ = \begin{pmatrix} 1 + \sum \gamma_j \delta_j \chi_j^2 P & 0 \\ 0 & 1 + \sum \gamma_j \delta_j \chi_j^2 Q \end{pmatrix},$$

which is an analytic factorization of the diagonal of the matrix $\prod V_j$. It follows that

$$\prod V_j = \Psi_- W \Psi_+,$$

where we have put

$$W = I - F, \quad F = \sum F_j, \quad F_j = - \begin{pmatrix} 0 & f_j T_j^{-1} \\ h_j T_j & 0 \end{pmatrix}.$$

Here

$$f_j = \gamma_j \chi_j \varphi_-^{-1} \psi_+^{-1}, \quad h_j = \delta_j \chi_j \varphi_+^{-1} \psi_-^{-1}.$$

We shall show that the conjugation problem for the matrix W reduces asymptotically to the additive summation of the contributions of the conjugation problems for the matrices $W_j = I - F_j$ in a domain not containing neighborhoods of the turning points.

3.2. Separation of contributions.

Alternating Schwarz method. Before passing to the fourth step, we recall the general algebraic scheme of the alternating Schwarz method. The details of this interpretation can be found in [2]. This method concerns an algebraic construction of the operator inverse to $I - \sum G_i$ provided that we know the inverses to the operators $I - G_i$.

Let G_i ($i = 1, \dots, n$) be linear operators in a complex vector space $\mathcal{X}$. Assume that all operators $I - G_i$ are bijective. Put

$$(3.10) \quad I - \Gamma_i = (I - G_i)^{-1}$$

and

$$(3.11) \quad G = \sum_{i=1}^n G_i.$$

The essence of the alternating Schwarz method will be shown in a series of claims.

First. If an operator matrix

$$(3.12) \quad \Phi = \begin{pmatrix} I & \Gamma_1 & \dots & \Gamma_1 \\ \Gamma_2 & I & \dots & \Gamma_2 \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_n & \Gamma_n & \dots & I \end{pmatrix}$$

is a bijection of the space $\mathcal{X}^n$, then $I - G$ is a bijection of the space $\mathcal{X}$.

Second. Let linear operators γ_{ij} in $\mathcal{X}$ be defined as solutions of the equation

$$(3.13) \quad \Phi \cdot \Gamma = \text{diag}(\Gamma_1, \dots, \Gamma_n),$$

where

$$(3.14) \quad \Gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn} \end{pmatrix}$$

and

$$(3.15) \quad \text{diag}(\Gamma_1, \dots, \Gamma_n) = \begin{pmatrix} \Gamma_1 & 0 & \cdots & 0 \\ 0 & \Gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_n \end{pmatrix}.$$

Then the operator Γ defined by

$$(3.16) \quad I - \Gamma = (I - G)^{-1},$$

is given in terms of the γ_{ij} by the formula

$$(3.17) \quad \Gamma = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \gamma_{ij}.$$

Note that the following identities are valid:

$$(3.18) \quad \gamma_{ij} = \Gamma_i \left(\delta_{ij} I - \sum_{k \neq i} \gamma_{kj} \right) = \left(\delta_{ij} I - \sum_{k \neq j} \gamma_{ik} \right) \Gamma_j.$$

Third. The components γ_{ij} can be recovered by the operator Γ by the formulas

$$(3.19) \quad \gamma_{ij} = -\delta_{ij} G_j - G_i (I - \Gamma) G_j.$$

Finally, we make a remark that will be important in what follows. Put $\Phi = I + \Delta$. Then

$$\Phi^{-1} = (I + \Delta)^{-1} = (I - \Delta^2)^{-1} (I - \Delta).$$

The components of the operator matrix Δ^2 are sums of operators of the form $\Gamma_i \Gamma_j$ ($i \neq j$). If the operator Δ is bounded in an appropriate Banach space and the norm of Δ^2 is less than 1, then

$$(3.20) \quad \Gamma = \sum \Gamma_i - \sum_{i \neq j} \Gamma_i \Gamma_j + \dots,$$

where the series of sums converges in norm.

Step 4: separation of contributions. As has already been mentioned, the conjugation problem

$$S_+ = S_- W$$

is equivalent to the inversion problem for the operator $I - F^T q$ with the multiplication operator $F = I - W$.

We employ the approach described above (the Schwarz method), putting $G_j = F_j^T q$. Observe that for $i \neq j$ the kernel of the product $G_i G_j$ is a diagonal matrix-valued function with diagonal entries of the form

$$\int_{\mathbb{R}} dz \frac{g_i(x) e^{it\kappa_i(x-k_i)^2}}{(x-z)} \cdot \frac{g_j(z) e^{it\kappa_j(z-k_j)^2}}{(z-y-i0)} = e^{it\kappa_i(x-k_i)^2} \chi_i(x) O(t^{-1/2}(1+|y|)^{-1}),$$

where the functions g_i and g_j are smooth and have disjoint supports (controlled by the cut-off functions χ_i and χ_j) lying in neighborhoods of the points k_i and k_j , respectively.

Since the contributions of the stationary points are localized, the role of the space $\mathcal{X}$ in the Schwarz method can be played by the space $H^\alpha = H^{\alpha,0}(\mathbb{R})$ of Hölder functions on the axis with an exponent α close to zero. It is well known that the singular operators p and q are bounded in H^α . As a consequence, the operators G_i and Γ_i will be bounded with norm of order of $O(t^\alpha)$. The operator products $G_i G_j, G_i \Gamma_j, \Gamma_i G_j, \Gamma_i \Gamma_j$ will have small norm of order of $O(t^{\alpha-1/2})$, which is implied by the above estimate for the kernel of the product $G_i G_j$.

Consequently, in this leading order in t we have

$$\Gamma = \sum \Gamma_j + \sum_{i \neq j} \Gamma_i \Gamma_j + O(t^{\alpha-1})$$

and

$$\begin{aligned} K^T &= (I - \Gamma) \sum F_i^T = \left(I - \sum \Gamma_j \right) \sum F_i^T + \sum_{j \neq i} \Gamma_j \Gamma_i \sum F_k^T + O(t^{\alpha-1}) \\ &= \sum (I - \Gamma_i) F_i^T - \sum_{i \neq j} \Gamma_j F_i^T + \sum_{j \neq i; k} \Gamma_j \Gamma_i F_k^T + O(t^{\alpha-1}) \\ &= \sum K_j^T + \mathcal{E} + O(t^{\alpha-1}), \end{aligned}$$

where we have put

$$\mathcal{E} = - \sum_{i \neq j} \Gamma_j F_i^T + \sum_{j \neq i; k} \Gamma_j \Gamma_i F_k^T.$$

We need to estimate $p(\mathcal{E})$ in the norm of H^α , see (2.18). For definiteness, we estimate the contribution of the term $\mathcal{E}_{ij} = \Gamma_j F_i^T$. Observe that

$$-\Gamma_j F_i^T = (I - \Gamma_j) G_j F_i^T = K_j^T q F_i^T.$$

This function is of the form

$$e^{it\kappa_j(x-k_j)^2} \chi_j(x) O(t^{-1/2}),$$

where, viewed as a function of x , $O(t^{-1/2})$ is differentiable without distorting the order in t . Therefore, as soon as the argument of the function $p(\mathcal{E}_{ij})$ goes beyond the domain controlled by the cut-off function $\chi = \sum \chi_j$, we can use the stationary phase method to obtain an estimate of the form $O(t^{-1})$ for the contribution in question. The contributions of the other terms are similar.

Thus, up to quantities of order of $O(t^{\alpha-1})$, where α can be takes as close to zero as we wish, the contribution to K^T of the stationary points of the phase function is additive.

Step 5: moving backward. To obtain explicit formulas, for us it will be convenient to return from factorization of the matrices

$$\begin{pmatrix} 1 & f_j T_j^{-1} \\ h_j T_j & 1 \end{pmatrix}$$

to that of the matrices of the form

$$\begin{pmatrix} 1 + m_j n_j P & m_j T_j^{-1} \\ n_j T_j & 1 + m_j n_j Q \end{pmatrix}.$$

The reason is that, as was mentioned above, in the leading order, the asymptotics of the factorizing factors only depends on the asymptotics of the functions m_j, n_j at the point k_j . Factorizing the diagonal of the last-written matrix, we get the following equation for m_j, n_j :

$$\begin{pmatrix} s_{j-} & 0 \\ 0 & t_{j-} \end{pmatrix} \begin{pmatrix} 1 & f_j T_j^{-1} \\ h_j T_j & 1 \end{pmatrix} \begin{pmatrix} s_{j+} & 0 \\ 0 & t_{j+} \end{pmatrix} = \begin{pmatrix} 1 + m_j n_j P & m_j T_j^{-1} \\ n_j T_j & 1 + m_j n_j Q \end{pmatrix},$$

where $s_{j+}s_{j-} = 1 + m_j n_j P$ and $t_{j+}t_{j-} = 1 + m_j n_j Q$. We have

$$f_j s_{j-} t_{j+} = m_j, \quad h_j s_{j+} t_{j-} = n_j,$$

whence

$$f_j h_j (1 + m_j n_j) = m_j n_j,$$

i.e.,

$$m_j n_j = \frac{f_j h_j}{1 - f_j h_j},$$

which agrees with the previous relations and determines the m_j, n_j .

Similarly, we get

$$f_j h_j = \frac{\gamma_j \delta_j \chi_j^2}{1 + \sum \gamma_k \delta_k \chi_k^2}.$$

Consequently,

$$m_j n_j = \frac{\gamma_j \delta_j \chi_j^2}{1 + \sum_{i \neq j} \gamma_i \delta_i \chi_i^2} = \gamma_j \delta_j \chi_j^2,$$

whence $\varphi_{j\pm} = s_{j\pm}$ and $\psi_{j\pm} = t_{j\pm}$. But this means that $m_j = \gamma_j \chi_j$ and $n_j = \delta_j \chi_j$. Thus, we arrive at factorization problems for the following matrix products:

$$M_{j-}^{-1} M_{j+} = \begin{pmatrix} \varphi_{j-}^{-1} & 0 \\ 0 & \psi_{j-}^{-1} \end{pmatrix} \begin{pmatrix} 1 + \gamma_j \delta_j \chi_j^2 P & \gamma_j \chi_j T_j^{-1} \\ \delta_j \chi_j T_j & 1 + \gamma_j \delta_j \chi_j^2 Q \end{pmatrix} \begin{pmatrix} \varphi_{j+}^{-1} & 0 \\ 0 & \psi_{j+}^{-1} \end{pmatrix}.$$

Putting $N_{j\pm} = M_{j\pm} - I$, we see that

$$(3.21) \quad H_+ \sim [I + \sum N_{j+}] \Psi_+ R_+ \Theta_+^{-1}, \quad H_- \sim [I + \sum N_{j-}] \Psi_-^{-1} L_-^{-1} \Theta_-.$$

As a result, we have the following statement.

Theorem 3.1. *For sufficiently large $t > 0$, the conjugation problem in question has a solution whose asymptotics has the form (3.21) with an error of $O(t^{\varepsilon-1})$, with $\varepsilon > 0$ arbitrarily small, relative to $H^{\alpha,1}(\mathbb{R})$, where $\alpha > 0$ is sufficiently small.*

3.3. Asymptotics of the off-diagonal entries of the matrix H at large $|x|$. As we have seen, the asymptotic behavior of the matrix H in the scattering domain is determined by that of its off-diagonal entries. At the same time, the asymptotics of the factors $\Psi_{\pm}$, L , R , and $\Theta_{\pm}$ as $x \rightarrow \infty$ show that the asymptotics of the off-diagonal entries of H is determined by the corresponding element of $\sum N_j$. With the same accuracy as above, we find

$$N_{j\pm} \sim O_{j\pm},$$

where the matrices $O_{j\pm}$ are given by the analytic factorization

$$(I + O_{j+}) = (I + O_{j-}) \begin{pmatrix} 1 + \gamma_j \delta_j P_j & \gamma_j T_j^{-1} \\ \delta_j T_j & 1 + \gamma_j \delta_j Q_j \end{pmatrix}.$$

However, this factorization problem was solved in the paper [1] devoted to the case of one turning point (see §2). We use formula (2.17), replacing, in accordance with (3.7), a_0, d_0, Δ_0 by a_j, d_j, Δ_j , b_0 and c_0 by $b_j e^{-it\Phi(k_j)}$ and $c_j e^{it\Phi(k_j)}$; and t by $t|\Phi''(k_j)|$. Also, we introduce the following indicator of a turning point:

$$(3.22) \quad \varkappa_j = \operatorname{sgn} \Phi''(k_j).$$

Then, by symmetry, the asymptotics of H_{12} as $x, t \rightarrow \infty$ is given by

$$(3.23) \quad H_{12} \sim \sum \frac{e^{-it\Phi(k_j)} a_j d_j}{c_j \Delta_j} \frac{\sqrt{2\pi}}{\Gamma(-i\nu_j \varkappa_j)} (t|\Phi''(k_j)|)^{-\frac{1}{2} - i\nu_j \varkappa_j} e^{-(A_{\varkappa_j j} + B_{-\varkappa_j j}) \varkappa_j} e^{-i\frac{\pi}{4} \varkappa_j} e^{-\frac{\pi \nu_j}{2}} \frac{\varkappa_j}{x}.$$

The change of the sign (the coefficient $\varkappa_j$ of $1/x$) for the contribution of a maximum point is explained naturally by the change of the direction of decreasing compared to the case of a point of minimum. We can also observe that, passing to the complex conjugate conjugation matrix, we get a conjugation problem of the form

$$\tilde{H}_- = \tilde{H}_+ \bar{V}.$$

The inverse to the matrix $\bar{V}$ is equal to

$$\frac{1}{\bar{\Delta}} \begin{pmatrix} \bar{d} & -\bar{b}T \\ -\bar{c}T^{-1} & \bar{a} \end{pmatrix}.$$

In essence, this means that a problem with a maximum point reduces to that with a minimum point if we replace b and c with $-b$ and $-c$, which, once again, leads to formula (3.23).

§4. APPLICATIONS

4.1. Modified Korteweg-de Vries equation. As an example, here we briefly discuss calculation of the asymptotics for the solution of the Cauchy problem for a well-known nonlinear equation of mathematical physics, namely, the modified Korteweg-de Vries equation.

This equation has the form

$$u_t - 6u^2u_x + u_{xxx} = 0$$

with the following initial condition:

$$u|_{t=0} \text{ is a smooth function decaying rapidly as } x \rightarrow \infty.$$

The matrix conjugation problem associated with this equation (see, e.g., [4]) has the standard form described above:

$$H_+ = H_- V, \quad V(k) = \begin{pmatrix} 1 - |r(k)|^2 & -\overline{r(k)}e^{-it\Phi(k)} \\ r(k)e^{it\Phi(k)} & 1 \end{pmatrix},$$

with the phase function

$$\Phi(k) = 8k^3 + 2lk,$$

where l has the meaning of the ratio $\frac{x}{t}$. The reflection coefficient $r(k)$ satisfies

$$(4.1) \quad \overline{r(-k)} = -r(k).$$

The relationship between the solution of the above Cauchy problem and the solution of the corresponding auxiliary matrix Riemann problem is given by the formula

$$u(x, t) = 2 \lim_{k \rightarrow \infty} k H_{12}(k).$$

The derivative Φ' has the roots $k_{1,2} = \pm k_0$, $k_0 = \sqrt{-\frac{l}{12}}$ (the square root is arithmetical). We have

$$(4.2) \quad \Phi(k_1) = -\Phi(k_2) = -16k_0^3, \quad \Phi''(k_1) = -\Phi''(k_2) = 48k_0.$$

Next, we put

$$(4.3) \quad r(k_1) = r_0, \quad r(k_2) = r(-k_1) = -\overline{r(k_1)} = -\overline{r_0}, \quad \nu_1 = \nu_2 = \nu_0 \equiv \frac{-1}{2\pi} \ln(1 - |r_0|^2).$$

Finally, to compute B_1 and B_2 we observe that

$$\begin{aligned} & \frac{-1}{2\pi i} \int_{-k_0}^{k_0} \frac{-\ln(1 - |r(y)|^2) dy}{x - y + i0} \\ &= \frac{i}{2\pi} \ln(1 - |r(y)|^2) \ln(x - y + i0) \Big|_{y=-k_0}^{y=k_0} - \frac{i}{2\pi} \int_{-k_0}^{k_0} \ln(x - y + i0) d\ln(1 - |r(y)|^2). \end{aligned}$$

Assuming that $\ln(x + i0)$ is defined as the principal branch on the plane with cut along the positive semiaxis and letting $x \rightarrow k_0$, we obtain

$$\begin{aligned} & \frac{-1}{2\pi i} \int_{-k_0}^{k_0} \frac{-\ln(1 - |r(y)|^2) dy}{x - y + i0} \\ & \sim -i\nu_0 \ln(x - k_0 + i0) + i\nu_0 \ln(2k_0) - \frac{i}{2\pi} \int_{-k_0}^{k_0} \ln |k_0 - y| d\ln(1 - |r(y)|^2). \end{aligned}$$

This means that

$$(4.4) \quad \begin{aligned} A_{1+} &= -iB + i\nu_0 \ln(2k_0), \\ B &= \frac{1}{2\pi} \int_{-k_0}^{k_0} \ln |k - y| d\ln(1 - |r(y)|^2), \quad B_{1-} = A_{1+} - 2\pi\nu_0. \end{aligned}$$

Letting $x \rightarrow -k_0$, we see that

$$\begin{aligned} & \frac{-1}{2\pi i} \int_{-k_0}^{k_0} \frac{-\ln(1 - |r(y)|^2) dy}{x - y + i0} \\ & \sim i\nu_0 \ln(x + k_0 + i0) - i\nu_0 \ln(-2k_0 + i0) - \frac{i}{2\pi} \int_{-k_0}^{k_0} \ln(-k_0 - y + i0) d\ln(1 - |r(y)|^2) \\ &= i\nu_0 \ln(x + k_0 + i0) - i\nu_0 \ln(2k_0) + \pi\nu_0 - \frac{i}{2\pi} \int_{-k_0}^{k_0} \ln |k_0 + y| d\ln(1 - |r(y)|^2) \\ &= i\nu_0 \ln(x + k_0 + i0) - i\nu_0 \ln(2k_0) + \pi\nu_0 + \frac{i}{2\pi} \int_{-k_0}^{k_0} \ln |k_0 - y| d\ln(1 - |r(y)|^2), \end{aligned}$$

whence

$$A_{2+} = -A_{1+} + \pi\nu_0$$

which implies

$$(4.5) \quad A_{2-} = A_{1+} + \pi\nu_0, \quad B_{2+} = A_{1-} + \pi\nu_0.$$

Thus, for the asymptotics of the solution of the Cauchy problem in the case of the modified Korteweg-de Vries equation we find

$$\begin{aligned} u &\sim 2 \left[\frac{e^{i16tk_0^3}}{r_0} \frac{\sqrt{2\pi}}{\Gamma(-i\nu_0)} (48k_0t)^{-\frac{1}{2}-i\nu_0} e^{2iB-2i\nu_0 \ln(2k_0)} e^{-i\frac{\pi}{4}} e^{-\frac{\pi\nu_0}{2}} \right. \\ &\quad \left. - \frac{e^{-i16tk_0^3}}{-\overline{r_0}} \frac{\sqrt{2\pi}}{\Gamma(i\nu_0)} (48k_0t)^{-\frac{1}{2}+i\nu_0} e^{-2iB+2i\nu_0 \ln(2k_0)} e^{i\frac{\pi}{4}} e^{-\frac{\pi\nu_0}{2}} \right] \\ &= A \cos \Psi, \\ A &= \frac{4\sqrt{2\pi}}{|r_0| |\Gamma(-i\nu_0)| \sqrt{48k_0t}} e^{-\frac{\pi\nu_0}{2}} = \sqrt{\frac{\nu_0}{3k_0t}}, \\ \Psi &= 16tk_0^3 - \nu_0 \ln(48k_0t) - \arg \Gamma(-i\nu_0) - \arg r_0 + 2B - 2\nu_0 \ln(2k_0) - \frac{\pi}{4} \\ &= 16k_0^3t - \nu_0 \ln t - \nu_0 \ln(192k_0^3) + \arg \Gamma(i\nu_0) - \arg r_0 + 2B - \frac{\pi}{4}. \end{aligned}$$

4.2. Remarks on comparison of approaches. For the first time, the strict justification of the asymptotics for the solution of the Cauchy problem for the modified Korteweg-de Vries equation, based on its relationship with the conjugation problem, was obtained in [5]. The authors of [5, 6] called their approach to exploring the asymptotics of solutions of the corresponding conjugation problems with oscillating matrices the fastest descent method. The main part of their calculations consists in solution of a series of auxiliary Riemann problems on cross-like contours. This idea is indeed borrowed from the standard steepest descent method. However, the relationship between the Riemann problems mentioned above is rather tricky. The difficulties related to this approach seem to be quite predictable. For the steepest descent method to be applicable to calculation of the asymptotics of the integral

$$\int_{\Gamma} F(z) e^{it\Phi(z)} dz,$$

it is necessary that the functions F and Φ be analytic. Change of variables can help in the case of a nonanalytic Φ , but the nonanalyticity of F is more important.

The choice in favor of the stationary phase method, made by V. C. Buslaev and the author of the present paper, makes it possible to avoid the above-mentioned difficulties under the approach of Deift and Zhou.

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