RESCALINGS AT POSSIBLE SINGULARITIES OF NAVIER–STOKES EQUATIONS IN HALF-SPACE

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ABSTRACT. The relationship is clarified between a possible blow-up for strong solutions of the initial boundary value problem for the incompressible Navier–Stokes equations in $\{x_3 > 0\}$, and the Liouville theorem for mild bounded ancient solutions.

§1. INTRODUCTION

Consider the initial boundary-value problem for the incompressible Navier–Stokes equations in half-space $\mathbb{R}_+^3 = \{x_3 > 0\}$:

$$(1.1) \quad \left. \begin{aligned} \partial_t v + v \cdot \nabla v - \Delta v &= -\nabla q \\ \operatorname{div} v &= 0 \end{aligned} \right\} \quad \text{in } \mathbb{R}_+^3 \times (0, \infty),$$

$$(1.2) \quad v(\cdot, t)|_{\partial\mathbb{R}_+^3} = 0 \quad \text{for } t > 0,$$

$$(1.3) \quad v(\cdot, 0) = v_0 \quad \text{in } \mathbb{R}^3,$$

where v_0 is a sufficiently regular div-free field in $\mathbb{R}_+^3$ with $v_0|_{\partial\mathbb{R}_+^3} = 0$ and with sufficiently fast decay as $x \rightarrow \infty$.

Our main goal is to understand the relationship between a possible blow-up for strong solutions and the Liouville theorem for mild bounded ancient solutions, in the spirit considered in [6] for the entire space. We recall that the local-in-time existence of strong solutions was proved in the entire space in [9], and for bounded domains similar results appeared in [5]¹. As to unbounded domains, we refer, for example, to [8] or [2].

Consider the local-in-time strong solution v on its maximal interval of existence $[0, T)$, which will be assumed finite. The time T is then the blow-up time. It is known that such solutions are unique (unlike the weak solutions, which on the other hand are global). We set

$$(1.4) \quad g(t) = \sup_{0 < \tau \leq t} M(\tau),$$

where

$$M(t) = \sup_{x \in \mathbb{R}_+^3} |v(x, t)|.$$

It is well known that $g(t) \rightarrow +\infty$ as $t \rightarrow T_-$, see for example [9].

Our considerations are motivated by methods used in the theory of geometric flows and based on scale-invariant transformations of the solution v when time is approaching T , see [6] for a more detailed discussion and references. In the case of the Navier–Stokes equations, the scale-invariant transformations have the form $v(x, t), q(x, t) \rightarrow \lambda v(\lambda x, \lambda^2 t), \lambda^2 q(\lambda x, \lambda^2 t)$. In the entire space, taking limits of sequences of suitably

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¹At the time of this writing it is unknown whether such solutions exist globally.

scaled solutions produces so-called bounded ancient (backward) solutions of the Navier–Stokes equations. They are defined on the semiinfinite time interval $]-\infty, 0]$ (backward in time), are bounded in L_∞ , and in fact belong to a subclass called mild bounded ancient solutions. In [6], it was shown that mild bounded ancient solutions are infinitely smooth both in space and in time. Moreover, if T is a blow-up time, the limiting mild bounded ancient solution cannot vanish. It was conjectured in [6] that *any mild bounded ancient solution is a constant*. This would rule out blow-up of Type I in the case of the Cauchy problem (whole space). We recall that Type I blow-up is usually defined by the inequality

$$(1.5) \quad |v(x, t)| \leq \frac{c}{\sqrt{T-t}}$$

for any $x \in \mathbb{R}^3$ and any $t < T$, although more general definitions are also possible.

To show a difference between bounded ancient solutions in the whole space and in the half-space, we recall their definitions. We say that u is a *bounded ancient solution* of the Navier–Stokes equations if u is bounded in $Q_- = \mathbb{R}^3 \times]-\infty, 0[$ and satisfies these equations in the sense of distributions with divergence free test functions, i.e.,

$$(1.6) \quad \int_{Q_-} (u \cdot (\partial_t \varphi + \Delta \varphi) + u \otimes u : \nabla \varphi) dz = 0$$

for any $\varphi \in C_{0,0}^\infty(Q_-) := \{\varphi \in C_0^\infty(Q_-) : \operatorname{div} \varphi = 0\}$;

$$(1.7) \quad \int_{Q_-} u \cdot \nabla q dz = 0$$

for any $q \in C_0^\infty(Q_-)$. By scaling, we may assume that $|u|$ is bounded by one. There are simple nontrivial bounded ancient solutions of the form

$$(1.8) \quad u(x, t) = a(t)$$

where a is an arbitrary bounded function of t only.

A vector field u is called a *mild bounded ancient solution* if u is a bounded ancient solution and there exists a pressure field $p \in L_\infty(-\infty, 0; BMO(\mathbb{R}^3))$ such that

$$(1.9) \quad \int_{Q_-} (u \cdot (\partial_t \varphi + \Delta \varphi) + u \otimes u : \nabla \varphi) dz = - \int_{Q_-} p \operatorname{div} \varphi dz$$

for any $\varphi \in C_0^\infty(Q_-)$. It is not so difficult to see that any solution of the form (1.8) is a mild bounded ancient solution if and only if $a(t) = \text{constant}$. As has already been mentioned, mild bounded ancient solutions are infinitely smooth and have the following property: for any $A < 0$, they can be presented in the form

$$(1.10) \quad \begin{aligned} u_i(x, t) = & \int_{\mathbb{R}^3} \Gamma(x - y, t - A) u_i(y, A) dy \\ & + \int_A^t \int_{\mathbb{R}^3} K_{ijm}(x, y, t - \tau) u_j(y, \tau) u_m(y, \tau) dy d\tau, \end{aligned}$$

where Γ is the well known heat kernel and K is obtained from the Oseen tensor by differentiation in spatial variables, see details in [14] and [6]. That is why those solutions are said to be mild. By the way, this is the original definition of mild bounded ancient solutions given in [6], and the equivalent definition in terms of the pressure appeared in [12] later on.

The case of the half-space is more complicated. First, it is not immediately clear how to understand the homogeneous Dirichlet boundary conditions when the velocity u is

only bounded in $Q_-^+ := \{z = (x, t) : x \in \mathbb{R}_+^3, -\infty < t < 0\}$. We shall use the following weak definition of (1.2):

$$(1.11) \quad \int_{Q_-^+} (u \cdot (\partial_t \varphi + \Delta \varphi) + u \otimes u : \nabla \varphi) dz = 0$$

for any $\varphi \in C_{0,0}^\infty(Q_-)$ with $\varphi(x', 0, t) = 0$ for any $x' \in \mathbb{R}^2$ and for any $-\infty < t < 0$;

$$(1.12) \quad \int_{Q_-^+} u \cdot \nabla q dz = 0$$

for any $q \in C_0^\infty(Q_-)$.

In [3] and [4], a class of simple nontrivial bounded ancient solutions to the Navier–Stokes equations was presented. They describe a bounded shear flow in the half-space and have the form

$$(1.13) \quad u(x, t) = (u_1(x_3, t), u_2(x_3, t), 0).$$

It was also shown in [3] and [4] that there are no other nontrivial solutions to the linear Stokes system in the half-space. It is unknown whether or not this is true for the nonlinear case.

We now define mild bounded ancient solutions in a half-space.

Definition 1.1. A bounded function u is a mild bounded ancient solution if and only if there exists a pressure p such that $p = p^1 + p^2$, where the even extension of p^1 to the whole $\mathbb{R}^3$ with respect to x_3 is an $L_\infty(-\infty, 0; BMO(\mathbb{R}^3))$ -function,

$$(1.14) \quad \Delta p^1 = -\operatorname{div} \operatorname{div} u \otimes u$$

in Q_-^+ with $p_3^1(x', 0, t) = 0$, and $p^2(\cdot, t)$ is a harmonic function in $\mathbb{R}_+^3$ whose gradient satisfies the estimate

$$(1.15) \quad |\nabla p^2(x, t)| \leq c \ln(2 + 1/x_3)$$

for all $(x, t) \in Q_-^+$ and has the property

$$(1.16) \quad \sup_{x' \in \mathbb{R}^2} |\nabla p^2(x, t)| \rightarrow 0$$

as $x_3 \rightarrow \infty$; u and p satisfy (1.12) and

$$(1.17) \quad \int_{Q_-^+} (u \cdot (\partial_t \varphi + \Delta \varphi) + u \otimes u : \nabla \varphi + p \operatorname{div} \varphi) dx dt = 0$$

for any $\varphi \in C_0^\infty(Q_-)$ with $\varphi(x', 0, t) = 0$ for any $x' \in \mathbb{R}^2$ and for any $t < 0$.

Remark 1.2. If u is a mild bounded ancient solution, then $\nabla u \in L_\infty(Q_-^+)$. The function u is infinitely smooth in spatial variables in the upper half-space $x_3 > 0$.

As in the case of the entire space, mild bounded ancient solutions can be defined by formula (1.10), in which $\mathbb{R}^3$ is replaced with $\mathbb{R}_+^3$ and kernels with their half-space analogs, see §2 for details. The corresponding statement might be called the equivalence theorem. The proof of such a result is more involved than its whole space version and will be published elsewhere.

It should be noticed that a nontrivial solution of the form (1.13) is a mild bounded ancient solutions if and only $u = 0$. Indeed, for solutions (1.13), the gradient of the pressure is a function of time only. By the above theorem, this is only possible if the gradient of the pressure is equal to zero. In its turn, this means that each component u_α , $\alpha = 1, 2$, is a bounded ancient solution to the heat equation in $\{x_3 > 0\} \times]-\infty, 0[$ with the boundary condition $u_\alpha(0, t) = 0$, which implies that $u_\alpha \equiv 0$.

We believe that the following is true.

Conjecture 1.3. *There is no nontrivial mild bounded ancient solution of the Navier–Stokes equations in the half-space.*

The validity of Conjecture 1.3 and the conjecture, made in [6] and mentioned above, would rule out Type I blow-up in the broad sense, when understood as blow-up solutions with a suitable bounded scale-invariant quantity.

We state our main result.

Theorem 1.4. *Assume that initial boundary value problem (1.1)–(1.3) has a solution that admits blow-up at time T . There exists at least one nontrivial (nonzero) mild bounded ancient solution either in the whole space or in the half-space.*

The appearance of mild bounded ancient solutions in the whole space is not surprising and should be expected if the solution v to the original problem (1.1)–(1.3) is smooth near the boundary $x_3 = 0$ near the blow-up time T . This scenario of blow-up can be called an interior blowup. All other mild bounded ancient solutions are then related to a boundary blowup. We note that a boundary blow-up could still lead to a mild bounded ancient solution in the whole space. This might happen when the velocity tends to infinity fast in comparison with the rate at which x is approaching the boundary.

An interesting consequence of the above theorem is the following statement.

Proposition 1.5. *There exists $\varepsilon > 0$ (independent of v) such that if*

$$(1.18) \quad |v(x, t)| \leq \frac{\varepsilon}{x_3}$$

for all $x \in \mathbb{R}_+^3$ and $t \in]0, T[$, the solution v does not blow-up.

§2. PRELIMARIES

Given $A < 0$, consider the following Stokes problem in half-space:

$$\partial_t u - \Delta u + \nabla p = -\operatorname{div} F, \quad \operatorname{div} u = 0$$

in $\mathbb{R}_+^3 \times]A, 0[$,

$$u(x', 0, t) = 0$$

for $(x', t) \in \mathbb{R}^2 \times]A, 0[$,

$$u(x, A) = u_0(x)$$

for $x \in \mathbb{R}_+^3$.

In addition, we assume that u_0 is divergence free and F and its first derivatives vanish at the boundary $x_3 = 0$. If F and u_0 are sufficiently smooth and decay sufficiently fast at infinity, a solution of the above problem can be presented in the following way, see [13],

$$(2.1) \quad u = u^1 + u^2,$$

where

$$(2.2) \quad u^1(x, t) = \int_{\mathbb{R}_+^3} G(x, y, t - A) u_0(y) dy$$

for any $x \in \mathbb{R}_+^3$ and any $t \in]A, 0[$, and

$$(2.3) \quad u^2(x, t) = - \int_A^t \int_{\mathbb{R}_+^3} G(x, y, t - \tau) \operatorname{div} H(y, \tau) dy d\tau$$

for the same x and t , where $H = F + p^1 \mathbb{I}$ and p^1 is a solution of the following Neumann boundary value problem:

$$\Delta p^1 = -\operatorname{div} \operatorname{div} F$$

in R_+^3 and $p_3^1 = 0$ at $x_3 = 0$. Here, G is the Green function for the Stokes system in the half-space that was studied in [13] and has the form

$$(2.4) \quad G = G^1 + G^2,$$

where

$$\begin{aligned} G_{ij}^1(x, y, t) &= \delta_{ij} (\Gamma(x - y, t) - \Gamma(x - y^*, t)), \\ G_{i\beta}^2(x, y, t) &= 4 \frac{\partial}{\partial x_\beta} \int_0^{x_3} \int_{\mathbb{R}^2} \frac{\partial E}{\partial x_i}(x - z) \Gamma(z - y^*, t) dz, \quad G_{i3}^2(x, y, t) = 0. \end{aligned}$$

As in the case of the entire space, it is convenient to present the function u^2 in the following equivalent way:

$$(2.5) \quad u_i^2(x, t) = \int_A^t \int_{\mathbb{R}_+^3} K_{ijm}(x, y, t - \tau) F_{jm}(y, \tau) dy d\tau$$

for the same x and t as above. The kernel K was introduced in [14] and has the following structure:

$$(2.6) \quad K_{ism}(x, z, t) = \bar{K}_{ism}(x, z, t) + \hat{K}_{ism}(x, z, t),$$

where $\bar{K}_{ism}(x, z, t)$ is a linear combination of the terms

$$\frac{\partial G_{ij}}{\partial z_k}(x, z, t),$$

and $\hat{K}_{ism}(x, z, t)$ is a linear combination of the terms

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta} \int_{\mathbb{R}_+^3} G_{ij}(x, y, t) \frac{\partial N^{(\pm)}}{\partial y_s}(y, z) dy.$$

Here, $N^{(\pm)}(x, y) = E(x - y) \pm E(x - y^*)$ with $y^* = (y', -y_3)$ and $E(x)$ is the fundamental solution of the Laplace equation in $\mathbb{R}^3$.

We outline how the above transformations can be done. Our arguments slightly differ from those used in the papers [7] and [14] mentioned above. Consider the following boundary value problems:

$$(2.7) \quad \Delta_y \Phi_{mn}(x, y, t) = G_{mn}(x, y, t)$$

with $\partial \Phi_{mn} / \partial y_3(x, y, t) = 0$ if $n < 3$ and with $\Phi_{mn}(x, y, t) = 0$ if $n = 3$ at $y_3 = 0$. Then integration by parts gives

$$K_{mjs}(x, y, t) = \frac{\partial^3 \Phi_{mj}}{\partial y_i \partial y_i \partial y_s}(x, y, t) - \frac{\partial^3 \Phi_{mn}}{\partial y_n \partial y_j \partial y_s}(x, y, t).$$

The above splitting for the potential K can easily be derived with help of the initial boundary value problems for function Φ , see (2.7), and some elementary properties of the Green functions G , N^+ , and N^- .

The following estimates for G^i and $\hat{K}$ were obtained in the papers [13, 14], and [15]:

$$(2.8) \quad \left| \frac{\partial^{|\alpha|+|\gamma|} G^2}{\partial x^\alpha \partial y^\gamma}(x, y, t - A) \right| \leq c(\alpha, \gamma) (t - A)^{-\frac{\gamma_3}{2}} (t - A + x_3^2)^{-\frac{\alpha_3}{2}} \times (|x - y^*|^2 + t - A)^{-\frac{3+|\alpha'|+|\gamma'|}{2}} \exp\left(-\frac{cy_3^2}{t - A}\right),$$

where $\alpha' = (\alpha_1, \alpha_2)$, $\gamma' = (\gamma_1, \gamma_2)$, and $|\gamma| = 0, 1$,

$$(2.9) \quad \left| \frac{\partial G_{ij}^1}{\partial y_i}(x, y, t) \right| + |\hat{K}_{ism}(x, y, t)| \leq \frac{c}{(|x - y|^2 + t)^2},$$

$$(2.10) \quad |\partial_t^l G^2(x, y, t)| \leq \frac{c}{t^l(|x' - y'|^2 + x_3^2 + y_3^2 + t)^{\frac{3}{2}}} \exp\left(-\frac{cy_3^2}{t}\right)$$

for $l = 0, 1$.

Let K^1 and K^2 be generated by G^1 and G^2 , respectively. In particular, we need the following estimate:

$$(2.11) \quad |\hat{K}^2(x, y, t)| \leq \frac{c}{(|x - y^*|^2 + t)^2},$$

which can be obtained by an elementary modification of the arguments used in the proof of Proposition 3.1 in [14].

§3. SCALING

It is not difficult to show that there exists a sequence $(x^{(k)}, t_k)$ with $x^{(k)} \in \mathbb{R}_+^3$ such that $t_k \rightarrow T - 0$ and

$$(3.1) \quad g(t_k) = M(t_k) = |v(x^{(k)}, t_k)| \rightarrow \infty.$$

Remark 3.1. From [1] and [10], it follows that the sequence $x^{(k)}$ is bounded.

We let $M_k = M(t_k)$. There are two main scenarios. In the first of the them

$$(3.2) \quad x_3^{(k)} M_k \rightarrow \infty,$$

and we scale v and q so that

$$(3.3) \quad u^{(k)}(y, s) = \frac{1}{M_k} v(x, t), \quad p_k(y, s) = \frac{1}{M_k^2} q(x, t),$$

where

$$(3.4) \quad y = M_k(x - x^{(k)}), \quad s = M_k^2(t - t_k).$$

By the above scaling, (1.1) and (1.2) are transformed into

$$(3.5) \quad \partial_s u^{(k)} + u^{(k)} \cdot \nabla u^{(k)} - \Delta u^{(k)} = -\nabla p_k, \quad \operatorname{div} u^{(k)} = 0$$

in $Q^k := \{y = (y', y_3) : y' \in \mathbb{R}^2, y_3 > -x_3^{(k)} M_k\} \times]-t_k M_k^2, 0[$,

$$(3.6) \quad u^{(k)}(y', -x_3^{(k)} M_k, t) = 0$$

for any $y' \in \mathbb{R}^2$ and $s \in]-t_k M_k^2, 0[$. And, moreover, by (3.1), we have

$$(3.7) \quad |u^{(k)}(0)| = 1.$$

In the second scenario,

$$(3.8) \quad x_3^{(k)} M_k \rightarrow a \in [0, \infty[.$$

This suggests the same scaling (3.3) but with a slightly different change of variables:

$$(3.9) \quad y' = M_k(x' - x'^{(k)}), \quad y_3 = M_k x_3, \quad s = M_k^2(t - t_k).$$

In this case, (1.1) and (1.2) are transformed into system (3.5), which is valid in

$$\mathbb{R}_+^3 \times]-t_k M_k^2, 0[$$

and into the boundary condition

$$(3.10) \quad u^{(k)}(y', 0, t) = 0$$

for any $y' \in \mathbb{R}^2$ and $s \in]-t_k M_k^2, 0[$. Condition (3.7) is replaced with

$$(3.11) \quad |u^{(k)}(0, x_3^{(k)} M_k, 0)| = 1.$$

Our aim is to understand what happens if $k \rightarrow \infty$.

Without loss of generality, we may assume that the following statements are true.

Scenario 1. There exists a divergence free function $u \in L_\infty(Q_-)$ such that $|u| \leq 1$ a.e. in Q_- and, for any $R > 0$,

$$(3.12) \quad \int_{Q(R)} u^{(k)} \cdot w \, dx \, dt \rightarrow \int_{Q(R)} u \cdot w \, dx \, dt$$

for any $w \in L_1(Q(R))$. Here, $Q(R) = B(R) \times]-R^2, 0[$.

Scenario 2. There exists a divergence free function $u \in L_\infty(Q_-^+)$ such that $|u| \leq 1$ a.e. in Q_-^+ and, for any $R > 0$,

$$(3.13) \quad \int_{Q_+(R)} u^{(k)} \cdot w \, dx \, dt \rightarrow \int_{Q_+(R)} u \cdot w \, dx \, dt$$

for any $w \in L_1(Q_+(R))$. Here, $Q_+(R) = B_+(R) \times]-R^2, 0[$.

§4. SCENARIO 1

Our goal is to show that the limit function u in (3.12) must be a mild bounded ancient solution in the entire space and satisfy the condition

$$|u(0, 0)| = 1.$$

Fix an arbitrary $A < 0$. Let k be sufficiently large so that $A > -t_k M_k^2$. We split $w := u^{(k)}$ into two parts

$$w = w^1 + w^2$$

so that

$$\partial_t w^1 - \Delta w^1 + \nabla r^1 = 0, \quad \operatorname{div} w^1 = 0$$

in $Q_A^k = \mathbb{R}^2 \times \{y_3 > -d_k\} \times]A, 0[$, where $d_k = x_3^{(k)} M_k \rightarrow \infty$,

$$w^1(y', -d_k, t) = 0$$

for $(y', t) \in \mathbb{R}^2 \times]A, 0[$, and

$$w^1(y, A) = w(y, A)$$

for $y \in \mathbb{R}^2 \times \{y_3 > -d_k\}$.

The second part of w is a solution to the following initial boundary value problem:

$$\partial_t w^2 - \Delta w^2 + \nabla r^2 = \operatorname{div} F + \nabla p^{1(k)}, \quad \operatorname{div} w^2 = 0$$

in Q_A^k ,

$$w^2(y', -d_k, t) = 0$$

for $(y', t) \in \mathbb{R}^2 \times]A, 0[$, and

$$w^2(y, A) = 0$$

for $y \in \mathbb{R}^2 \times \{y_3 > -d_k\}$. Here, $F = w \otimes w$ and $p^{1(k)}$ is defined by the following Neumann boundary value problem

$$\Delta p^{1(k)}(y, t) = -\operatorname{div} \operatorname{div} F(y, t), \quad (y, t) \in Q_A^k,$$

$$p_{,3}^{1(k)}(y', -d_k, t) = 0 \quad (y', t) \in \mathbb{R}^2 \times]A, 0[.$$

Keeping (2.4) in mind, we can present w^1 in the form $w^1 = w^{1,1} + w^{1,2}$, where

$$w^{1,1}(y, t) = \int_{\mathbb{R}^2 \times \{y_3 > -d_k\}} G^1(y + d_k e_3, z + d_k e_3, t - A) w(z, A) dz$$

and

$$w^{1,2}(y, t) = \int_{\mathbb{R}^2 \times \{y_3 > -d_k\}} G^2(y + d_k e_3, z + d_k e_3, t - A) w(z, A) dz.$$

Elementary calculations, estimate (2.8), and the inequality $|w(y, A)| \leq 1$ ensure an upper bound for $w^{1,2}$:

$$(4.1) \quad |w^{1,2}(y, t)| \leq c \frac{\sqrt{t - A}}{y_3 + d_k}.$$

A similar decomposition can be exploited in order to evaluate w^2 . We have $w^2(y, t) = w^{2,1} + w^{2,2}$, where

$$w^{2,1}(y, t) = - \int_A^0 \int_{\mathbb{R}^2 \times \{y_3 > -d_k\}} G^1(y + d_k e_3, z + d_k e_3, t - \tau) (\operatorname{div} F + \nabla p^{1(k)})(z, \tau) dz d\tau$$

and

$$w^{2,2}(y, t) = - \int_A^0 \int_{\mathbb{R}^2 \times \{y_3 > -d_k\}} G^2(y + d_k e_3, z + d_k e_3, t - \tau) (\operatorname{div} F + \nabla p^{1(k)})(z, \tau) dz d\tau.$$

In fact, the velocity $w^{2,2}$ obeys the same estimate as (4.1)

$$(4.2) \quad |w^{2,2}(y, t)| \leq c \frac{\sqrt{t - A}}{y_3 + d_k}.$$

To prove (4.2), we need the following lemma.

Lemma 4.1. *Let*

$$V_i(x, t) = \int_0^t \int_{\mathbb{R}_+^3} K_{ijm}^2(x, y, t - \tau) H_{km}(y, \tau) dy d\tau$$

with $K_{ijm}^2 = \bar{K}_{ijm}^2 + \hat{K}_{ijm}^2$, where $\bar{K}_{ism}^2(x, z, t)$ is a linear combination of the terms $\frac{\partial G_{ij}^2}{\partial y_k}(x, z, t)$ and $\hat{K}_{ism}^2(x, z, t)$ is a linear combination of the terms

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta} \int_{\mathbb{R}_+^3} G_{ij}^2(x, y, t) \frac{\partial N^{(\pm)}}{\partial y_s}(y, z) dy.$$

Then $|V(x, t)| \leq c \|H\|_\infty \frac{t}{x_3}$.

Proof. By (2.8), we find

$$|\bar{K}^2(x, y, t)| \leq \frac{c}{t^{\frac{1}{2}}(|x - y^*|^2 + t)^{\frac{3}{2}}} \exp\left(-\frac{cy_3^2}{t}\right).$$

For the second term $\widehat{K}^2$, we are going to make use of estimate (2.11). So, we have

$$\begin{aligned} |V(x, t)| &\leq c\|H\|_\infty \int_0^t \int_{\mathbb{R}_+^3} |K^2(x, z, t - \tau)| dz d\tau \\ &\leq c\|H\|_\infty \int_0^t \int_{\mathbb{R}_+^3} \left(\frac{1}{\tau^{\frac{1}{2}}(|x - z^*|^2 + \tau)^{\frac{3}{2}}} \exp\left(-\frac{cz_3^2}{\tau}\right) + \frac{1}{(|x - z^*|^2 + \tau)^2} \right) dz d\tau \\ &\leq c\|H\|_\infty \int_0^t \int_0^\infty \left(\frac{1}{\tau^{\frac{1}{2}}(|x_3 + z_3|^2 + \tau)^{\frac{1}{2}}} \exp\left(-\frac{cz_3^2}{\tau}\right) + \frac{1}{|x_3 + z_3|^2 + \tau} \right) dz_3 d\tau \\ &\leq c\|H\|_\infty \left(\frac{1}{x_3} \int_0^t \int_0^\infty \frac{1}{\tau^{\frac{1}{2}}} \exp\left(-\frac{cz_3^2}{\tau}\right) dz_3 d\tau + \int_0^t \int_{x_3}^\infty \frac{1}{|u|^2 + \tau} du d\tau \right) \\ &\leq c\|H\|_\infty \left(\frac{t}{x_3} \int_0^\infty \exp(-cu^2) du + \int_0^t \frac{1}{\tau^{\frac{1}{2}}} \left(\frac{\pi}{2} - \arctan\left(\frac{x_3}{\tau^{\frac{1}{2}}}\right) \right) d\tau \right). \end{aligned}$$

To estimate the last integral, we do the following:

$$\int_0^t \frac{1}{\tau^{\frac{1}{2}}} \left(\frac{\pi}{2} - \arctan\left(\frac{x_3}{\tau^{\frac{1}{2}}}\right) \right) d\tau = \int_0^t \frac{1}{\tau^{\frac{1}{2}}} \arctan\left(\frac{\tau^{\frac{1}{2}}}{x_3}\right) d\tau \leq \frac{t}{x_3}.$$

Lemma 4.1 is proved. $\square$

Now, the upper bound (4.2) follows from Lemma 4.1 and the identity

$$w_i^{2,2}(y, t) = \int_A \int_{\mathbb{R}^2 \times \{y_3 > -d_k\}} K_{ijm}^2(y + d_k e_3, z + d_k e_3, t - \tau) F_{jm}(z, \tau) dz d\tau$$

with the potential K^2 derived from the boundary value problem (2.7), in which G is replaced with G^2 .

If we let $w^0(y, t) = w^{1,1} + w^{2,1}$, then the new function is a solution of the following initial boundary value problem:

$$\partial_t w^0 - \Delta w^0 = \operatorname{div} F + \nabla p^{1(k)}$$

in Q_A^k ,

$$w^0(y', -d_k, t) = 0$$

for $(y', t) \in \mathbb{R}^2 \times]A, 0[$, and

$$w^0(y, A) = w(y, A)$$

for $y \in \mathbb{R}^2 \times \{y_3 > -d_k\}$. Using exact representation formulas for $w^{1,1}$ and $w^{2,1}$, we may assume that w^0 is bounded by a constant c , which is independent of k . On the other hand, we know that the function $p^{1(k)}$, being extended to the whole $\mathbb{R}^3$ so that $p^{1(k)}(y', y_3, t) = p^{1(k)}(y', y_3 + 2d_k, t)$ for $y_3 < -d_k$, belongs to the space $L_\infty(A, 0; BMO(\mathbb{R}^3))$ and the corresponding norm is bounded by a constant independent of k . So, we have

$$\sup_{A \leq t \leq 0} \int_{B(x, \sqrt{-A})} |p^{1(k)}(y, t) - [p^{1(k)}]_{B(x, \sqrt{-A})}(t)|^m dy \leq c(m)(-A)^{\frac{3}{2}}.$$

This means, see Appendix II, that the sequence w^0 is precompact in $C(K \times [A/2, 0])$, where K is an arbitrary compact subset of $\mathbb{R}^3$. Now, it remains to make use of estimates for $w^{1,2}$ and $w^{2,2}$, pass to the limit in the equation for w^0 taking into account that $F = w \otimes w$, and conclude that, by the arbitrariness of A , u is a mild bounded ancient solution to the Navier–Stokes equations satisfying $|u(0, 0)| = 1$.

Remark 4.2. In fact, we have shown that

$$u^{(k)} \rightarrow u$$

uniformly on the closure of the set $Q(R)$ for any $R > 0$.

§5. SCENARIO 2

Here, we are going to prove the following statement.

Proposition 5.1. *We have*

$$u^{(k)} \rightarrow u$$

uniformly on the closure of the set $Q_+(R) = B_+(R) \times]-R^2, 0[$ with $B_+(R) = \{x \in B(R) : x_3 > 0\}$ for any $R > 0$. The limit function u is equal to zero at $y_3 = 0$ and is nontrivial in the sense that

$$|u(0, a, 0)| = 1$$

with $a > 0$.

Proof. As in the previous section, we split w into two parts

$$w = w^1 + w^2,$$

where

$$\partial_t w^1 - \Delta w^1 + \nabla r^1 = 0, \quad \operatorname{div} w^1 = 0$$

in $Q_A = \mathbb{R}_+^3 \times]A, 0[$,

$$w^1(y', 0, t) = 0$$

for $(y', t) \in \mathbb{R}^2 \times]A, 0[$, and

$$w^1(y, A) = w(y, A)$$

for $y \in \mathbb{R}_+^3$.

The second part is a solution of the following problem:

$$\partial_t w^2 - \Delta w^2 + \nabla r^2 = \operatorname{div} F + \nabla p^{1(k)}, \quad \operatorname{div} w^2 = 0$$

in Q_A ,

$$w^2(y', 0, t) = 0$$

for $(y', t) \in \mathbb{R}^2 \times]A, 0[$, and

$$w^2(y, A) = 0$$

for $y \in \mathbb{R}_+^3$. Here, $F = w \otimes w$ and $p^{1(k)}$ is defined by the following Neumann boundary value problem:

$$\Delta p^{1(k)}(y, t) = -\operatorname{div} \operatorname{div} F(y, t), \quad (y, t) \in Q_A,$$

$$p_{,3}^{1(k)}(y', 0, t) = 0, \quad (y', t) \in \mathbb{R}^2 \times]A, 0[.$$

First, we discuss the precompactness of w^2 in $C(\bar{B}_+(R) \times [A/2, 0])$ for any positive R . Indeed, by (2.8), we have

$$\left| \frac{\partial G_{ij}^2}{\partial z_i}(x, z, t) \right| \leq \frac{c}{t^{\frac{1}{2}}(|x - z^*|^2 + t)^{\frac{3}{2}}} \exp\left(-\frac{cz_3^2}{t}\right)$$

and, using estimates (2.9) and the definition of the kernel K , it is not difficult to show that

$$\int_{\mathbb{R}_+^3} |K_{ism}(x, z, t)| dz d\tau \leq \frac{c}{\sqrt{t}}.$$

Next, first, assuming $A \leq t_1 < t_2 \leq 0$ and $x^1, x^2 \in \mathbb{R}_+^3$, we obtain

$$\begin{aligned} |w^2(x^1, t_1) - w^2(x^2, t_2)| &\leq \left| \int_A^{t_1} \int_{\mathbb{R}_+^3} (K(x^1, y, t_1 - \tau) - K(x^2, y, t_2 - \tau)) F(y, \tau) dy d\tau \right| \\ &\quad + \left| \int_{t_1}^{t_2} \int_{\mathbb{R}_+^3} K(x^2, y, t_2 - \tau) F(y, \tau) dy d\tau \right| = I_1 + I_2. \end{aligned}$$

For the second term, we have

$$I_2 \leq c \|F\|_\infty \sqrt{t_2 - t_1}.$$

Next, in the first term, we make a change of variables:

$$\begin{aligned} I_1 &\leq \|F\|_\infty \int_0^{t_1-A} \int_{\mathbb{R}_+^3} |K(x^1, y, \tau) - K(x^2, y, t_2 - t_1 + \tau)| dy d\tau \\ &\leq \|F\|_\infty \int_0^{-A} \int_{\mathbb{R}_+^3} |K(x^1, y, \tau) - K(x^2, y, t_2 - t_1 + \tau)| dy d\tau \\ &= \|F\|_\infty J(x^1, x^2; t_1, t_2). \end{aligned}$$

Using the above estimates of the Green function, it is not difficult to show that, given $\varepsilon > 0$, there exists $\kappa(\varepsilon, A, R)$ such that if $|x^1 - x^2| < \kappa$, $0 < t_2 - t_1 < \kappa$, $x^1, x^2 \in \bar{B}_+(R)$, and $A \leq t_1 < t_2 \leq 0$, then $J(x^1, x^2; t_1, t_2) < \varepsilon$. So, the required precompactness of w^2 follows from the Arzela–Ascoli theorem and from the inequality

$$|w^2(x^1, t_1) - w^2(x^2, t_2)| \leq \|F\|_\infty J(x^1, x^2; t_1, t_2) + c \|F\|_\infty \sqrt{t_2 - t_1}.$$

The precompactness of w^1 on the same sets is based on similar arguments and the following fact:

$$\int_{\mathbb{R}_+^3} |G^2(x, y, t)| dy \leq c \sqrt{\frac{t}{x_3^2 + t}} \leq c.$$

So, as usual, applying the diagonal Cantor procedure, we select a subsequence still denoted by $u^{(k)}$ that converges uniformly on sets $\bar{Q}_+(n)$ for any natural n . On the other hand, $u^{(k)}(0, d_k, 0) \rightarrow u(0, a, 0)$ so that $|u(0, a, 0)| = 1$. The latter actually implies that $a > 0$. $\square$

The last arguments in the proof of Proposition 5.1 allow us to prove Proposition 1.5.

Proof of Proposition 1.5. If we assume that T is a blow-up time, then, since $a \leq \varepsilon$, we have scenario 2 if one blows-up our solution v and gets a function u of Proposition 5.1. It is not so difficult to see that the modulus of continuity of u in $\bar{Q}_+(2)$ depends only on the integral modulus of continuity of the above Green functions. So, there exists $\delta_0 > 0$ such that $|u(0, 0, 0) - u(0, a, 0)| < 1/2$ provided $a \leq \min(1, \delta_0)$ but in fact $|u(0, 0, 0) - u(0, a, 0)| = |u(0, a, 0)| = 1$. It remains to take $\varepsilon = \min(1, \delta_0)$. $\square$

Proposition 5.2. *The limit function u is a mild bounded ancient solution.*

Proof. We have already proved that

$$(5.1) \quad u^{(k)} \rightarrow u$$

in $C(\bar{Q}_+(R))$.

We define $p^{1(k)}$ as a solution of the following Neumann boundary value problem:

$$\Delta p^{1(k)}(x, t) = -\operatorname{div} \operatorname{div} (u^{(k)} \otimes u^{(k)})(x, t)$$

for $(x, t) \in Q_-^+$ and

$$p_{,3}^{1(k)}(x', 0, t) = 0$$

for $(x', t) \in \mathbb{R}^2 \times]-\infty, 0[$. We extend $p^{1(k)}$ to the entire space $\mathbb{R}^3$ in the even way with respect to x_3 , and $u^{(k)}$ is assumed to be extended by zero. Then $H^{(k)} := u^{(k)} \otimes u^{(k)} + \mathbb{I} p^{1(k)} \in L_\infty(-\infty, 0; BMO(\mathbb{R}^3))$, so that

$$\sup_k (\|H^{(k)}\|_{L_\infty(BMO)} + \|u^{(k)}\|_\infty) = d < \infty.$$

We know that

$$u^{(k)} = \int_{\mathbb{R}_+^3} G(x, y, t - A) u^{(k)}(y, A) dy + \int_A^t \int_{\mathbb{R}_+^3} K(x, y, t - \tau) F^{(k)}(y, \tau) dy d\tau,$$

where $F^{(k)} = u^{(k)} \otimes u^{(k)}$. All the norms, being bounded by $\|u^{(k)}\|_\infty$, $\|F^{(k)}\|_\infty$, or $\|H^{(k)}\|_{L^\infty(BMO)}$ only, remain to be bounded for the limit functions.

We may transform the above formula by integration by parts in the second term on the right hand side:

$$\begin{aligned} u_i^{(k)}(x, t) &= \int_{\mathbb{R}_+^3} G_{ij}(x, y, t - A) u_j^{(k)}(y, A) dy \\ &\quad + \int_A^t \int_{\mathbb{R}_+^3} \frac{\partial G_{ij}}{\partial y_l}(x, y, t - \tau) (H_{jl}^{(k)}(y, \tau) - [H_{jl}^{(k)}]_{B((x', 0), a)}(\tau)) dy d\tau, \end{aligned}$$

where $B((x', 0), a)$ is a ball of radius $a = (x_3^2 + t - \tau)^{\frac{1}{2}}$ centered at the point $(x', 0)$.

Then we split $u^{(k)} = u_{(k)}^1 + u_{(k)}^2$ according to the decomposition of the Green function: $G = G^1 + G^2$, see (2.4). Next, let $u_{(k)}^2 = u_{(k)}^{2,1} + u_{(k)}^{2,2}$ so that

$$u_{i(k)}^{2,2}(x, t) = \int_A^t \int_{\mathbb{R}_+^3} \frac{\partial G_{ij}^2}{\partial y_l}(x, y, t - \tau) (H_{jl}^{(k)}(y, \tau) - [H_{jl}^{(k)}]_{B((x', 0), a)}(\tau)) dy d\tau.$$

From (2.8), it follows that

$$\begin{aligned} (5.2) \quad \left| \frac{\partial^{|\alpha|+|\gamma|} G^2}{\partial x^\alpha \partial y^\gamma}(x, y, t - A) \right| &\leq c(\alpha, \gamma) (t - A)^{-\frac{|\gamma|}{2}} (t - A + x_3^2)^{-\frac{|\alpha|}{2}} \\ &\quad \times \exp\left(-\frac{cy_3^2}{t - A}\right) (|x' - y'|^2 + y_3^2 + x_3^2 + t - A)^{-\frac{3}{2}}. \end{aligned}$$

Now, we have

$$\begin{aligned} |\nabla^{|\alpha|} u_{(k)}^{2,1}(x, t)| &\leq c(\alpha) (t - A + x_3^2)^{-\frac{|\alpha|}{2}} \\ &\quad \times \int_{\mathbb{R}_+^3} (|x' - y'|^2 + y_3^2 + x_3^2 + t - A)^{-\frac{3}{2}} \exp\left(-\frac{cy_3^2}{t - A}\right) |u^{(k)}(y, A)| dy \end{aligned}$$

and

$$\begin{aligned} |\nabla^{|\alpha|} u_{(k)}^{2,2}(x, t)| &\leq c(\alpha) \int_A^t (t - \tau)^{-\frac{1}{2}} (t - \tau + x_3^2)^{-\frac{|\alpha|}{2}} \\ &\quad \times \int_{\mathbb{R}_+^3} (|x' - y'|^2 + y_3^2 + x_3^2 + t - \tau)^{-\frac{3}{2}} \exp\left(-\frac{cy_3^2}{t - \tau}\right) |H^{(k)}(y, \tau) - [H^{(k)}]_{B((x', 0), a)}(\tau)| dy d\tau. \end{aligned}$$

So, we find

$$(5.3) \quad |\nabla^{|\alpha|} u_{(k)}^{2,1}(x, t)| \leq c(\alpha) d(x_3^2 + t - A)^{-\frac{|\alpha|}{2}} \leq c(\alpha) d(x_3^2 - A)^{-\frac{|\alpha|}{2}}$$

for all $(x, t) \in \mathbb{R}_+^3 \times]A/2, 0[$ and, by Lemma 6.1 with $\beta = \frac{t-\tau}{c}$ and $a^2 = x_3^2 + t - \tau$,

$$\begin{aligned} (5.4) \quad |\nabla^{|\alpha|} u_{(k)}^{2,2}(x, t)| &\leq c(\alpha) d \int_A^t (t - \tau)^{-\frac{1}{2}} (x_3^2 + t - \tau)^{-\frac{|\alpha|}{2}} d\tau \\ &\leq c(\alpha) d \int_0^{\sqrt{t-A}} \frac{dq}{(x_3^2 + q^2)^{\frac{|\alpha|}{2}}} \leq c(\alpha) d \int_0^{\sqrt{-A}} \frac{dq}{(x_3^2 + q^2)^{\frac{|\alpha|}{2}}} \end{aligned}$$

for all $(x, t) \in \mathbb{R}_+^3 \times]A, 0[$.

Next, we observe that $u_{(k)}^1$ is a solution of the following initial boundary value problem:

$$(5.5) \quad \partial_t u_{(k)}^1 - \Delta u_{(k)}^1 = -\operatorname{div} H^{(k)}$$

for any $(x, t) \in \mathbb{R}_+^3 \times]A, 0[$,

$$(5.6) \quad u_{(k)}^1(x', 0, t) = 0$$

for any (x', t) from $\mathbb{R}^2 \times]A, 0[$,

$$(5.7) \quad u_{(k)}^1(x, A) = u^{(k)}(x, A)$$

for all $x \in \mathbb{R}_+^3$.

Regarding $u_{(k)}^{2,2}(x, t)$, we can say the following: for an appropriate extension of it to the whole $\mathbb{R}^3$, we can claim that $\nabla u_{(k)}^{2,2} \in L_\infty(A, 0; BMO)$ and $\nabla u_{(k)}^{2,2} \in L_\infty(A, 0; L_{q,\text{unif}}(\mathbb{R}^3))$ for any finite $q \geq 1$.

Since $H^{(k)}$ is bounded in $L_\infty(A, 0; BMO(\mathbb{R}^3))$, by estimates (5.3) and (5.4), we have (from the energy inequality)

$$(5.8) \quad \sup_k \int_{A/2}^0 \int_{\mathbb{R}_+^3} \varphi^2 |\nabla u_{(k)}^1|^2 dx dt < \infty$$

for any test function $\varphi \in C_0^\infty(\mathbb{R}^3 \times]-A/2, A/2[)$. Then, from the equation for the pressure $p^{1(k)}$, we derive a similar estimate for the gradient of $p^{1(k)}$:

$$(5.9) \quad \sup_k \int_{A/2}^0 \int_{\mathbb{R}_+^3} \varphi^2 |\nabla p^{1(k)}|^2 dx dt < \infty$$

for any test function $\varphi \in C_0^\infty(\mathbb{R}^3 \times]-A/2, A/2[)$. Using coercive estimates for the heat equation, we then get

$$(5.10) \quad \sup_k \int_{A/2}^0 \int_{\mathbb{R}_+^3} \varphi^2 (|\partial_t u_{(k)}^1|^2 + |\nabla^2 u_{(k)}^1|^2) dx dt < \infty$$

for any test function $\varphi \in C_0^\infty(\mathbb{R}^3 \times]-A/2, A/2[)$.

Next, we may use a parabolic embedding theorem to show that

$$\sup_k \int_{A/2}^0 \int_{\mathbb{R}_+^3} \varphi^2 |\nabla u_{(k)}^1|^3 dx dt < \infty$$

for the same test function φ . An estimate of the same type is valid for the gradient of $p^{1(k)}$. Repeating these arguments several times, we can state that $\nabla u_{(k)}^1$ is bounded in the domains $\mathbb{R}_+^3 \times]3A/8, 0[$. The boundedness of $\nabla p^{1(k)}$ follows from local regularity and the equation $\Delta p^{1(k)} = -u_{i,j}^{(k)} u_{j,i}^{(k)}$, which is valid in $\mathbb{R}^3$ for appropriate extensions of $p^{1(k)}$ and $u_{i,j}^{(k)} u_{j,i}^{(k)}$ from $\mathbb{R}_+^3$ to $\mathbb{R}^3$. Summarizing all these estimates, taking the arbitrariness of A into account, and using a shift in time, we see that

$$(5.11) \quad \sup_k |\nabla u^{(k)}(x, t)| \leq c \ln(2 + x_3^{-1})$$

for $(x, t) \in Q_-^+$, and

$$(5.12) \quad \sup_k \sup_{(x,t) \in Q_-^+} |\nabla p^{1(k)}(x, t)| \leq C < \infty.$$

Now, we can go back to evaluation of the function $u_{(k)}^2$. By (5.11) and (5.12), we do not need integrating by parts in the expression for $u_{(k)}^{2,2}$ any more, so that

$$u_{i(k)}^{2,2}(x, t) = - \int_A^t \int_{\mathbb{R}_+^3} G_{ij}^2(x, y, t - \tau) \left[\frac{\partial}{\partial y_l} F_{jl}^{(k)}(y, \tau) + \frac{\partial}{\partial y_j} p^{1(k)}(y, \tau) \right] dy d\tau.$$

Next, by (5.11) and (5.12) and by estimates of the Green function G^2 , we find the following bound:

$$\begin{aligned} |\nabla u_{(k)}^{2,2}(x, t)| &\leq c \int_A^t \frac{d\tau}{(x_3^2 + t - \tau)^{\frac{3}{2}}} \int_0^\infty \frac{\exp\left(-\frac{cy_3^2}{t-\tau}\right)}{(x_3^2 + y_3^2 + t - \tau)^{\frac{1}{2}}} \ln(2 + 1/y_3) dy_3 \\ &\leq c \int_A^t \frac{d\tau}{(x_3^2 + t - \tau)^{\frac{3}{4}}} \int_0^\infty \frac{\exp\left(-\frac{cy_3^2}{t-\tau}\right)}{(x_3^2 + y_3^2 + t - \tau)^{\frac{1}{4}}} \ln(2 + 1/y_3) dy_3 \\ &\leq c \int_A^t \frac{d\tau}{(x_3^2 + t - \tau)^{\frac{3}{4}}} \left[\int_0^1 y_3^{-\frac{1}{2}} \ln(2 + 1/y_3) dy_3 + \int_1^\infty \exp\left(-\frac{cy_3^2}{t-\tau}\right) dy_3 \right] \\ &\leq c(1 + \sqrt{-A})(-A)^{\frac{1}{4}}. \end{aligned}$$

This implies that, in fact,

$$(5.13) \quad \sup_k \sup_{(x,t) \in Q_-^+} |\nabla u^{(k)}(x, t)| \leq C < \infty.$$

Using estimates (2.8), (5.2), (5.12), and bootstrap arguments, we show that

$$(5.14) \quad \sup_{(x,t) \in Q_-^+, x_3 \geq \delta} |\nabla^l u^{(k)}(x, t)| + |\nabla^{l+1} p^{1(k)}(x, t)| \leq C(l, \delta) < \infty$$

for any $l \geq 0$ and for any $\delta > 0$.

As to the derivatives in t , by (2.10), we find

$$\begin{aligned} &|\partial_t u_{(k)}^{2,2}(x, t)| \\ &\leq c \int_A^t \int_0^\infty \int_{\mathbb{R}^2} \frac{1}{t - \tau} \frac{1}{(|x' - y'|^2 + x_3^2 + y_3^2 + t - \tau)^{\frac{3}{2}}} \exp\left(-\frac{cy_3^2}{t-\tau}\right) |\operatorname{div} H^{(k)}(y, \tau)| dy d\tau \\ &\leq c \int_A^t \frac{1}{(t - \tau)^{\frac{1}{2}}} \frac{1}{(x_3^2 + t - \tau)^{\frac{1}{2}}} d\tau. \end{aligned}$$

So,

$$(5.15) \quad \sup_{x' \in \mathbb{R}^2} |\partial_t u_{(k)}^{2,2}(x', x_3, t)| \leq c \int_0^{t-A} \frac{1}{\vartheta^{\frac{1}{2}}} \frac{1}{(x_3^2 + \vartheta)^{\frac{1}{2}}} d\vartheta.$$

For the first term, we show in the same way that

$$|\partial_t u_{(k)}^{2,1}(x, t)| \leq c \int_{\mathbb{R}_+^3} |\partial_t G^2(x, y, t - A)| |u^{(k)}(y, A)| dy$$

and thus

$$(5.16) \quad |\partial_t u_{(k)}^{2,1}(x, t)| \leq \frac{1}{(t - A)^{\frac{1}{2}}} \frac{1}{(x_3^2 + t - A)^{\frac{1}{2}}}.$$

From equations (5.5), (5.15), and (5.16), we can deduce that

$$(5.17) \quad \sup_{(x,t) \in Q_-^+, x_3 \geq \delta} |\partial_t u^{(k)}(x, t)| \leq c(\delta).$$

After passing to the limit as $k \rightarrow \infty$, we see that u satisfies the standard integral identity with divergence free test functions. Then one can claim that there exists p^2 such that

$$(5.18) \quad \partial_t u - \Delta u + \nabla p^2 = -\operatorname{div} H$$

in Q_-^+ .

Obviously, $p^2(\cdot, t)$ is a harmonic function in a half-space whose gradient is bounded in t , in x' , and in $x_3 \geq \delta$ for any $\delta > 0$. Taking the limit in (5.5), we show that ∇p^2 satisfies

$$\partial_t u^2 - \Delta u^2 + \nabla p^2 = 0$$

in $\mathbb{R}_+^3 \times]A, 0[$. This, together with (5.3), (5.4) and (5.15), (5.16), implies that

$$\sup_{x' \in \mathbb{R}^2} |\nabla p^2(x', x_3, t)| \rightarrow 0$$

as $x_3 \rightarrow \infty$ for each $-\infty < t < 0$. Moreover, a more detailed analysis of the above estimates for Green functions allows us to state that

$$|\nabla p^2(x, t)| \leq c \ln(2 + 1/x_3)$$

for all $(x, t) \in Q_-^+$. Proposition 5.2 is proved. $\square$

Proof of Theorem 1.4 easily follows from the above arguments.

§6. APPENDIX I

Lemma 6.1. *Assume that numbers m and α_0 satisfy the condition*

$$(6.1) \quad 0 < \alpha_0 < \frac{m-1}{3}.$$

Let a and β be positive numbers, and let

$$I(a, \beta) := \int_{\mathbb{R}^3} \frac{|f(x) - [f]_{B(a)}|}{(|x|^2 + a^2)^{\frac{3}{2}}} \exp\left(-\frac{x_3^2}{\beta}\right) dx,$$

where $[f]_{B(R)}$ is the mean value of f over the ball $B(R)$ of radius R centered at the origin.

Let further $n = \frac{m}{m-1}$, $m_1 = \frac{m}{1+\alpha_0}$, and $n_1 = \frac{m_1}{m_1-1}$. Then

$$(6.2) \quad |I(a, \beta)| \leq c(m, \alpha_0) \|f\|_{BMO(\mathbb{R}^3)} a^{-\frac{3\alpha_0}{m}} \times \left(\sqrt{\beta} \int_0^\infty \exp(-u^2) \frac{du}{(u^2 \beta n_1/n + a^2)^{\frac{3n}{2n_1}-1}} \right)^{\frac{1}{n}}.$$

Proof. Without loss of generality, we may assume that $[f]_{B(a)} = 0$. We also put

$$K(x) := \frac{1}{(|x|^2 + a^2)^{\frac{3}{2}}} \exp\left(-\frac{x_3^2}{\beta}\right), \quad A := \|f\|_{BMO(\mathbb{R}^3)}.$$

Then, by the Hölder inequality,

$$\begin{aligned} |I(a, \beta)| &\leq \int_{\mathbb{R}^3} K^{\frac{1}{m_1} + \frac{1}{n_1}}(x) |f(x)| dx \\ &\leq \left(\int_{\mathbb{R}^3} K^{\frac{m}{m_1}}(x) |f(x)|^m dx \right)^{\frac{1}{m}} \left(\int_{\mathbb{R}^3} K^{\frac{n}{n_1}}(x) dx \right)^{\frac{1}{n}}. \end{aligned}$$

For the second factor on the right-hand side of the last inequality, we find

$$\begin{aligned} \int_{\mathbb{R}^3} K^{\frac{n}{n_1}}(x) dx &= \int_{-\infty}^\infty \exp\left(-\frac{x_3^2 n}{\beta n_1}\right) dx_3 \int_0^\infty \frac{\varrho d\varrho}{(\varrho^2 + x_3^2 + a^2)^{\frac{3n}{2n_1}}} \\ &= 2\pi \frac{1}{\frac{3n}{2n_1} - 1} \int_0^\infty \exp\left(-\frac{x_3^2 n}{\beta n_1}\right) dx_3 \frac{1}{(x_3^2 + a^2)^{\frac{3n}{2n_1}-1}}. \end{aligned}$$

Here, we have used condition (6.1) that ensures the inequality $\frac{3n}{2n_1} - 1 > 0$. After changing variables, we have

$$\left(\int_{\mathbb{R}^3} K^{\frac{n}{n_1}}(x) dx \right)^{\frac{1}{n}} \leq c(m, \alpha) \left(\sqrt{\beta} \int_0^\infty \exp(-u^2) \frac{du}{(u^2 \beta n_1 / n + a^2)^{\frac{3n}{2n_1} - 1}} \right)^{\frac{1}{n}}.$$

Next,

$$\begin{aligned} \int_{\mathbb{R}^3} K^{\frac{m}{m_1}}(x) |f(x)|^m dx &= \int_{B(a)} K^{\frac{m}{m_1}}(x) |f(x)|^m dx \\ &\quad + \sum_{k=0}^{\infty} \int_{B(a2^{k+1}) \setminus B(a2^k)} K^{\frac{m}{m_1}}(x) |f(x)|^m dx = I_1 + I_2. \end{aligned}$$

Obviously,

$$I_1 \leq c(m) A^m a^{3 - \frac{3m}{m_1}}.$$

For I_2 , we are going to use the fact that $m/m_1 = 1 + \alpha_0$ and the inequalities

$$|f|^m \leq 2^{m-1} (|f - [f]_{B(a2^{k+1})}|^m + |[f]_{B(a2^{k+1})}|^m)$$

and

$$|[f]_{B(a2^{k+1})}| \leq c(k+1)A.$$

The latter inequality can be found for example in [16, p. 141]. Then, we have

$$\begin{aligned} I_2 &\leq c(m) \sum_{k=0}^{\infty} \frac{|B(a2^{k+1})|}{(a2^k)^{3(1+\alpha_0)}} \frac{1}{|B(a2^{k+1})|} \int_{B(a2^{k+1})} |f - [f]_{B(a2^{k+1})}|^m \\ &\quad + c(m) \sum_{k=0}^{\infty} \frac{|B(a2^{k+1})|}{(a2^k)^{3(1+\alpha_0)}} |[f]_{B(a2^{k+1})}|^m \\ &\leq c(m) A^m \sum_{k=0}^{\infty} \frac{1}{(a2^k)^{3\alpha_0}} (k+1)^m \leq c(m, \alpha_0) a^{-3\alpha_0} A^m. \end{aligned}$$

This completes the proof of Lemma 6.1. □

§7. APPENDIX II

Consider the following problem:

$$\partial_t u - \Delta u = -\operatorname{div} g$$

in $\mathcal{Q}(a) = Q(a) + (0, a^2)$. Assuming that u and g are sufficiently smooth in $\mathcal{Q}(a)$, we wish to estimate the modulus of continuity of u in the closure of the set $B(a/2) \times]3(a/2)^2, a^2[$ in terms of $\|u\|_{\infty, \mathcal{Q}(a)}$ and $\|g - [g]_{B(a)}\|_{L_{m,\infty}(\mathcal{Q}(a))}$ for sufficiently large $m \geq 1$.

First, we take a positive cut-off function $\varphi \in C_0^\infty(B(a) \times]0, 2a^2])$ and extend it by zero. We let $w = \varphi u$. This function solves the following Cauchy problem:

$$\partial_t w - \Delta w = -\operatorname{div}(\varphi f) + f \nabla \varphi + w(\partial_t \varphi + \Delta \varphi) - 2 \operatorname{div}(u \nabla \varphi)$$

in $\mathbb{R}^3 \times]0, a^2[$, where $f(x, t) = g(x, t) - [g]_{B(a)}(t)$, and $w(x, 0) = 0$ for $x \in \mathbb{R}^3$.

We split w into two parts $w = w^1 + w^2$, where

$$\partial_t w^1 - \Delta w^1 = -\operatorname{div}(\varphi f) - 2 \operatorname{div}(u \nabla \varphi)$$

in $\mathbb{R}^3 \times]0, a^2[$ with $w^1(\cdot, 0) = 0$ in $\mathbb{R}^3$, and

$$\partial_t w^2 - \Delta w^2 = f \nabla \varphi + w(\partial_t \varphi + \Delta \varphi)$$

in $\mathbb{R}^3 \times]0, a^2[$ with $w^2(\cdot, 0) = 0$ in $\mathbb{R}^3$. The second part can be estimated with the help of coercive estimates:

$$\|w^2\|_{W_{m,1}^{2,1}(\mathbb{R}^3 \times]0, a^2[)} \leq C(a, m, \|u\|_{\infty, \mathcal{Q}(a)}, \|f\|_{L_{m,\infty}(\mathcal{Q}(a))}).$$

For sufficiently large m , they give an estimate for a Hölder norm of u ; thus, we get a required estimate for the modulus of continuity of w^2 .

For the first part, we make use of the solution formula

$$\begin{aligned} w^1(x, t) &= - \int_0^t \int_{\mathbb{R}^3} \Gamma(x - y, t - \tau) (\operatorname{div}(\varphi f)(y, \tau) + 2 \operatorname{div}(u \nabla \varphi))(y, \tau) dy d\tau \\ &= - \int_0^t \int_{\mathbb{R}^3} \nabla_x \Gamma(x - y, t - \tau) \cdot ((\varphi f)(y, \tau) + 2(u \nabla \varphi)(y, \tau)) dy d\tau. \end{aligned}$$

We first assume that $m > 2012$ and observe that the following simple estimate is true:

$$\int_{\mathbb{R}^3} |K(x - y, t - \tau)| dy \leq \frac{c}{\sqrt{t - \tau}},$$

where $K(x - y, t) = \nabla_x \Gamma(x - y, t)$. Now, we wish to show that, for a given positive ε , there exists a positive $\delta(m, a)$ such that if $(x^1, t_1), (x^2, t_2) \in B(a) \times]0, a^2[$ with $t_2 > t_1$ and $|x^1 - x^2| + t_2 - t_1 < \delta$, then

$$\int_0^{a^2} \left(\int_{\mathbb{R}^3} |K(x^2 - y, t_2 - t_1 + \vartheta) - K(x^1 - y, \vartheta)| dy \right)^{\frac{1}{m'}} \frac{1}{\vartheta^{\frac{2}{m}}} d\vartheta < \varepsilon.$$

Assume that this is false. Then there exists $\varepsilon_0 > 0$ and sequences

$$(x^{1,n}, t_{1,n}), (x^{2,n}, t_{2,n}) \in B(a) \times]0, a^2[$$

such that $t_{2,n} > t_{1,n}$ and $|x^{1,n} - x^{2,n}| + t_{2,n} - t_{1,n} \rightarrow 0$, but

$$\int_0^{a^2} \left(\int_{\mathbb{R}^3} |K(x^{2,n} - y, t_{2,n} - t_{1,n} + \vartheta) - K(x^{1,n} - y, \vartheta)| dy \right)^{\frac{1}{m'}} \frac{1}{\vartheta^{\frac{2}{m}}} d\vartheta \geq \varepsilon_0.$$

Indeed, for any positive ϑ ,

$$\int_{\mathbb{R}^3} |K(x^{2,n} - y, t_{2,n} - t_{1,n} + \vartheta) - K(x^{1,n} - y, \vartheta)| dy \rightarrow 0$$

by the Lebesgue theorem. But then

$$\int_0^{a^2} \left(\int_{\mathbb{R}^3} |K(x^{2,n} - y, t_{2,n} - t_{1,n} + \vartheta) - K(x^{1,n} - y, \vartheta)| dy \right)^{\frac{1}{m'}} \frac{1}{\vartheta^{\frac{2}{m}}} d\vartheta \rightarrow 0,$$

because

$$\left(\int_{\mathbb{R}^3} |K(x^{2,n} - y, t_{2,n} - t_{1,n} + \vartheta) - K(x^{1,n} - y, \vartheta)| dy \right)^{\frac{1}{m'}} \frac{1}{\vartheta^{\frac{2}{m}}} \leq \frac{c}{\vartheta^{\frac{1}{2m'}}} \frac{1}{\vartheta^{\frac{2}{m}}} \leq \frac{c}{\vartheta^{\frac{1}{2} + \frac{3}{2m}}}.$$

Since the function on the right-hand side is integrable under our assumption on m , we arrive at a contradiction again, by the Lebesgue dominated convergence theorem. The rest goes as follows:

$$\begin{aligned} &|w^1(x^2, t_2) - w^1(x^1, t_1)| \\ &\leq \left| \int_{t_1}^{t_2} \int_{\mathbb{R}^3} K(x_2 - y, t_2 - \tau) \cdot ((\varphi f)(y, \tau) + 2(u \nabla \varphi)(y, \tau)) dy d\tau \right| \\ &+ \left| \int_0^{t_1} \int_{\mathbb{R}^3} (K(x_2 - y, t_2 - \tau) - K(x_1, t_1 - \tau)) ((\varphi f)(y, \tau) + 2(u \nabla \varphi)(y, \tau)) dy d\tau \right|. \end{aligned}$$

Then we apply the Hölder inequality to obtain

$$\begin{aligned}
|w^1(x^2, t_2) - w^1(x^1, t_1)| &\leq \int_{t_1}^{t_2} \left(\int_{\mathbb{R}^3} |K(x_2 - y, t_2 - \tau)| dy \right)^{\frac{1}{m'}} \\
&\quad \times \left(\int_{B(a)} |K(x_2 - y, t_2 - \tau)| (|f|^m + |u|^m) dy d\tau \right)^{\frac{1}{m}} \\
&\quad + \int_0^{a^2} \left(\int_{\mathbb{R}^3} |K(x_2 - y, t_2 - \tau) - K(x_1, t_1 - \tau)| dy \right)^{\frac{1}{m'}} \\
&\quad \times \left(\int_{B(a)} |K(x_2 - y, t_2 - \tau) - K(x_1, t_1 - \tau)| (|f|^m + |u|^m) dy d\tau \right)^{\frac{1}{m}} \\
&\leq C(a, m, \|u\|_{\infty, \mathcal{Q}(a)}, \|f\|_{L_{m, \infty}(\mathcal{Q}(a))}) \left(\int_{t_1}^{t_2} \frac{1}{(t_2 - \tau)^{\frac{1}{2} + \frac{3}{2m}}} d\tau \right. \\
&\quad \left. + \int_0^{a^2} \left(\int_{\mathbb{R}^3} |K(x^2 - y, t_2 - t_1 + \vartheta) - K(x^1 - y, \vartheta)| dy \right)^{\frac{1}{m'}} \frac{1}{\vartheta^{\frac{2}{m}}} d\vartheta \right) \\
&= C(a, m, \|u\|_{\infty, \mathcal{Q}(a)}, \|f\|_{L_{m, \infty}(\mathcal{Q}(a))}) \left((t_2 - t_1)^{\frac{1}{2} - \frac{3}{2m}} d \right. \\
&\quad \left. + \int_0^{a^2} \left(\int_{\mathbb{R}^3} |K(x^2 - y, t_2 - t_1 + \vartheta) - K(x^1 - y, \vartheta)| dy \right)^{\frac{1}{m'}} \frac{1}{\vartheta^{\frac{2}{m}}} d\vartheta \right).
\end{aligned}$$

Together with the above statements, this gives a required estimate for the modulus of continuity in the closure of $B(a/2) \times]3(a/2)^2, a^2[$.

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