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LOCAL SMOOTHNESS OF AN ANALYTIC FUNCTION COMPARED TO THE SMOOTHNESS OF ITS MODULUS

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Dedicated to
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Abstract. Let Φ be a function analytic in the disk and continuous up to the boundary, and let its modulus of continuity satisfy the Hölder condition of order α, $0 < \alpha < 2$, at a single boundary point. Under standard assumptions on the zeros of Φ, this function must be then at least $\alpha/2$-Hölder (in a certain integral sense) at the same point. There are generalizations to not necessarily power-type Hölder smoothness.

§0. Introduction

0.1. Consider a function Φ analytic in the unit disk and continuous up to the boundary. What is the relationship between the smoothness of Φ and of $\varphi = |\Phi|$? Surely, it suffices to study this question for the restrictions of Φ and φ to the unit circle. The answer is well known: under some natural assumptions, Φ must be at least one half as smooth as φ, and this is best possible.

This natural assumptions should be imposed on the zeros of Φ. Consider the canonical factorization (see [8] for the details) $\Phi = F\theta B$, where F is the outer function constructed by φ, θ is a singular inner function, and B is the Blaschke product over the zeros of Φ. We remind the reader that for the boundary values of F (also denoted by the same letter F) we have

$$F = \varphi e^{i\mathcal{H}(\log \varphi)},$$

where $\mathcal{H}$ is the operator of harmonic conjugation. Next, θ is generated by a certain positive singular measure on the circle, and the boundary values of θ coincide a.e. with the function $e^{-i\mathcal{H}\mu}$. It is well known (see [8]) that if Φ is continuous up to the boundary, then the support of μ is included in the set $\{t \in \mathbb{T} : \varphi(t) = 0\}$; moreover, the zeros of B also may accumulate only to points of this set.

No lower bound for the smoothness drop is available without further assumptions about the zeros of Φ (see an explanation in [6]), and the simplest way out is to forbid them radically in the disk, i.e., to assume that $B = \theta = 1$. In this case, in the 1950s, Carleson and Jakobs proved that if $\varphi \in \mathrm{Lip}_\alpha$, $0 < \alpha < 1$, then $\Phi = F \in \mathrm{Lip}_{\alpha/2}(\mathbb{T})$. The proof was not published, and later the result was rediscovered by Havin and Shamoyan (see [7]), who also included the case of $\alpha = 1$. The story goes that Carleson extended the result to an arbitrary positive power-like smoothness, but the proof also did not appear in print. The only available proof of the fact that $F \in \mathrm{Lip}_{\alpha/2}$ for all positive α is due to Shirokov, see [11].

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Now we dwell on Havin's paper [6]. There it was not assumed that $\theta = B = 1$, but an additional condition was imposed on the zeros of the Blaschke product B. Specifically, they were forbidden to approach the unit circle tangentially. We present a way to express this requirement, to be employed in the sequel. Suppose that a point $t \in \mathbb{T}$ satisfies $\varphi(t) > 0$, and let I be an arc centered at t and such that $\varphi(\zeta) \geq \varphi(t)/2$ on I; we require that

$$\text{(NT)} \qquad B(r\zeta) \neq 0 \quad \text{for} \quad \zeta \in I, \quad r > 1 - c|I|,$$

where c does not depend on t and I.

Under these assumptions, the function $\Phi = F\theta B$ must again be at least one half as smooth as φ. Moreover, the moduli of continuity considered in [6] were not necessarily of power type: it was assumed that $|\varphi(t) - \varphi(s)| \leq \omega(|t-s|)$ and, under certain regularity conditions on ω, it was ensured that $|\Phi(t) - \Phi(s)| \leq c\omega(\sqrt{|t-s|})$. We mention two principal regularity conditions often invoked in similar situations (strictly speaking, they were used in [6] only implicitly):

$$\text{(R1)} \qquad \int_0^\delta \frac{\omega(u)}{u}\, du \leq c\omega(\delta);$$

$$\text{(R2)} \qquad \delta \int_\delta^{2\pi} \frac{\omega(u)}{u^2}\, du \leq c\omega(\delta).$$

Substantially, these conditions are related to "smoothness below 1", though it is difficult to formalize this statement. Next, in [6] certain doubling conditions were imposed on some functions related to ω, which were also assumed quasimonotone. All these condition, including (R1) and (R2), are fulfilled if $\omega(\delta) = \delta^\alpha$, $0 < \alpha < 1$.

We mention yet another condition, which is very important but has been involved only implicitly up to this point:

$$\text{(LOG)} \qquad B_1 = \int_{\mathbb{T}} |\log \varphi| < \infty.$$

The boundary values of an arbitrary analytic function bounded in the disk satisfy it *a fortiori* (see, e.g., [8]). In the quite recent paper [12], Shirokov proved that the smoothness of order $\frac{p}{p+1}\alpha$ can be guaranteed for an *outer* function F with modulus $\varphi \in \mathrm{Lip}_\alpha$ $(\alpha > 0)$ provided

$$(\mathrm{LOG_p}) \qquad B_p = \left(\int_{\mathbb{T}} |\log \varphi|^p \right)^{1/p} < \infty.$$

He also proved that smoothness does not drop at all if $\log \varphi \in \mathrm{BMO}$ (for $0 < \alpha < 1$, the last statement was established in [1]; see also [4] on this matter).

In the present paper, we want to show that the phenomenon described above is of *local* nature. This local nature will be ensured in the strongest form: a Hölder condition *at only one point* for φ implies that Φ is at least one half as smooth at the same point. Unfortunately, for the time being we have to restrict ourselves to fairly low smoothness, specifically, lower than 2. The authors hope to return to the case of arbitrary smoothness later. As a matter of fact, we do not understand at the moment whether a general answer should be sought in a form similar to that in the present paper. But from a technical point of view, it is not clear to us what should substitute the (simple!) Lemma 2 in §1 if smoothness is high. It should, however, be mentioned that the "global" theory also survived a period when the result was known only for smoothness lower than 2; see [2].

0.2. How to measure smoothness? For our purposes, this is convenient to do either in terms of mean oscillation or in terms of averaged finite differences. The mean oscillation of a (2π-periodic) function f over an interval I is the quantity

$$\nu_r(f;I) = \inf_c \left(\frac{1}{|I|} \int_I |f-c|^r \right)^{1/r}, \tag{1}$$

where the infimum is taken over all constants c; the number r, $r \in [1,\infty)$ can be fixed arbitrarily. Often, it will be convenient to assume that $r > 1$. Clearly, the quantity $\nu_r(f,I)$ grows with r.

The "mean" smoothness of a function f at a point x can be described by the condition $\nu_r(f,I) \le \omega(|I|)$ for every interval I that contains x (since f is 2π-periodic, I also should be assumed not excessively long, say, $|I| \le 4\pi$). Here ω is some nonnegative continuous monotone increasing function on $[0,+\infty)$ equal to zero at 0 and strictly positive elsewhere.

If we turn to the power scale $\omega(t) = Ct^\alpha$, $\alpha > 0$, then this condition reflects smoothness "properly" only when $\alpha \le 1$; for $\alpha > 1$ it can be fulfilled at certain points but if, for instance, it is true on an interval with one and the same number C, then f is constant on this interval. An appropriate similar description of "smoothness in the mean" for $\alpha > 1$ is obtained as follows: instead of a constant c in (1), we subtract a polynomial of degree $[\alpha]$ and take the infimum over all such polynomials. Two nonequivalent versions exist if α is an integer: we may subtract polynomials of degree α or, alternatively, of degree $\alpha - 1$. Both ways are reasonable. See, e.g., the monographs [5] and [9] for more details. However, we shall not need all this here.

Next, the nth difference $\Delta^n f(x;t)$ of a function f at a point x is defined as follows:

$$\Delta^1 f(x,t) = f(x+t) - f(x),$$
$$\Delta^{n+1} f(x,t) = \Delta^n f(x+t,t) - \Delta^n f(x,t) \text{ for } n \ge 1.$$

The "mean" smoothness of a function f at a point x can also be expressed by the condition

$$\left(\frac{1}{2h} \int_{-h}^{h} |\Delta^n f(x,t)|^r \, dt \right)^{1/r} \le \omega(h), \tag{2}$$

where ω is as above and r is fixed. For every specific ω, the choice of n is not arbitrary: if n is too small, degeneration may occur, but in the nondegenerate case raising n does not lead usually to new function classes and can be useful only technically. It is well known that, in the power scale $\omega(t) = ct^\alpha$, the natural values are $n = 1$ for $0 < \alpha \le 1$ and $n = 2$ for $1 \le \alpha < 2$ (as in the case of mean oscillations, the two variants fit if $\alpha = 1$, and then they are not equivalent). Since we mainly deal with smoothness "up to order 2", in the sequel we restrict ourselves by first and second differences. However, for example, if $n = 2$, in principle it is not forbidden for ω to tend to zero faster than t^2 as $t \to 0$ – in that case we talk about "degeneration at the point x".

We observe that condition (2) with $n = 1$ implies the condition $\nu_r(f,I) \le A\omega(A|I|)$ for mean oscillations (indeed, taking $c = f(x)$ in (1) and assuming that the center of I coincides with x, we obtain the first difference).

"Smoothness up to 1" (the term is used with some reservations) will be analyzed in this paper with the help of estimates for mean oscillations, and "smoothness between 1 and 2" will be treated by means of averaged second differences. In fact, it would be possible to do otherwise: to use averaged first differences for smoothness up to 1 and – surely, not mean oscillations, but local approximations by polynomials of degree at most 1 for smoothness not exceeding 2. This would require slightly stronger regularity conditions on ω in the first case and would lead to a less sharp result in the second. To estimate mean oscillations is somewhat easier than to estimate differences, though, basically, the methods are the same. Nevertheless, there are certain technical differences, which we find amusing and which should not be abandoned, to our mind. That is why we have decided to retain this "dualism".

Now, we describe the layout and the main results of the paper. §1 contains some preparatory material. In §2, we present a pointwise version of the main results of [6], measuring smoothness in terms of mean oscillations. Let us discuss some consequences of the estimates established there. We start with a spectacular but "improper" statement (it hides some essential details).

As in the beginning of the Introduction, let $\Phi = F\theta B$ be a function analytic in the unit disk and continuous up to the boundary. The smoothness of its modulus $\varphi = |\Phi|$ (φ is identified with a 2π-periodic function on $\mathbb{R}$ in a standard way) will be measured in the usual (nonintegral) sense: for every $x \in \mathbb{R}$, let

$$\Omega_{\varphi,x}(h) = \sup_{|x-s|\le h} |\varphi(x) - \varphi(s)|, \quad 0 < h \le 4\pi,$$

be the standard *local* modulus of continuity of φ at a point x. To tell something substantial, we need to introduce sufficiently regular majorants for $\Omega_{\varphi,x}$: let $\Omega_{\varphi,x}(h) \le \omega_x(h)$, where ω_x is a continuous monotone decreasing function on $[0, +\infty)$ that is equal to zero only at zero. We assume that these majorants satisfy a doubling condition:

$$\text{(DB)} \qquad \omega_x(2h) \le D_x \omega_x(h),$$

and also that the function $\frac{\omega_x(h)}{h}$ is almost monotone decreasing (the relevant constant may depend on x). We remind the reader that a function $\varkappa$ is said to be almost monotone decreasing with constant C if

$$\text{(QM)} \qquad \varkappa(h_1) \ge C\varkappa(h_2) \quad \text{for} \quad h_1 \le h_2.$$

Also, we assume that condition (NT) is fulfilled.

Theorem A. *Let $r > 1$. Under the above assumptions, for every x and every interval $I \ni x$, $|I| \le 4\pi$, we have $\nu_r(\Phi; I) \le C_x \omega_x(\sqrt{|I|})$, where C_x depends on r, on the constants in the regularity conditions for ω_x, on the constant c in* (NT)*, on B_1 (see* (LOG)*), on certain properties of the Blaschke product, and on the norm of the singular measure corresponding to the singular inner function θ.*

It should be noted that the functions $t \mapsto at^\alpha$, $0 < \alpha < 1$, satisfy all the regularity conditions listed above.

The next theorem claims that, when analyzed "quite locally", smoothness does not drop at all. Apparently, this phenomenon remained somewhat in shade in the course of smoothness estimates uniform over the entire boundary. However, the statement is quite obvious at the points x where $\varphi(x) = 0$. For other x, the following is true.

Theorem B. *Under the above assumptions, let condition* (R2) *be fulfilled as well. If $\varphi(x) > 0$, then there exists a number $h_x > 0$ (depending on the quantity $\varphi(x)$) such that $\nu_r(\Phi, I) \le C_x \omega_x(I)$ if $|I| \le h_x$ and $I \ni x$.*

Clearly, Theorem B applies in the case where the role of ω_x is played by power functions ct^α, $0 < \alpha < 1$. In what follows, we shall see that, as $\varphi(x)$ approaches zero, the constants C_x grow unlimitedly and the constants h_x tend to zero (it is assumed that all other constants involved in the hypotheses remain bounded). Theorem B shows that the smoothness drop by one half is a purely quantitative phenomenon, visible only at "medium" distances to x. So, in spite of the paper's title, the entire theory is "not quite local" in principle, though it is of pointwise nature.

Now, we pass to "proper" statements. The drawback of the said above is in the fact that, to estimate the smoothness ("in the mean") of Φ at x, it is not necessary to avail any smoothness of φ anywhere except at x (it may even be discontinuous elsewhere). In this case, however, the outer and inner factors F and θB are not related so intimately as in the case of a continuous Φ; so, such a relationship should be required forcibly. In all previous results about smoothness drop, the main difficulty has been in estimating the outer function F. As we shall see, the same feacher occurs when we are interested in the behavior at a single point: the incorporation of an inner factor under *a priori* "coordination conditions" is an absolutely mechanical procedure. Therefore, in the remaining part of the Introduction, we restrict ourselves to outer functions.

So, let φ be a nonnegative measurable function on the circle (a 2π-periodic function on $\mathbb{R}$) satisfying (LOG) and continuous at a point x; moreover, let $|\varphi(x) - \varphi(s)| \le \omega(|x-s|)$ for $|x-s| \le 4\pi$, where ω is continuous on $[0, +\infty)$, is monotone increasing and vanishes only at zero. Let $F = \varphi e^{i\mathcal{H}(\log\varphi)}$ be the outer function constructed by φ. Then analogs of Theorems A and B are fulfilled at x. However, we give a precise statement under other regularity conditions, in order to present a generalization (for low smoothness) of Shirokov's result from [12]. Let $\varkappa(\delta) = \delta\int_\delta^{2\pi}\frac{\omega(t)}{t^2}\,dt$ be the left-hand side in (R2).

Theorem C. *Suppose that* ($\mathrm{LOG_p}$) *is fulfilled,* ω *satisfies a doubling condition,*[1] *and the function* $\omega(t)t^{-(p+1)/p}$ *is almost monotone decreasing; then for every* $r \ge 1$ *and every interval* $I \ni x$ *with* $|I| \le 4\pi$ *we have* $\nu_r(F; I) \le C\big(\omega\big(|I|^{\frac{p}{p+1}}\big) + \varkappa(|I|)\big)$, *where* C *depends on* r, *the constant* (B_p) *in* ($\mathrm{LOG_p}$), *and the constants in the regularity conditions for* ω.

A $\frac{p+1}{p}$ times smoothness drop emerges if we require additionally that $\varkappa(h) \le C'\omega\big(h^{\frac{p}{p+1}}\big)$. Avoiding the analysis of this condition, we simply look at Theorem C for power-type smoothness moduli $\omega(t) = at^\alpha$. The requirement that $\omega(t)t^{(p+1)/p}$ be (almost) monotone decreasing is equivalent to the condition $\alpha \le \frac{p+1}{p}$. So, certain $\alpha > 1$ are admissible; in this case a sort of degeneration occurs for φ at x. Next,

$$\varkappa(\delta) \asymp \begin{cases} \delta & \text{if } \alpha > 1, \\ \delta|\log\delta| & \text{if } \alpha = 1, \\ \delta^\alpha & \text{if } \alpha < 1. \end{cases}$$

So, for $\alpha \le \frac{p+1}{p}$ smoothness always drops $\frac{p+1}{p}$ times.

In §2, it will also be shown that smoothness does not drop at all if, instead of ($\mathrm{LOG_p}$), we require that $\log\varphi$ satisfy the condition in the definition of BMO, but again at the point x only:

$$(\mathrm{LOG}_\infty) \qquad \frac{1}{|I|}\int_I |\log\varphi - (\log\varphi)_I| \le B_\infty$$

for all intervals I containing x. Here and below, g_I is the average of g over I. It is natural to treat the above statement as the case of $p = \infty$ in Theorem C.

Finally, we turn to the content of §3, where, as has already been said, we analyze smoothness "between 1 and 2" for the original function φ, at a single point as before. Again, we restrict ourselves to outer functions. Next, it would have been possible to write out estimates with an arbitrary "smoothness modulus" ω (as in §2), but definitely the result would not bring about any sort of merry mood. Furthermore, to make the formulas transparent would have required certain regularity conditions that involve logarithmic factors, which arise inevitably in abundance. Therefore, we restrict ourselves to the power

[1] This assumption is redundant because it is implied by the assumption after it. (Added in translation.)

scale. The smoothness of φ at (generally speaking, only one) point x will be measured in terms of a "global" approximation by a polynomial of degree at most 1:

$$|\varphi(t) - \varphi(x) - b(t-x)| \le d|t-x|^{\alpha}, \tag{3}$$

where $\alpha > 1$ is a fixed number. We impose condition ($\mathrm{LOG_p}$) with $p < \infty$ on φ, and then require that $\alpha \le 2\left(\frac{p+1}{p}\right)$ (thus, $\alpha \le 4$ for $p = 1$). Again, for $\alpha > 2$ the smoothness of φ at x is "degenerate", but it will drop $\frac{p+1}{p}$ times in any case. In principle, we could also include condition ($\mathrm{LOG_\infty}$), basically by the same arguments as in §2, but we omit these details.

Theorem D. *Let F be the outer function constructed by φ, and let $r > 1$. Then*

$$\left(\frac{1}{2h}\int_{-h}^{h} |\Delta^2 F(x,t)|^r\, dt\right)^{1/r} \le A h^{\frac{\alpha p}{p+1}}, \quad |h| \le 4\pi,$$

where the constant A depends on d in (3)*, on B_p in* ($\mathrm{LOG_p}$)*, on α, and on r.*

It should be noted that the definition of the second difference involves the number $F(x)$, which is well defined in our case (see §3 for the explanations), though F is only defined almost everywhere.

0.3. How to pass to "genuine" smoothness? It is fairly well known that, whenever integral conditions like those mentioned in the preceding section are fulfilled uniformly at all points x, the function is smooth "in the right way". This gives additional value to the results described here.

Let us understand why this general principle applies to our situation.

0.3.1. Estimates of mean oscillation. Let Ω be a nonnegative monotone increasing function on $[0, \infty)$ that vanishes only at zero.

Proposition 1. *Let g be a 2π-periodic measurable function, and let Δ be an interval with $|\Delta| \le 2\pi$. Suppose that $\nu_r(g, I) \le \Omega(|I|)$ for all intervals I with $|I| \le 4\pi$ that intersect Δ. If Ω satisfies condition* (R1)*, then $|g(x_1) - g(x_2)| \le C\Omega(|x_1 - x_2|)$ for all $x_1, x_2 \in \Delta$.*

This statement, proved in [13], was inspired by the result for $\Omega(t) = ct^{\alpha}$, $0 < \alpha \le 1$, obtained in [3] and [10]. See also the presentation in the monograph [9], which is also restricted to power-type continuity moduli. Now it is no longer surprising that in [6] the two regularity conditions (R1) and (R2) were involved (in fact), whereas here we only use something like (R2): condition (R1) is required in the passage to global Hölder classes.

0.3.2. Estimates of averaged second differences. Now, suppose that the condition

$$\left(\frac{1}{2h}\int_{-h}^{h} |\Delta^2 g(x,t)|^r\, dt\right)^{1/r} \le D h^{\beta}, \quad 0 \le h \le 4\pi, \tag{4}$$

is fulfilled for all x with the same constant and the same β. (As above, this may only be assumed for x in a fixed interval Δ, but we leave this aside.) In our context, β arises as $\frac{p\alpha}{p+1}$ (see Theorem D), so it is sometimes smaller and sometimes greater than 1. In the first case, we can reduce the matter to estimates for first differences, as in the classical Zygmund lemma [15, Volume 1, Chapter 2, Theorem 3.4] or Marchaud inequality (see [14]). Specifically, we have the following statement. The only point in it that may happen to be original (to a certain extent) is the observation that this technique is applicable to smoothness in the mean at a unique point.

Proposition 2. *Let g be a 2π-periodic measurable function, and let $|g| \le L$ everywhere. Fixing x, we put for brevity*

$$\Delta(h) = \left(\frac{1}{2h}\int_{-h}^{h} |\Delta^2 g(x,t)|^r\, dt\right)^{1/r}, \qquad \varkappa(h) = \left(\frac{1}{2h}\int_{-h}^{h} |\Delta^1 g(x,t)|^r\, dt\right)^{1/r}.$$

Then

$$\varkappa(\xi) \le \frac{L}{2^{k-1}} + 2^{-1}\sum_{s=0}^{k-1}\frac{\Delta(2^s\xi)}{2^s} \quad \textit{for} \quad 0 \le \xi \le \frac{\pi}{2},$$

where k is the greatest integer with $2^k\xi \le \pi$.

The functions g that arise in (4) by the intermediance of Theorem D, are uniformly bounded (see Corollary 2 in §1 about the parameters determining this bound). So, if $\Delta(\xi) = c\xi^\beta$ with $0 < \beta < 1$, uniformly in x, then Proposition 2 gives the estimate $\varkappa(\xi) \le C\xi^\beta$ for averaged first differences of g at an arbitrary point x (C does not depend on x). After that, the genuine Lipschitz property for g follows again from Proposition 1 (it has already been mentioned that the averaged first difference dominates the mean oscillation).

Before we prove Proposition 2, let us find out what to do if (4) is fulfilled with some $\beta \in [1,2]$. It turns out that if this happens uniformly in x, we can *directly* deduce the existence of good local *uniform* approximations for g by polynomials of degree at most one. The method of proof is standard in essence, but we include the arguments for completeness because we are not sure that they can be found in the literature precisely in this context.

Proposition 3. *Let condition* (4) *be fulfilled with some $r \ge 1$ and $\beta \in [1,2]$, uniformly for all x with $|x| \le 4\pi$. If $g \in C^2$, then for every interval $|I|$, $|I| < 2\pi$, there exists a linear polynomial ρ with* $\sup_{x\in I}|g(x) - \rho(x)| \le C|I|^\beta$.

If Proposition 3 is proved, we do the following. If g is C^2-smooth, Proposition 3 easily shows that $|\Delta^2 g(x,t)| \le C'|t|^\beta$ for all x (and, say, for $|t| \le \pi/2$). In the general case, we convolve g, for example, with the Féjer kernels. This will result in a sequence g_n of infinitely smooth functions that satisfy (4) uniformly in n and converge to g a.e. It remains to pass to the limit as $n \to \infty$ in the inequality $|\Delta^2 g_n(x,t)| \le C'|t|^\beta$ and deduce that g coincides a.e. with a β-Hölder function (if $\beta = 1$, the Zygmund class arises rather than Lip_1).

Proof of Proposition 2. Observe that

$$g(x+2t) - 2g(x+t) - g(x) = [g(x+2t) - g(x)] - 2[g(x+t) - g(x)],$$

whence

$$\begin{aligned}\Delta(h) &= \left(\frac{1}{2h}\int_{|t|\le h} |g(x+2t) - 2g(x+t) + g(x)|^r\, dt\right)^{1/r} \\ &\ge \left|\left(\frac{1}{2h}\int_{|t|\le h} |g(x+2t) - g(x)|^r\, dt\right)^{1/r} - 2\left(\frac{1}{h}\int_{|t|\le h} |g(x+t) - g(x)|^r\, dt\right)^{1/r}\right|.\end{aligned}$$

Introducing the new variable $2t$ in the first term under the modulus sign on the right, we rewrite this inequality in the following way: $|\varkappa(2h) - 2\varkappa(h)| \le \Delta(h)$. Substituting $\frac{h}{2^j}$ for h and multiplying by 2^{j-1} ($j = 1, \dots, k$), we obtain

$$\left|2^{j-1}\varkappa\left(\frac{h}{2^{j-1}}\right) - 2^j\varkappa\left(\frac{h}{2j}\right)\right| \le 2^{j-1}\Delta\left(\frac{h}{2^j}\right), \quad j = 1, \dots, k,$$

whence

$$\left|\varkappa(h) - 2^k \varkappa\left(\frac{h}{2^k}\right)\right| \le \sum_{j=1}^{k} 2^{j-1} \Delta\left(\frac{h}{2^j}\right).$$

Now, let $0 \le \xi \le \pi/2$, and let k be the greatest positive integer with $2^k\xi \le \pi$. Taking $h = 2^k\xi$ in the last inequality and dividing by 2^k, we arrive at the required estimate

$$\varkappa(\xi) \le \frac{\varkappa(2^k\xi)}{2^k} + 2^{-k}\sum_{j=1}^{k} 2^{j-1}\Delta(\xi \cdot 2^{k-j}) \le \frac{L}{2^{k-1}} + 2^{-1}\sum_{s=0}^{k-1}\frac{\Delta(2^s\xi)}{2^s}. \qquad \square$$

Proof of Proposition 3. Clearly, (4) implies the inequality

$$\frac{1}{h}\int_{h/2\le|t|\le h} |\Delta^2 g(x,t)|\, dt \le C' h^\beta, \quad 0 < h \le 4\pi.$$

It is also clear that

$$\Delta^2 g(x,t) = \int_0^t \int_0^t g''(x+\sigma+\tau)\, d\sigma\, d\tau. \tag{5}$$

We integrate the identity $g(x) = \Delta^2 g(x, \sigma+\tau) + (2g(x+\sigma+\tau) - g(x+2\sigma+2\tau))$ in σ and in τ from 0 to t and divide by t^2, obtaining

$$g(x) = \psi(x,t) + \frac{1}{t^2}\int_0^t\int_0^t \Delta^2 g(x,\sigma,+\tau)\, d\sigma\, d\tau,$$

where

$$\psi(x,t) = \frac{1}{t^2}\int_0^t\int_0^t (2g(x+\sigma+\tau) - g(x+2\sigma+2\tau))\, d\sigma\, d\tau.$$

Yet another integration yields

$$g(x) = \varphi_h(x) + \frac{1}{h}\int_{h/2\le|t|\le h}\frac{1}{t^2}\int_0^t\int_0^t \Delta^2 g(x,\sigma,+\tau)\, d\sigma\, d\tau\, dt,$$

where $\varphi_h(x) = \frac{1}{h}\int_{h/2\le t\le h}\psi(x,t)\, dt$. The second term on the right in the last formula splits into two integrals, by positive and by negative t; their estimates are similar. For example, for the integral $I_1(x)$ over the interval $h/2 \le t \le h$, we have

$$\begin{aligned}
|I_1(x)| &\le \frac{C}{h^3}\int_{h/2\le|t|\le h}\left(\int_0^h\int_0^h |\Delta^2 g(x,\sigma+\tau)|\, d\sigma\, d\tau\right) dh \\
&\le \frac{C'}{h^2}\int_0^h\int_0^h |\Delta^2 g(x,\sigma+\tau)|\, d\sigma\, d\tau = \frac{C'}{h^2}\int_0^h\int_\tau^{\tau+h} |\Delta^2 g(x,u)|\, du\, d\tau \\
&= \frac{C'}{h^2}\left[\int_0^h\int_0^u |\Delta^2 g(x,u)|\, d\sigma\, du + \int_h^{2h}\int_{u-h}^h |\Delta^2 g(x,u)|\, d\sigma\, du\right] \le C'' h^\beta
\end{aligned}$$

by (4). Therefore, $|g(x) - \varphi_h(x)| \le C h^\beta$ for all x and h.

Now we estimate the second derivative (in x) for the function $\varphi_h(x)$. First,

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\psi(x,t) &= \frac{1}{t^2}\int_0^t\int_0^t (2g''(x+\sigma+\tau) - g''(x+2\sigma+2\tau))\, d\sigma\, d\tau \\
&= \frac{2}{t^2}\Delta^2 g(x,t) - \frac{4}{t^2}\Delta^2 g(x,2t)
\end{aligned}$$

by (5). Consequently,

$$|\varphi_h''(x)| \le \frac{1}{h}\int_{h/2\le t\le h} |\frac{\partial^2}{\partial x^2}\psi(x,t)\,dh|$$
$$\le \frac{C}{h^2}\Big(\frac{1}{h}\int_{h/2\le |t|\le h}|\Delta^2 g(x,t)|\,dt + \frac{1}{h}\int_{h/2\le |t|\le h}|\Delta^2 g(x,2t)|\,dt\Big) \le C'h^{\beta-2}$$

by (4).

Now, let I be an interval, x_0 it center, and h half its length. For $x\in I$ we have

$$\varphi_h(x) = \varphi_h(x_0) + \frac{1}{2}\varphi_h'(x_0)(x-x_0) + \frac{1}{2}\int_{x_0}^{x}\varphi_h''(u)(x-u)\,du,$$

so that, putting $\rho(x) = \varphi_h(x_0) + \frac{1}{2}\varphi_h'(x_0)(x-x_0)$, we obtain

$$|\varphi_h(x) - \rho(x)| \le ch^{\beta-2}|x-x_0|^2 \le ch^{\beta}, \quad x\in I.$$

It remains to combine this with the inequality

$$|g(x) - \varphi_h(x)| \le ch^{\beta}$$

proved above. □

§1. Auxiliary statements

As before, we denote by φ a nonnegative function on the circle satisfying condition (LOG). We shall assume that φ is smooth at a point x and, at the first place, we shall be interested in the smoothness of the outer function $F = \varphi e^{i\mathcal{H}\log\varphi}$ at the same point x. As has already been mentioned, we identify φ with its 2π-periodic extension to the reals. On 2π-periodic functions, the harmonic conjugation operator is given by the formula

$$(\mathcal{H}u)(t) = \frac{1}{2\pi}\,\text{p.v.}\int_{-\pi}^{\pi}\cot\frac{t-s}{2}u(s)\,ds, \quad t\in\mathbb{R}.$$

In what follows, all similar integrals are tacitly understood in the principal value sense, so we shall omit the symbol v.p.

Without loss of generality, we assume that the point x where we measure smoothness in 0. Generally speaking, φ is defined up to values on a set of measure zero, but we assume that it has a definite value $\varphi(0)$ at zero and

$$|\varphi(t) - \varphi(0) - bt| \le \omega(|t|) \tag{6}$$

for some constant b and some function ω. It will be $b = 0$ in §2, ω will be of power type in §3, but it is convenient to do preliminary preparations in the general form. For the moment, we only assume that ω is a monotone increasing continuous function on $[0,+\infty)$ vanishing only at 0.

We introduce the near inverse function $\widetilde{\omega}$: $\widetilde{\omega}(s) = \min\{t : \omega(t) = s\}$. It is defined on the range of ω (so, definitely near zero), is monotone increasing, and its limit at zero is 0. Clearly,

$$\omega(\widetilde{\omega}(s)) = s, \quad \widetilde{\omega}(\omega(t)) \le t, \quad \text{but} \quad \widetilde{\omega}((1+\varepsilon)\omega(t)) \ge t \quad \text{for} \quad \varepsilon > 0.$$

Lemma 1. *Suppose that for some $\gamma > 0$ the function $t \mapsto \frac{\omega(t)}{t^\gamma}$ is almost monotone decreasing. Then the function $s \mapsto \frac{s}{\widetilde{\omega}(s)^\gamma}$ is almost monotone decreasing.*

Proof. Let $s_1 \le s_2$, and let $t_i = \widetilde{\omega}(s_i)$, $i = 1,2$. Then $t_1 \le t_2$. We have $\frac{\omega(t_1)}{t_1^\gamma} \ge c\frac{\omega(t_2)}{t_2^\gamma}$ by assumption. Since $\omega(t_i) = s_i$, the claim follows. □

As has already been mentioned, our results are not of quite local nature ("an effect of medium distances" mentioned in the Introduction). So, we cannot proceed without global conditions on φ. One of them has already been imposed: this is (LOG_1). We find it convenient to formulate another condition as follows: we assume that (6) is true for (almost) all t with $|t| \le 4\pi$. Soon we shall see that this is equivalent[2] to the assumption that (6) is fulfilled only near zero but φ is bounded.

Lemma 2. *Let* $\varphi(0) > 0$*, then* $|b| \le \max\left(\frac{2\varphi(0)}{\widetilde{\omega}(\varphi(0))}, \frac{\varphi(0)}{\pi}\right)$.

Remark. For small values of $\varphi(0)$, the maximum is attained at the first quantity. More precisely, if it is attained at the second quantity, then $\varphi(0) \ge \omega(2\pi)$.

Proof of the lemma. If $b = 0$, there is nothing to prove. If $b \ne 0$, we plug $t_0 = -\frac{2\varphi(0)}{b}$ in (6). This is surely possible provided $\frac{2\varphi(0)}{|b|} \le 2\pi$, but, again, there is nothing to prove in the opposite case.

So, we obtain $\varphi(t_0) + \varphi(0) \le \omega(|t_0|)$ and, *a fortiori*, $\varphi(0) \le \omega(|t_0|)$. Therefore, $\widetilde{\omega}(\varphi(0)) \le |t_0| = \frac{2\varphi(0)}{|b|}$, whence $|b| \le \frac{2\varphi(0)}{\widetilde{\omega}(\varphi(0))}$. ☐

Lemma 3. *Let* $\varphi(0) > 0$*. If* $|t| \le 20^{-1} \min\left(\widetilde{\omega}\left(\frac{\varphi(0)}{20}\right), \frac{\pi}{4}\right)$*, then* $\varphi(t) \ge \frac{\varphi(0)}{2}$.

Proof. Suppose that the maximum in the inequality in Lemma 2 is attained at the first quantity, then

$$\begin{aligned}\varphi(t) &\ge \varphi(0) - |b|\,|t| - \omega(|t|)\\ &\ge \varphi(0) - 20^{-1}\widetilde{\omega}\Big(\frac{\varphi(0)}{20}\Big)\frac{2\varphi(0)}{\widetilde{\omega}(\varphi(0))} - \omega\Big(20^{-1}\widetilde{\omega}\Big(\frac{\varphi(0)}{20}\Big)\Big)\\ &\ge \varphi(0) - 10^{-1}\varphi(0) - \omega\Big(\widetilde{\omega}\Big(\frac{\varphi(0)}{20}\Big)\Big) \ge \varphi(0) - \frac{3}{20}\varphi(0) \ge \varphi(0)/2.\end{aligned}$$

But if that maximum is attained at the second quantity, we have

$$\varphi(t) \ge \Big(\varphi(0) - |t|\frac{\varphi(0)}{\pi}\Big) - \omega(|t|).$$

Next, $\varphi(0) - |t|\frac{\varphi(0)}{\pi} \ge \frac{79}{80}\varphi(0)$, because $|t| \le \frac{\pi}{80}$ by assumption. Since also $|t| \le 20^{-1}\widetilde{\omega}\big(\frac{\varphi(0)}{20}\big) \le \widetilde{\omega}\big(\frac{\varphi(0)}{20}\big)$, we have $\omega(|t|) \le \frac{\varphi(0)}{20}$. So, again, the inequality $\varphi(t) \ge \frac{\varphi(0)}{2}$ is guaranteed. ☐

Corollary 1. *Under the above assumptions on* φ*, the number* $\varphi(0)$ *cannot be too large*: $\varphi(0) \le D = D(B_1, \omega)$*, where* $B_1 = \int_{-\pi}^{\pi} |\log\varphi|$*, see* (LOG).

Indeed, Lemma 3 shows that otherwise φ may be as large as we wish on an interval of fixed length, with is not compatible with (LOG).

There exists also a "universal" upper estimate for b, depending neither on $\varphi(0)$ nor on B_1.

Lemma 4. *We have* $|b| \le \omega(4\pi)/4\pi$.

Proof. Let I be the integral of φ over any interval of length 2π (all such integrals are equal). Integration of (6) over $[-2\pi, 0]$ and then over $[0, 2\pi]$ yields

$$|I - 2\pi\varphi(0) + 2\pi^2 b| \le 2\pi\omega(2\pi),$$
$$|I - 2\pi\varphi(0) - 2\pi^2 b| \le 2\pi\omega(2\pi),$$

whence $4\pi^2|b| \le 4\pi\omega(2\pi)$. ☐

[2]We mean control for constants both ways.

Corollary 2. *The function φ is uniformly bounded by a constant $D_1 = D_1(B_1, \omega)$ that only depends on B_1 and ω.*

Remark. If (6) is only true near zero, but it is known beforehand that φ is bounded, we can restore (6) for $|t| \le 4\pi$ by an unessential modification of ω.

Some calculations in §§ 2 and 3 become much simpler if $\varphi(0) \le 1$. This is definitely true if the quantity $D(B_1, \varphi)$ in Corollary 1 does not exceed 1, but in the opposite case we may simply change φ, b, and ω slightly, dividing (6) by $D(B_1, \varphi)$. This leads to a controlled change of B_1 (or B_p, see ($\mathrm{LOG_p}$)), so does not lead to any loss of generality. It should be noted that the normalization by division by $\varphi(0)$ (to obtain $\varphi(0) = 1$) may lead to an incontrollable change of the constants B_p and should be avoided.

For references, we write out the obvious inequality

$$|\varphi(t) - \varphi(0)| \le |b|\,|t| + \omega(|t|), \tag{7}$$

and also note that if $\omega(h) = o(h)$ as $h \to 0$, then $b = 0$ whenever $\varphi(0) = 0$ (because φ is nonnegative).

Also for references, we need some formulas for first and second differences. Let G be a function at least of class C^2 on $\mathbb{R}$, and let $x_0, x_1, x_2 \in \mathbb{R}$. We put $\delta_0 = x_0$, $\delta_1 = x_1 - x_0$, $\delta_2 = (x_2 - x_1) - (x_1 - x_0)$. Clearly,

$$G(x_1) - G(x_0) = \int_0^1 \frac{\partial}{\partial t} G\left(x_0 + t(x_1 - x_0)\right)\, dt.$$

Repeating the same observation, we obtain

$$\begin{aligned}
&[G(x_2) - G(x_1)] - [G(x_1) - G(x_0)] \\
&\quad = \int_0^1 \int_0^1 \frac{\partial}{\partial s}\frac{\partial}{\partial t} G(x_0 + t[x_1 - x_0] + s[x_1 - x_0 + t((x_2 - x_1) - (x_1 - x_0))])\, dt\, ds \\
&\quad = \int_0^1 \int_0^1 \frac{\partial}{\partial s}\frac{\partial}{\partial t} G(\delta_0 + t\delta_1 + s\delta_1 + ts\delta_2)\, dt\, ds.
\end{aligned}$$

All values of the argument of G in the last integral lie in the convex hull of x_0, x_1, and x_2, i.e., in the smallest interval containing these points. Suppose that $|G'| \le u$, $|G''| \le v$ on this interval. Calculating the second derivative in the integrand, we arrive at the estimate

$$|G(x_2) - 2G(x_1) + G(x_0)| \le v(|\delta_1| + |\delta_2|)^2 + u|\delta_2|. \tag{8}$$

It the nodes are equidistant, i.e., $\delta_2 = 0$, we simply obtain

$$|G(x_2) - 2G(x_1) + G(x_0)| \le v|\delta_1|^2. \tag{9}$$

§2. Smoothness below 1: estimates of mean oscillation

In this section, we assume that φ satisfies (6) with $b = 0$ for $|t| \le 4\pi$. Regularity conditions will be imposed on ω at a due time. For the moment, we only assume that a doubling condition, see (DB), is fulfilled. Throughout, we assume that (LOG) is true and $\varphi(0) \le 1$. The latter leads to no loss of generality, as we saw in §1.

2.1. The case where φ satisfies $\mathrm{LOG_p}$ with $p < \infty$. First, we study the smoothness at 0 for the outer function $F = \varphi e^{i\mathcal{H}(\log\varphi)}$ (an inner part will be adjoined later). We shall estimate the quantity

$$\nu_r(F; I) = \inf_a \left(\frac{1}{|I|} \int |F - a|^r \right)^{1/r}$$

(see (1)) with fixed $r > 1$, where I is an interval containing 0 in its interior and included in $[-\pi/2, \pi/2]$.

Lemma 5. *Suppose that* (LOG$_{\mathrm{p}}$) *is true with some* $p \in [1, +\infty)$. *Then*:

(a) $\nu_r(F, I) \le \omega(|I|) + 2\varphi(0)$;

(b) *if* $\varphi(0) > 0$ *and* $|I| \le 100^{-1} \min\left(\widetilde{\omega}\left(\frac{\varphi(0)}{20}\right), \frac{\pi}{4}\right)$, *and* l *is the greatest positive integer with* $2^l I \subset [-\pi, \pi]$, *then*

$$\nu_r(F; I) \le c\omega(|I|) + c \sum_{j \le l} \frac{\omega(2^{j+1}|I|)}{2^j} + \frac{c|I|\varphi(0)B_p}{\min\left(\widetilde{\omega}\left(\frac{\varphi(0)}{20}\right), \frac{\pi}{4}\right)^{\frac{p+1}{p}}}.$$

The constant c *depends only on* r *and the behavior of* ω.

As usual, aI denotes the a times dilation of I relative to its center.

It should be noted at once that, if $\frac{\omega(t)}{t^2}$ is almost monotone decreasing, the regularity condition (R2) says simply that the sum in the second term is dominated by $\omega(c|I|)$. Thus, we arrive at Theorem B in the Introduction for *outer* functions: at small distances from 0, smoothness does not drop. If $\varphi(0) = 0$, this can be read in (a), and if $\varphi(0) > 0$, this is implied by (b). We postpone the further analysis of consequences of Lemma 5 and prove it.

Proof of Lemma 5. In the formula for $\nu_r(F; I)$, we take the infimum only over the constants of the form $a = \varphi(0)e^{ic}$, where c is real. We have

$$\nu_r(F; I) \le \left(\frac{1}{|I|}\int_I |\varphi - \varphi(0)|^r\right)^{1/r} + \varphi(0)\left(\frac{1}{|I|}\int_I |e^{i\mathcal{H}(\log\varphi)} - e^{ic}|\right)^{1/r}.$$

The factor at $\varphi(0)$ in the second summand does not exceed 2, so (a) follows. In order to obtain (b) for $\varphi(0) > 0$, we need sharper estimates of the second summand. We choose c in the following way:

$$c = \frac{1}{2\pi}\int_{[-\pi,\pi]\setminus 2I} \cot\left(-\frac{s}{2}\right)(\log\varphi(s) - \log\varphi(0))\,ds.$$

We have

$$\begin{aligned}\varphi(0)\left(\frac{1}{|I|}\int_I |e^{i\mathcal{H}(\log\varphi)} - e^{ic}|^r\right)^{1/r} &\le \varphi(0)\left(\frac{1}{|I|}\int_I |\mathcal{H}(\log\varphi) - c|^r\right)^{1/r} \\ &= \varphi(0)\left(\frac{1}{|I|}\int_I |\mathcal{H}(\log\varphi - \log\varphi(0)) - c|^r\right)^{1/r},\end{aligned}$$

because $\mathcal{H}$ takes the constants to zero. We continue the estimate:

$$\begin{aligned}&\dots \le \varphi(0)\Big(\frac{1}{|I|}\int_I |\mathcal{H}(\chi_{2I}\cdot(\log\varphi - \log\varphi(0)))|^r\Big)^{1/r} \\ &+ \varphi(0)\Big(\frac{1}{|I|}\int_I \Big|\frac{1}{2\pi}\int_{[-\pi,\pi]\setminus 2I}\Big(\cot\frac{t-s}{2} - \cot\Big(-\frac{s}{2}\Big)\Big)(\log\varphi(s) - \log\varphi(0))\,ds\Big|^r dt\Big)^{1/r} \\ &= \varphi(0)A + \varphi(0)B.\end{aligned} \tag{10}$$

By Lemma 3 and the restriction on the length of I in (b), we have $\varphi(t) \ge \varphi(0)/2$ on $2I$. Since $\mathcal{H}$ is bounded on L^r, we estimate the quantity A in (10) as follows:

$$A \le \left(\frac{1}{|I|}\int_{-\pi}^{\pi} |\mathcal{H}(\chi_{2I}\cdot(\log\varphi - \log\varphi(0))|^r\right)^{1/r} \le C_r\left(\frac{1}{|I|}\int_{2I} |\log\varphi - \log\varphi(0)|^r\right)^{1/r}.$$

The Lagrange formula for the logarithm yields $|\log\varphi(t)-\log(\varphi(0)|\le 2|\varphi(t)-\varphi(0)|\varphi(0)^{-1}$ for $t\in 2I$, so, finally,

$$\varphi(0)A\le C\omega(2|I|)\le C'\omega(|I|). \tag{11}$$

Now, we estimate the quantity B in (10). Let l be the greatest positive integer with $2^lI\subset[-\pi,\pi]$. We use the estimate

$$\Big|\cot\frac{t-s}{2}-\cot\Big(-\frac{s}{2}\Big)\Big|\le\frac{C|t|}{s^2},\quad s\notin 2I,\quad t\in I,$$

where C is independent of I:

$$B\le C\Big(\frac{1}{|I|}\int_I\Big(\sum_{j=1}^{l}\int_{2^{j+1}I\setminus 2^jI}\frac{|t|}{s^2}|\log\varphi(s)-\log\varphi(0)|\,ds\Big)^r dt\Big)^{1/r}.$$

(Strictly speaking, in the summand with $j=l$, the integral over $2^{l+1}I\setminus 2^lI\cap[-\pi,\pi]$ arises, but after putting the modulus in the integrand it can be extended over $2^{l+1}I\setminus 2^lI$.) It follows that

$$B\le C|I|\sum_{j=1}^{l}|2^jI|^{-2}\int_{2^{j+1}I\setminus 2^jI}|\log\varphi(s)-\log\varphi(0)|\,ds. \tag{12}$$

Let $E_j=\{s\in 2^{j+1}I\setminus 2^jI:\varphi(s)\ge\varphi(0)/2\}$, and let $F_j=(2^{j+1}I\setminus 2^jI)\setminus E_j$. We know that $F_j=\varnothing$ for $j\le k$, where k is the greatest positive integer with $2^j|I|\le 50^{-1}\min\big(\widetilde{\omega}\big(\frac{\varphi(0)}{20}\big),\frac{\pi}{4}\big)$ (see Lemma 3). Next, on E_j we have

$$|\log\varphi(s)-\log\varphi(0)|\le\frac{2\omega(|s|)}{\varphi(0)},$$

and, since $\varphi(0)\le 1$, on F_j we have $0\le\log\frac{1}{\varphi(0)}\le\log\frac{2}{\varphi(0)}\le\log\frac{1}{\varphi(s)}$, so that

$$|\log\varphi(s)-\log\varphi(0)|=\log\frac{1}{\varphi(s)}-\log\frac{1}{\varphi(0)}\le\log\frac{1}{\varphi(s)}.$$

Thus,

$$B\le C\sum_{j\le l}\frac{\omega(2^{j+1}|I|)}{2^j\varphi(0)}+C|I|\sum_{j=k+1}^{l}|2^jI|^{\frac{1}{q}-2}\Big(\int_{-\pi}^{\pi}|\log\varphi|^p\Big)^{1/p}, \tag{13}$$

where q is the exponent conjugate to p in condition $(\mathrm{LOG_p})$ (surely, if $p=1$, we agree that $\frac{1}{q}=0$.) In the second sum we have

$$|2^{k+1}I|\ge 50^{-1}\min\Big(\widetilde{\omega}\Big(\frac{\varphi(0)}{20}\Big),\frac{\pi}{4}\Big),$$

so, finally, we obtain

$$\varphi(0)B\le C\sum_{j\le l}\frac{\omega(2^{j+1}|I|)}{2^j}+\frac{CB_p|I|\varphi(0)}{\min\big(\widetilde{\omega}\big(\frac{\varphi(0)}{20}\big),\frac{\pi}{4}\big)^{\frac{p+1}{p}}}.$$

Combined with (11), this proves statement (b) of the lemma. □

2.2. The case where φ satisfies (LOG_∞)**.** Let us find out what can be obtained in this case in place of (b) in Lemma 5. Up to formula (12) (inclusive), we proceed as before. The summands with $j \le k$ on the right in (12) are also estimated as previously. For $k < j \le l$ we argue differently, i.e., we do not split the integral into two (over E_j and F_j). To begin with, in each of these summands we replace $\log\varphi(0)$ by the average $(\log\varphi)_I = \frac{1}{|I|}\int_I \log\varphi$. This leads to an "error" of at most

$$\frac{1}{|I|}\int_I |\log\varphi(s) - \log\varphi(0)|\,ds \le \frac{c\omega(|I|)}{\varphi(0)}$$

(we recall that the interval I is "short", we have $\varphi(s) \ge \frac{\varphi(0)}{2}$) on it). Consequently,

$$\begin{aligned} S &\stackrel{\text{def}}{=} |I| \sum_{j=k+1}^{l} |2^j I|^{-2} \int_{2^{j+1}I\setminus 2^j I} |\log\varphi(s) - \log\varphi(0)|\,ds \\ &\le C \sum_{j=k+1}^{l} 2^{-j} \frac{1}{|2^{j+1}I|}\int_{2^{j+1}I} |\log\varphi(s) - (\log\varphi)_I| + c\sum_{j=k+1}^{l} 2^{-j}\frac{\omega(|I|)}{\varphi(0)}. \end{aligned}$$

In each integral, we want to replace $(\log\varphi)_I$ by $(\log\varphi)_{2^{j+1}I}$. For this, we observe that for every function u and every interval J we have

$$|u_J - u_{2J}| \le \frac{1}{|J|}\int_J |u - u_{2J}| \le 2\frac{1}{|2J|}\int_{2J} |u - u_{2J}|,$$

so that

$$|(\log\varphi)_I - (\log\varphi)_{2^{j+1}I}| \le \sum_{s=0}^{j} |(\log\varphi)_{2^s I} - (\log\varphi)_{2^{s+1}I}| \le 2jB_\infty.$$

We continue estimating the quantity S:

$$S \le C\Big(\sum_{j=k+1}^{l} (j+1)2^{-j}B_\infty + 2^{-k}\frac{\omega(|I|)}{\varphi(0)}\Big) \le C'\frac{k}{2^k}B_\infty + \frac{\omega(|I|)}{2^k\varphi(0)}.$$

Since $k \asymp \log_2 |I|^{-1} 50^{-1}\min\big(\widetilde{\omega}\big(\frac{\varphi(0)}{20}\big), \frac{\pi}{4}\big)$, we arrive at the following statement.

Lemma 6. *If φ satisfies* LOG_∞*, then inequality* (b) *in Lemma* 5 *can be replaced by the estimate*

$$\begin{aligned} \nu_r(F;I) \le C\Bigg[\omega(|I|) + \sum_{j\le k}\frac{\omega(2^{j+1}|I|)}{2^j} + \frac{|I|\omega(|I|)}{\min\big(\widetilde{\omega}\big(\frac{\varphi(0)}{20}\big), \frac{\pi}{4}\big)} \\ + \log_2\left(\frac{\min\big(\widetilde{\omega}\big(\frac{\varphi(0)}{20}\big), \frac{\pi}{4}\big)}{50|I|}\right)\cdot\frac{|I|\varphi(0)}{\min\big(\widetilde{\omega}\big(\frac{\varphi(0)}{20}\big), \frac{\pi}{4}\big)}\Bigg]. \end{aligned}$$

2.3. Proof of Theorem C and of its extension to the case where φ satisfies (LOG_∞)**.** To prove Theorem C from the Introduction, it suffices to trace the interrelation of statements (a) and (b) in Lemma 5. We assume that if the function $t \mapsto \varphi(t)t^{-\frac{p+1}{p}}$ is almost monotone decreasing, then *a fortiori*, $\frac{\varphi(t)}{t^2}$ is almost monotone decreasing, and then the sum in the inequality in (b) is dominated by the quantity $\varkappa(|I|)$ from Theorem C. By assumption, the interval I in statement (b) satisfies $20\omega(100|I|) \le \varphi(0)$. However, we shall apply (b) or (a) depending on whether or not I obeys a stronger condition, namely,

$$20\omega\big(100|I|^{\frac{p}{p+1}}\big) \le \varphi(0) \qquad (14)$$

(this condition is stronger because also $|I| < 1$ in (b) automatically). If (14) is violated, statement (a) in Lemma 5 yields

$$\nu_r(F, I) \le \omega(|I|) + c\omega\big(|I|^{\frac{p}{p+1}}\big).$$

But if (14) is true, we use inequality (b) in Lemma 5, in which we need to estimate the last term on the right, i.e., the quantity

$$|I| \frac{\varphi(0)}{\min\left(\widetilde{\omega}\big(\frac{\varphi(0)}{20}\big), \frac{\pi}{4}\right)^{\frac{p+1}{p}}}. \tag{15}$$

This quantity does not exceed $c|I|$ if the minimum in the denominator is equal to $\pi/4$. But if it is equal to $\widetilde{\omega}\big(\frac{\varphi(0)}{20}\big)$, we apply Lemma 1 and see that the quantity (15) does not exceed the quantity

$$C|I| \frac{\omega\big(100|I|^{\frac{p}{p+1}}\big)}{\widetilde{\omega}\big(20\omega(100|I|^{\frac{p}{p+1}})\big)^{\frac{p+1}{p}}} \le C'\omega\big(|I|^{\frac{p}{p+1}}\big).$$

Collecting the estimates, we obtain Theorem C. □

In order to extend Theorem C to the case of $p = \infty$, we must use the estimate in Lemma 6, and we are not enthusiastic of doing that for an arbitrary continuity modulus ω, if for no other reason than that this would require certain "regularity conditions with a logarithm". So, we restrict ourselves to the case of a power-type function $\omega(t) = ct^\alpha$, $0 < \alpha < 1$. It can easily be calculated that, applying (a) in Lemma 5 if $|I| \le d\varphi(0)^{1/\alpha}$ and Lemma 6 otherwise (d is a sufficiently small positive number), we obtain $\nu_r(F; I) \le C|I|^\alpha$.

2.4. Adjoining an inner function. Now, we consider an arbitrary function $\Phi = F\theta B$, which is the boundary function of an analytic function with outer part F, F being constructed by the same function φ as before. Now one cannot expect a relationship between F, B, and θ, like that described in the beginning of the Introduction. We do not see a better issue than to impose deliberately the same conditions on θ and B as if Φ were continuous everywhere up to the boundary. In that case, we must ensure an analog of Lemma 5, and it will soon become transparent that there are good reasons to restrict ourselves to the case of $p = 1$.

2.4.1. Adjoining a singular inner factor. Let θ be a singular inner function corresponding to a finite positive singular measure μ. It is well known that $\theta = e^{-i\mathcal{H}\mu}$ a.e. on the boundary of the disk. In accordance with the said above, in the case where $\varphi(0) > 0$ we assume that $\mu\{t : |t| \le 20^{-1}\min\left(\widetilde{\omega}\big(\frac{\varphi(0)}{20}\big), \frac{\pi}{4}\right\} = 0$. We recall that φ is strictly positive on this "arc", so that the conditions would definitely be fulfilled if φ were continuous up to the boundary.

Recall that in Lemma 5 we estimated the quantity $\left(\frac{1}{|I|}\int_I |F - a|^r\right)^{1/r}$ in two ways. Here $a = \varphi(0)e^{ic}$ and c is certain fixed real constant. We introduce another real constant w and write

$$\left(\frac{1}{|I|}\int_I |F\theta - ae^{iw}|^r\right)^{1/r} \le \left(\frac{1}{|I|}\int_I |\Phi - a|^r\right)^{1/r} + \varphi(0)\left(\frac{1}{|I|}\int_I |\theta - e^{iw}|^r\right)^{1/r}, \tag{16}$$

because $|a| = \varphi(0)$. This yields immediately an analog of (a) in Lemma 5:

$$\nu_r(F\theta; I) \le \omega(|I|) + 4\varphi(0).$$

In an instant, we shall see that, if $p = 1$, the second term on the right in (16) does not contribute anything new in the estimate in (b) in Lemma 5. Indeed, put

$$w = -\frac{1}{2\pi}\int_{|s| \ge 20^{-1}\min(\widetilde{\omega}(\frac{\varphi(0)}{20}), \pi/4)} \cot\Big(-\frac{s}{2}\Big)\, d\mu(s).$$

Clearly, for any interval I as in statement (b) of Lemma 5, we have

$$
\begin{aligned}
&\varphi(0)\Big(\frac{1}{|I|}\int_I |\theta - e^{iw}|^r\Big)^{1/r}\\
&\quad\le \varphi(0)\Big(\frac{1}{|I|}\int_I \Big|\frac{1}{2\pi}\int_{|s|\ge 20^{-1}\min(\tilde\omega(\frac{\varphi(0)}{20}),\frac{\pi}{4})}\Big(\cot\frac{t-s}{2}-\cot\Big(-\frac{s}{2}\Big)\Big)\,d\mu(s)\Big|^r dt\Big)^{1/r}\\
&\quad\le C\varphi(0)\Big(\frac{1}{|I|}\int_I\Big|\int_{|s|\ge 20^{-1}\min(\tilde\omega(\frac{\varphi(0)}{20}),\frac{\pi}{4})}\frac{|t|}{s^2}\,d\mu(s)\Big|^r dr \le \frac{c\varphi(0)|I|\,\|\mu\|}{\min\big(\omega\big(\frac{\varphi(0)}{20}\big),\frac{\pi}{4}\big)^2}.
\end{aligned}
$$

So, an analog of Lemma 5 with $p = 1$ is true for $\theta\Phi$, and all consequences of that lemma are retained.

2.4.2. Adjoining a Blaschke product. Now, let F and θ be as above, and let $B = z^m\prod_k \frac{z-z_k}{1-z\bar z_k}\frac{\bar z_k}{|z_k|}$ be a Blaschke product, where the z_k are points of the open unit disk satisfying $\beta = \sum_k(1-|z_k|) < +\infty$. In accordance with the general ideology presented above, we "need" to impose on B the condition (NT) near the point $0 \in \mathbb{R}$ (i.e., the point 1 on the unit circle): if $\varphi(0) > 0$, then there is a constant ρ such that all zeros $z_k = re^{it_k}$ lie outside the "cell"

$$
\Big\{re^{it} : |t| \le 40^{-1}\min\Big(\tilde\omega\Big(\frac{\varphi(0)}{20}\Big),\frac{\pi}{4}\Big),\ \ 1-r \le \rho\min\Big(\tilde\omega\Big(\frac{\varphi(0)}{20}\Big),\frac{\pi}{4}\Big)\Big\}.
$$

It follows that, if $|I|$ contains 0 and, as before, $|I| \le 100^{-1}\min\big(\tilde\omega\big(\frac{\varphi(0)}{20}\big),\frac{\pi}{4}\big)$, then for every $s \in I$ and every k we have $|z_k - e^{is}| \ge C(\rho)\min\big(\tilde\omega\big(\frac{\varphi(0)}{20}\big),\frac{\pi}{4}\big)$. Next, for such s we have

$$
\Big|\frac{d}{ds}B(e^{is})\Big| \le m + \sum_k \frac{|1-z_k|^2}{|z_k - e^{is}|^2} \le m + \frac{2\beta}{c(\rho)\min\big(\tilde\omega\big(\frac{\varphi(0)}{20}\big),\frac{\pi}{4}\big)^2}.
$$

In an estimate like in (b) with $p = 1$ in Lemma 5 for $B\theta F$, this will lead to yet another summand similar to one already present (specifically, to the last) and to an additional summand of order in $|I|$, and so on.

2.5. Theorem A and B in the Introduction. At this point, no additional comment on Theorem B is needed any longer. As to Theorem A. we have promised to tell something in it under the mere condition that the function $\frac{\omega_x(h)}{h}$ is almost monotone decreasing (and without explicit reference to anything like (R2)). For this, we again turn to Lemma 5 and its "explanation" in Subsection 2.3. If $\frac{\varphi(h)}{h}$ is almost monotone decreasing, then $\frac{\varphi(h)}{h^2}$ also has this property, but this is precisely what is required in the analysis of competing terms in (a) and (b) of Lemma 5 for $p = 1$. It remains to cope with the sum $\sum_{j\le l}\frac{\omega(2^{j+1}|I|)}{2^j}$. Let m be the greatest among the indices j with $2^{j+1}|I| \le \sqrt{|I|}$, then $2^j \asymp 1/\sqrt{|I|}$, and the sum in question is dominated by

$$
\Big(\sum_{j\le m}\frac{1}{2^j}\Big)\omega(\sqrt{I}) + \Big(\sum_{j>m}\frac{1}{2^j}\Big)\omega(4\pi) \le c\big(\omega\big(\sqrt{|I|}\big) + \sqrt{|I|}\big).
$$

Since the function $\frac{\omega(h)}{h}$ is almost monotone decreasing, we have $\frac{\omega(\sqrt{|I|})}{\sqrt{|I|}} \ge C\frac{\omega(2\pi)}{2\pi}$, whence $\sqrt{|I|} \le C\omega(\sqrt{|I|})$.

Thus, Theorem A is also proved. Surely, various statements in the spirit of Theorem A are possible that involve other regularity conditions.

§3. Smoothness up to 2: averaged second differences

In this section, we impose condition (6) on φ, where b is not necessarily zero. Intuitively, this corresponds to smoothness up to 2. Next, for simplicity we restrict ourselves to power-type functions ω in (6). As will be apparent, the result of calculations with an arbitrary function ω would have been exceedingly bulky.

So, we assume that $\omega(h) = dh^\alpha$ with $\alpha > 1$. The methods allow us to tell something also in the degenerate case, when $\alpha > 2$ (some upper restrictions on α will emerge, however). In this section, we only deal with outer functions $F = \varphi e^{i\mathcal{H}(\log\varphi)}$ and impose on φ some condition ($\mathrm{LOG_p}$) with $1 \le p < \infty$. We leave the case of $p = \infty$ aside. All these "trifles" are omitted because, among other things, we wanted to concentrate sooner on the use of averaged second differences in this range of problems. By the way, for $\alpha \le 1$ we also could employ second differences roughly in the same spirit, but we also omit this because, on the one hand, the nature of calculations given below changes sometimes when we pass through the point $\alpha = 1$, and, on the other hand, the case of $\alpha \le 1$ was treated in §2.

Note that now $\widetilde{\omega}(s) = d^{-1/\alpha}s^{1/\alpha}$. Inequality (7) takes the form $|\varphi(t) - \varphi(0)| \le |b|\,|t| + d|t|^\alpha$. We shall often combine it with the estimate for b in Lemma 2 and much less often with the estimate in Lemma 4. In particular, it follows that if $\varphi(0) \ne 0$, then the value $\mathcal{H}(\log\varphi)(0)$ is well defined (the corresponding principal value integral converges). If $\varphi(0) = 0$, we still assign some value to $\mathcal{H}(\log\varphi)$ at the point zero, nothing will depend on a particular choice of it. For brevity, we put $\psi = \mathcal{H}\log\varphi$. We want to estimate the quantity

$$\Big(\frac{1}{2h}\int_{-h}^{h} |\Delta^2(\varphi e^{i\psi})(0,t)|^r\,dt\Big)^{1/r}$$

with fixed $r > 1$. We have

$$\begin{aligned}\Delta^2(\varphi e^{i\psi})(0,t) &= \Delta^2\varphi(0,t)e^{i\psi(2t)} + 2\Delta^1\varphi(0,t)\Delta^1(e^{i\psi})(t,t) + \varphi(0)\Delta^2(e^{i\psi})(0,t)\\ &= S_1(t) + S_2(t) + S_3(t).\end{aligned}$$

We shall estimate the averages of these three summands separately.

Lemma 7. $|S_1(t)| \le C_{d,\alpha}|t|^\alpha$.

Proof. Let $u(s) = \varphi(0) + bs$, then

$$|\Delta^2\varphi(0,t)| = |\Delta^2(\varphi - u)(0,t)| = |(\varphi-u)(2t) - 2(\varphi-u)(t)| \le d(2^\alpha + 2)|t|^\alpha. \qquad \square$$

Lemma 8. (a) $|S_2(t)| \le 2(|b|\,|t| + d|t|^\alpha)$.

(b) *If* $\varphi(0) > 0$ *and* $|h| \le 100^{-1}\min((20d)^{-1/\alpha}\varphi(0)^{1/\alpha}, \pi/4)$, *then*

$$\Big(\frac{1}{2h}\int_{-h}^{h}|S_2(t)|^r dt\Big)^{1/r} \le C\Big(h^\alpha + \frac{h^{1+\alpha}}{\varphi(0)^{1/\alpha}} + \frac{h^2|\log\frac{1}{h}|}{\varphi(0)^{2/\alpha-1}} + \frac{h^2}{\varphi(0)^{\frac{1}{\alpha}\left(\frac{p+1}{p}+1\right)-1}} + h^2\frac{|\log\varphi(0)|}{\varphi(0)^{2/\alpha-1}}\Big).$$

Proof. Inequality (a) follows because $|\varphi(t) - \varphi(0)| \le |b|\,|t| + d|t|^\alpha$ and $|\Delta_1(e^{i\psi})(0,t)| \le 2$. In the case of (b), using the same two estimates, we argue slightly subtler:

$$\Big(\frac{1}{2h}\int_{-h}^{h}|S_2(t)|^r\,dt\Big)^{1/t} \le 2dh^\alpha + 2|b|h\Big(\frac{1}{2h}\int_{-h}^{h}|e^{i\psi(2t)} - e^{i\psi(t)}|^r dt\Big)^{1/r}. \tag{17}$$

The integral on the right in (17) does not exceed the sum

$$\Big(\frac{1}{2h}\int_{-h}^{h}|e^{i\psi(2t)} - e^{ic}|^r\,dt\Big)^{1/r} + \Big(\frac{1}{2h}\int_{-h}^{h}|e^{i\psi(t)} - e^{ic}|^r\,dt\Big)^{1/r}, \tag{18}$$

where c is chosen roughly as in §2:

$$c = \frac{1}{2\pi} \int_{[-\pi,\pi]\setminus[-4h,4h]} \cot\Big(-\frac{s}{2}\Big)(\log\varphi(s) - \log\varphi(0))\, ds.$$

Next, we repeat some calculations of §2 with slight modifications. Let positive integers k and l play the same role as in §2, then $l \asymp \log_2 \frac{1}{h}$, $2^k h \asymp 50^{-1}\min((20d)^{-1/\alpha}\varphi(0)^{1/\alpha}, \frac{\pi}{4})$. As in §2 (see (11) and (12)), we write

$$\begin{aligned}(19)\qquad \Big(\frac{1}{2h}\int_{-h}^{h} |e^{i\psi(t)} - e^{ic}|^r\, dt\Big)^{1/r} &\le C\Big(\frac{1}{2h}\int_{-4h}^{4h} |\log\varphi(s) - \log\varphi(0)|^r ds\Big)^{1/r} \\ &\quad + Ch\sum_{j=1}^{l} (2^j h)^{-2} \int_{2^{j+1}h \le |s| \le 2^{j+2}h} |\log\varphi(s) - \log\varphi(0)|\, ds\end{aligned}$$

(the second summand in (18) is treated similarly, we do not write out the calculation for it). The first distinction from §2 is that now we use the inequality $|\log\varphi(s) - \log\varphi(0)| \le c\frac{|b|\,|s| + d|s|^\alpha}{\varphi(0)}$ for $\varphi(s) \ge \varphi(0)/2$. The second distinction is that the summands with $k+1 \le j \le l$ will be treated in a less refined manner than in §2, but now this will yield a better result. Specifically, we simply write

$$\int_{2^{j+1}h \le |s| \le 2^{j+2}h} |\log\varphi(s) - \log\varphi(0)|\, ds \le C(2^j h)^{1-\frac{1}{p}} \|\log\varphi\|_{L^p} + C2^j h|\log\varphi(0)|.$$

Then (19) can be continued in the following way:

$$\begin{aligned}(20)\qquad \dots \le C\Big(&\frac{|b|h + h^\alpha}{\varphi(0)} + \frac{1}{\varphi(0)}\sum_{j\le k}(2^{(\alpha-1)j}h^\alpha + |b|h) \\ &+ h\sum_{j=k+1}^{l} |2^j h|^{-\frac{p+1}{p}} B_p + h\sum_{j=k+1}^{l} \frac{|\log\varphi(0)|}{|2^j h|}\Big).\end{aligned}$$

In this expression, we want to estimate $|b|$ by Lemma 2, namely, to write $\frac{1}{\varphi(0)}|b| \le \max\big(\frac{2d^{1/\alpha}}{\varphi(0)^{1/\alpha}}, \frac{1}{\pi}\big)$, and to use, in the two last sums, the fact that

$$|2^j h| \ge C\min((20d)^{1/\alpha}\varphi(0)^{1/\alpha}, \pi/4).$$

To expedite matters considerably, we allow ourselves a slight inaccuracy, which will also be repeated in the sequel. After these substitutions, all terms in (20) will involve some power of h in the numerator (which is good) and a "hampering" power of the "small" number $\varphi(0)$ in the denominator (logarithmic factors do not influence the entire picture). If either the maximum or the minimum written above is attained at the second quantity, then a certain universal lower estimate for $\varphi(0)$ is available, so that these "small" denominators do not present an obstruction. This means that we can continue the calculations under the assumption that both the maximum and the minimum are attained at the first quantity (otherwise we do insure a better estimate). In accordance with that, we continue (20), taking into account the relation

$$\sum_{j\le k} 2^{(\alpha-1)j}h^\alpha \le ch^\alpha(2^k)^{\alpha-1} \le c'h^\alpha\Big(\frac{\varphi(0)^{1/\alpha}}{h}\Big)^{\alpha-1} = c'\frac{h}{\varphi(0)^{1/\alpha-1}}$$

and the fact that the sum of $|b|h$ k times gives a certain term with a logarithm:

$$\begin{aligned}\dots &\le C'\Big(\frac{h}{\varphi(0)^{1/\alpha}}+\frac{h^\alpha}{\varphi(0)}+\frac{h}{\varphi(0)^{1/\alpha}}\Big|\log\frac{\varphi(0)^{1/\alpha}}{h}\Big|+\frac{h}{\varphi(0)^{\frac{1}{\alpha}\frac{p+1}{p}}}+h\frac{|\log\varphi(0)|}{\varphi(0)^{1/\alpha}}\Big)\\ &\le C''\Big(\frac{h^\alpha}{\varphi(0)}+\frac{h\log\frac{1}{h}}{\varphi(0)^{1/\alpha}}+\frac{h}{\varphi(0)^{\frac{1}{\alpha}\frac{p+1}{p}}}+\frac{h|\log\varphi(0)|}{\varphi(0)^{1/\alpha}}\Big).\end{aligned}\tag{21}$$

Multiplying (21) by $2|b|h$ and substituting the result in (17) for the second term on the right, after some simplifications we arrive at

$$\begin{aligned}&\Big(\frac{1}{2h}\int_{-h}^{h}|S_2(t)|^r\,dt\Big)^{1/r}\\ &\quad\le C\Big(h^\alpha+\frac{h^{1+\alpha}}{\varphi(0)^{1/\alpha}}+\frac{h^2|\log\frac{1}{h}|}{\varphi(0)^{2/\alpha-1}}+\frac{h^2}{\varphi(0)^{\frac{1}{\alpha}(\frac{p+1}{p}+1)-1}}+\frac{h^2|\log\varphi(0)|}{\varphi(0)^{2/\alpha-1}}\Big).\qquad\square\end{aligned}$$

Lemma 9. (a) $|S_3(t)|\le 4\varphi(0)$.

(b) *If* $\varphi(0)>0$ *and* $h\le 100^{-1}\min((20d)^{-1/\alpha}\varphi(0)^{1/\alpha},\pi/4)$, *then*

$$\begin{aligned}\Big(\frac{1}{2h}\int_{-h}^{h}|S_3(t)|^r\,dt\Big)^{1/r} &\le C\Big(h^2\log\frac{c}{h}+\frac{h^2}{\varphi(0)^{2/\alpha-1}}\log\frac{c}{h}+\frac{h^{2\alpha}}{\varphi(0)}+\frac{h^2}{\varphi(0)^{\frac{2}{\alpha}\frac{p+1}{p}-1}}\\ &\quad+h^\alpha+h^2|\log\varphi(0)|+\frac{h^2|\log\varphi(0)|}{\varphi(0)^{2/\alpha-1}}+\frac{h^2}{\varphi(0)^{(2+\frac{1}{p})\frac{1}{\alpha}-1}}\Big).\end{aligned}$$

Proof. Statement (a) is obvious, we only need to verify (b). We have

$$\Big(\frac{1}{2h}\int_{-h}^{h}|S_3(t)|^r\,dt\Big)^{1/r}=\varphi(0)\Big(\frac{1}{2h}\int_{-h}^{h}|\Delta^2(e^{i\psi})(0,t)|^r\,dt\Big)^{1/r}.$$

Using formula (8) with $G(u)=e^{iu}$, we see that

$$|\Delta^2(e^{i\mathcal{H}(\log\varphi)})(0,t)|\le(|\Delta^2(\mathcal{H}(\log\varphi))(0,t)|+|\Delta^2(\mathcal{H}(\log\varphi))(0,t)|)^2+|\Delta^2(\mathcal{H}(\log\varphi))(0,t)|.$$

Inside the parentheses (i.e., in the term that is squared), we estimate the second difference in terms of the first in accordance with the general formula $f(x+2t)-2f(x+t)+f(x)=(f(x+2t)-f(x))-2(f(x+t)-f(x))$. This yields

$$\begin{aligned}&|\Delta^2(e^{i\mathcal{H}(\log\varphi)})(0,t)|\\ &\quad\le C(|\Delta^1(\mathcal{H}(\log\varphi))(0,t)|^2+|\Delta^1(\mathcal{H}(\log\varphi))(0,2t)|^2+|\Delta^2(\mathcal{H}(\log\varphi))(0,t)|).\end{aligned}\tag{22}$$

The contributions of the first and the second summand on the right to the averaged second difference are estimated similarly, we only present the calculations for the first summand. They are nearly the same as in the proof of (b) in Lemma 8 with the difference that the results are squared and with an additional small alteration.

So, as in §2 and (implicitly) in Lemma 8, we replace the function $\log\varphi$ under the sign of $\mathcal{H}$ by $u=\log\varphi-\log\varphi(0)$ and write $u=u\chi_{[-2h,2h]}+v$; then

$$\begin{aligned}A&\overset{\text{def}}{=}\Big(\frac{1}{2h}\int_{|t|\le h}|\mathcal{H}(\log\varphi)(t)-\mathcal{H}(\log\varphi)(0)|^{2r}\,dt\Big)^{1/r}\\ &\le C\Big[\Big(\frac{1}{2h}\int_{|t|\le h}|\mathcal{H}(\chi_{-[2h,2h]}\cdot u)(t)|^{2r}\,dt\Big)^{1/r}+|\mathcal{H}(\chi_{[-2h,2h]}\cdot u)(0)|^2\\ &+\frac{1}{2h}\int_{|t|\le h}\Big(\Big|\int_{[-\pi,\pi]\setminus[-2h,2h]}\Big(\cot\Big(\frac{t-s}{2}\Big)-\cot\Big(-\frac{s}{2}\Big)\Big)(\log\varphi(s)-\log\varphi(0))\,ds\Big|^{2r}dt\Big)^{1/r}\Big].\end{aligned}$$

Compared to calculations done before, a new term (the second) arose here, which is estimated as follows:

$$\begin{aligned}\left|\mathcal{H}(u\chi_{[-2h,2h]})(0)\right|^2 &= \frac{1}{4\pi^2}\left|\int_{-2h}^{2h}(\log\varphi(s)-\log\varphi(0))\cot\Big(-\frac{s}{2}\Big)\,ds\right|^2\\ &\le C\Big(\frac{1}{\varphi(0)}\int_{-2h}^{2h}\frac{|b|\,|s|+|s|^\alpha}{|s|}\,ds\Big)^2 \le C'\Big(\frac{|b|h+h^\alpha}{\varphi(0)}\Big)^2\\ &\le C''\Big(\frac{h^2}{\varphi(0)^{2/\alpha}}+\frac{h^{2\alpha}}{\varphi(0)^2}\Big).\end{aligned}$$

The same quantity dominates the first terms on the right (see the corresponding place in the proof of Lemma 8). Thus, we see that the sum of the squares of the terms in (21) dominates A, and, when multiplied by $\varphi(0)$, this sum estimates the contribution of the first two terms in (22) to the average $\big(\frac{1}{2h}\int_{-h}^{h}|S_3(t)|^r\,dt\big)^{1/r}$. Here is an explicit expression:

$$C\bigg[\frac{h^{2\alpha}}{\varphi(0)}+\frac{h^2\big(\log\frac{1}{h}\big)^2}{\varphi(0)^{\frac{2}{\alpha}-1}}+\frac{h^2}{\varphi(0)^{\frac{2}{\alpha}\frac{p+1}{p}-1}}+\frac{h^2}{\varphi(0)^2\alpha\frac{p+1}{p}-1}+\frac{h^2|\log\varphi(0)|^2}{\varphi(0)^{2/\alpha-1}}\bigg].$$

It remains to estimate the contribution to the average in question of the third summand on the right in (22), i.e., the quantity $\varphi(0)D$, where

$$D=\Big(\frac{1}{2h}\int_{-h}^{h}|\Delta^2(\mathcal{H}(\log\varphi)(0,t)|^r\,dt\Big)^{1/r}.$$

We would like to subtract from $\log\varphi$ under the sigh of $\mathcal{H}$ an appropriate linear polynomial, which, however, should be modified to a 2π-periodic function. So, we write

$$D\le\Big(\frac{1}{2h}\int_{|t|\le h}|\Delta^2[\mathcal{H}(\log\varphi-u)](0,t)|^r dt\Big)^{1/r}+\Big(\frac{1}{2h}\int_{-h}^{h}|\Delta^2[\mathcal{H}u(0,t)]|^r dt\Big)^{1/r}\overset{\mathrm{def}}{=}W_1+W_2,$$

where $u(s)=\xi(s)\tau(s)$, ξ is a cut-off function of class C^∞ that is equal to 1 for $|s|\le\pi/4$ and to 0 for $|s|\in[\pi/2,\pi]$, and τ is the Taylor polynomial of order 1 for the function $s\mapsto\log(\varphi(0)+bs)$ (the function $\varphi(0)+bs$ itself will be denoted by $\varkappa(s)$) for brevity). Namely, $\tau(s)=\log\varphi(0)+\frac{bs}{\varphi(0)}$. The second derivative of $\log\varkappa(s)$ is $-\frac{b^2}{\varkappa(s)^2}$, and from Lemma 2 it follows that $\varkappa(s)\ge\varphi(0)/2$ for

$$|s|\le 40^{-1}\min\Big((20d)^{-1/\alpha}\varphi(0)^{1/\alpha},\frac{\pi}{4}\Big)\tag{23}$$

(this is done roughly as in Lemma 3, but with simplifications). This allows us to estimate the second derivative of $\log\varkappa(s)$ from above and to conclude that, for s satisfying (23), we have

$$|\log\varkappa(s)-\tau(s)|\le 4\frac{b^2}{\varphi(0)}|s|^2\le C\frac{|s|^2}{\varphi(0)^{2/\alpha}}\tag{24}$$

(we have used Lemma 2 and the observation that the case where the maximum in the inequality in that lemma is attained at $\frac{\varphi(0)}{\pi}$ can be disregarded). We assume here that u is defined originally on $[-\pi,\pi]$ and then extended to become 2π-periodic.

Now, observe that $\mathcal{H}$ commutes with the second difference operator,

$$\Delta^2\mathcal{H}(u)(0,t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cot\Big(-\frac{s}{2}\Big)\Delta^2u(s,t)\,ds.$$

By construction, $\Delta^2u(s,t)=0$ at least for $|s|\le\pi/8$ if

$$|t|\le h\le 40^{-1}\min\Big((20d)^{-1/\alpha}\varphi(0)^{1/\alpha},\frac{\pi}{4}\Big).$$

On the complement of the interval $\{s : |s| \leq \pi/8\}$, the cotangent is bounded by a universal constant; therefore,

$$|\Delta^2 \mathcal{H}(u)(0,t)| \leq C \int_{-\pi}^{\pi} |\Delta^2 u(s,t)|\, ds.$$

Next, $u''(s) = \xi''(s)\tau(s) + 2\xi'(s)\tau'(s)$, whence $|u''(s)| \leq c\big(|\log\varphi(0)| + \frac{1}{\varphi(0)}\big)$. By (9), we obtain

$$|\Delta^2 u(s,t)| \leq C\Big(|\log\varphi(0)| + \frac{1}{\varphi(0)}\Big)h^2 \ \text{ for } |t| \leq h,$$

so

$$\varphi(0)W_2 \leq C'(1 + \varphi(0)|\log\varphi(0)|)h^2 \leq C''h^2.$$

It remains to estimate the quantity

$$\varphi(0)W_1 = \varphi(0)\Big(\frac{1}{2h}\int_{-h}^{h} |\Delta^2[\mathcal{H}(\log\varphi - u)](0,t)|^r\, dt\Big)^{1/r}. \tag{25}$$

We split the function under the sign of $\mathcal{H}$ into 2 summands:

$$\log\varphi - u = \chi_{[-4h,4h]} \cdot (\log\varphi - u) + v = v_0 + v.$$

When treating the first summand, we extend the integral to $[-\pi, \pi]$ and use the continuity of $\mathcal{H}$ on L^r:

$$\begin{aligned}
&\frac{1}{2h}\int_{-h}^{h} |\Delta^2[\mathcal{H}(v_0)](0,t)|^r\, dt)^{1/r} \\
&\quad\leq \frac{1}{2h}\int_{-\pi}^{\pi} \Big(|\mathcal{H}(v_0)(2t)| + 2|\mathcal{H}v_0(t)|\big)^r\, dt\Big)^{1/r} + |\mathcal{H}v_0(0)| \\
&\quad\leq C\Big(\frac{1}{2h}\int_{-4h}^{4h} |v_0(s)|^r\, ds\Big)^{1/r} + C|\mathcal{H}v_0(0)|.
\end{aligned} \tag{26}$$

Next, for $|s| \leq 4h$ by (24) we have

$$\begin{aligned}
|v_0(s)| &= |\log\varphi(s) - \tau(s)| \leq |\log\varphi(s) - \log\varkappa(s)| + |\log\varkappa(s) - \tau(s)| \\
&\leq C\frac{|\varphi(s) - \varkappa(s)|}{\varphi(0)} + C\frac{|s|^2}{\varphi(0)^{2/\alpha}} \leq C'\Big(\frac{|s|^\alpha}{\varphi(0)} + \frac{|s|^2}{\varphi(0)^{2/\alpha}}\Big),
\end{aligned}$$

and the same quantity (with s replaced by h and with a different constant) majorizes the first summand on the right in (26). Furthermore,

$$\begin{aligned}
|\mathcal{H}v_0(0)| &= \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} \cot\Big(-\frac{s}{2}\Big)v_0(s)\, ds\Big| \\
&\leq C\int_{-4h}^{4h}\Big(\frac{1}{\varphi(0)}|s|^{\alpha-1} + \frac{1}{\varphi(0)^{2/\alpha}}|s|\Big)\, ds \leq C'\Big(\frac{h^\alpha}{\varphi(0)} + \frac{h^2}{\varphi(0)^{2/\alpha}}\Big).
\end{aligned}$$

Thus, the contribution of v_0 to (25) is majorized by

$$C''\Big(h^\alpha + \frac{h^2}{\varphi(0)^{2/\alpha - 1}}\Big).$$

The contribution of v to (25) looks like this:

$$\begin{aligned}
\varphi(0)\Big(\frac{1}{2\pi}\int_{-h}^{h}\Big|\frac{1}{2\pi}\int_{[-\pi,\pi]\setminus[-4h,4h]}&\Big(\cot\frac{2t-s}{2} - 2\cot\frac{t-s}{2} + \cot\Big(-\frac{s}{2}\Big)\Big) \\
&\times (\log\varphi(s) - u(s))\, ds\Big|^r dt\Big)^{1/r}.
\end{aligned}$$

By (9), the second difference of the cotangent on the domain of integration is dominated by $\frac{ct^2}{|s|^3}$, so the last expression is majorized by

$$\varphi(0)\Big(\frac{l}{2h}\int_{-h}^{h}\Big(\sum_{j=1}^{l}\int_{2^{j+1}h\le|s|\le 2^{j+2}h}\frac{t^2}{|s|^3}|\log\varphi(s)-u(s)|\,ds\Big)^r dt\Big)^{1/r}$$
$$\le C\varphi(0)h^2\sum_{j=1}^{l}(2^jh)^{-3}\int_{2^{j+1}h\le|s|\le 2^jh}|\log\varphi(s)-u(s)|\,ds,$$

where l is the same natural number as before. We also introduce the natural number k as before, and first treat the terms with $j\le k$ in the last sum (observe that $u(s)=\tau(s)$ for these terms):

$$\int_{2^{j+1}h\le|s|\le 2^{j+2}h}|\log\varphi(s)-u(s)|\,ds$$
$$\le\int_{2^{j+1}h\le|s|\le 2^{j+2}h}|\log\varphi(s)-\log\varkappa(s)|\,ds+\int_{2^{j+1}h\le|s|\le 2^{j+2}h}|\log\varkappa(s)-\tau(s)|\,ds$$
$$\le C2^jh\frac{(2^{jh})^\alpha}{\varphi(0)}+C\frac{(2^jh)^3}{\varphi(0)^{2/\alpha}},$$

so that

$$\varphi(0)h^2\sum_{j=1}^{k}(2^jh)^{-3}\int_{2^{j+1}h\le|s|\le 2^{j+2}h}|\log\varphi(s)-u(s)|\,ds$$
$$\le C\Big(\sum_{j=1}^{k}\frac{(2^jh)}{2^{2j}}+k\frac{h^2}{\varphi(0)^{2/\alpha-1}}\Big)\le C\begin{cases}h^\alpha+|\log_2\frac{\varphi(0)^{1/\alpha}}{h}|\frac{h^2}{\varphi(0)^{2/\alpha-1}} & \text{if } \alpha<2;\\ h^2|\log_2\frac{\varphi(0)^{1/2}}{h}| & \text{if } \alpha=2;\\ \varphi(0)^{1-\frac{2}{\alpha}}h^2 & \text{if } \alpha>2.\end{cases}$$

The remaining terms with $k<j\le l$ are again estimated quite roughly:

$$\int_{2^{j+1}h\le|s|\le 2^{j+2}h}|\log\varphi(s)-u(s)|\,ds$$
$$\le\int_{2^{j+1}h\le|s|\le 2^{j+2}h}|\log\varphi(s)|\,ds+\int_{2^{j+1}h\le|s|\le 2^{j+2}h}|\tau(s)|\,ds$$
$$\le C\Big(B_p(2^jh)^{1-1/p}+|\log\varphi(0)|2^jh+\frac{|b|}{\varphi(0)}2^jh\Big),$$

so that

$$\varphi(0)h^2\sum_{j=k+1}^{l}(2^jh)^{-3}\int_{2^{j+1}h\le|s|\le 2^{j+2}h}|\log\varphi(s)-u(s)|\,ds$$
$$\le C\Big(\frac{h^2}{\varphi(0)^{\left(2+\frac{1}{p}\right)\frac{1}{\alpha}-1}}+\frac{h^2|\log\varphi(0)|}{\varphi(0)^{2/\alpha-1}}\Big).$$

Collecting the estimates, we obtain statement (b) in Lemma 9. □

Now, we combine Lemma 7 and statements (a) in Lemmas 8 on the one hand and statements (b) in Lemmas 8 and 9 on the other. This will result in two inequalities:
(I) $\big(\frac{1}{2h}\int_{-h}^{h}|\Delta^2(\varphi e^{i\psi})(0,t)|^r\,dt\big)^{1/r}\le C\big(h^\alpha+\varphi(0)^{1-\frac{1}{\alpha}}h+\boxed{\varphi(0)}\big)$;

(II) if $|h| \le 100^{-1}\min((20d)^{-1/\alpha}\varphi(0)^{1/\alpha}, \pi/4)$, then

$$\left(\frac{1}{2h}\int_{-h}^{h}|\Delta^2(\varphi e^{i\psi})(0,t)|^r\,dt\right)^{1/r}$$
$$\le C\left(h^\alpha+\frac{h^{1+\alpha}}{\varphi(0)^{1/\alpha}}+\frac{h^2|\log\frac{1}{h}|}{\varphi(0)^{2/\alpha-1}}+\frac{h^2}{\varphi(0)^{\frac{1}{\alpha}(2+\frac{1}{p})-1}}+\frac{h^2|\log\varphi(0)|}{\varphi(0)^{2/\alpha-1}}+\frac{h^{2\alpha}}{\varphi(0)}+\boxed{\frac{h^2}{\varphi(0)^{\frac{2}{\alpha}\frac{p+1}{p}-1}}}\right).$$

It turns out that only the boxed terms in these formulas really compete, and the other terms are less important. The guideline for the last step is that the boxed terms are roughly equal if $\varphi(0) \asymp h^{\alpha\frac{p}{p+1}}$. So, we finish the proof of Theorem D from the Introduction as follows.

1°. If $\varphi(0) = 0$, then (I) yields more than it was claimed: smoothness does not drop.

2°. Let $\varphi(0) > 0$. Since $\frac{p+1}{p} > 1$ and $\varphi(0) \le 1$, for a sufficiently small $\delta > 0$ independent of $\varphi(0)$ we obtain

$$\delta\varphi(0)^{\frac{1}{\alpha}\frac{p+1}{p}} \le 200^{-1}\Big(\min(20d)^{-1/\alpha}\varphi(0)^{1/\alpha}, \frac{\pi}{4}\Big).$$

Fixing such δ, we assume first that $h > \delta\varphi(0)^{\frac{1}{\alpha}\frac{p+1}{p}}$, then the right-hand side of (I) is dominated by

$$C'\big(h^\alpha + h^{\frac{\alpha p}{p+1}(1-\frac{1}{\alpha})+1} + h^{\frac{\alpha p}{p+1}}\big).$$

The exponent in the second term is $\frac{\alpha p}{p+1} + 1 - \frac{p}{p+1} > \frac{\alpha p}{p+1}$, and the entire expression is majorized by $C''h^{\frac{\alpha p}{p+1}}$.

3°. Let $h \le \delta\varphi(0)^{\frac{1}{\alpha}\frac{p+1}{p}}$, then we can use (II). We observe that the terms where $\varphi(0)$ occurs in the denominator are of order not worse that $c_\alpha h^2\log\frac{1}{h}$ *provided that the exponent of* $\varphi(0)$ *is nonpositive*, and in this case they present no trouble. In the boxed term, this exponent becomes nonpositive when $\alpha \ge 2 + \frac{2}{p}$, and there are two other cases where it also becomes nonpositive, but earlier: respectively, for $\alpha \ge 2 + \frac{1}{p}$, and for $\alpha \ge 2$.

We leave the case where $\alpha \ge 2 + \frac{2}{p}$ aside (some estimate can be deduced under this condition, but this is not very interesting): suppose that we have the opposite inequality. In all terms where the denominator involves $\varphi(0)$ in a positive power, we can replace $\varphi(0)$ with the smaller quantity $\left(\frac{h}{\delta}\right)^{\frac{\alpha p}{p+1}}$. For the boxed term, this yields an estimate of the form $ch^{2-2+\frac{\alpha p}{p+1}} = h^{\frac{\alpha p}{p+1}}$. It can easily be calculated that the other terms are dominated by powers of h with greater exponent.

It should be noted that for $p = 1$ the restriction $\alpha < 2 + \frac{2}{p}$ means that $\alpha < 4$.

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