NONDISPERSIVE VANISHING AND BLOW UP AT INFINITY FOR THE ENERGY CRITICAL NONLINEAR SCHRÖDINGER EQUATION IN $\mathbb{R}^3$

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Dedicated to the memory of Vladimir Savelievich Buslaev

ABSTRACT. The following energy critical focusing nonlinear Schrödinger equation in $\mathbb{R}^3$ is considered: $i\psi_t = -\Delta\psi - |\psi|^4\psi$; it is proved that, for any ν and α_0 sufficiently small, there exist radial finite energy solutions of the form $\psi(x, t) = e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x) + e^{i\Delta t}\zeta^* + o_{\dot{H}^1}(1)$ as $t \rightarrow +\infty$, where $\alpha(t) = \alpha_0 \ln t$, $\lambda(t) = t^\nu$, $W(x) = (1 + \frac{1}{3}|x|^2)^{-1/2}$ is the ground state, and ζ^* is arbitrary small in $\dot{H}^1$.

§1. INTRODUCTION

1.1. Setting of the problem and statement of the result. In this paper we consider the energy critical focusing nonlinear Schrödinger equation

$$(1.1) \quad \begin{aligned} i\psi_t &= -\Delta\psi - |\psi|^4\psi, \quad x \in \mathbb{R}^3, \\ \psi|_{t=0} &= \psi_0 \in \dot{H}^1(\mathbb{R}^3). \end{aligned}$$

The Cauchy problem (1.1) is locally well posed and the solutions during their life span satisfy conservation of energy:

$$(1.2) \quad E(\psi(t)) \equiv \int \left(|\nabla\psi(x, t)|^2 - \frac{1}{3}|\psi(x, t)|^6 \right) dx = E(\psi_0).$$

The problem is energy critical in the sense that both (1.1) and (1.2) are invariant with respect to the scaling $\psi(x, t) \rightarrow \lambda^{1/2}\psi(\lambda x, \lambda^2 t)$, $\lambda \in \mathbb{R}_+$. For $\dot{H}^1$ small data, one has global existence and scattering. In the case of large data, blow up may occur. Indeed, the classical virial identity

$$\frac{d^2}{dt^2} \int |x|^2 |\psi(x, t)|^2 dx = 8 \int (|\nabla\psi(x, t)|^2 - |\psi(x, t)|^6) dx$$

shows that if $x\psi_0 \in L^2(\mathbb{R}^3)$ and $E(\psi_0) < 0$, then the solution breaks down in finite time.

Furthermore, equation (1.1) admits an explicit stationary solution (ground state):

$$W(x) = \left(1 + \frac{1}{3}|x|^2\right)^{-1/2}, \quad \Delta W + W^5 = 0,$$

so that scattering cannot always occur even for solutions that exist globally in time.

The ground state W plays an important role in the dynamics of (1.1). It was proved by Kenig and Merle [6] that $E(W)$ is an energy threshold for the dynamics in the following sense. If ψ_0 is radial and $E(\psi_0) < E(W)$, then:

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(i) the solution of (1.1) is global and scatters to zero as a free wave in both directions, provided $\|\nabla\psi_0\|_{L^2} < \|\nabla W\|_{L^2}$;

(ii) the solution blows up in finite time in both directions, provided $\psi_0 \in L^2$ and $\|\nabla\psi_0\|_{L^2} > \|\nabla W\|_{L^2}$.

The behavior of radial solutions with critical energy $E(\psi_0) = E(W)$ was classified by Duyckaerts and Merle [5]. In this case, in addition to the finite time blow up and scattering to zero (and W itself), there exist solutions that converge as $t \rightarrow \infty$ to a rescaled ground state. In the case of energy slightly greater than $E(W)$, the dynamics is expected to be more rich and to include solutions that behave as $t \rightarrow \infty$ like $e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x)$ with fairly general $\alpha(t)$ and $\lambda(t)$. For a closely related model of the critical wave equation, the existence of solutions of this type with $\lambda(t) \rightarrow \infty$ (blow up at infinity) and $\lambda(t) \rightarrow 0$, $t\lambda(t) \rightarrow \infty$ (nondispersive vanishing) was recently proved by Donninger and Krieger [4]. Our objective in this paper is to obtain a similar result for NLS (1.1). More precisely, we prove the following.

Theorem 1.1. *There exists $\beta_0 > 0$ such that for any $\nu, \alpha_0 \in \mathbb{R}$ with $|\nu| + |\alpha_0| \leq \beta_0$ and any $\delta > 0$, there exists $T > 0$ and a radial solution $\psi \in C([T, +\infty), \dot{H}^1 \cap \dot{H}^2)$ of (1.1) of the form*

$$(1.3) \quad \psi(x, t) = e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x) + \zeta(x, t),$$

where $\lambda(t) = t^\nu$, $\alpha(t) = \alpha_0 \ln t$, and $\zeta(t)$ satisfies:

$$(1.4) \quad \begin{aligned} \|\zeta(t)\|_{\dot{H}^1 \cap \dot{H}^2} &\leq \delta, \\ \|\zeta(t)\|_{L^\infty} &\leq Ct^{-\frac{1+\nu}{2}}, \\ \|\langle \lambda(t)x \rangle^{-1}\zeta(t)\|_{L^\infty} &\leq Ct^{-1-\frac{3}{2}\nu} \end{aligned}$$

for all $t \geq T$. The constants C here and below are independent of ν, α_0 , and δ . Furthermore, there exists $\zeta^* \in \dot{H}^s$, for any $s > \frac{1}{2} - \nu$, such that, as $t \rightarrow +\infty$, $\zeta(t) - e^{it\Delta}\zeta^* \rightarrow 0$ in $\dot{H}^1 \cap \dot{H}^2$.

Remark 1.2. In fact, Theorem 1.1 remains valid with $\dot{H}^2$ replaced by $\dot{H}^k$ for any $k \geq 2$ (with β_0 depending on k).

Remark 1.3. The restriction on ν and α_0 that appears in Theorem 1.1 seems to be technical. One might expect the same result to be true for any $\nu > -1/2$ and any $\alpha_0 \in \mathbb{R}$.

Remark 1.4. The solutions we construct to prove the theorem belong, in fact, to $\dot{H}^{\frac{1}{2}-\nu+}$.

Remark 1.5. Using the techniques developed in this paper one can prove the existence of radial finite time blow up solutions of the form

$$\begin{aligned} \psi(x, t) &= e^{i\alpha(t)}\lambda^{1/2}(t)W(\lambda(t)x) + \zeta(x, t), \\ \lambda(t) &= (T - t)^{-1/2-\nu}, \quad \alpha(t) = \alpha_0 \ln(T - t), \end{aligned}$$

where $\zeta(t)$ is arbitrarily small in $\dot{H}^1 \cap \dot{H}^2$, and $\nu > 1$, and $\alpha_0 \in \mathbb{R}$ can be chosen arbitrarily. For the critical wave equation an analogous result was proved by Krieger, Schlag, Tataru in [8], see also [9] for a similar construction in the context of the critical Schrödinger map equation.

1.2. Outline of the paper. The paper is organized as follows. In §2, we construct (Proposition 2.1) a sufficiently good approximate solution of (1.1) very much in the spirit of [4, 8, 9]. In §3, we build up an exact solution by solving the problem for the small remainder with zero initial data at infinity, the main technical tool of the construction

being some suitable energy type estimates for the linearized evolution. These estimates are proved in §4.

§2. APPROXIMATE SOLUTIONS

In this section we prove the following result.

Proposition 2.1. *For any ν and α_0 sufficiently small and any $0 < \delta \leq 1$, there exists a radial approximate solution $\psi^{ap} \in C^\infty(\mathbb{R}^3 \times \mathbb{R}_+^*)$ of (1.1) such that the following holds true for $t \geq T$ with some $T = T(\nu, \alpha_0, \delta) > 0$.*

(i) *ψ^{ap} has the form $\psi^{ap}(x, t) = e^{i\alpha(t)}\lambda^{1/2}(t)(W(\lambda(t)x) + \chi^{ap}(\lambda(t)x, t))$, where the function $\chi^{ap}(y, t)$ with $y = \lambda(t)x$ satisfies*

$$(2.1) \quad \|\chi^{ap}(t)\|_{\dot{H}^k} \leq C\delta^{\nu+k-1/2}t^{-\nu(k-1)}, \quad k = 1, 2,$$

$$(2.2) \quad \|\chi^{ap}(t)\|_{L^\infty} \leq Ct^{-(1+2\nu)/2},$$

$$(2.3) \quad \| |y|^{-1}\chi^{ap}(t) \|_{L^\infty} + \|\nabla\chi^{ap}(t)\|_{L^\infty} \leq Ct^{-1-2\nu},$$

$$(2.4) \quad \| |y|^{-2}\chi^{ap}(t) \|_{L^\infty} + \| |y|^{-1}\nabla_y\chi^{ap}(t) \|_{L^\infty} \leq C(|\nu| + |\alpha_0|)t^{-1-2\nu},$$

$$(2.5) \quad \|\nabla^2\chi^{ap}(t)\|_{L^\infty} \leq C(|\nu| + |\alpha_0|)t^{-1-2\nu}.$$

Furthermore, there exists $\zeta^* \in \dot{H}^s$, for any $s > \frac{1}{2} - \nu$, such that, as $t \rightarrow +\infty$,

$$e^{i\alpha(t)}\lambda^{1/2}(t)\chi^{ap}(\lambda(t)\cdot, t) - e^{it\Delta}\zeta^* \rightarrow 0$$

in $\dot{H}^1 \cap \dot{H}^2$.

(ii) *The corresponding error $R = -i\psi_t^{ap} - \Delta\psi^{ap} - |\psi^{ap}|^4\psi^{ap}$ satisfies*

$$(2.6) \quad \|R(t)\|_{\dot{H}^k} \leq t^{-(2+\frac{1}{8})(1+2\nu)+\nu(k+1)}, \quad k = 0, 1, 2.$$

The construction of $\psi^{ap}(t)$ will be achieved by considering separately the three regions that correspond to three different space scales: the inner region with the scale $t^\nu|x| \lesssim 1$, the self-similar region where $|x| = O(t^{1/2})$, and finally, the remote region where $|x| = O(t)$. In the inner region the solution will be constructed as a perturbation of the profile $e^{i\alpha_0 \ln t}t^{\nu/2}W(t^\nu x)$. The self-similar and remote regions are regions where the solution is small and is described essentially by the linear equation $i\psi_t = -\Delta\psi$. In the self-similar region, the profile of the solution will be determined uniquely by the matching conditions coming out from the inner region, while in the remote region the profile remains essentially a free parameter of the construction, only the limiting behavior at the origin being prescribed by the matching procedure.

2.1. The inner region. We start by treating the inner region $0 \leq t^\nu|x| \leq 10t^{1/2+\nu-\epsilon_1}$, with $0 < \epsilon_1 < 1/2 + \nu$ to be fixed later. Writing $\psi(x, t)$ as $\psi(x, t) = e^{i\alpha(t)}\lambda^{1/2}(t)u(\rho, t)$, $\rho = \lambda(t)|x|$, from (1.1) we get

$$(2.7) \quad it^{-2\nu}u_t - \alpha_0 t^{-(1+2\nu)}u + i\nu t^{-(1+2\nu)}\left(\frac{1}{2} + \rho\partial_\rho\right)u = -\Delta u - |u|^4u.$$

Write $u(\rho, t) = W(\rho) + \chi(\rho, t)$. Then $\vec{\chi}(t) = \begin{pmatrix} \chi(t) \\ \vec{\chi}(t) \end{pmatrix}$ solves

$$(2.8) \quad it^{-2\nu}\vec{\chi}_t = H\vec{\chi} + \mathcal{N}(\chi),$$

where

$$\begin{aligned} H &= -\Delta\sigma_3 - 3W^4\sigma_3 - 2W^4\sigma_3\sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \mathcal{N}(\chi) &= \begin{pmatrix} N(\chi) \\ -N(\chi) \end{pmatrix}, \quad N(\chi) = N_0 + N_1(\chi) + N_2(\chi), \\ N_0 &= \alpha_0 t^{-(1+2\nu)}W - i\nu t^{-(1+2\nu)}W_1, \quad W_1(\rho) = \left(\frac{1}{2} + \rho\partial_\rho\right)W(\rho) \\ N_1(\chi) &= \alpha_0 t^{-(1+2\nu)}\chi - i\nu t^{-(1+2\nu)}\left(\frac{1}{2} + \rho\partial_\rho\right)\chi, \\ N_2(\chi) &= -|W + \chi|^4(W + \chi) + W^5 + 3W^4\chi + 2W^4\bar{\chi}. \end{aligned}$$

We look for a solution of (2.8) of the form

$$(2.9) \quad \chi(\rho, t) = \sum_{k=1}^{\infty} t^{-k(1+2\nu)}\chi_k(\rho).$$

Substituting (2.9) in (2.8) and identifying the terms with the same powers of t , we get the following system for $\{\chi_k\}_{k \geq 1}$:

$$(2.10) \quad H\vec{\chi}_k = \mathcal{D}_k, \quad k \geq 1,$$

$$\text{where } \mathcal{D}_k = \begin{pmatrix} D_k \\ -D_k \end{pmatrix},$$

$$\begin{aligned} D_1 &= -\alpha_0 W + i\nu W_1, \\ D_k &= D_k^{(1)} + D_k^{(2)}, \quad k \geq 2, \end{aligned}$$

$D_k^{(1)}$ and $D_k^{(2)}$ being the contributions of $it^{-2\nu}\chi_t - N_1(\chi)$ and $-N_2(\chi)$, respectively:

$$\begin{aligned} D_k^{(1)} &= -i(1+2\nu)(k-1)\chi_{k-1} - \alpha_0\nu\chi_{k-1} + i\nu\left(\frac{1}{2} + \rho\partial_\rho\right)\chi_{k-1}, \\ N_2(\chi) &= -\sum_{k=2}^{\infty} t^{-k(1+2\nu)}D_k^{(2)}(\rho). \end{aligned}$$

Note that D_k depends on χ_p , $1 \leq p \leq k-1$ only:

$$D_k = D_k(\rho; \chi_p, 1 \leq p \leq k-1).$$

We subject (2.10) to zero initial conditions at 0: $\chi_k(0) = \partial_\rho\chi_k(0) = 0$.

Lemma 2.2. *System (2.10) has a unique solution $\{\chi_k\}_{k \geq 1}$ such that:*

i) *for any $k \geq 1$, χ_k is a C^∞ -function with an even Taylor expansion at $\rho = 0$ that starts at order $2k$;*

ii) *as $\rho \rightarrow +\infty$, the function χ_k , $k \geq 1$, has an asymptotic expansion of the form*

$$(2.11) \quad \chi_k(\rho) = \sum_{l=0}^k \sum_{j \leq 2k-2l-1} \alpha_{l,j}^{(k)} (\ln \rho)^l \rho^j,$$

with some coefficients $\alpha_{l,j}^{(k)}$ verifying $\alpha_{l,j}^{(k)} = 0$ for all k, m . The asymptotic expansion (2.11) can be differentiated any number of times with respect to ρ .

Proof. It will be convenient for us to rewrite (2.10) as

$$(2.12) \quad L_+ v_k^+ = G_k^+, \quad L_- v_k^- = G_k^-, \quad k \geq 1,$$

where

$$\begin{aligned} v_k^+ &= \operatorname{Re} \chi_k, & v_k^- &= \operatorname{Im} \chi_k, \\ G_k^+ &= \operatorname{Re} D_k, & G_k^- &= \operatorname{Im} D_k, \\ L_+ &= -\Delta - 5W^4, & L_- &= -\Delta - W^4. \end{aligned}$$

For $k = 1$, (2.12) gives

$$(2.13) \quad L_+ v_1^+ = -\alpha_0 W, \quad L_- v_1^- = \nu W_1.$$

The homogeneous equation $L_{\pm} f = 0$ has two explicit solutions $\Phi_{\pm}$, $\Theta_{\pm}$ given by

$$(2.14) \quad \begin{aligned} \Phi_-(\rho) &= W(\rho), & \Theta_-(\rho) &= \left(1 + \frac{\rho^2}{3}\right)^{-1/2} \left(\frac{\rho}{3} - \frac{1}{\rho}\right), \\ \Phi_+(\rho) &= W_1(\rho), & \Theta_+(\rho) &= -2 \left(1 + \frac{\rho^2}{3}\right)^{-3/2} \left(\frac{1}{\rho} - 2\rho + \frac{\rho^3}{9}\right). \end{aligned}$$

Therefore, solving (2.13) with zero initial conditions at the origin, we obtain

$$(2.15) \quad \begin{aligned} v_1^+(\rho) &= \alpha_0 \int_0^{\rho} s^2 (\Theta_+(\rho) \Phi_+(s) - \Theta_+(s) \Phi_+(\rho)) W(s) ds, \\ v_1^-(\rho) &= -\nu \int_0^{\rho} s^2 (\Theta_-(\rho) \Phi_-(s) - \Theta_-(s) \Phi_-(\rho)) W_1(s) ds. \end{aligned}$$

Since W , W_1 are even C^∞ -functions, v_1^+ and v_1^- are also C^∞ -functions with even Taylor expansion at $\rho = 0$ that starts at order 2. Furthermore, the asymptotic expansions of v_1^+ and v_1^- as $\rho \rightarrow +\infty$ can be obtained directly from (2.15). As claimed, we have

$$v_1^+(\rho) + i v_1^-(\rho) = \sum_{j \leq 1} \alpha_{0,j}^{(1)} \rho^j + \sum_{j \leq 0} \alpha_{1,j}^{(1)} \rho^{2j-1} \ln \rho \quad \text{as } \rho \rightarrow +\infty.$$

Next we proceed by induction. Consider $k > 1$ and assume that we have found χ_i , $i = 1, \dots, k-1$, that satisfy i), ii). Then we can check easily that D_k is an even C^∞ -function with a Taylor series at 0 starting at order $2(k-1)$, and that, as $\rho \rightarrow +\infty$, D_k admits an asymptotic expansion of the form

$$D_k(\rho) = \sum_{l=0}^{k-1} \sum_{j \leq 2k-2l-3} d_{j,l}^{(k)} (\ln \rho)^l \rho^j + (\ln \rho)^k \sum_{j \leq -5} d_{j,k}^{(k)} \rho^j,$$

where $d_{-2,k-1}^{(k)} = 0$ and $d_{2m,k}^{(k)} = 0$, for all m . Therefore, solving $L_{\pm} v_k^{\pm} = G_k^{\pm}$ with zero conditions at $\rho = 0$ we get a C^∞ even solution $v_k^{\pm}$ which is $O(\rho^{2k})$ at the origin. Finally, the asymptotic expansion at infinity follows directly from the representation

$$v_k^{\pm}(\rho) = - \int_0^{\rho} s^2 (\Theta_{\pm}(\rho) \Phi_{\pm}(s) - \Theta_{\pm}(s) \Phi_{\pm}(\rho)) G_k^{\pm}(s) ds. \quad \square$$

Remark 2.3. Clearly, for any k , the function χ_k is a polynomial with respect to α_0 and ν of the form

$$\chi_k = \sum_{1 \leq m+n \leq k} \alpha_0^m \nu^n \chi_{m,n}^k(\rho),$$

where the coefficients $\chi_{m,n}^k$ are C^∞ -functions of ρ with an even Taylor expansion at 0 that starts at order $2k$. As $\rho \rightarrow +\infty$, $\chi_{m,n}^k$ admits an asymptotic expansion of the form (2.11).

For any $N \geq 2$, define

$$\chi^{(N)}(\rho, t) = \sum_{k=1}^N t^{-k(1+2\nu)} \chi_k(\rho).$$

From our construction it follows that $\chi^{(N)}$ verifies

$$(2.16) \quad |\rho^{-k} \partial_\rho^l (-it^{-2\nu} \bar{\chi}_t^{(N)} + H \bar{\chi}^{(N)} + \mathcal{N}(\chi^{(N)}))| \leq C_{N,l,k} t^{-(N+1)(1+2\nu)} \langle \rho \rangle^{2N-1-l-k},$$

for any $k, l \in \mathbb{N}$, with $k+l \leq 2N$, $0 \leq \rho \leq 10t^{\frac{1}{2}+\nu-\epsilon_1}$, $t \geq 1$.

Fix $N = 27$, $\epsilon_1 = \frac{1+2\nu}{27}$,¹ and set

$$u_{in}^{ap} = W + \chi_{in}^{ap}, \quad \chi_{in}^{ap} = \chi^{(27)},$$

$$\mathcal{R}_{in} = -it^{-2\nu} \partial_t u_{in}^{ap} - \Delta u_{in}^{ap} + \alpha_0 t^{-1-2\nu} u_{in}^{ap} - i\nu t^{-1-2\nu} \left(\frac{1}{2} + \rho \partial_\rho \right) u_{in}^{ap} - |u_{in}^{ap}|^4 u_{in}^{ap}.$$

As a direct consequence of Lemma 2.2 and estimate (2.16), we obtain the following result.

Lemma 2.4. *For any $\alpha_0 \in \mathbb{R}$ and any $\nu > -\frac{1}{2}$, there exists $T = T(\alpha_0, \nu) > 0$ such that for $t \geq T$ the following holds true.*

(i) *The profile $\chi_{in}^{ap}(t)$ satisfies*

$$(2.17) \quad \|\chi_{in}^{ap}\|_{L^\infty(0 \leq \rho \leq 10t^{\frac{1}{2}+\nu-\epsilon_1})} \leq C(|\nu| + |\alpha_0|) t^{-\frac{1}{2}-\nu},$$

$$(2.18) \quad \|\rho^{-k} \partial_\rho^l \chi_{in}^{ap}\|_{L^\infty(0 \leq \rho \leq 10t^{\frac{1}{2}+\nu-\epsilon_1})} \leq C(|\nu| + |\alpha_0|) t^{-1-2\nu}, \quad 1 \leq k+l \leq 2,$$

$$(2.19) \quad \|\rho^{-k} \partial_\rho^l \chi_{in}^{ap}\|_{L^2(\rho^2 d\rho, 0 \leq \rho \leq 10t^{\frac{1}{2}+\nu-\epsilon_1})} \leq C(|\nu| + |\alpha_0|) t^{-(\frac{1}{2}+\nu)(k+l-\frac{1}{2})}, \quad k+l \leq 2.$$

(ii) *The error $\mathcal{R}_{in}(t)$ admits the estimate*

$$(2.20) \quad \|\rho^{-k} \partial_\rho^l \mathcal{R}_{in}(t)\|_{L^2(\rho^2 d\rho, 0 \leq \rho \leq 10t^{\frac{1}{2}+\nu-\epsilon_1})} \leq t^{-3(1+2\nu)/4-\epsilon_1(2N+1/2)}, \quad k+l \leq 2.$$

2.2. The self-similar region. We next consider the self-similar region

$$\frac{1}{10} t^{-\epsilon_1} \leq |x| t^{-1/2} \leq 10 t^{\epsilon_2},$$

with $0 < \epsilon_2 < 1/2$ to be fixed later. Write $\psi(x, t) = e^{i\alpha_0 \ln t} t^{-1/4} w(y, t)$, $y = t^{-1/2}|x|$. Then $w(t)$ solves

$$(2.21) \quad itw_t = (\mathcal{L} + \alpha_0)w - |w|^4 w,$$

where $\mathcal{L} = -\Delta + \frac{i}{2}(\frac{1}{2} + y \partial_y)$.

Note that in the limit $\rho \rightarrow +\infty$, $y \rightarrow 0$ at least formally we have

$$(2.22) \quad \begin{aligned} & t^{\nu/2} (W(\rho) + \sum_{k \geq 1} t^{-k(1+2\nu)} \chi_k(\rho)) \\ &= t^{-1/4} \sum_{n \geq 0} \sum_{0 \leq l \leq \frac{n}{2}} t^{-\frac{1}{4}(2n+1)(1+2\nu)} \left(\ln y + \left(\frac{1}{2} + \nu \right) \ln t \right)^l \sum_{k \geq l} \alpha_{l, 2k-n-1}^{(k)} y^{2k-n-1}, \end{aligned}$$

where the $\alpha_{l,j}^{(k)}$, $k \neq 0$, are given by Lemma 2.2, and the $\alpha_{l,j}^{(0)}$ come from the expansion of $W(\rho)$ as $\rho \rightarrow \infty$:

$$W(\rho) = \sum_{j \leq 0} \alpha_{0,j}^{(0)} \rho^j, \quad \alpha_{0,2m}^{(0)} = 0, \quad \text{for all } m \in \mathbb{Z}.$$

¹This choice has no specific meaning here. To produce an approximate solution with an error satisfying (2.6), it suffices to require $(2N+3)\epsilon_1 > 3(1+2\nu)/2$, $0 < \epsilon_1 < \frac{1+2\nu}{20}$, see (2.20) and (2.41), (2.42).

Equation (2.22) suggests the following Ansatz for w :

$$(2.23) \quad w(y, t) = \sum_{n \geq 0} \sum_{0 \leq l \leq \frac{n}{2}} t^{-\frac{1}{4}(2n+1)(1+2\nu)} \left(\ln y + \left(\frac{1}{2} + \nu \right) \ln t \right)^l A_{n,l}(y).$$

As it will become clear later, to prove Proposition 2.1 it suffices to consider only first three terms in (2.23). Therefore, we look for an approximate solution of the form

$$w_{ss}^{ap}(y, t) = t^{-(1+2\nu)/4} A_{0,0}(y) + t^{-3(1+2\nu)/4} A_{1,0}(y) \\ + t^{-5(1+2\nu)/4} \left(A_{2,0}(y) + \left(\ln y + \left(\frac{1}{2} + \nu \right) \ln t \right) A_{2,1}(y) \right).$$

Substituting this in the expression $-itw_t + (\mathcal{L} + \alpha_0)w - |w|^4w$, we get

$$(2.24) \quad -it\partial_t w_{ss}^{ap} + (\mathcal{L} + \alpha_0)w_{ss}^{ap} - |w_{ss}^{ap}|^4 w_{ss}^{ap} = t^{-(1+2\nu)/4} S_{0,0}(y) + t^{-3(1+2\nu)/4} S_{1,0}(y) \\ + t^{-5(1+2\nu)/4} \left(S_{2,0}(y) + \left(\ln y + \left(\frac{1}{2} + \nu \right) \ln t \right) S_{2,1}(y) \right) + S(y, t),$$

where

$$S_{n,0}(y) = (\mathcal{L} + \mu_n)A_{n,0}(y), \quad n = 0, 1, \\ S_{2,1}(y) = (\mathcal{L} + \mu_2)A_{2,1}(y), \\ S_{2,0}(y) = (\mathcal{L} + \mu_2)A_{2,0}(y) - i\nu A_{2,1}(y) - \frac{2}{y} \partial_y A_{2,1}(y) - \frac{A_{2,1}(y)}{y^2} - |A_{0,0}(y)|^4 A_{0,0}(y), \\ S(y, t) = -|w_{ss}^{ap}(y, t)|^4 w_{ss}^{ap}(y, t) + t^{-5(1+2\nu)/4} |A_{0,0}(y)|^4 A_{0,0}(y).$$

Here $\mu_n = \alpha_0 + \frac{i}{4}(2n+1)(1+2\nu)$.

We require that $S_{n,l} = 0$, $n = 0, 1, 2$, $l = 0, 1$, which means that the corresponding $A_{n,l}$ must solve

$$(2.25) \quad \begin{cases} (\mathcal{L} + \mu_n)A_{n,0} = 0, & n = 0, 1, \\ (\mathcal{L} + \mu_2)A_{2,1} = 0, \\ (\mathcal{L} + \mu_2)A_{2,0} = i\nu A_{2,1} + \frac{2}{y} \partial_y A_{2,1} + \frac{A_{2,1}}{y^2} + |A_{0,0}|^4 A_{0,0}. \end{cases}$$

Moreover, in order to have the matching with the inner region, the $A_{n,l}$ must satisfy

$$(2.26) \quad A_{n,l}(y) = \sum_{k \geq l} \alpha_{l,2k-n-1}^{(k)} y^{2k-n-1}, \quad y \rightarrow 0.$$

Lemma 2.5. *There exists a unique solution of (2.25) such that, as $y \rightarrow 0$, it admits an asymptotic expansion of the form*

$$(2.27) \quad A_{n,l}(y) = \sum_{k \geq l} d_{n,k,l} y^{2k-n-1},$$

with $d_{0,0,0} = \alpha_{0,-1}^{(0)}$, $d_{1,1,0} = \alpha_{0,0}^{(1)}$ and $d_{2,1,0} = \alpha_{0,-1}^{(1)}$.

Proof. First, we note that the equation $(\mathcal{L} + \mu)f = 0$ has a basis of solutions $e_1(y, \mu)$, $e_2(y, \mu)$ such that

(i) $e_1(y, \mu) = \frac{1}{y} + (\mu - \frac{i}{4})\tilde{e}_1(y, \mu)$, where $\tilde{e}_1$ is an entire function of y and μ , odd with respect to y ;

(ii) e_2 is an entire function of y and μ even with respect to y , and $e_2(y, \mu) = 1 + O(y^2)$ as $y \rightarrow 0$.

The first two equations in (2.25) together with (2.27) give

$$(2.28) \quad A_{0,0}(y) = \alpha_{0,-1}^{(0)} e_1(y, \mu_0), \quad A_{1,0}(y) = \alpha_{0,0}^{(1)} e_2(y, \mu_1).$$

Next we consider the remaining equations in (2.25). The equation $(\mathcal{L} + \mu_2)A_{2,1}(y) = 0$ and (2.27) yield $A_{2,1}(y) = c_0 e_1(y, \mu_2)$ with some constant c_0 . Then, for $A_{2,0}$ we have $(\mathcal{L} + \mu_2)A_{2,0} = F$, where

$$F = c_0 \left(i\nu + \frac{2}{y} \partial_y + y^{-2} \right) e_1(y, \mu_2) + |A_{0,0}|^4 A_{0,0}.$$

As $y \rightarrow 0$, F has an asymptotic expansion of the form

$$F(y) = \sum_{i \geq -2} \kappa_i y^{2i-1},$$

with some coefficients κ_i , κ_{-2} and $\kappa_{-1} + c_0$ being independent of c_0 .

Write $A_{2,0}(y) = -\frac{\kappa_{-2}}{6y^3} + \tilde{A}_{2,0}(y)$. Then $\tilde{A}_{2,0}$ solves

$$(2.29) \quad (\mathcal{L} + \mu_2) \tilde{A}_{2,0} = \tilde{F},$$

where $\tilde{F} = F + \frac{\kappa_{-2}}{6} (\mathcal{L} + \mu_2) \frac{1}{y^3}$ has the following asymptotics as $y \rightarrow 0$:

$$\tilde{F}(y) = \sum_{i \geq -1} \tilde{\kappa}_i y^{2i-1}, \quad \tilde{\kappa}_{-1} = \tilde{\kappa}_{-1}^0 - c_0,$$

with $\tilde{\kappa}_{-1}^0$ independent of c_0 . Take $c_0 = \tilde{\kappa}_{-1}^0$. Then (2.29) has a unique solution of the form

$$\tilde{A}_{2,0}(y) = \alpha_{0,-1}^{(1)} e_1(y, \mu_2) + \text{an odd } C^\infty\text{-function.} \quad \square$$

Remark 2.6. By uniqueness, the functions $A_{n,l}$ given by Lemma 2.5 verify the matching conditions (2.26). Note also that all $A_{n,l}$ are entire functions of α_0 and ν .

Next we study the behavior of $A_{n,l}$ as $y \rightarrow +\infty$. For this, notice that for any $\mu \in \mathbb{C}$, equation $(\mathcal{L} + \mu)f = 0$ has a basis of solutions $f_1(y, \mu)$, $f_2(y, \mu)$ such that yf_1 , yf_2 are smooth functions in both variables, and as $y \rightarrow +\infty$ we have

$$(2.30) \quad f_1(y, \mu) = y^{-1/2+2i\mu} (1 + O(y^{-2})), \quad f_2(y, \mu) = e^{i\frac{y^2}{4}} y^{-5/2-2i\mu} (1 + O(y^{-2})).$$

These asymptotics are uniform in μ on the compact subsets of $\mathbb{C}$ and can be differentiated any number of times with respect to y .

Decomposing $A_{1,0}$, $A_{2,0}$, $A_{2,1}$ in the basis f_1 , f_2 , we get

$$(2.31) \quad \begin{aligned} A_{n,0}(y) &= d_1^n f_1(y, \mu_n) + d_2^n f_2(y, \mu_n), \quad n = 0, 1, \\ A_{2,1}(y) &= d_1^2 f_1(y, \mu_2) + d_2^2 f_2(y, \mu_2), \end{aligned}$$

with some coefficients d_j^n , $j = 1, 2$, $n = 0, 1, 2$. As a consequence, as $y \rightarrow +\infty$, we have

$$(2.32) \quad \begin{aligned} A_{0,0}(y) &= d_1^0 y^{2i\alpha_0-1-\nu} (1 + O(y^{-2})) + d_2^0 e^{iy^2/4} y^{-2i\alpha_0-2+\nu} (1 + O(y^{-2})), \\ A_{1,0}(y) &= d_1^1 y^{2i\alpha_0-2-3\nu} (1 + O(y^{-2})) + d_2^1 e^{iy^2/4} y^{-2i\alpha_0-1+3\nu} (1 + O(y^{-2})), \\ A_{2,1}(y) &= d_1^2 y^{2i\alpha_0-3-5\nu} (1 + O(y^{-2})) + d_2^2 e^{iy^2/4} y^{-2i\alpha_0+5\nu} (1 + O(y^{-2})). \end{aligned}$$

The asymptotics (2.32) can be differentiated any number of times with respect to y . Now, we consider $A_{2,0}$ and write it as

$$(2.33) \quad A_{2,0}(y) = 2d_1^2 \nu \ln y f_1(y, \mu_2) - 2(\nu + 1) d_2^2 \ln y f_2(y, \mu_2) + \hat{A}_{2,0}(y).$$

Then $\hat{A}_{2,0}(y)$ solves

$$(2.34) \quad (\mathcal{L} + \mu_2) \hat{A}_{2,0} = G$$

with $G = d_2^2 G_1 + G_2$, where

$$\begin{aligned} G_1 &= -d_2^2(1+2\nu)(2y^{-1}\partial_y + y^{-2} - i)f_2(y, \mu_2), \\ G_2 &= |A_{0,0}|^4 A_{0,0} + d_1^2(1+2\nu)(2y^{-1}\partial_y + y^{-2})f_1(y, \mu_2). \end{aligned}$$

The asymptotics (2.30) and (2.32) show that G_j , $j = 1, 2$, has the following behavior as $y \rightarrow +\infty$:

$$\begin{aligned} G_1(y) &= e^{iy^2/4} y^{-2i\alpha_0} G_{1,1}(y), \quad G_2(y) = \sum_{m=-2}^3 e^{imy^2/4} y^{-2i\alpha_0\nu(2m-1)} G_{2,m}(y), \\ \partial_y^l G_{1,1}(y) &= O(y^{-2+5\nu-l}), \\ \partial_y^l G_{2,m}(y) &= O(y^{-5-5\nu-|m|(1-2\nu)-l}), \quad -2 \leq m \leq 3, \end{aligned}$$

for any $l \geq 0$, provided ν is sufficiently small.

Integrating (2.34), we get

$$(2.35) \quad \hat{A}_{2,0}(y) = \lambda_1 f_1(y, \mu_2) + \lambda_2 f_2(y, \mu_2) + d_2^2 g_1(y) + g_2(y).$$

Here each λ_i , $i = 1, 2$, is a constant, and g_i , $i = 1, 2$, is the solution of $(\mathcal{L} + \mu_2)g_i = G_i$ with the following behavior as $y \rightarrow +\infty$:

$$\begin{aligned} g_1(y) &= e^{iy^2/4} y^{-2i\alpha_0} g_{1,1}(y), \\ g_2(y) &= \sum_{m=-2}^3 e^{imy^2/4} y^{-2i\alpha_0\nu(2m-1)} g_{2,m}(y), \\ \partial_y^l g_{1,1}(y) &= O(y^{-2+5\nu-l}), \\ \partial_y^l g_{2,m}(y) &= O(y^{-5-5\nu-m(1-2\nu)-l}), \quad m = 0, 1, \\ \partial_y^l g_{2,m}(y) &= O(y^{-7-5\nu-|m|(1-2\nu)-l}), \quad m = -2, -1, 2, 3, \end{aligned} \quad (2.36)$$

for any $l \geq 0$.

Denote

$$\begin{aligned} u_{ss}^{ap}(\rho, t) &= t^{-(1+2\nu)/4} w_{ss}^{ap}(t^{-(1+2\nu)/2} \rho, t), \\ \chi_{ss}^{ap}(\rho, t) &= u_{ss}^{ap}(\rho, t) - W(\rho), \\ \mathcal{R}_{ss}(\rho, t) &= t^{-5(1+2\nu)/4} S(t^{-(1+2\nu)/2} \rho, t). \end{aligned}$$

The next lemma is a direct consequence of (2.26), (2.30), (2.32), (2.33), (2.35), and (2.36).

Lemma 2.7. *For any $\alpha_0, \nu \in \mathbb{R}$ sufficiently small, there exists $T(\alpha_0, \nu) > 0$ such that for $t \geq T(\alpha_0, \nu)$ the following holds true.*

(i) $\chi_{ss}^{ap}(t)$ verifies

$$(2.37) \quad \|\chi_{ss}^{ap}(t)\|_{L^\infty(\frac{1}{10}t^{\frac{1}{2}+\nu-\epsilon_1} \leq \rho \leq 10t^{\frac{1}{2}+\nu+\epsilon_2})} \leq Ct^{-\frac{1}{2}-\nu},$$

$$(2.38) \quad \|\rho^{-k} \partial_\rho^l \chi_{ss}^{ap}(t)\|_{L^\infty(\frac{1}{10}t^{\frac{1}{2}+\nu-\epsilon_1} \leq \rho \leq 10t^{\frac{1}{2}+\nu+\epsilon_2})} \leq Ct^{-1-2\nu}, \quad k+l=1,$$

$$(2.39) \quad \|\rho^{-k} \partial_\rho^l \chi_{ss}^{ap}(t)\|_{L^\infty(\frac{1}{10}t^{\frac{1}{2}+\nu-\epsilon_1} \leq \rho \leq 10t^{\frac{1}{2}+\nu+\epsilon_2})} \leq C(|\alpha_0| + |\nu|)t^{-1-2\nu}, \quad k+l=2,$$

$$(2.40) \quad \|\rho^{-k} \partial_\rho^l \chi_{ss}^{ap}(t)\|_{L^2(\rho^2 d\rho, \frac{1}{10}t^{\frac{1}{2}+\nu-\epsilon_1} \leq \rho \leq 10t^{\frac{1}{2}+\nu+\epsilon_2})} \leq Ct^{-(1+2\nu)(1-2\epsilon_2)/4},$$

$$1 \leq k+l \leq 2.$$

(ii) The error $\mathcal{R}_{ss}(t)$ admits the estimate

$$(2.41) \quad \|\rho^{-k} \partial_\rho^l \mathcal{R}_{ss}(t)\|_{L^2(\rho^2 d\rho, \frac{1}{10}t^{\frac{1}{2}+\nu-\epsilon_1} \leq \rho \leq 10t^{\frac{1}{2}+\nu+\epsilon_2})} \leq Ct^{-(2+\frac{1}{4})(1+2\nu)+5\epsilon_1/2},$$

$$0 \leq k+l \leq 2.$$

(iii) The difference $u_{in}^{ap}(\rho, t) - u_{ss}^{ap}(\rho, t)$ verifies

$$(2.42) \quad \left| \partial_\rho^l (u_{in}^{ap}(t) - u_{ss}^{ap}(t)) \right| \leq C \rho^{-2-l} t^{-(1+2\nu)} (\ln t + t^{3(1+2\nu)/2 - (2N+3)\varepsilon_1}),$$

for any $l \geq 0$ and $\frac{1}{10} t^{\frac{1}{2} + \nu - \varepsilon_1} \leq \rho \leq 10 t^{\frac{1}{2} + \nu - \varepsilon_1}$.

2.3. The remote region. Next we consider the remote region $|x| \geq \frac{1}{10} t^{1/2 + \varepsilon_2}$. In this region, as an approximate solution of (1.1) we take the following radial profile:

$$\psi_{out}^{ap}(x, t) = v_1(x, t) + v_2(x, t) + v_3(x, t),$$

where

$$\begin{aligned} v_1(x, t) &= e^{i\alpha_0 \ln t} [d_1^0 t^{-(1+\nu)/2} f_1(y, \mu_0) + d_1^1 t^{-(2+3\nu)/2} f_1(y, \mu_1)], \quad y = t^{-1/2} |x|, \\ v_2(x, t) &= \Theta_\delta \left(\frac{x}{t} \right) e^{i\alpha_0 \ln t} \left[d_2^0 t^{-(1+\nu)/2} f_2(y, \mu_0) + d_2^1 t^{-(2+3\nu)/2} f_2(y, \mu_1) \right. \\ &\quad \left. + t^{-(3+5\nu)/2} \left(d_2^2 g_1(y) - (d_2^2(2\nu+1) \ln \left(\frac{|x|}{t} \right) - \lambda_2) f_2(y, \mu_2) \right) \right], \end{aligned}$$

$$\Theta_\delta(\xi) = \Theta\left(\frac{\xi}{\delta}\right), \quad \Theta \in C_0^\infty(\mathbb{R}^3) \text{ is radial, } \Theta(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1, \\ 0 & \text{if } |\xi| \geq 2. \end{cases}$$

Finally, $v_3(x, t)$ is given by

$$v_3(x, t) = \frac{e^{i \frac{|x|^2}{4t}}}{t^{5/2}} \hat{v}_3 \left(\frac{x}{t} \right), \quad \hat{v}_3 = -iz \Delta \Theta_\delta - 2i \nabla z \cdot \nabla \Theta_\delta,$$

where

$$z(\xi) = d_2^0 |\xi|^{-2i\alpha_0 - 2 + \nu} + d_2^1 |\xi|^{-2i\alpha_0 - 1 + 3\nu} - (d_2^2(2\nu+1) \ln |\xi| - \lambda_2) |\xi|^{-2i\alpha_0 + 5\nu}.$$

The asymptotics (2.30) show that, for $t \geq T$ with some $T = T(\delta) > 0$ and for any $l \geq 0$, we have

$$(2.43) \quad \begin{aligned} |\nabla^l v_1(x, t)| &\leq C_l |x|^{-l-1-\nu}, \quad \frac{1}{10} t^{1/2 + \varepsilon_2} \leq |x|, \\ |\nabla^l v_2(x, t)| &\leq \frac{C_l}{t^{3/2}} \left| \frac{x}{t} \right|^{l-2+\nu}, \quad \frac{1}{10} t^{1/2 + \varepsilon_2} \leq |x| \leq 2\delta t. \end{aligned}$$

Furthermore, v_2 can be written as

$$(2.44) \quad \begin{aligned} v_2(x, t) &= v_{2,0}(x, t) + v_{2,1}(x, t), \\ v_{2,0}(x, t) &= \frac{e^{i \frac{|x|^2}{4t}}}{t^{3/2}} \Theta_\delta \left(\frac{x}{t} \right) z \left(\frac{x}{t} \right), \quad v_{2,1}(x, t) = \frac{e^{i \frac{|x|^2}{4t}}}{t^{3/2}} \Theta_\delta \left(\frac{x}{t} \right) \hat{v}_{2,1}(x, t), \end{aligned}$$

with $\hat{v}_{2,1}$ verifying

$$(2.45) \quad |\nabla^l \hat{v}_{2,1}(x, t)| \leq C_l t^{3-\nu} |x|^{-l-4+\nu}, \quad \frac{1}{10} t^{1/2 + \varepsilon_2} \leq |x| \leq 2\delta t,$$

for any $l \geq 0$.

Next we address v_3 . We have

$$(2.46) \quad \begin{aligned} \|\nabla^l v_3(t)\|_{L^\infty(|x| \geq \frac{1}{10} t^{1/2 + \varepsilon_2})} &\leq C_l t^{-5/2} \delta^{-4+l+\nu}, \\ \|\nabla^l v_3(t)\|_{L^2(|x| \geq \frac{1}{10} t^{1/2 + \varepsilon_2})} &\leq C_l t^{-1} \delta^{-5/2+l+\nu}, \end{aligned}$$

for any $l \geq 0$ and $t \geq T(\delta)$.

As a direct consequence of estimates (2.43), (2.45), (2.46), we obtain

$$\begin{aligned}
(2.47) \quad & \|\psi_{\text{out}}^{ap}(t)\|_{L^\infty(|x| \geq \frac{1}{10}t^{\frac{1}{2}+\varepsilon_2})} \leq Ct^{-(\frac{1}{2}+\varepsilon_2)(1+\nu)}, \\
& \| |x|^{-1} \psi_{\text{out}}^{ap}(t) \|_{L^\infty(|x| \geq \frac{1}{10}t^{\frac{1}{2}+\varepsilon_2})} \leq Ct^{-5/4}, \\
& \|\nabla^l \psi_{\text{out}}^{ap}(t)\|_{L^\infty(|x| \geq \frac{1}{10}t^{\frac{1}{2}+\varepsilon_2})} \leq Ct^{-5/4}, \quad l = 1, 2, \\
& \|\nabla^l \psi_{\text{out}}^{ap}(t)\|_{L^2(|x| \geq \frac{1}{10}t^{\frac{1}{2}+\varepsilon_2})} \leq C\delta^{\nu+l-1/2}, \quad l = 1, 2, \\
& \|\nabla^l (\psi_{\text{out}}^{ap}(t) - v_{2,0}(t))\|_{L^2(|x| \geq \frac{1}{10}t^{\frac{1}{2}+\varepsilon_2})} \leq Ct^{-\frac{1}{2}(\frac{1}{2}+\varepsilon_2)(1+2\nu)}, \quad l = 1, 2, \\
& \| |x|^{-1} (\psi_{\text{out}}^{ap}(t) - v_{2,0}(t)) \|_{L^2(|x| \geq \frac{1}{10}t^{\frac{1}{2}+\varepsilon_2})} \leq Ct^{-\frac{1}{2}(\frac{1}{2}+\varepsilon_2)(1+2\nu)},
\end{aligned}$$

provided $\frac{3}{8} \leq \varepsilon_2 < \frac{1}{2}$, ν is sufficiently small, and $t \geq T(\delta)$.

Denote

$$\psi_{ss}^{ap}(x, t) = e^{i\alpha_0 \ln t} t^{-1/4} w_{ss}^{ap}(t^{-1/2}|x|, t),$$

and consider the difference $\psi_{ss}^{ap}(x, t) - \psi_{\text{out}}^{ap}(x, t)$. If $\frac{1}{10}t^{1/2+\varepsilon_2} \leq |x| \leq 10t^{1/2+\varepsilon_2}$, then

$$(2.48) \quad \psi_{ss}^{ap}(x, t) - \psi_{\text{out}}^{ap}(x, t) = e^{i\alpha_0 \ln t} t^{-(3+5\nu)/2} ((d_1^2(1+2\nu) \ln |x| + \lambda_1) f_1(y, \mu_2) + g_2(y)),$$

which together with (2.30) and (2.36) implies that

$$(2.49) \quad |\nabla^l (\psi_{\text{out}}^{ap} - \psi_{ss}^{ap})| \leq C_l (|\ln t| t^{-(\frac{1}{2}+\varepsilon_2)(3+5\nu+l)} + t^{-(\frac{1}{2}+\varepsilon_2)(3+5\nu+1)}),$$

for any $l \geq 0$ and $\frac{1}{10}t^{1/2+\varepsilon_2} \leq |x| \leq 10t^{1/2+\varepsilon_2}$, provided ν is sufficiently small.

Now we analyze the error $R_{\text{out}}(t) = -i\partial_t \psi_{\text{out}}^{ap}(t) - \Delta \psi_{\text{out}}^{ap}(t) - |\psi_{\text{out}}^{ap}(t)|^4 \psi_{\text{out}}^{ap}(t)$. It has the form

$$\begin{aligned}
(2.50) \quad R_{\text{out}}(x, t) = & -\frac{e^{i\frac{|x|^2}{4t}}}{t^{9/2}} \left[t \widehat{v}_{2,1}(x, t) \Delta \Theta_\delta \left(\frac{x}{t} \right) + 2t^2 \nabla \widehat{v}_{2,1}(x, t) \cdot \nabla \Theta_\delta \left(\frac{x}{t} \right) + \Delta \widehat{v}_3 \left(\frac{x}{t} \right) \right] \\
& - |\psi_{\text{out}}^{ap}|^4 \psi_{\text{out}}^{ap}.
\end{aligned}$$

Combined with (2.43), (2.45), (2.46), formula (2.50) shows that if $\frac{3}{8} \leq \varepsilon_2 < \frac{1}{2}$ and ν is sufficiently small, then

$$(2.51) \quad \|\nabla^l R_{\text{out}}(t)\|_{L^2(|x| \geq \frac{1}{10}t^{1/2+\varepsilon_2})} \leq Ct^{-\frac{9}{4}(1+2\nu)}, \quad t \geq T(\delta), \quad l = 0, 1, 2.$$

2.4. Proof of Proposition 2.1. Now we are in a position to conclude the proof of Proposition 2.1. Fix ε_2 such that $\frac{3}{8} \leq \varepsilon_2 < \frac{1}{2}$ and consider the radial profile $\psi^{ap}(x, t)$ defined by

$$\begin{aligned}
\psi^{ap}(x, t) = & \Theta(t^{-1/2+\varepsilon_1}x) \psi_{in}^{ap}(x, t) + (1 - \Theta(t^{-1/2+\varepsilon_1}x)) \Theta(t^{-1/2-\varepsilon_2}x) \psi_{ss}^{ap}(x, t) \\
& + (1 - \Theta(t^{-1/2-\varepsilon_2}x)) \psi_{\text{out}}^{ap}(x, t), \quad x \in \mathbb{R}^3,
\end{aligned}$$

where $\psi_{in}^{ap}(x, t) = e^{i\alpha_0 \ln t} t^{\nu/2} u_{in}^{ap}(t^\nu|x|, t)$. Write ψ^{ap} as

$$\psi^{ap}(x, t) = e^{i\alpha_0 \ln t} t^{\nu/2} (W(y) + \chi^{ap}(y, t)), \quad y = t^\nu x.$$

By Lemma 2.4 (estimates (2.17), (2.18)), Lemma 2.7 (estimates (2.37), (2.38), (2.39)), and (2.47), we have

$$(2.52) \quad \|\chi^{ap}(t)\|_{L^\infty} \leq Ct^{-(1+2\nu)/2}$$

$$(2.53) \quad \| |y|^{-1} \chi^{ap}(t) \|_{L^\infty} + \|\nabla \chi^{ap}(t)\|_{L^\infty} \leq Ct^{-1-2\nu},$$

$$(2.54) \quad \| |y|^{-2} \chi^{ap}(t) \|_{L^\infty} + \| |y|^{-1} \nabla_y \chi^{ap}(t) \|_{L^\infty} \leq C(|\nu| + |\alpha_0|) t^{-1-2\nu},$$

$$(2.55) \quad \|\nabla^2 \chi^{ap}(t)\|_{L^\infty} \leq C(|\nu| + |\alpha_0|) t^{-1-2\nu}.$$

All the estimates stated in this subsection are valid for ν sufficiently small and $t \geq T(\alpha_0, \nu, \delta)$.

Futhermore, from Lemma 2.4 (estimate (2.19)), Lemma 2.7 (estimate (2.39)) and the last two inequalities in (2.47) it follows that

$$(2.56) \quad \begin{aligned} \|\nabla^l \chi^{ap}(t)\|_{L^2(|y| \leq 10t^{1/2+\nu+\varepsilon_2})} &\leq Ct^{-(1+2\nu)(1-2\varepsilon_2)/4}, \quad l = 1, 2, \\ \|\nabla^l(\chi^{ap}(t) - \chi_0^{ap}(t))\|_{L^2(|y| \geq t^{1/2+\nu+\varepsilon_2})} &\leq Ct^{-(1+2\nu)/4}, \quad l = 1, 2, \end{aligned}$$

where $\chi_0^{ap}(y, t) = e^{-i\alpha_0 \ln t} t^{-\nu/2} v_{2,0}(t^{-\nu} y, t)$.

In particular, inequalities (2.56) imply

$$\|\nabla^l \chi^{ap}(t)\|_{L^2(\mathbb{R}^3)} \leq Ct^{-\nu(l-1)} \delta^{\nu+l-1/2}, \quad l = 1, 2.$$

Moreover, introducing $\zeta^*(x) = \pi^{-3/2} e^{3i\pi/4} \int_{\mathbb{R}^3} d\xi e^{ix \cdot \xi} \Theta_\delta(2\xi) z(2\xi)$ and observing that $\zeta^* \in \dot{H}^s(\mathbb{R}^3)$ for any $s > 1/2 - \nu$, and $\|\nabla^l(v_{2,0} - e^{i\Delta t} \zeta^*)\|_{L^2(|x| \geq t^\gamma)} \rightarrow 0$ as $t \rightarrow +\infty$ for any $\gamma > \frac{1-2\nu}{3-2\nu}$ and any $l \geq 1$, we get that

$$e^{i\alpha(t)} \lambda^{1/2}(t) \chi^{ap}(\lambda(t) \cdot, t) - e^{it\Delta} \zeta^* \rightarrow 0 \text{ in } \dot{H}^1 \cap \dot{H}^2 \text{ as } t \rightarrow +\infty.$$

This concludes the proof of the first part of Proposition 2.1.

Next, we consider the error $R = -i\psi_t^{ap} - \Delta\psi^{ap} - |\psi^{ap}|^4 \psi^{ap}$. It has the form

$$R = E_1 + E_2 + E_3 + E_4,$$

where

$$\begin{aligned} E_1 &= i\left(\frac{1}{2} - \varepsilon_1\right) t^{-1} (\psi_{in}^{ap}(x, t) - \psi_{ss}^{ap}(x, t)) \tilde{\Theta}(t^{-1/2+\varepsilon_1} x) \\ &\quad - 2t^{-1/2+\varepsilon_1} (\nabla \psi_{in}^{ap}(x, t) - \nabla \psi_{ss}^{ap}(x, t)) \cdot \nabla \Theta(t^{-1/2+\varepsilon_1} x) \\ &\quad - t^{-1+2\varepsilon_1} (\psi_{in}^{ap}(x, t) - \psi_{ss}^{ap}(x, t)) \Delta \Theta(t^{-1/2+\varepsilon_1} x), \\ E_2 &= i\left(\frac{1}{2} + \varepsilon_2\right) t^{-1} (\psi_{ss}^{ap}(x, t) - \psi_{out}^{ap}(x, t)) \tilde{\Theta}(t^{-1/2-\varepsilon_2} x) \\ &\quad - 2t^{-1/2-\varepsilon_2} (\nabla \psi_{ss}^{ap}(x, t) - \nabla \psi_{out}^{ap}(x, t)) \cdot \nabla \Theta(t^{-1/2-\varepsilon_2} x) \\ &\quad - t^{-1-2\varepsilon_2} (\psi_{ss}^{ap}(x, t) - \psi_{out}^{ap}(x, t)) \Delta \Theta(t^{-1/2-\varepsilon_2} x), \\ \tilde{\Theta}(\xi) &= \xi \cdot \nabla \Theta(\xi), \end{aligned}$$

and E_3, E_4 are given by

$$\begin{aligned} E_3 &= \Theta(t^{-1/2+\varepsilon_1} x) R_{in}(x, t) + (1 - \Theta(t^{-1/2+\varepsilon_1} x)) \Theta(t^{-1/2-\varepsilon_2} x) R_{ss}(x, t) \\ &\quad + (1 - \Theta(t^{-1/2-\varepsilon_2} x)) R_{out}(x, t), \\ E_4 &= \Theta(t^{-1/2+\varepsilon_1} x) (|\psi_{in}^{ap}|^4 \psi_{in}^{ap} - |\psi^{ap}|^4 \psi^{ap}) \\ &\quad + (1 - \Theta(t^{-1/2+\varepsilon_1} x)) \Theta(t^{-1/2-\varepsilon_2} x) (|\psi_{ss}^{ap}|^4 \psi_{ss}^{ap} - |\psi^{ap}|^4 \psi^{ap}) \\ &\quad + (1 - \Theta(t^{-1/2-\varepsilon_2} x)) (|\psi_{out}^{ap}|^4 \psi_{out}^{ap} - |\psi^{ap}|^4 \psi^{ap}). \end{aligned}$$

Here

$$R_{in}(x, t) = e^{i\alpha_0 \ln t} t^{5\nu/2} \mathcal{R}_{in}(t^\nu |x|, t), \quad R_{ss}(x, t) = e^{i\alpha_0 \ln t} t^{5\nu/2} \mathcal{R}_{ss}(t^\nu |x|, t).$$

First, we address E_1 . By Lemma 2.7 (iii), we have

$$(2.57) \quad \|E_1\|_{H^2} \leq Ct^{-9(1+2\nu)/4+\nu+5\varepsilon_1/2} \ln t \leq Ct^{-(2+\frac{3}{20})(1+2\nu)}.$$

Similarly, by (2.49), for E_2 we get

$$(2.58) \quad \|E_2\|_{H^2} \leq Ct^{-1-(\frac{1}{2}+\varepsilon_2)(\frac{3}{2}+5\nu)} \ln t \leq Ct^{-(2+\frac{1}{4})(1+2\nu)}.$$

Next, consider E_3 . From Lemma 2.4 (ii), Lemma 2.7 (ii), and (2.51) it is clear that

$$(2.59) \quad \|E_3\|_{H^2} \leq Ct^{-\frac{9}{4}(1+2\nu)+5\varepsilon_1/2} \leq Ct^{-(2+\frac{3}{20})(1+2\nu)}.$$

Finally, applying Lemma 2.4 (estimates (2.17), (2.18)), Lemma 2.7 (estimates (2.37), (2.38), (2.39), (2.42)), and (2.47), (2.49), it is not difficult to check that

$$(2.60) \quad \|E_4\|_{H^2} \leq Ct^{-3(1+2\nu)}.$$

Combining (2.57), (2.58), (2.59), (2.60), we get (2.6), which concludes the proof of Proposition 2.1.

§3. CONSTRUCTION OF AN EXACT SOLUTION

Now we are in a position to prove Theorem 1.1. Consider (1.1) and write $\psi(x, t) = e^{i\alpha_0 \ln t} t^{\nu/2} U(y, \tau)$, where $y = t^\nu x$ and $\tau = \frac{t^{1+2\nu}}{1+2\nu}$. Further decomposing U as

$$U(y, \tau) = U^{ap}(y, \tau) + f(y, \tau), \quad U^{ap}(y, \tau) = e^{-i\alpha_0 \ln t} t^{-\nu/2} \psi^{ap}(x, t),$$

where ψ^{ap} is the approximate solution of (1.1) given by Proposition 2.1, we get the following equation for the remainder f :

$$(3.1) \quad i\vec{f}_\tau = \mathcal{H}(\tau)\vec{f} + \mathcal{F}(f) + r, \quad \vec{f} = \begin{pmatrix} f \\ \bar{f} \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{H}(\tau) &= H + \tau^{-1}l, \\ H &= -\Delta\sigma_3 - 3W^4\sigma_3 - 2W^4\sigma_3\sigma_1, \quad l = \frac{\alpha_0}{2\nu+1}\sigma_3 - i\frac{\nu}{2\nu+1}\left(\frac{1}{2} + y \cdot \nabla\right), \\ \mathcal{F}(f) &= \begin{pmatrix} F(f) \\ -\bar{F}(f) \end{pmatrix}, \quad F(f) = F_1(f) + F_2(f), \\ F_1(f) &= \mathcal{V}_1(\tau)f + \mathcal{V}_2(\tau)\bar{f}, \\ \mathcal{V}_1(\tau) &= 3(W^4 - |U^{ap}(\tau)|^4), \quad \mathcal{V}_2(\tau) = 2(W^4 - (U^{ap}(\tau))^2|U^{ap}(\tau)|^2), \\ F_2(f) &= -|U^{ap} + f|^4(U^{ap} + f) + |U^{ap}|^4U^{ap} + 3|U^{ap}|^4f + 2(U^{ap})^2|U^{ap}|^2\bar{f}, \\ r &= \begin{pmatrix} \mathbf{r} \\ -\bar{\mathbf{r}} \end{pmatrix}, \quad \mathbf{r}(y, \tau) = t^{-5\nu/2}e^{-i\alpha_0 \ln t}R(x, t), \end{aligned}$$

R being the error given by Proposition 2.1. Note that by Proposition 2.1 we have

$$(3.2) \quad \|\mathcal{V}_i(\tau)\|_{W^{2,\infty}(\mathbb{R}^3)} \leq C(|\alpha_0| + |\nu|)\tau^{-1}, \quad i = 1, 2,$$

$$(3.3) \quad \|U^{ap}(\tau)\|_{W^{2,\infty}(\mathbb{R}^3)} \leq C,$$

$$(3.4) \quad \|r(\tau)\|_{H^2(\mathbb{R}^3)} \leq C\tau^{-2-\frac{1}{8}},$$

for any $\tau \geq \tau_0$ with some $\tau_0 > 0$.

Our intention is to solve (3.1) with zero condition at $\tau = +\infty$ by a fixed point argument. To carry out this analysis, we shall need some energy type estimates for the linearized equation $i\vec{f}_\tau = \mathcal{H}(\tau)\vec{f}$. The required estimates are collected in the next subsection, their proofs can be found in §4.

3.1. Linear estimates. We start by recalling some basic spectral properties of the operator H (see, e.g., [5] for a more detailed discussion and proofs). Since we are considering only radial solutions, we shall view H as an operator on $L^2_{\text{rad}}(\mathbb{R}^3; \mathbb{C}^2)$ with domain $D(H) = H^2_{\text{rad}}(\mathbb{R}^3; \mathbb{C}^3)$. H satisfies the relations

$$\sigma_3 H \sigma_3 = H^*, \quad \sigma_1 H \sigma_1 = -H.$$

The essential spectrum of H fills up the real axis. The discrete spectrum of H consists of two simple purely imaginary eigenvalues $i\lambda_0, -i\lambda_0$, $\lambda_0 > 0$. The corresponding eigenfunctions ζ_+, ζ_- are in $\mathcal{S}(\mathbb{R}^3)$ and can be chosen in such a way that $\zeta_- = \sigma_1 \zeta_+ = \bar{\zeta}_+$. Observe also that $HW\begin{pmatrix} 1 \\ -1 \end{pmatrix} = HW_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$, which means that H has a resonance at zero.

Consider the projection of the linearized equation $if_\tau = \mathcal{H}(\tau)\vec{f}$ to the essential spectrum of H :

$$(3.5) \quad if_\tau = P\mathcal{H}(\tau)P\vec{f}.$$

Here P is the spectral projection of H to the essential spectrum given by

$$P = I - P_+ - P_-, \quad P_\pm = \frac{\langle \cdot, \sigma_3 \zeta_\mp \rangle}{\langle \zeta_\pm, \sigma_3 \zeta_\mp \rangle} \zeta_\pm,$$

$\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\mathbb{R}^3, \mathbb{C}^2)$.

Let $U(\tau, s)$ be the propagator associated with equation (3.5). In §4 we prove the following results.

Proposition 3.1. *There exists a constant $C > 0$ such that*

$$\|U(\tau, s)f\|_{H^2} \leq C \left(\frac{s}{\tau}\right)^{C(|\alpha_1| + |\nu_1|)} \|f\|_{H^2}$$

for any $s \geq \tau > 0$ and any $f \in H^2_{\text{rad}}$. Here $\alpha_1 = \frac{\alpha_0}{1+2\nu}$, $\nu_1 = \frac{\nu}{1+2\nu}$.

3.2. Contraction argument. Now we transform (3.1) into a fixed point problem. Rewrite (3.1) in the following integral form:

$$(3.6) \quad f(\tau) = J(f)(\tau),$$

where

$$\begin{aligned} J(f)(\tau) &= J_0(f)(\tau) + J_+(f)(\tau) + J_-(f)(\tau), \\ J_0(f)(\tau) &= i \int_\tau^{+\infty} ds U(\tau, s) P(\mathcal{F}_1(f(s)) + r(s)), \\ J_+(f)(\tau) &= i \int_\tau^{+\infty} ds e^{\lambda_0(\tau-s)} P_+(\mathcal{F}_2(f(s)) + r(s)), \\ J_-(f)(\tau) &= -i \int_{\tau_1}^\tau ds e^{-\lambda_0(\tau-s)} P_-(\mathcal{F}_2(f(s)) + r(s)), \\ \mathcal{F}_1(f) &= \mathcal{F}(f) + s^{-1}l(P_+ + P_-)\vec{f}, \\ \mathcal{F}_2(f) &= \mathcal{F}(f) + s^{-1}l\vec{f}, \end{aligned}$$

with $\tau_1 \geq \max\{\tau_0, 1\}$ to be fixed later (slightly abusing notation, in (3.6) we identify $\mathbb{C}^2$ -vectors of the form $\begin{pmatrix} f \\ \vec{f} \end{pmatrix}$ with their first component f).

Our intention is to view J as a mapping in the space $C([\tau_1, +\infty), H^2_{\text{rad}})$ equipped with the norm $\|f\| = \sup_{\tau \geq \tau_1} \|f(\tau)\|_{H^2} \tau^{1+1/16}$, and to show that J is a contraction of the unite ball $\|f\| \leq 1$ into itself provided $|\alpha_0| + |\nu|$ is sufficiently small and τ_1 is chosen

sufficiently large. Indeed, by (3.3) and (3.2) we have, for any $f, g \in H^2$ with $\|f\|_{H^2} \leq 1$, $\|g\|_{H^2} \leq 1$,

$$\begin{aligned}\|\mathcal{F}_1(f) - \mathcal{F}_1(g)\|_{H^2} &\leq C(\|f\|_{H^2} + \|g\|_{H^2} + (|\alpha_0| + |\nu|)\tau^{-1})\|f - g\|_{H^2}, \\ \|P_{\pm}(\mathcal{F}_2(f) - \mathcal{F}_2(g))\| &\leq C(\|f\|_{H^2} + \|g\|_{H^2} + (|\alpha_0| + |\nu|)\tau^{-1})\|f - g\|_{H^2},\end{aligned}$$

which, together with (3.4) and Proposition 3.1, gives

$$\|J(f)\| \leq \frac{1}{2} + C\tau_1^{-1/16}, \quad \|J(f) - J(g)\| \leq \left(\frac{1}{2} + C\tau_1^{-1/16}\right)\|f - g\|$$

for any $f, g \in \{\|h\| \leq 1\}$, provided $|\alpha_0| + |\nu|$ is sufficiently small. This means that if τ_1 is sufficiently large, then J is a contraction of the unit ball $\|f\| \leq 1$ into itself, and consequently, has a unique fixed point f satisfying

$$\|f(\tau)\|_{H^2} \leq \tau^{-1-1/16}, \quad \text{for all } \tau \geq \tau_1.$$

Together with Proposition 2.1, this gives Theorem 1.1.

§4. LINEARIZED EVOLUTION

In this section we prove Proposition 3.1. The proof will be achieved by combining the results of [5] with a careful spectral analysis of the operator H around the zero energy. The section is organized as follows. In Subsection 1 we consider the operator H as before, restricted to the subspace of radial functions, and construct a basis of Jost solutions of the equation $H\zeta = E\zeta$. In Subsection 2 we study the spectral decomposition of H near $E = 0$. In Subsection 3 we prove Proposition 3.1 by combining the results of the previous two subsections with the coercivity properties of H established in [5].

4.1. Solutions of the equation $H\zeta = E\zeta$. In this subsection we construct a basis of Jost solutions of the equation $H\zeta = E\zeta$, $E \in \mathbb{R}$. Since the subject is completely standard we shall only briefly sketch the proofs (see also [1] and [7] for a closely related construction in the context of energy subcritical NLS). Recall that

$$\begin{aligned}H &= -(\partial_\rho^2 + 2\rho^{-1}\partial_\rho)\sigma_3 + V(\rho), \quad V = \begin{pmatrix} V_1 & V_2 \\ -V_2 & -V_1 \end{pmatrix}, \\ V_1(\rho) &= -3W^4(\rho), \quad V_2(\rho) = -2W^4(\rho), \quad W(\rho) = (1 + \rho^2/3)^{-1/2}.\end{aligned}$$

We emphasize that $V(\rho)$ is a smooth function of ρ that decays as ρ^{-4} as $\rho \rightarrow \infty$. Since $\sigma_1 H = -H\sigma_1$, it suffices to consider the case where $E \geq 0$, so we write $E = k^2$, $k \geq 0$. It will be convenient for us to remove the first derivative in H . Setting $f = \rho\zeta$, we get

$$(4.1) \quad \tilde{H}f = Ef, \quad \tilde{H} = -\partial_\rho^2\sigma_3 + V(\rho).$$

We shall consider the operator $\tilde{H}$ on $\mathbb{R}$; to recover the original radial $\mathbb{R}^3$ problem, it suffices to restrict $\tilde{H}$ to the subspace of odd functions.

We start with constructing the most rapidly decaying solution of (4.1).

Lemma 4.1. *For all $k \geq 0$, there exists a real solution $f_3(\rho, k)$ of the equation*

$$(4.2) \quad \tilde{H}f = k^2 f$$

such that $f_3(\rho, k) = e^{-k\rho}\chi_3(\rho, k)$, where χ_3 is C^∞ -function of $(\rho, k) \in \mathbb{R} \times \mathbb{R}_+$ verifying $\chi_3(\rho, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + a(\rho, k)$,

$$(4.3) \quad \begin{aligned}|\partial_\rho^l \partial_k^m a(\rho, k)| &\leq C_l \langle \rho \rangle^{-2-l+m} (1 + k \langle \rho \rangle)^{-1-m}, \quad m = 0, 1, \\ |\partial_\rho^l \partial_k^2 a(\rho, k)| &\leq C_l \langle \rho \rangle^{-l} (1 + k \langle \rho \rangle)^{-3} \ln \left(\frac{1}{k \langle \rho \rangle} + 2 \right),\end{aligned}$$

for all $\rho \geq 0$, $k > 0$, and $l \geq 0$.

Proof. We write the following integral equation for χ_3 :

$$\begin{aligned}\chi_3(\rho, k) &= \binom{0}{1} - \int_{\rho}^{+\infty} K(\rho - s, k) \sigma_3 V(s) \chi_3(s, k) ds, \\ K(\xi, k) &= \begin{pmatrix} \frac{\sin k\xi}{k} & 0 \\ 0 & \frac{\sinh k\xi}{k} \end{pmatrix} e^{k\xi}.\end{aligned}$$

Then, the claim follows from the estimate

$$|\partial_k^l K(\xi, k)| \leq C_l \frac{|\xi|^{l+1}}{\langle k\xi \rangle^{l+1}}, \quad \xi \leq 0, \quad k \geq 0, \quad l \geq 0,$$

and the decay properties of V :

$$|\partial_{\rho}^l V(\rho)| \leq C_l \langle \rho \rangle^{-4-l}, \quad \rho \in \mathbb{R}, \quad l \geq 0,$$

by the standard Volterra iterations. □

Next we construct oscillating solutions of (4.2).

Lemma 4.2. *For all $k \geq 0$, there exists a solution $f_1(\rho, k)$ of equation (4.2) such that f_1 is a smooth function of $(\rho, k) \in \mathbb{R} \times \mathbb{R}_+^*$ of the form $f_1(\rho, k) = e^{ik\rho} \left(\binom{1}{0} + b(\rho, k) \right)$, where b verifies*

$$\begin{aligned}(4.4) \quad & |b(\rho, k)| \leq C(\langle \rho \rangle^{-2} + k e^{-k\rho}), \\ & |\partial_{\rho} b(\rho, k)| \leq C(\langle \rho \rangle^{-3} + k^2 e^{-k\rho}), \\ & |\partial_k b(\rho, k)| \leq C(\langle \rho \rangle^{-1} + \langle k\rho \rangle e^{-k\rho}), \\ & |\partial_{\rho k}^2 b(\rho, k)| \leq C(\langle \rho \rangle^{-2} + k \langle k\rho \rangle e^{-k\rho}),\end{aligned}$$

for all $\rho \geq 0$, $0 \leq k \lesssim 1$. Moreover, we have

$$|\partial_k^2 b(\rho, k)| \leq C \ln \left(\frac{1}{k} + 1 \right)$$

for all $0 \leq \rho \lesssim 1$, $0 < k \lesssim 1$.

Proof. To construct f_1 we reduce the order of system (4.2) via the substitution $f_1 = z_0 f_3 + z_1 \binom{1}{0}$. Further setting $z_2 = z'_0 f_{3,2}$, $f_3 = \begin{pmatrix} f_{3,1} \\ f_{3,2} \end{pmatrix}$, we see that $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ solves

$$\begin{aligned}(4.5) \quad & -z_1'' - k^2 z_1 + V_{11} z_1 + V_{12} z_2 = 0, \\ & -z_2' + k z_2 + V_{21} z_1 + V_{22} z_2 = 0.\end{aligned}$$

Here

$$\begin{aligned}V_{11} &= V_1 - V_2 \frac{f_{3,1}}{f_{3,2}}, \quad V_{12} = \frac{2}{f_{3,2}^2} (f_{3,1} f'_{3,2} - f'_{3,1} f_{3,2}), \\ V_{21} &= V_2, \quad V_{22} = -\frac{1}{f_{3,2}} (f'_{3,2} + k f_{3,2}).\end{aligned}$$

By Lemma 4.1, there exists $R > 0$ independent of k such that the functions $V_{ij}(\rho, k)$, $i, j = 1, 2$, are smooth in both variables for $k > 0$ and $\rho \geq R$ and verify, for all $l \geq 0$,

$\rho \geq R$, and $k > 0$,

$$\begin{aligned}
 |\partial_\rho^l V_{j1}(\rho, k)| &\leq C_l \langle \rho \rangle^{-4-l}, \quad j = 1, 2, \\
 |\partial_\rho^l \partial_k V_{11}(\rho, k)| &\leq C_l \langle \rho \rangle^{-5-l} \langle k\rho \rangle^{-2}, \\
 |\partial_\rho^l \partial_k^2 V_{11}(\rho, k)| &\leq C_l \langle \rho \rangle^{-4-l} \langle k\rho \rangle^{-3} \ln \left(\frac{1}{k\rho} + 2 \right), \\
 |\partial_\rho^l \partial_k^m V_{j2}(\rho, k)| &\leq C_l \langle \rho \rangle^{-3-l+m} \langle k\rho \rangle^{-1-m}, \quad j = 1, 2, \quad m = 0, 1, \\
 |\partial_\rho^l \partial_k^2 V_{22}(\rho, k)| &\leq C_l \langle \rho \rangle^{-1-l} \langle k\rho \rangle^{-3} \ln \left(\frac{1}{k\rho} + 2 \right).
 \end{aligned}
 \tag{4.6}$$

Writing for z the integral equation

$$z(\rho, k) = e^{ik\rho} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \int_\rho^\infty \begin{pmatrix} \frac{\sin k(\rho-s)}{k} & 0 \\ 0 & e^{-k(s-\rho)} \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} z(s, k) ds,$$

and taking (4.6) into account, we easily prove the existence of a smooth solution satisfying

$$\begin{aligned}
 |\partial_\rho^l \partial_k^m (e^{-ik\rho} z_1 - 1)| + \langle \rho \rangle |\partial_\rho^l \partial_k^m (e^{-ik\rho} z_2)| &\leq C_l \langle \rho \rangle^{-2-l+m} \langle k\rho \rangle^{-1-m}, \quad m = 0, 1, \\
 |\partial_\rho^n \partial_k^2 (e^{-ik\rho} z_1 - 1)| + |\partial_\rho^n \partial_k^2 (e^{-ik\rho} z_2)| &\leq C \ln \left(\frac{1}{k\rho} + 2 \right), \quad n = 0, 1,
 \end{aligned}
 \tag{4.7}$$

for all $\rho \geq R$, $k > 0$, $l \geq 0$.

To recover f_1 , we set

$$z_0(\rho, k) = \int_R^\rho \frac{z_2(s, k)}{f_{3,2}(s, k)} ds - \int_R^{+\infty} \frac{z_2(s, 0)}{f_{3,2}(s, 0)} ds.$$

Then, for $\rho \geq R$, the claim of Lemma 4.2 follows directly from (4.7) and Lemma 4.1. To cover the case where $x \leq R$, we can invoke the Cauchy problem with initial data at $\rho = R$. $\square$

Note that, since $k^2 \in \mathbb{R}$, the function $f_2(\cdot, k) = \overline{f_1(\cdot, k)}$ is also a solution of (4.2).

Remark 4.3. Recall that the equation $\tilde{H}f = 0$ has a basis of explicit solutions $\rho\Phi_\pm(\rho)\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$, $\rho\Theta_\pm(\rho)\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$, with $\Phi_\pm$, $\Theta_\pm$ given by (2.14). Comparing the behavior of $\rho\Phi_\pm$, $\rho\Theta_\pm$ with the asymptotics of $f_1(\rho, 0)$, $f_3(\rho, 0)$, we see that

$$f_1(\rho, 0) = \frac{1}{2}\rho(\xi_0(\rho) + \xi_1(\rho)), \quad f_3(\rho, 0) = \frac{1}{2}\rho(\xi_1(\rho) - \xi_0(\rho)),$$

where $\xi_0 = \frac{1}{\sqrt{3}}W\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\xi_1 = -\frac{2}{\sqrt{3}}W_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Next, we construct an exponentially growing solution at $+\infty$.

Lemma 4.4. *For any $k > 0$, there exists a solution $f_4(\rho, k)$ of (4.2) such that $f_4 = e^{k\rho}\chi_4$ with χ_4 verifying*

$$\partial_\rho^l \left(\chi_4(\rho, k) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = O_k(\rho^{-3-l}), \quad \rho \rightarrow +\infty.$$

Proof. We construct f_4 with the help of the following integral equation:

$$\begin{aligned}
 \chi_4(\rho, k) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_\rho^{+\infty} \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2k} \end{pmatrix} V\chi_4(s, k) ds \\
 &+ \int_{R_1}^\rho \begin{pmatrix} \frac{e^{k(s-\rho)} \sin k(\rho-s)}{k} & 0 \\ 0 & \frac{e^{2k(s-\rho)}}{2k} \end{pmatrix} V\chi_4(s, k) ds.
 \end{aligned}
 \tag{4.9}$$

For $k > 0$ and R_1 sufficiently large (depending on k), the operator generating (4.9) is small on the space of bounded continuous functions. Therefore, (4.9) has a solution χ_4 verifying

$|\chi_4(\rho, k)| \leq C$, $\rho \geq R_1$. Iterating this bound, we see that $\chi_4(\rho, k) - \binom{0}{1} = O_k(\rho^{-3})$ as $\rho \rightarrow \infty$. Finally, estimates for the derivatives can be obtained by differentiating (4.9). $\square$

Now we briefly describe some properties of the solutions f_j , $j = 1, \dots, 4$, that we will need later. Recall that the Wronskian $w(f, g) = \langle f', g \rangle_{\mathbb{R}^2} - \langle f, g' \rangle_{\mathbb{R}^2}$ does not depend on ρ if f and g are solutions of (4.1).

The estimates of Lemmas 4.1, 4.2, 4.4 lead to the relations

$$(4.10) \quad w(f_1, f_2) = 2ik, \quad w(f_1, f_3) = w(f_2, f_3) = 0, \quad w(f_3, f_4) = -2k, \quad k > 0,$$

the three first relations being valid also for $k = 0$. Observe that, by Lemmas 4.1 and 4.2, $\partial_k f_1(\rho, 0)$, $\partial_k f_3(\rho, 0)$ are solutions of the equation $\tilde{H}f = 0$ verifying

$$\begin{aligned} \left| \partial_k f_1(\rho, 0) - \binom{i\rho}{0} \right| &\leq C, & \left| \partial_{k\rho}^2 f_1(\rho, 0) - \binom{i}{0} \right| &\leq \frac{C}{\langle \rho \rangle^2}, \\ \left| \partial_k f_3(\rho, 0) + \binom{0}{\rho} \right| &\leq \frac{C}{\langle \rho \rangle}, & \left| \partial_{k\rho}^2 f_3(\rho, 0) + \binom{0}{1} \right| &\leq \frac{C}{\langle \rho \rangle^2} \end{aligned}$$

for $\rho \geq 0$. As a consequence,

$$(4.11) \quad \begin{aligned} w(\partial_k f_1|_{k=0}, f_1|_{k=0}) &= i, & w(\partial_k f_1|_{k=0}, f_3|_{k=0}) &= 0, \\ w(\partial_k f_3|_{k=0}, f_1|_{k=0}) &= 0, & w(\partial_k f_3|_{k=0}, f_3|_{k=0}) &= -1. \end{aligned}$$

In addition to the scalar Wronskian, we shall use matrix Wronskians. If F, G are (2×2) matrix solutions of (4.2), their matrix Wronskian

$$W(F, G) = F^{t'} G - F^t G'$$

is independent of ρ .

Set $g_j(\rho, k) = f_j(-\rho, k)$, $j = 1, \dots, 4$. Since the potential V is even, g_j , $j = 1, \dots, 4$, are again solutions of (4.2), and, as $\rho \rightarrow -\infty$, they have the same asymptotic behavior as f_j as $\rho \rightarrow +\infty$.

Consider the matrix solutions F, G defined by

$$F = (f_1, f_3), \quad G = (g_1, g_3).$$

Denote $D(k) = W(F, G)$. From Lemmas 4.1, 4.2 it follows that D is smooth for $k > 0$ and admits the estimate

$$(4.12) \quad |\partial_k^2 D(k)| \leq C \ln \left(\frac{1}{k} + 1 \right), \quad 0 < k \lesssim 1.$$

Moreover, by (4.8), (4.10), and (4.11) we have

$$(4.13) \quad D(0) = 0, \quad \partial_k D(0) = \begin{pmatrix} -2i & 0 \\ 0 & 2 \end{pmatrix}.$$

4.2. Scattering solutions and the distorted Fourier transform in the vicinity of zero energy. Set

$$(4.14) \quad \mathcal{F}(\rho, k) = F(\rho, k)s(k),$$

where $s(k) = D^{t-1}(k) \binom{2ik}{0}$. By (4.12) and (4.13), $s = \binom{s_1}{s_2}$ is a smooth function of k for $0 < k < k_0$ (k_0 sufficiently small), continuous up to $k = 0$, and verifying

$$(4.15) \quad \begin{aligned} s_1(0) &= -1, & s_2(0) &= 0, \\ |\partial_k s(k)| &\leq C |\ln k|, & 0 < k &\leq k_0. \end{aligned}$$

By construction,

$$w(\mathcal{F}, g_1) = 2ik, \quad w(\mathcal{F}, g_3) = 0$$

for any $0 \leq k < k_0$. As a consequence,

$$(4.16) \quad \mathcal{F}(\rho, k) = r_1(k)g_1(\rho, k) + g_2(\rho, k) + r_2(k)g_3(\rho, k), \quad 0 \leq k < k_0,$$

with some coefficients $r_1(k)$, $r_2(k)$ for which, by (4.8) and (4.15), we have

$$(4.17) \quad r_1(0) = r_2(0) = 0.$$

Computing the Wronskians $w(\mathcal{F}, \bar{\mathcal{F}})$ and $w(\mathcal{F}, \bar{\mathcal{G}})$, where $\mathcal{G}(\rho, k) = \mathcal{F}(-\rho, k)$, we get

$$|s_1(k)|^2 + |r_1(k)|^2 = 1, \quad r_1(k)\overline{s_1(k)} + \overline{r_1(k)}s_1(k) = 0, \quad 0 \leq k < k_0.$$

We can write the following Wronskian representation for r_1 :

$$(4.18) \quad r_1(k) = s_1(k)\frac{w(g_2, f_1)}{2ik} + s_2(k)\frac{w(g_2, f_3)}{2ik}, \quad k \neq 0.$$

Using (4.15) and the relations

$$w(g_2, f_3)|_{k=0} = w(g_2, f_1)|_{k=0} = \partial_k w(g_2, f_1)|_{k=0},$$

from (4.18) we deduce easily that r_1 is smooth for $0 < k < k_0$ and continuous up to $k = 0$, and verifies

$$(4.19) \quad |\partial_k r_1(k)| \leq C|\ln k|, \quad 0 < k < k_0;$$

in its turn, this implies that r_2 is smooth for $0 < k < k_0$ and continuous up to $k = 0$, and admits a similar estimate:

$$(4.20) \quad |\partial_k r_2(k)| \leq C|\ln k|, \quad 0 < k < k_0.$$

Introduce the following odd solution of (4.2):

$$e(\rho, k) = \mathcal{F}(-\rho, k) - \mathcal{F}(\rho, k).$$

By (4.14), (4.16),

$$(4.21) \quad e = a_1 f_1 + f_2 + a_2 f_3, \quad a_j = r_j - s_j, \quad j = 1, 2.$$

From (4.15), (4.17), (4.19), (4.20) it follows that

$$(4.22) \quad a_1(0) = 1, \quad a_2(0) = 0,$$

and

$$(4.23) \quad |\partial_k a_j| \leq C|\ln k|, \quad 0 < k < k_0, \quad j = 1, 2.$$

Together with Lemmas 4.1 and 4.2, this implies the following result.

Lemma 4.5. (i) $e(\rho, k) = e_0(\rho, k) + e_1(\rho, k)$, where $e_0(\rho, k) = a_1(k)e^{ik\rho}\binom{1}{0} + e^{-ik\rho}\binom{1}{0}$ and the remainder $e_1(\rho, k)$ admits the estimates

$$(4.24) \quad \begin{aligned} |e_1(\rho, k)| &\leq C(\langle \rho \rangle^{-2} + k|\ln k|e^{-k\rho}), \quad \rho \geq 0, \\ |\partial_k e_1(\rho, k)| &\leq C|\ln k|(\langle \rho \rangle^{-1} + e^{-k\rho/2}), \quad \rho \geq 0, \\ \|e_1(\cdot, k)\|_{L^2(\mathbb{R}_+)} &\leq C, \\ \|\rho e_1(\cdot, k)\|_{L^2(\mathbb{R}_+)} + \|\partial_k e_1(\cdot, k)\|_{L^2(\mathbb{R}_+)} &\leq Ck^{-1/2}|\ln k| \end{aligned}$$

for any $0 < k \leq k_0$.

(ii) $(\rho\partial_\rho - k\partial_k)e(\rho, k) = e^{ik\rho}\binom{1}{0}k\partial_k a_1(k) + e_2(\rho, k)$, with $e_2(\rho, k)$ verifying

$$(4.25) \quad \begin{aligned} |e_2(\rho, k)| &\leq C(\langle \rho \rangle^{-1} + k|\ln k|e^{-k\rho/2}), \quad \rho \geq 0, \\ \|e_2(\cdot, k)\|_{L^2(\mathbb{R}_+)} &\leq C \end{aligned}$$

for any $0 < k \leq k_0$.

For $0 < \kappa \leq k_0$, introduce the operators $\mathbb{E}_\kappa : L^2(\mathbb{R}_+, \mathbb{C}^2) \rightarrow L^2(\mathbb{R}^3, \mathbb{C}^2)$,

$$(\mathbb{E}_\kappa \Phi)(y) = \frac{1}{2^{3/2}\pi} \int_{\mathbb{R}_+} dk \theta_\kappa(k) \mathcal{E}(y, k) \Phi(k), \quad \Phi \in L^2(\mathbb{R}_+, \mathbb{C}^2),$$

where $\mathcal{E}(y, k)$ is a (2×2) matrix given by

$$\mathcal{E}(y, k) = \rho^{-1}(e(\rho, k), \sigma_1 \overline{e(\rho, k)}), \quad \rho = |y|,$$

$$\theta_\kappa(k) = \theta(\kappa^{-1}k), \quad \theta \text{ is an even } C^\infty\text{-function satisfying } \theta(k) = \begin{cases} 1 & \text{if } |k| \leq 1/4, \\ 0 & \text{if } |k| \geq 1/2. \end{cases}$$

Since $e(\rho, k)$ is a solution of the equation $\tilde{H}e = k^2 e$, we have $H\mathbb{E}_\kappa = \mathbb{E}_\kappa k^2 \sigma_3$.

By Lemma 4.5 (i), the operators $\mathbb{E}_\kappa$ are bounded uniformly with respect to $\kappa \leq k_0$. The action of the adjoint operators $\mathbb{E}_\kappa^* : L^2(\mathbb{R}^3, \mathbb{C}^2) \rightarrow L^2(\mathbb{R}_+, \mathbb{C}^2)$ is given by

$$(\mathbb{E}_\kappa^* \psi)(k) = \frac{1}{2^{3/2}\pi} \theta_\kappa(k) \int_{\mathbb{R}^3} dy \mathcal{E}^*(y, k) \psi(y), \quad \psi \in L^2(\mathbb{R}^3, \mathbb{C}^2).$$

Clearly,

$$(4.26) \quad \mathbb{E}_\kappa^* \sigma_3 \zeta_\pm = 0$$

for any $0 < \kappa \leq k_0$.

The following relation is a standard consequence of the asymptotics given by Lemma 4.5 (i):

$$(4.27) \quad \mathbb{E}_{\kappa_2}^* \sigma_3 \mathbb{E}_{\kappa_1} \sigma_3 = \theta_{\kappa_1}(k) \theta_{\kappa_2}(k)$$

for any $0 < \kappa_1, \kappa_2 \leq k_0$.

Remark 4.6. Notice that because of the presence of the cut-off function θ_κ , $\mathbb{E}_\kappa$ is bounded as an operator from $L^2([0, k_0])$ to $H^m(\mathbb{R}^3)$ for any $m \geq 0$, uniformly in $\kappa \leq k_0$.

Next we introduce quairesonant functions $h_\kappa(y)$, $0 < \kappa \leq k_0$, by setting

$$h_\kappa = \sqrt{2} \mathbb{E}_\kappa \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Lemma 4.7. *For any $0 < \kappa \leq k_0$, $h_\kappa \in \langle y \rangle^{-1} L^2(\mathbb{R}^3)$, and as $\kappa \rightarrow 0$ we have*

$$(4.28) \quad \|h_\kappa\|_{L^2(\mathbb{R}^3)} = O(\kappa^{1/2}), \quad \|y h_\kappa\|_{L^2(\mathbb{R}^3)} = O(\kappa^{-1/2}),$$

$$(4.29) \quad \langle h_\kappa, \sigma_3(\xi_0 + \xi_1) \rangle = 4\pi + O(\kappa^{1/2} \ln \kappa), \quad \langle h_\kappa, \sigma_3(\xi_1 - \xi_0) \rangle = O(\kappa^{1/2} \ln \kappa).$$

Proof. Applying Lemma 4.5 (i), we decompose h_κ as follows:

$$(4.30) \quad \begin{aligned} h_\kappa(y) &= h_{\kappa,0}(y) + h_{\kappa,1}(y) + h_{\kappa,2}(y), \\ h_{\kappa,0}(y) &= \frac{1}{2\pi\rho} \kappa \hat{\theta}(\kappa\rho) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ h_{\kappa,1}(y) &= \frac{1}{2\pi\rho} \int_{\mathbb{R}_+} dk e^{ik\rho} (a_1(k) - 1) \theta_\kappa(k) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ h_{\kappa,2}(y) &= \frac{1}{2\pi\rho} \int_{\mathbb{R}_+} dk \theta_\kappa(k) e_1(\rho, k), \end{aligned}$$

where $\hat{\theta}(\rho) = \int_{\mathbb{R}} e^{ik\rho} \theta(k) dk$, $\rho = |y|$.

Clearly, $h_{\kappa,0} \in \langle y \rangle^{-1} L^2(\mathbb{R}^3)$, and

$$(4.31) \quad \|h_{\kappa,0}\|_{L^2(\mathbb{R}^3)} \leq C\kappa^{1/2}, \quad \|y h_{\kappa,0}\|_{L^2(\mathbb{R}^3)} \leq C\kappa^{-1/2}.$$

Consider $h_{\kappa,i}$, $i = 1, 2$. From (4.22), (4.23), (4.24) it follows that

$$(4.32) \quad \|h_{\kappa,i}\|_{L^2(\mathbb{R}^3)} \leq C\kappa, \quad \|y h_{\kappa,i}\|_{L^2(\mathbb{R}^3)} \leq C\kappa^{1/2} |\ln \kappa|, \quad i = 1, 2,$$

which together with (4.31) leads to the estimates

$$(4.33) \quad \|h_\kappa\|_{L^2(\mathbb{R}^3)} \leq C\kappa^{1/2}, \quad \|yh_\kappa\|_{L^2(\mathbb{R}^3)} \leq C\kappa^{-1/2}.$$

Next we compute $\langle h_\kappa, \sigma_3(\xi_1 \pm \xi_0) \rangle$. By (4.31) and (4.32), as $\kappa \rightarrow 0$ we have

$$(4.34) \quad \begin{aligned} \langle h_\kappa, \sigma_3(\xi_1 \pm \xi_0) \rangle &= \langle h_{\kappa,0}, \sigma_3(\xi_1 \pm \xi_0) \rangle + O(\kappa^{1/2} \ln \kappa), \\ \langle h_{\kappa,0}, \sigma_3(\xi_1 - \xi_0) \rangle &= O(\kappa), \\ \langle h_{\kappa,0}, \sigma_3(\xi_1 + \xi_0) \rangle &= 2\kappa \int_{\mathbb{R}} d\rho \hat{\theta}(\kappa\rho) + O(\kappa) = 4\pi + O(\kappa), \end{aligned}$$

which gives (4.29). $\square$

4.3. Proof of Proposition 3.1. We start with deriving some coercivity bounds for the operator H .

Lemma 4.8. *There exists κ_0 , $0 < \kappa_0 \leq k_0$, and $C > 0$ such that*

$$(4.35) \quad \langle Hf, \sigma_3 f \rangle \geq C\kappa \|\nabla f\|_{L^2(\mathbb{R}^3)}^2$$

for any $0 < \kappa \leq \kappa_0$ and any $f \in \dot{H}_{\text{rad}}^1(\mathbb{R}^3, \mathbb{C}^2)$ verifying

$$(4.36) \quad \langle f, \sigma_3 \zeta_- \rangle = \langle f, \sigma_3 \zeta_+ \rangle = \langle f, \sigma_3 h_\kappa \rangle = \langle f, \sigma_3 \sigma_1 \bar{h}_\kappa \rangle = 0.$$

Remark 4.9. Notice that, since $\zeta_\pm, h_\kappa \in \langle y \rangle^{-1} L^2(\mathbb{R}^3)$, the scalar products involved in (4.36) are well defined for any $f \in \dot{H}^1$.

Proof. The proof of Lemma 4.8 is based on the following result, which is due to Duyckaerts and Merle.

Lemma 4.10. *There exists $c_0 > 0$ such that*

$$\langle Hf, \sigma_3 f \rangle \geq c_0 \|\nabla f\|_{L^2(\mathbb{R}^3)}^2$$

for any $f \in \dot{H}_{\text{rad}}^1(\mathbb{R}^3, \mathbb{C}^2)$ verifying

$$\langle f, \sigma_3 \zeta_- \rangle = \langle f, \sigma_3 \zeta_+ \rangle = \langle f, \Delta \xi_0 \rangle = \langle f, \Delta \xi_1 \rangle = 0,$$

see [5] for the proof.

Let $f \in \dot{H}_{\text{rad}}^1$ be such that (4.36) is fulfilled. We can write f as

$$f = \alpha_0 \xi_0 + \alpha_1 \xi_1 + g,$$

where

$$\alpha_j = -\frac{\langle f, \Delta \xi_j \rangle}{\|\nabla \xi_j\|_{L^2(\mathbb{R}^3)}^2}, \quad j = 0, 1,$$

and $g \in \dot{H}_{\text{rad}}^1$ verifies

$$\langle g, \sigma_3 \zeta_- \rangle = \langle g, \sigma_3 \zeta_+ \rangle = \langle g, \Delta \xi_0 \rangle = \langle g, \Delta \xi_1 \rangle = 0.$$

Therefore, by Lemma 4.10,

$$(4.37) \quad \langle Hf, \sigma_3 f \rangle = \langle Hg, \sigma_3 g \rangle \geq c_0 \|\nabla g\|_{L^2(\mathbb{R}^3)}^2.$$

Furthermore, since f satisfies (4.36), we have

$$A(\kappa) \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \langle g, \sigma_3 h_\kappa \rangle \\ \langle g, \sigma_3 \sigma_1 \bar{h}_\kappa \rangle \end{pmatrix},$$

where

$$A(\kappa) = - \begin{pmatrix} \langle \xi_0, \sigma_3 h_\kappa \rangle & \langle \xi_1, \sigma_3 h_\kappa \rangle \\ \langle h_\kappa, \sigma_3 \xi_0 \rangle & -\langle h_\kappa, \sigma_3 \xi_1 \rangle \end{pmatrix}.$$

By (4.29),

$$A(\kappa) = -2\pi \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + O(\kappa^{1/2} \ln \kappa), \quad \kappa \rightarrow 0.$$

Therefore, if κ is sufficiently small, then

$$|\alpha_0| + |\alpha_1| \leq C \|\nabla g\|_{L^2(\mathbb{R}^3)} \|\langle y \rangle h_\kappa\|_{L^2(\mathbb{R}^3)} \leq C \kappa^{-1/2} \|\nabla g\|_{L^2(\mathbb{R}^3)}.$$

As a consequence,

$$\|\nabla f\|_{L^2(\mathbb{R}^3)} \leq C \kappa^{-1/2} \|\nabla g\|_{L^2(\mathbb{R}^3)}.$$

Combining this inequality with (4.37), we get (4.35). $\square$

Lemma 4.11. *There exists κ_1 , $0 < \kappa_1 \leq k_0$, and $C > 0$ such that for any $0 < \kappa \leq \kappa_1$ we have*

$$\|f\|_{H^1(\mathbb{R}^3)} \leq \frac{C}{\kappa} \|\nabla f\|_{L^2(\mathbb{R}^3)}$$

for all $f \in H_{\text{rad}}^1(\mathbb{R}^3)$ with $\mathbb{E}_\kappa^* f = 0$.

Proof. By (4.22), (4.23), and Lemma 4.5 (i), $\mathbb{E}_\kappa^* f$ can be written as

$$(\mathbb{E}_\kappa^* f)(k) = \Phi_0(k) + \Phi_r(k),$$

where

$$\Phi_0(k) = \frac{1}{2^{3/2}\pi} \theta_\kappa(k) \check{f}(k),$$

$\check{f}(k) = 2 \int_{\mathbb{R}^3} dy \frac{\cos k|y|}{|y|} f(y)$, and the remainder Φ_r satisfies

$$\|\Phi_r\|_{L^2(\mathbb{R}_+)} \leq C \kappa^{1/2} \|f\|_{L^2(\mathbb{R}^3)}.$$

Therefore, $\mathbb{E}_\kappa^* f = 0$ implies

$$(4.38) \quad \|\check{f}\|_{L^2(0, \kappa/4)} \leq C \kappa^{1/2} \|f\|_{L^2(\mathbb{R}^3)}.$$

Notice also that for any $f \in H_{\text{rad}}^1$ and any $0 < \kappa \leq 1$ we have

$$\|f\|_{H^1(\mathbb{R}^3)} \leq C(\|\check{f}\|_{L^2(0, \kappa/4)} + \kappa^{-1} \|\nabla f\|_{L^2(\mathbb{R}^3)}).$$

Combining this inequality with (4.38), we get

$$\|f\|_{H^1(\mathbb{R}^3)} \leq \frac{C}{\kappa} \|\nabla f\|_{L^2(\mathbb{R}^3)},$$

provided κ is sufficiently small. $\square$

Finally, we combine Lemmas 4.8 and 4.11 to derive the following result, which will be in the heart of the proof of Proposition 3.1

Lemma 4.12. *There exists κ_2 , $0 < \kappa_2 \leq k_0$, and $C > 0$ such that for any $0 < \kappa \leq \kappa_2$ we have*

$$(4.39) \quad \langle Hf, \sigma_3 f \rangle \geq C \kappa^3 \|f\|_{H^1}^2 - \frac{\kappa}{C} \|\mathbb{E}_\kappa^* \sigma_3 f\|_{L^2(\mathbb{R}_+)}^2$$

for any $f \in H_{\text{rad}}^1(\mathbb{R}^3, \mathbb{C}^2)$ with $\langle f, \sigma_3 \zeta_\pm \rangle = 0$.

Proof. Write $f = f_1 + f_2$, where $f_1 = \mathbb{E}_\kappa \sigma_3 \mathbb{E}_\kappa^* \sigma_3 f$ and $f_2 = f - f_1$. Clearly,

$$(4.40) \quad \|f_1\|_{H^1(\mathbb{R}^3)} \leq C \|\mathbb{E}_\kappa^* \sigma_3 f\|_{L^2(\mathbb{R}_+)}, \quad \|Hf_1\|_{L^2(\mathbb{R}^3)} \leq C \kappa^2 \|\mathbb{E}_\kappa^* \sigma_3 f\|_{L^2(\mathbb{R}_+)},$$

for any $0 < \kappa \leq k_0$.

Consider f_2 . From (4.26) and (4.27) it follows that for any $\kappa' \leq \kappa/2$,

- $\langle f_2, \sigma_3 \zeta_\pm \rangle = 0$;
- $\mathbb{E}_{\kappa'}^* \sigma_3 f_2 = 0$;
- $\langle f_2, \sigma_3 h_{\kappa'} \rangle = \langle f_2, \sigma_3 \sigma_1 \bar{h}_{\kappa'} \rangle = 0$.

Hence, by Lemmas 4.8 and 4.11, we have

$$(4.41) \quad \langle Hf_2, \sigma_3 f_2 \rangle \geq C\kappa^3 \|f_2\|_{H^1(\mathbb{R}^3)}^2,$$

provided κ is sufficiently small.

Combining (4.40) and (4.41), we arrive at (4.39). $\square$

Now we are in a position to prove Proposition 3.1. Consider the equation

$$(4.42) \quad \begin{aligned} i\psi_\tau &= P\mathcal{H}(\tau)P\psi, \\ \psi|_{\tau=s} &= f, \end{aligned}$$

where

$$\mathcal{H}(\tau) = H + \tau^{-1}l, \quad l = \alpha_1\sigma_3 - i\nu_1\left(\frac{1}{2} + y \cdot \nabla\right),$$

$\alpha_1, \nu_1 \in \mathbb{R}$, $s > 0$, and $f \in \mathcal{S}(\mathbb{R}^3)$ satisfies $\langle f, \sigma_3 \zeta_\pm \rangle = 0$.

Fix κ such that $0 < \kappa \leq \kappa_2$ and consider the functional $G_1(\tau) = \langle H\psi, \sigma_3 \psi \rangle + c_0 \|\mathbb{E}_\kappa^* \sigma_3 \psi\|_{L^2(\mathbb{R}_+)}^2$. Clearly,

$$(4.43) \quad G_1(\tau) \leq C\|\psi(\tau)\|_{H^1(\mathbb{R}^3)}^2.$$

Moreover, since $\langle \psi(\tau), \sigma_3 \zeta_\pm \rangle = 0$, choosing c_0 sufficiently large, we get

$$(4.44) \quad G_1(\tau) \geq c_1 \|\psi(\tau)\|_{H^1(\mathbb{R}^3)}^2.$$

Next we compute the derivative $\frac{d}{d\tau}G_1$. We have

$$i\frac{d}{d\tau}\langle H\psi, \sigma_3 \psi \rangle = \frac{2i}{\tau} \operatorname{Im} \langle l\psi, \sigma_3 H\psi \rangle,$$

which implies

$$(4.45) \quad \left| \frac{d}{d\tau} \langle H\psi, \sigma_3 \psi \rangle \right| \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) \|\nabla \psi(\tau)\|_{L^2(\mathbb{R}^3)}^2.$$

Now, we address $\|\mathbb{E}_\kappa^* \sigma_3 \psi\|_{L^2(\mathbb{R}_+)}^2$. Denote $\Phi(\tau) = \mathbb{E}_\kappa^* \sigma_3 \psi(\tau)$. Then $\Phi(k, \tau)$ solves

$$(4.46) \quad i\Phi_\tau = k^2 \sigma_3 \Phi + \frac{1}{\tau} Y,$$

where

$$Y = \mathbb{E}_\kappa^* \sigma_3 l\psi.$$

Integrating by parts and applying Lemma 4.5 (ii), we can rewrite Y in the form

$$Y(k, \tau) = Y_0(k, \tau) + Y_1(k, \tau),$$

where

$$Y_0(k, \tau) = i\nu_1 k \partial_k \Phi(k, \tau),$$

and $Y_1(k, \tau)$ admits the estimate

$$\|Y_1(\tau)\|_{L^2(\mathbb{R}_+)} \leq C(|\alpha_1| + |\nu_1|) \|\psi(\tau)\|_{L^2(\mathbb{R}^3)}.$$

Therefore, (4.46) gives

$$\left| \frac{d}{d\tau} \|\Phi(\tau)\|_{L^2(\mathbb{R}_+)}^2 \right| \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) \|\psi(\tau)\|_{L^2(\mathbb{R}^3)}^2.$$

Combining this inequality with (4.46) and taking (4.44) into account, we get

$$(4.47) \quad \left| \frac{d}{d\tau} G_1(\tau) \right| \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) \|\psi(\tau)\|_{H^1(\mathbb{R}^3)}^2 \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) G_1(\tau).$$

Integration yields

$$G_1(\tau) \leq C \left(\frac{s}{\tau} \right)^{C(|\alpha_1| + |\nu_1|)} G_1(s), \quad 0 < \tau \leq s,$$

which, by (4.43) and (4.44), leads to the bound

$$(4.48) \quad \|U(\tau, s)f\|_{H^1(\mathbb{R}^3)} \leq C \left(\frac{s}{\tau}\right)^{C(|\alpha_1|+|\nu_1|)} \|f\|_{H^1(\mathbb{R}^3)}$$

for any $0 < \tau \leq s$ and any $f \in H_{\text{rad}}^1$.

To control higher regularity, consider the functional $G_2(\tau) = \langle H^2\psi, \sigma_3 H\psi \rangle + c_2 G_1(\tau)$. We have

$$C^{-1} \|\psi\|_{H^3(\mathbb{R}^3)}^2 \leq G_2 \leq C \|\psi\|_{H^3(\mathbb{R}^3)}^2,$$

provided c_2 is chosen sufficiently large.

Computing the derivative $\frac{d}{d\tau} \langle H^2\psi(\tau), \sigma_3 H\psi(\tau) \rangle$ and using (4.47), we get

$$(4.49) \quad \left| \frac{d}{d\tau} G_2(\tau) \right| \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) \|\psi(\tau)\|_{H^3(\mathbb{R}^3)}^2 \leq \frac{C}{\tau} (|\alpha_1| + |\nu_1|) G_2(\tau),$$

which implies

$$(4.50) \quad \|U(\tau, s)f\|_{H^3(\mathbb{R}^3)} \leq C \left(\frac{s}{\tau}\right)^{C(|\alpha_1|+|\nu_1|)} \|f\|_{H^3(\mathbb{R}^3)}$$

for any $0 < \tau \leq s$.

The H^2 bound stated in Proposition 3.1 follows from (4.48) and (4.50) by interpolation.

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